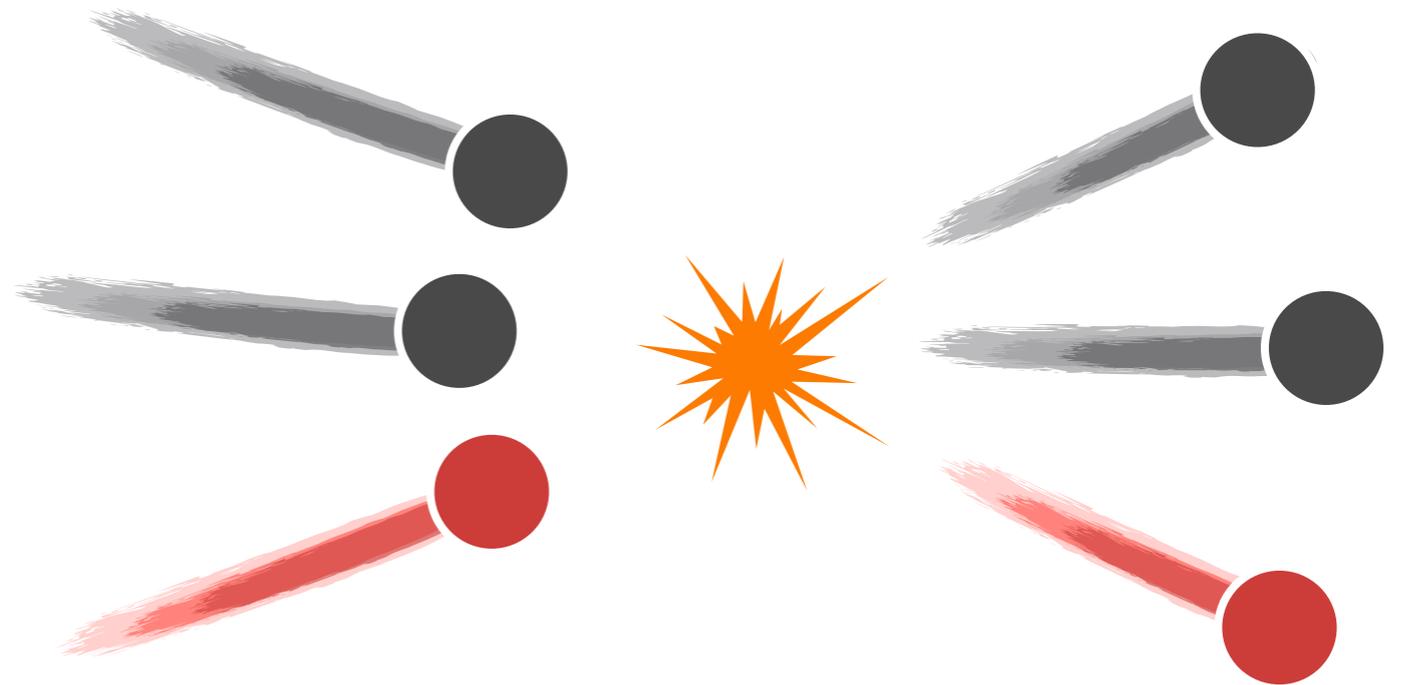


# Few-Body Dynamics from QCD

Andrew W. Jackura

The 9th International Conference on Quarks and Nuclear Physics (QNP2022)

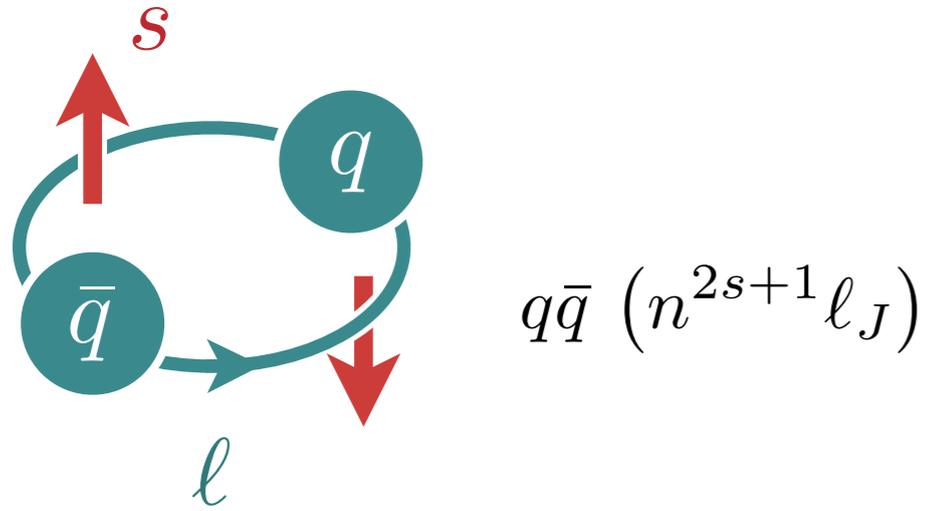
Thursday, September 8th, 2022



# The Hadron Spectrum

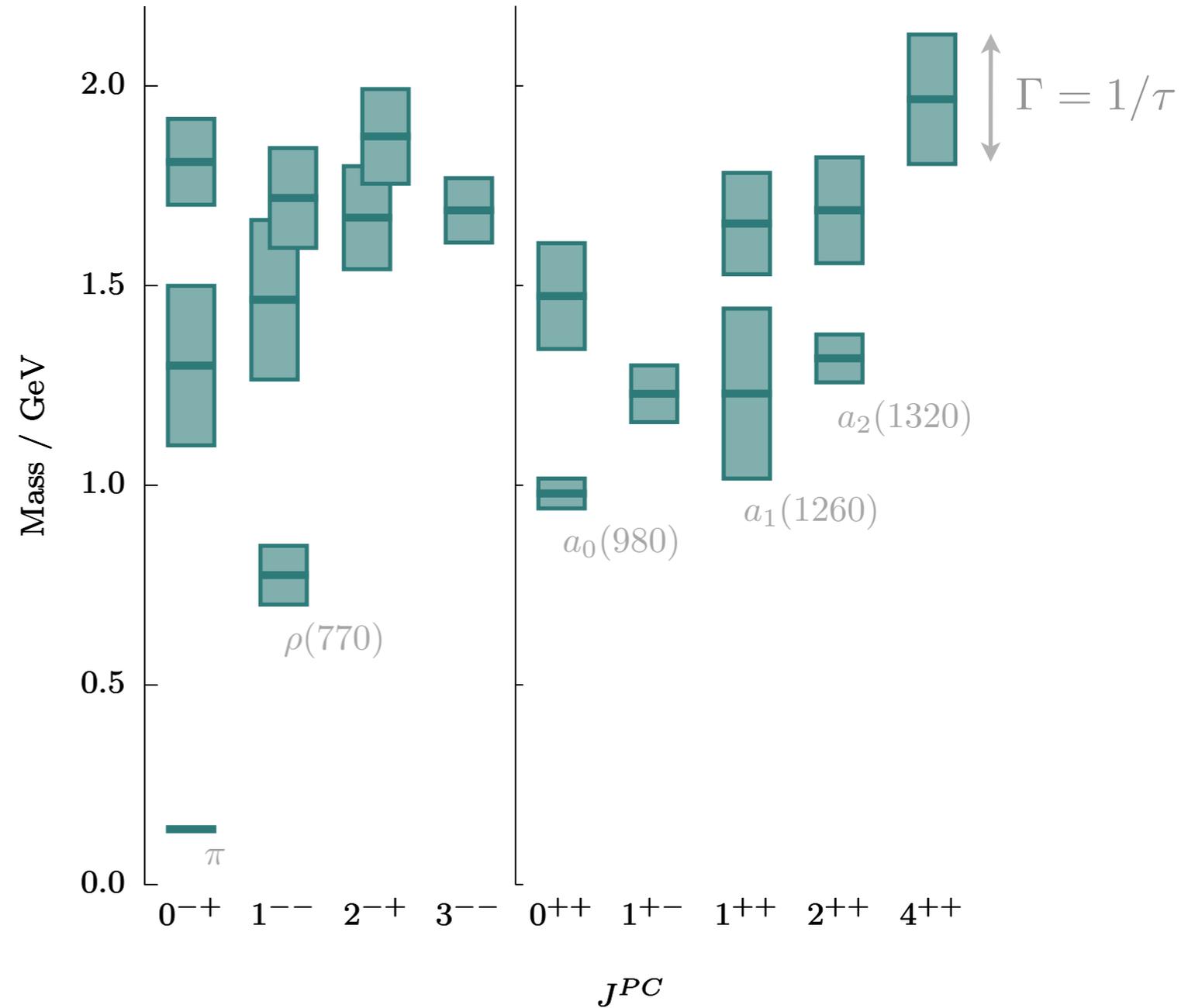
Quark models give gross structure of the hadron spectrum

*e.g. light isovector mesons*



$$q\bar{q} \ (n^{2s+1} l_J)$$

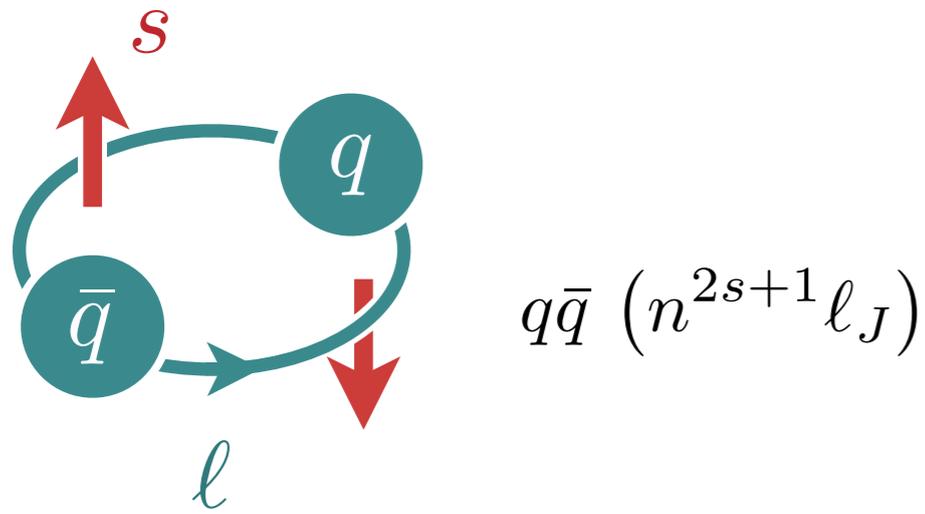
	$J^{PC}$	
	singlet	triplet
$l = 0$	$0^{-+}$	$1^{--}$
$l = 1$	$1^{+-}$	$(0, 1, 2)^{++}$
$l = 2$	$2^{-+}$	$(1, 2, 3)^{--}$
$\vdots$	$\vdots$	$\vdots$



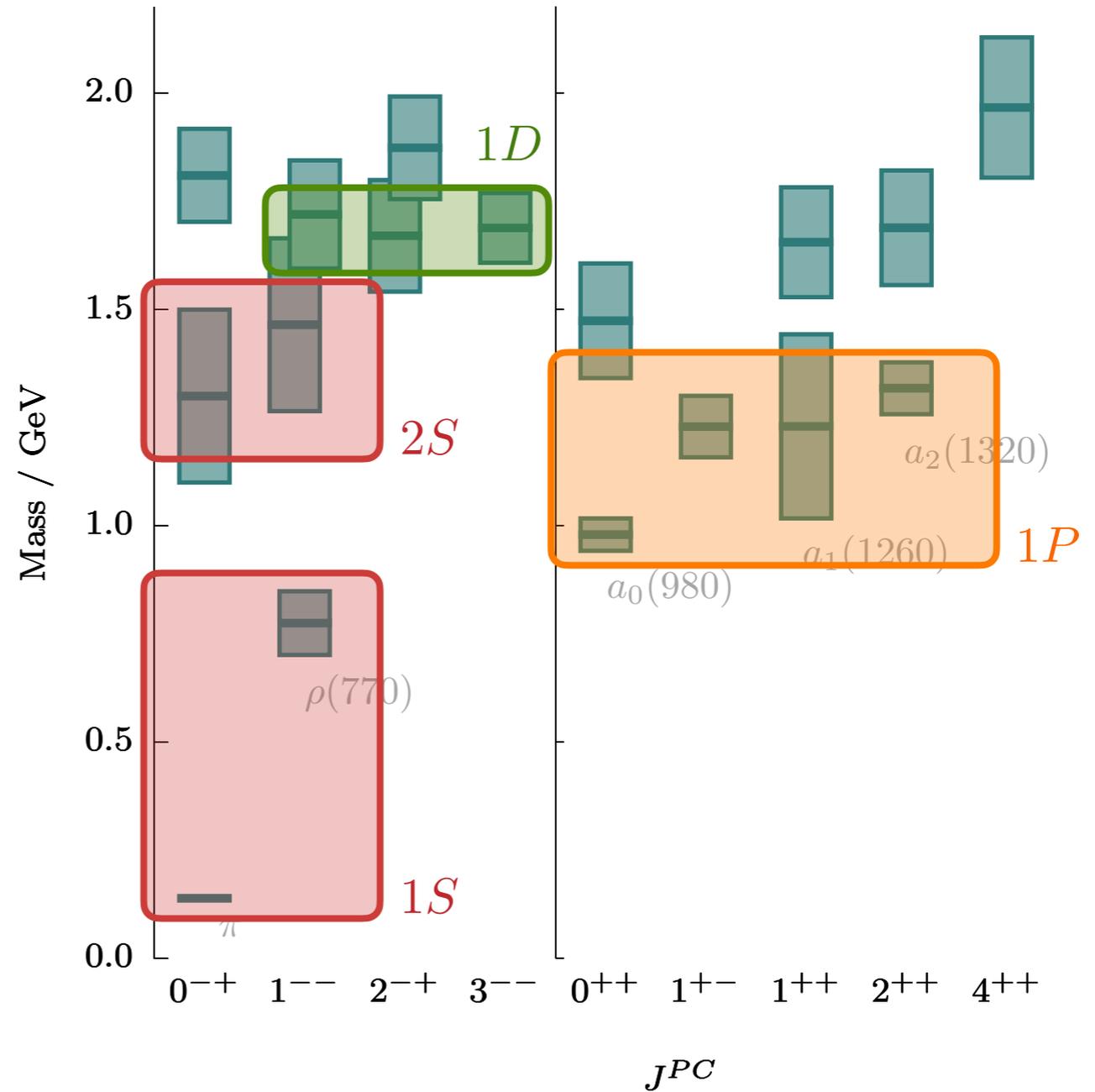
# The Hadron Spectrum

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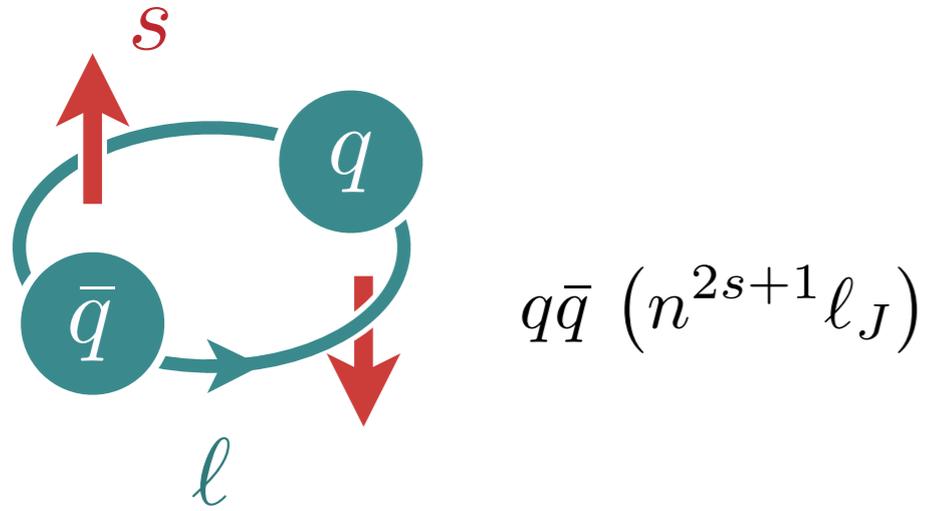
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# The Hadron Spectrum

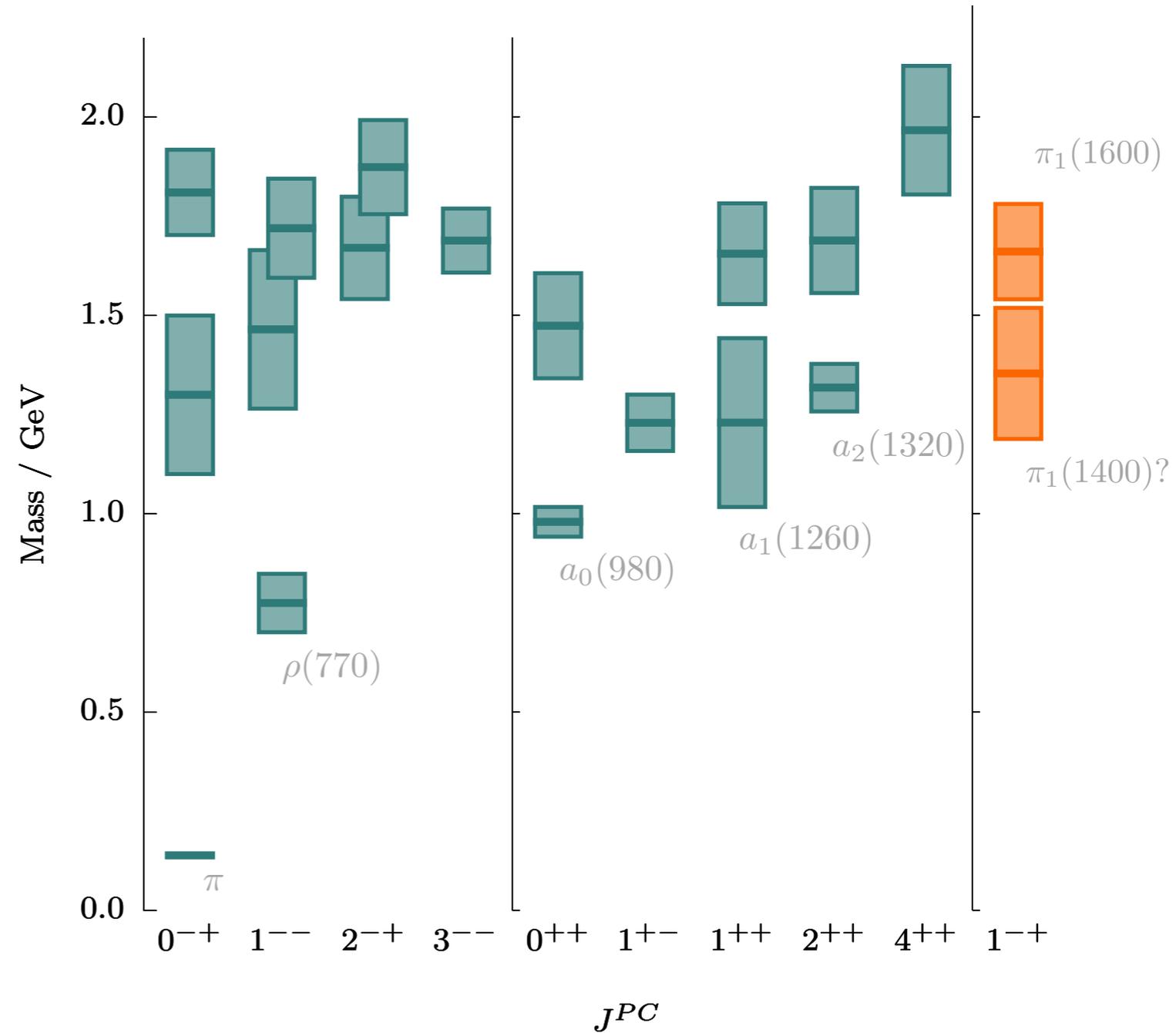
Quark models give gross structure of the hadron spectrum

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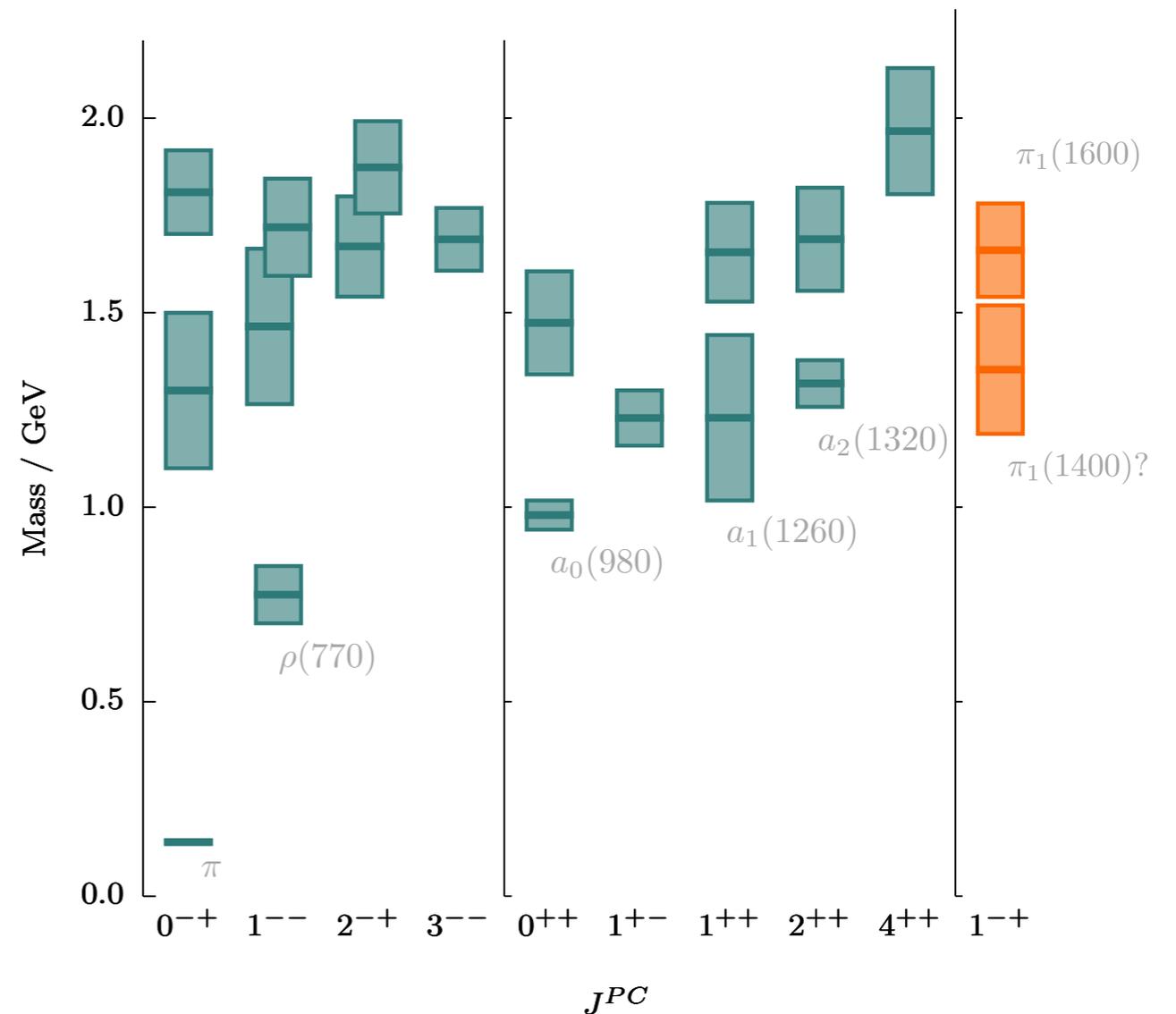
**Forbidden quantum numbers :**  $0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, \dots$

# The Hadron Spectrum

How to connect QCD to the hadron spectrum?

- Need to understand how to quantify what the hadrons are in nature
- Need to find non-perturbative approach to access these hadrons *rigorously*

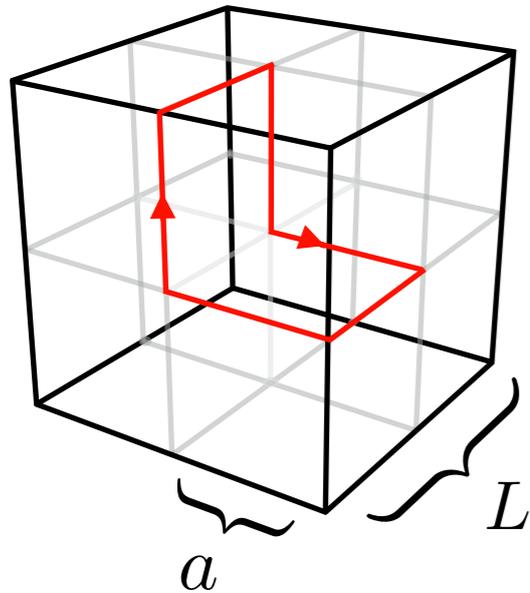
$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\not{D} - m_f) \psi_f - \frac{1}{2} \text{tr} (\mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu})$$



# Few-Body Physics from QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



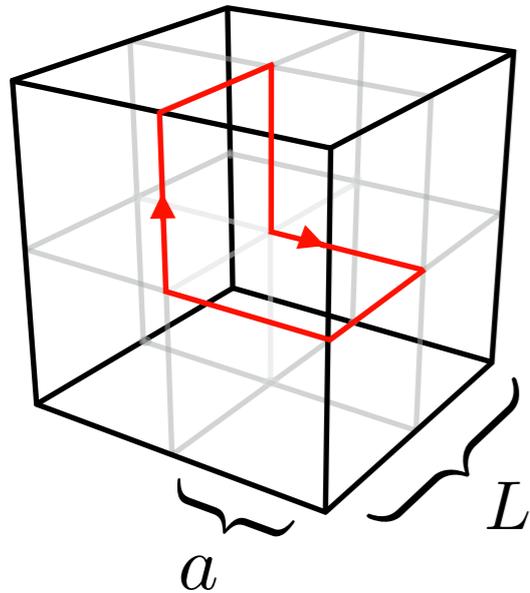
$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_{\mu} e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_{\mu})}$$

- Euclidean spacetime,  $t \rightarrow -i\tau$
- Finite volume,  $L$
- Discrete spacetime,  $a$
- Heavier than physical quark mass,  $m > m_{\text{phys}}$ .

# Few-Body Physics from QCD

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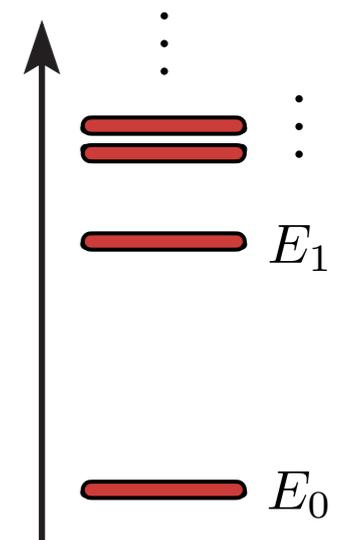


$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

- Euclidean spacetime,  $t \rightarrow -i\tau$
- Finite volume,  $L$
- Discrete spacetime,  $a$
- Heavier than physical quark mass,  $m$

Correlation functions yield discrete spectrum

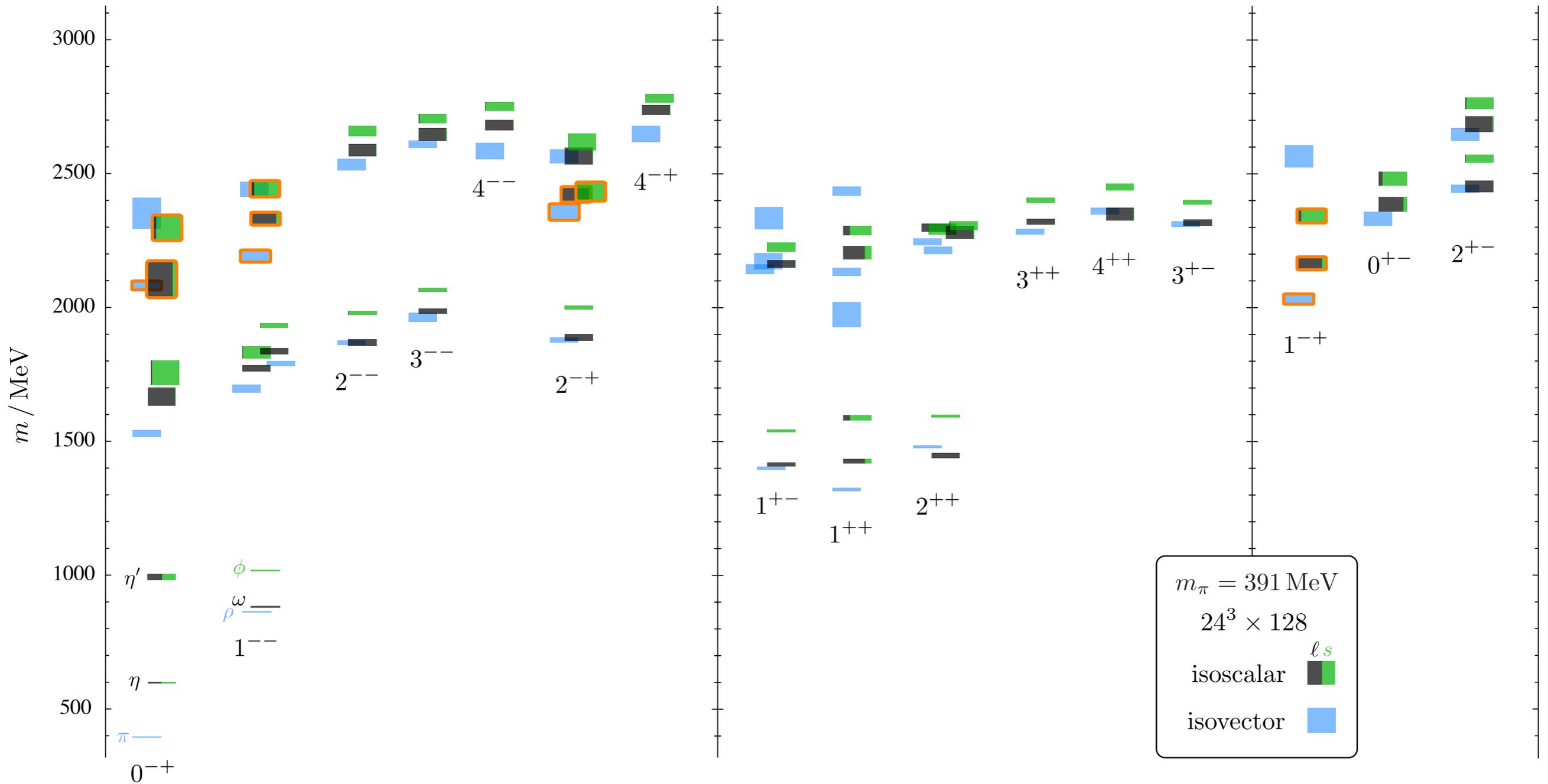
$$\langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle = \sum_{\mathbf{n}} |\langle 0 | \mathcal{O} | \mathbf{n} \rangle|^2 e^{-E_{\mathbf{n}} \tau}$$



# Few-Body Physics from QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

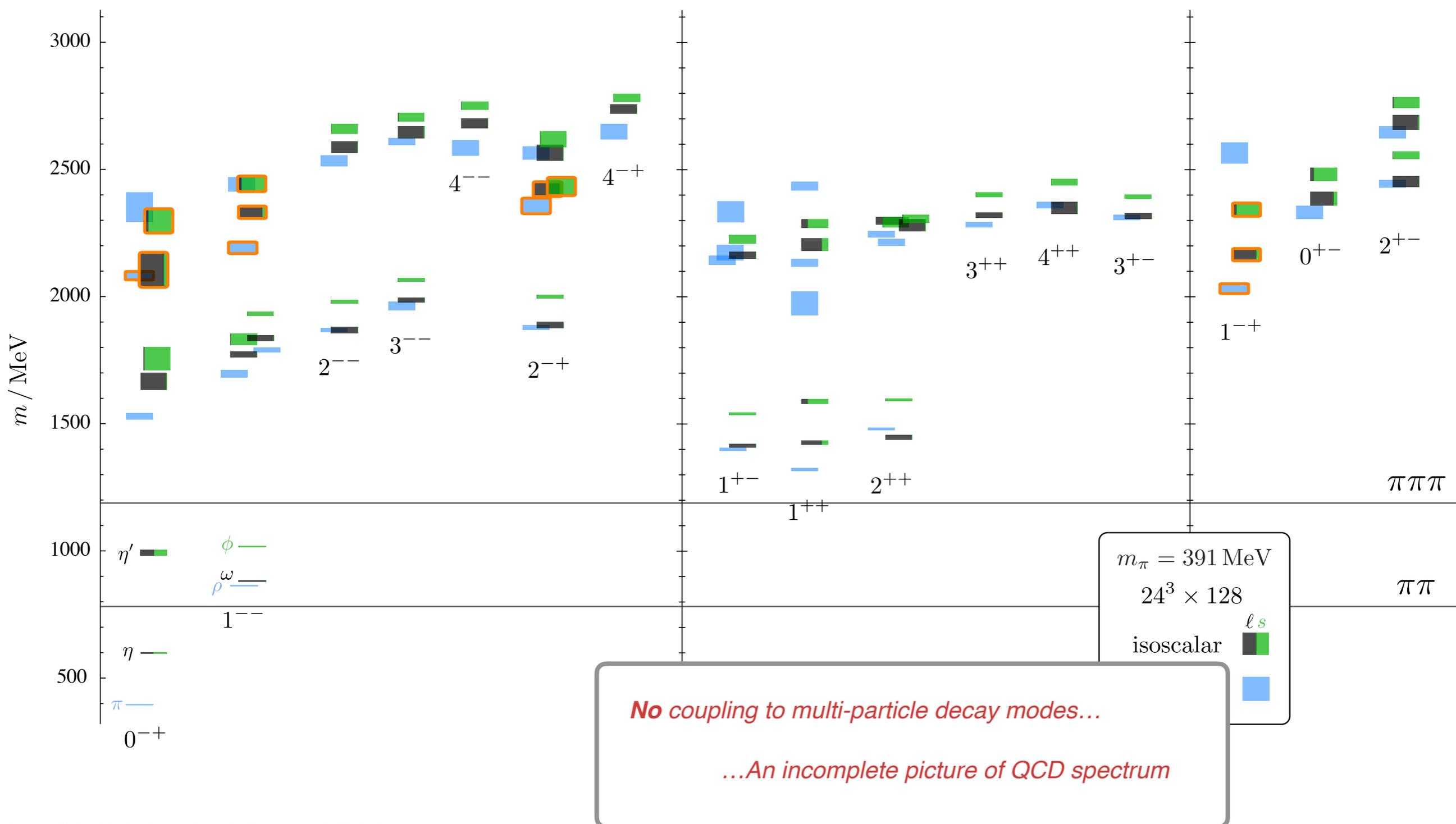
- Numerically evaluate QCD path integral via Monte Carlo sampling



# Few-Body Physics from QCD

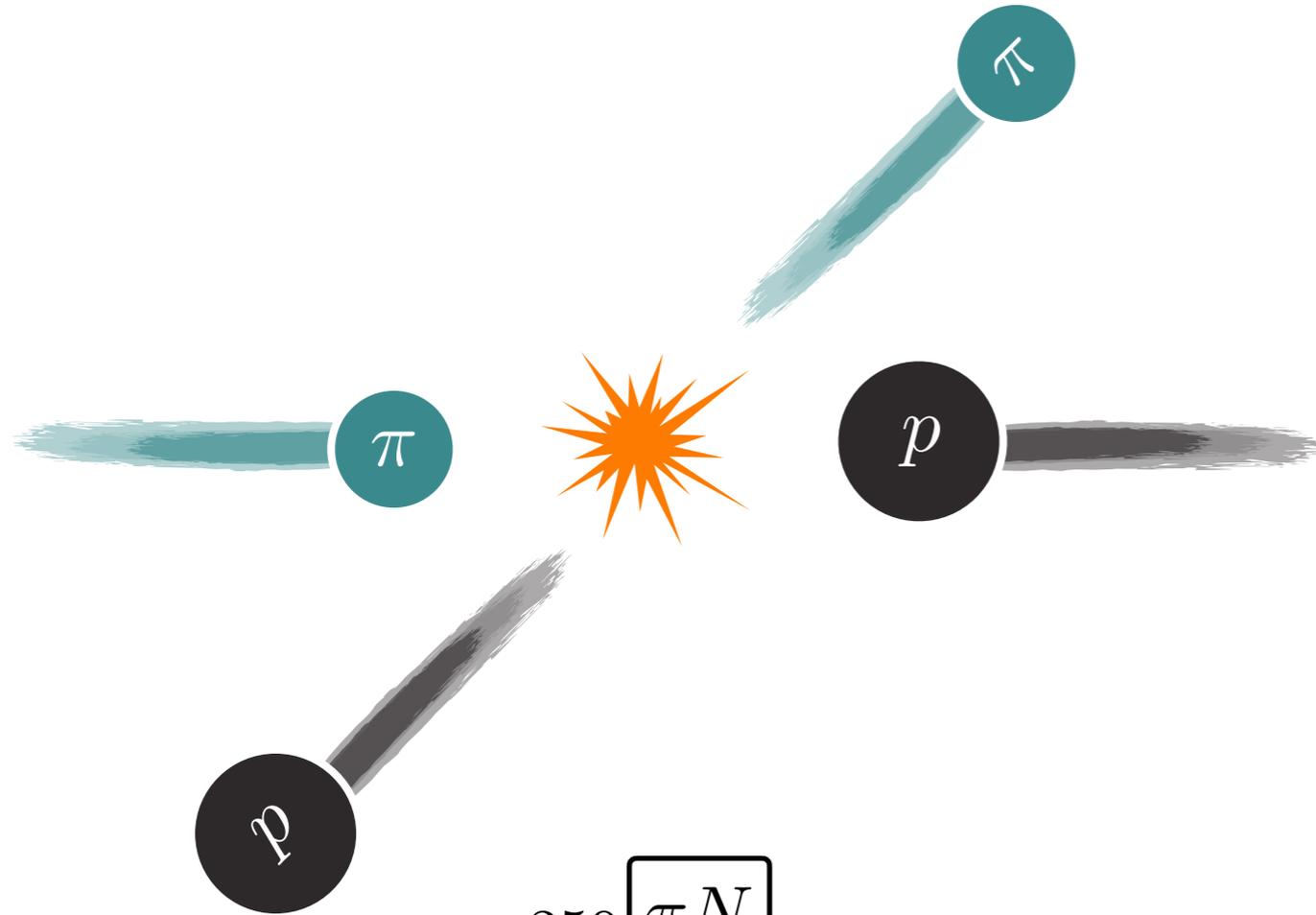
Lattice QCD offers a systematic approach to compute hadrons from QCD

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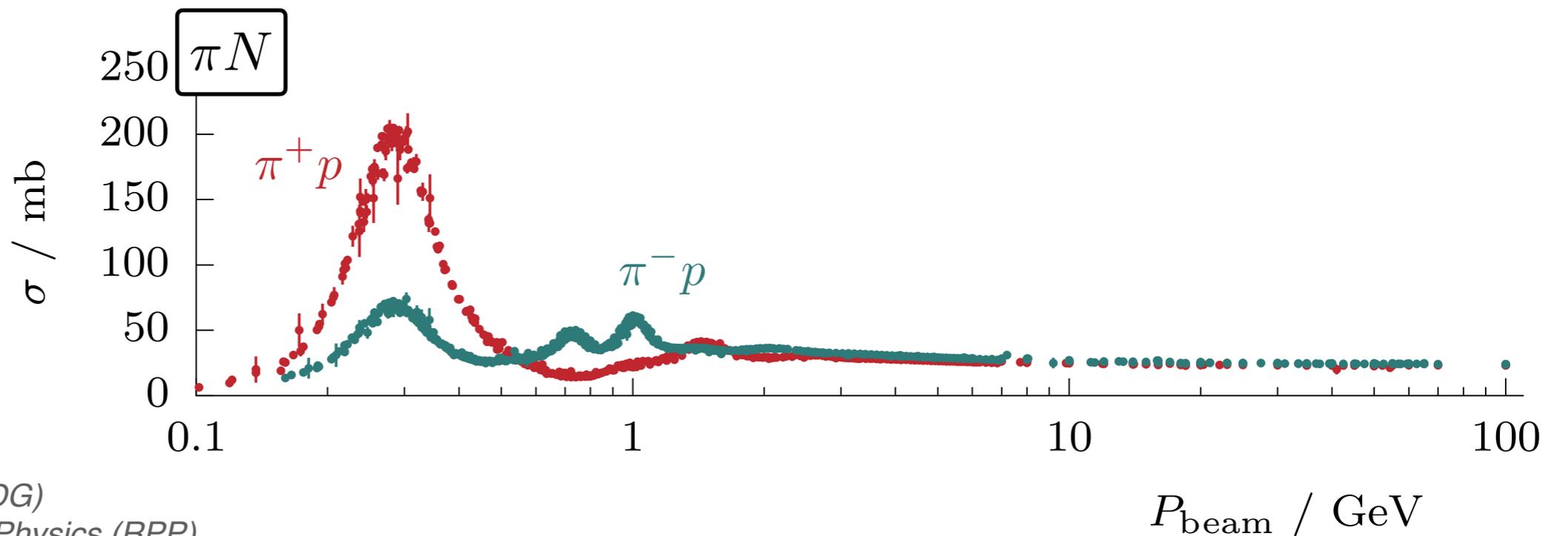


# Few-Body Physics from QCD

All our knowledge of the hadrons comes from *scattering experiments*



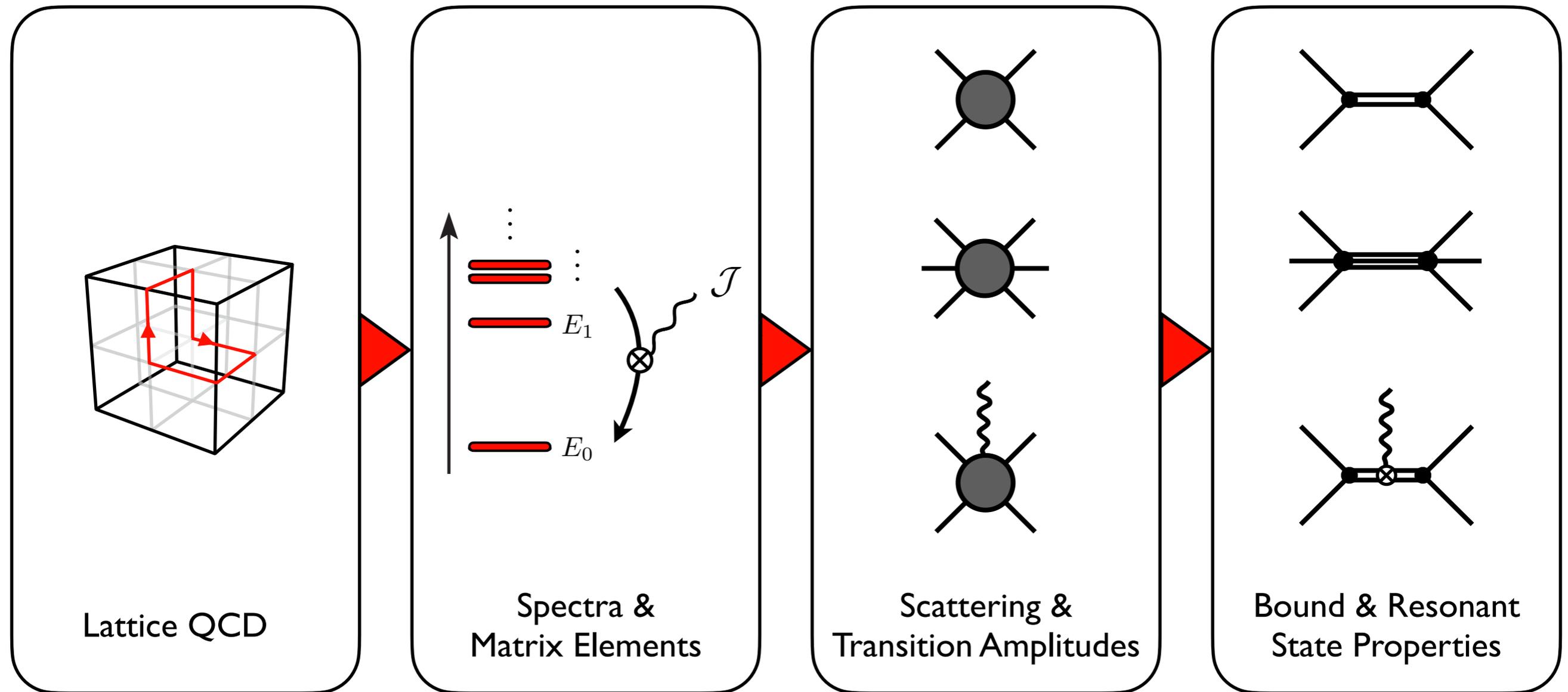
How to quantify such a process with Lattice QCD?



# Few-Body Physics from QCD

Path to few-body physics from QCD

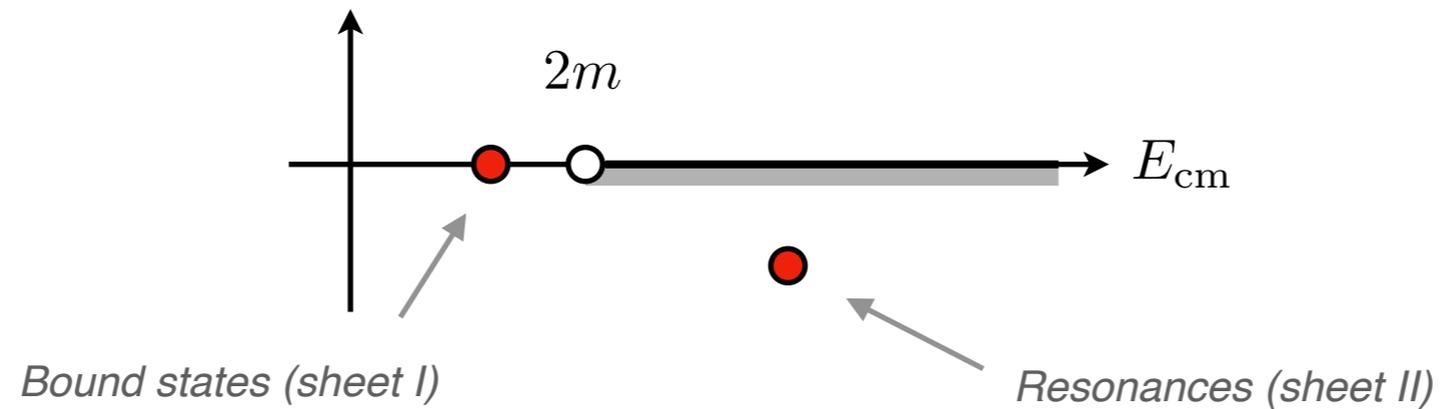
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Scattering Theory & QCD Spectrum

Resonances & Bound states are pole singularities of scattering amplitudes

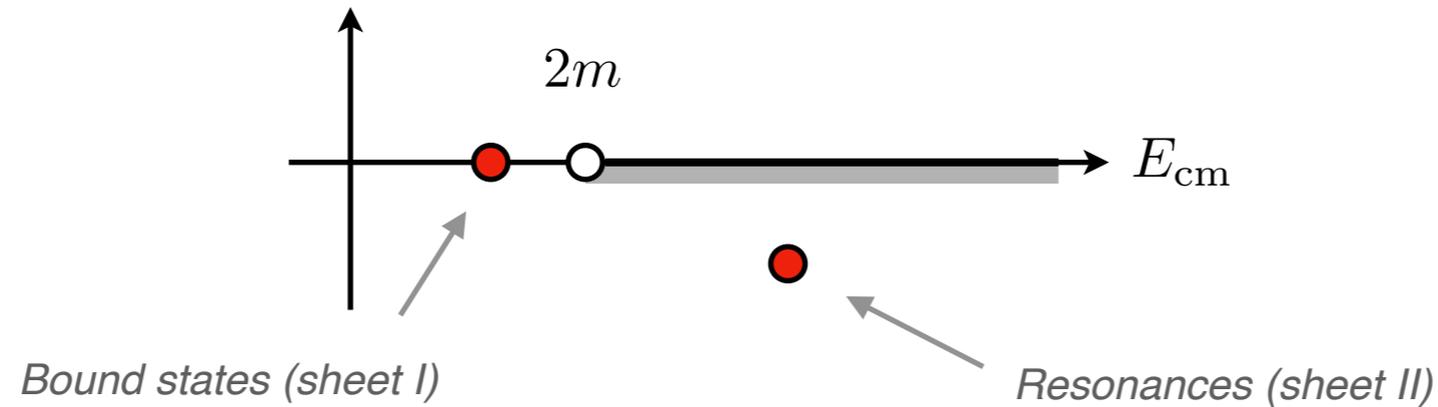
$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$



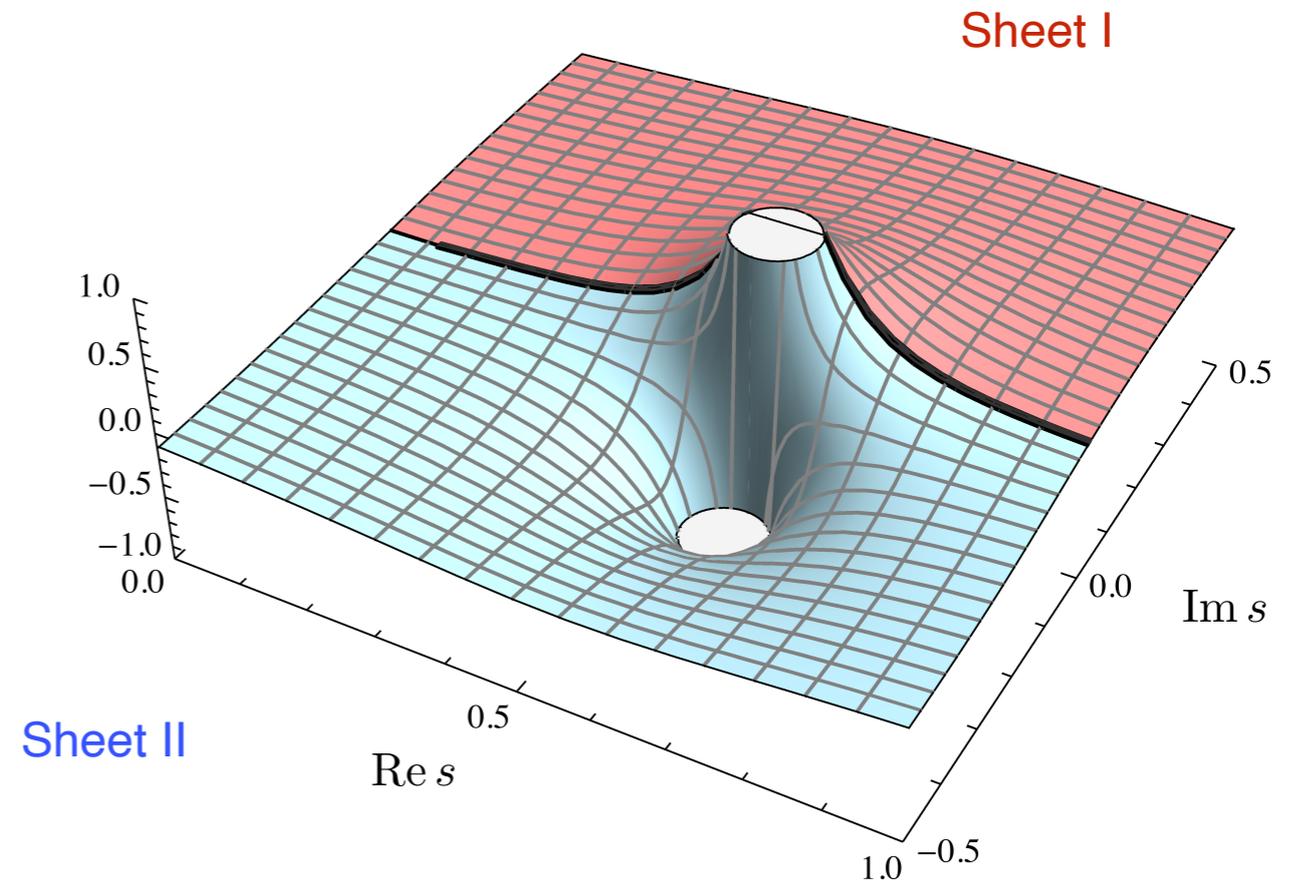
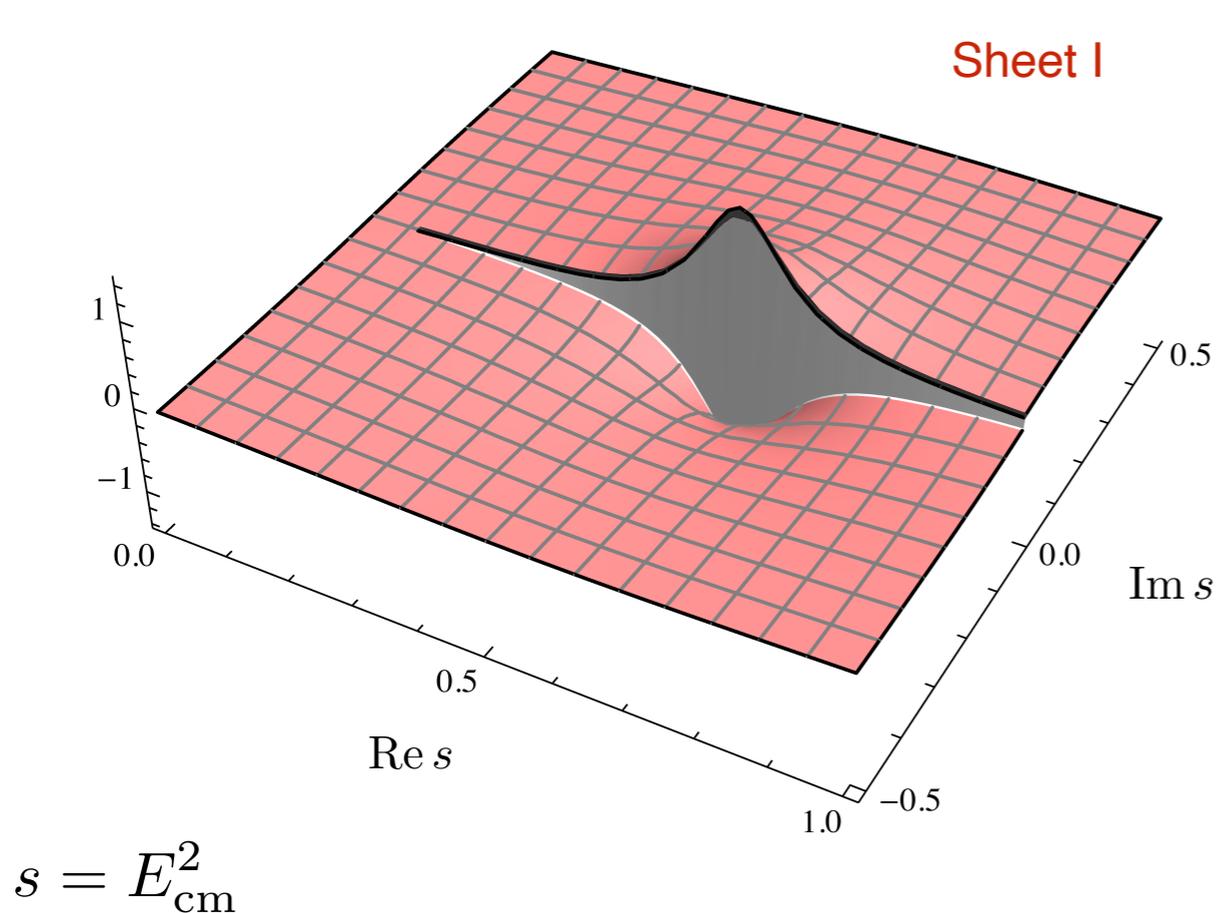
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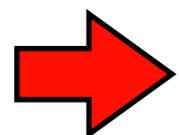
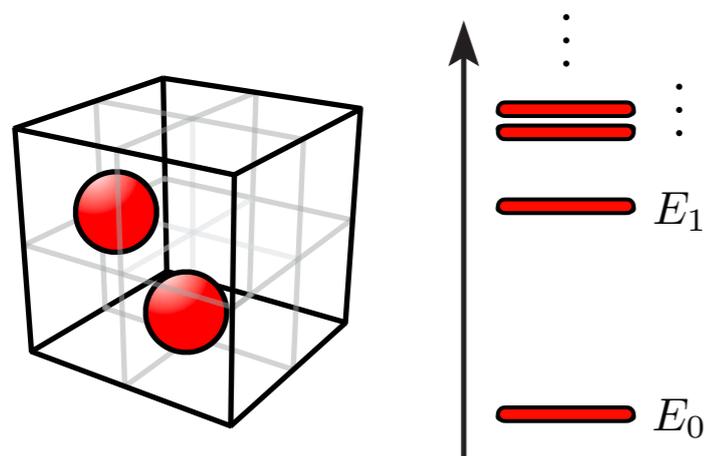


e.g., Narrow resonance

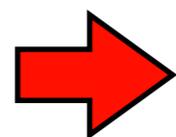


# Connecting Scattering Physics to QCD

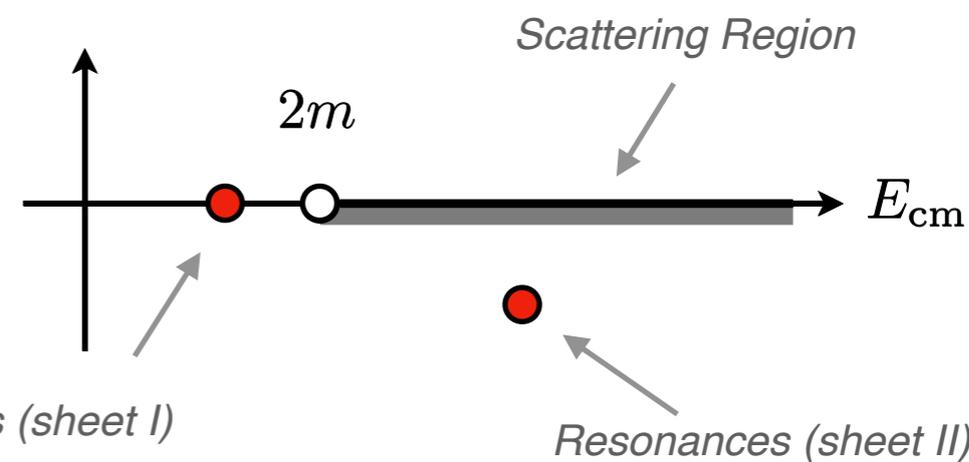
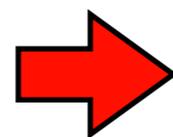
Employing scattering theory and EFTs to all-orders  
connects lattice QCD spectra to scattering observables



$$\det ( 1 + \mathcal{K}_2 F_L )_{E=E_n} = 0$$



$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$

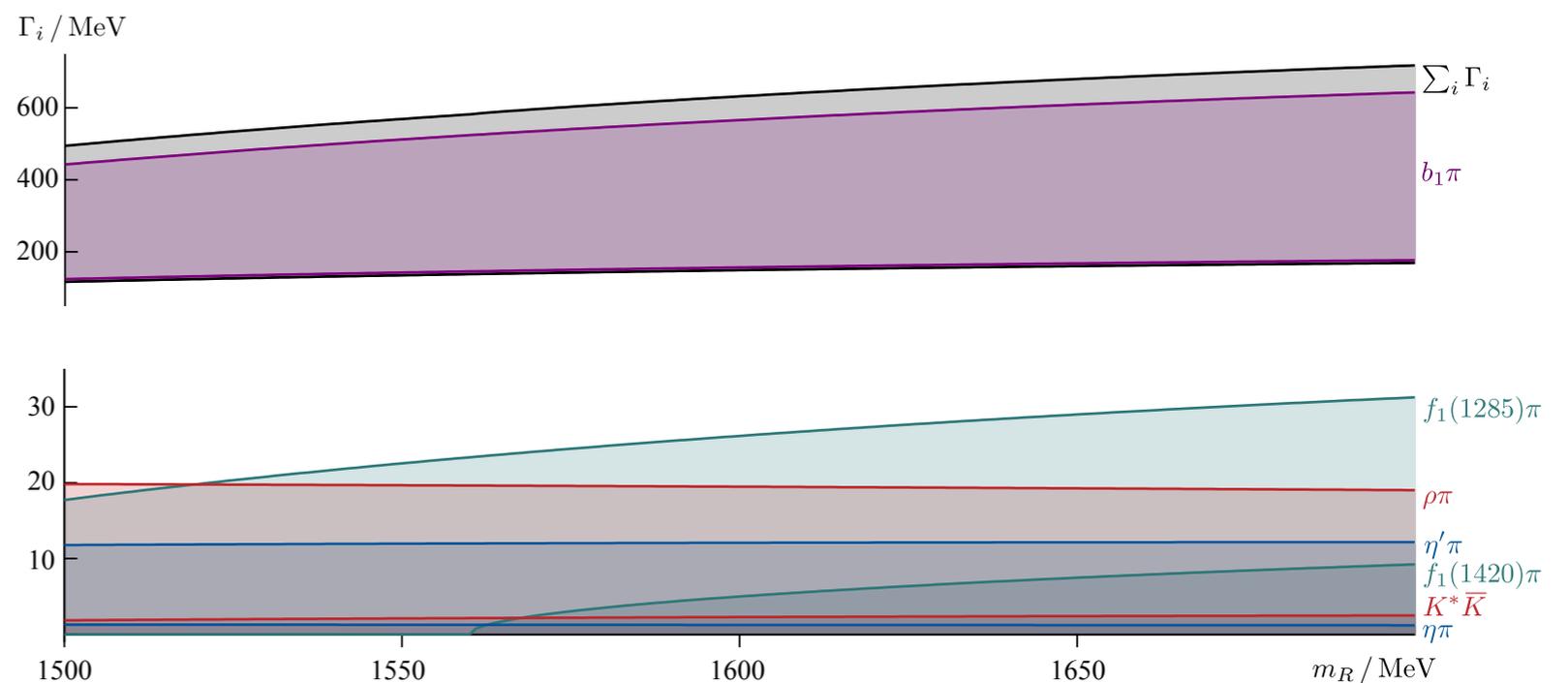
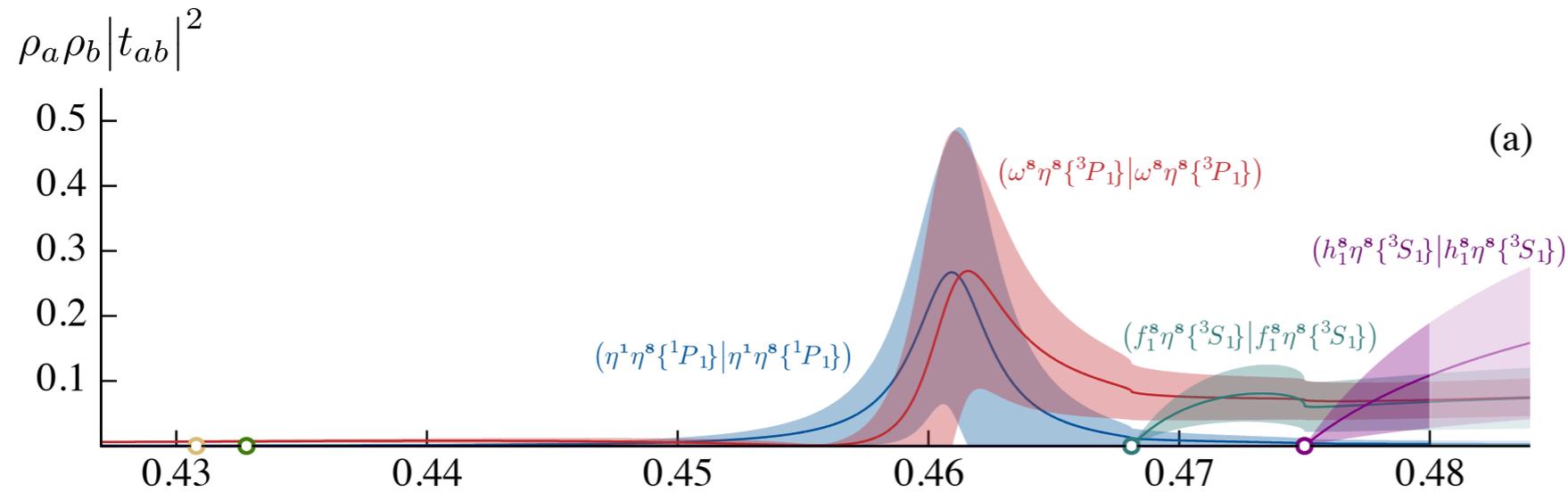


M. Lüscher  
Commun.Math.Phys. **105**, 153 (1986)  
Nucl.Phys. **B354**, 531 (1991)

Many others...

# Connecting Scattering Physics to QCD

First determination of hybrid candidate,  $J^{PC} = 1^{-+}$ ,  $m_\pi \sim 700$  MeV

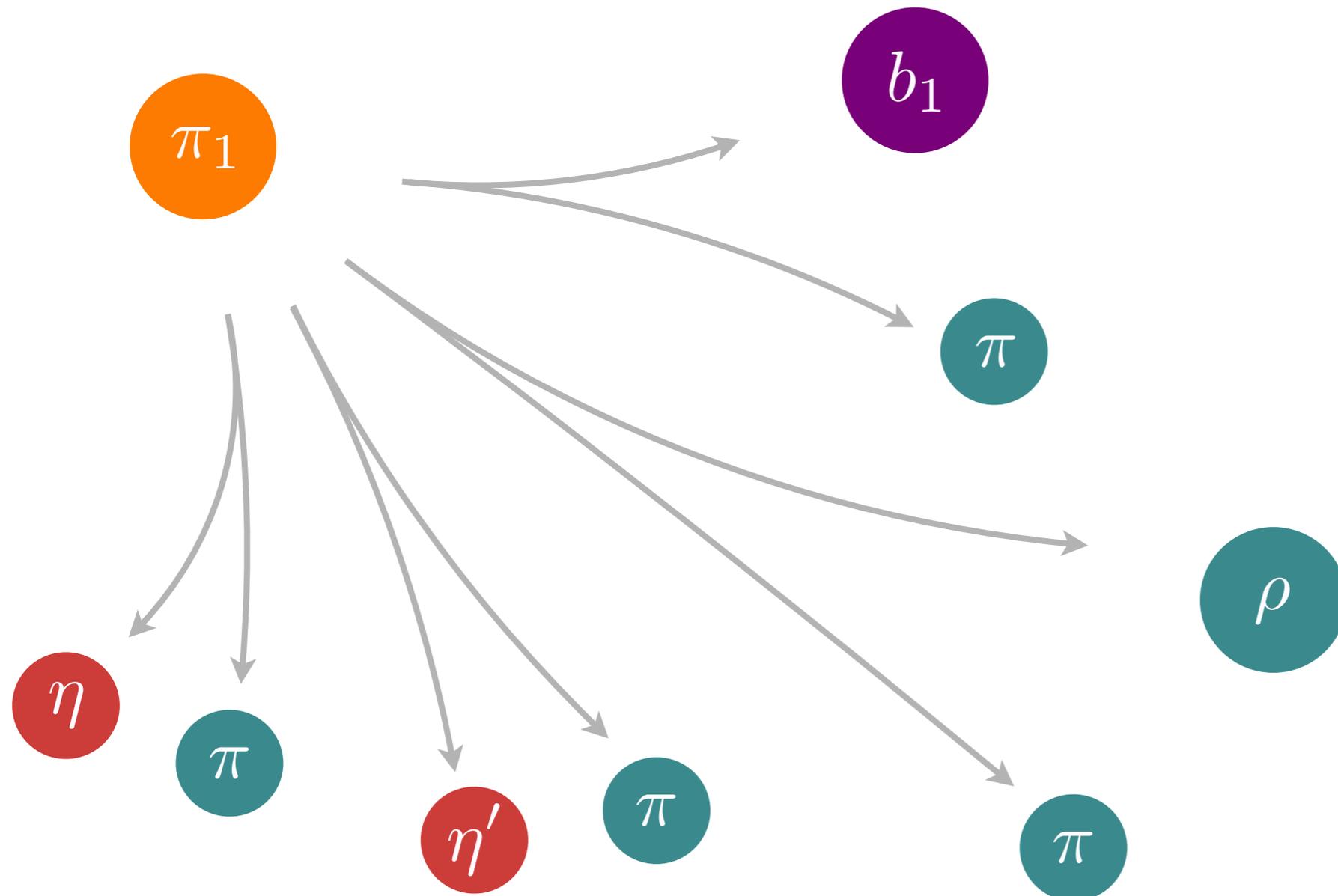


A.J. Woss et al. [HadSpec]  
Phys. Rev. **D103**, 054502 (2021)

had spec

# Connecting Scattering Physics to QCD

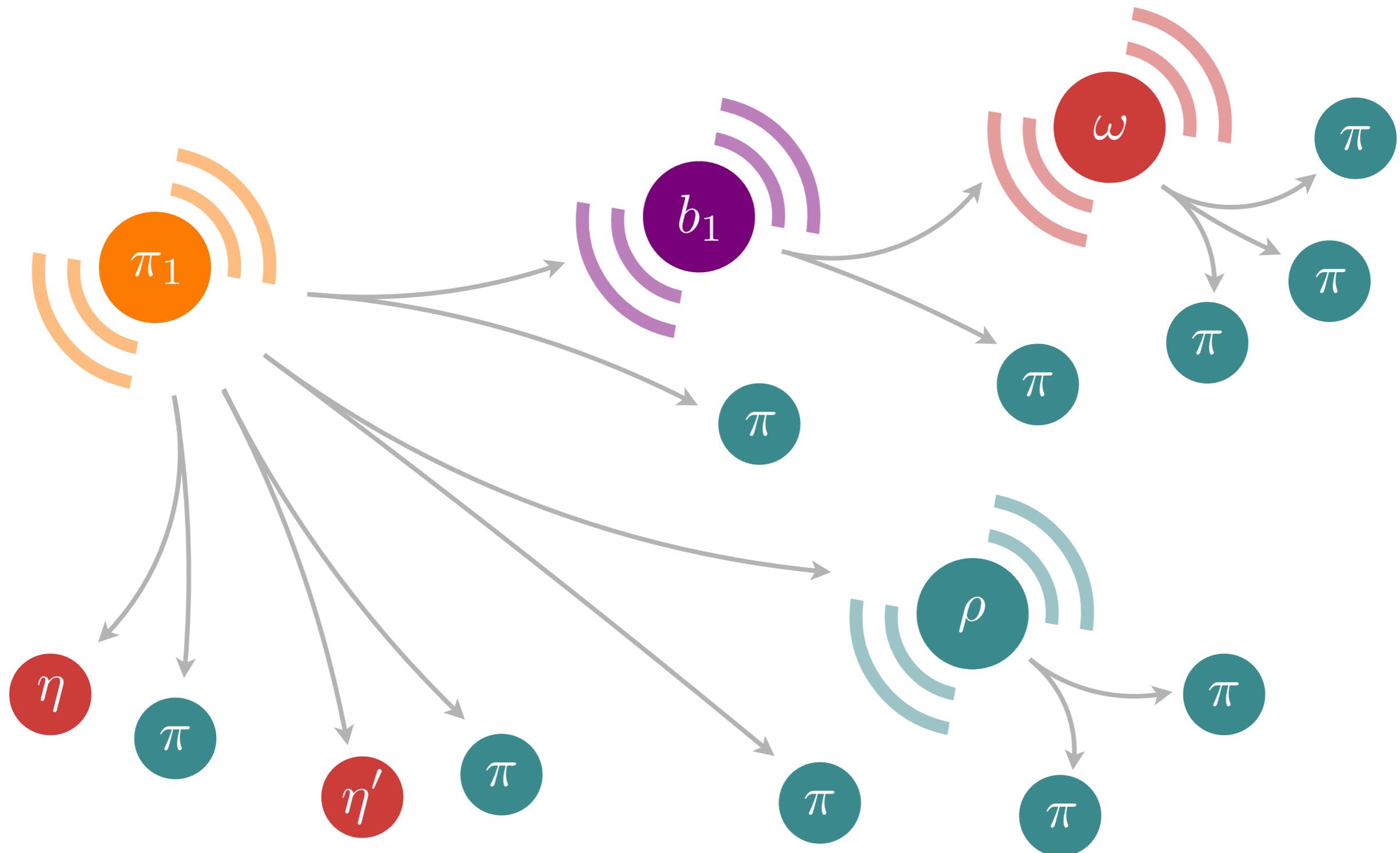
First determination of hybrid candidate,  $J^{PC} = 1^{-+}$ ,  $m_{\pi} \sim 700$  MeV



# Connecting Scattering Physics to QCD

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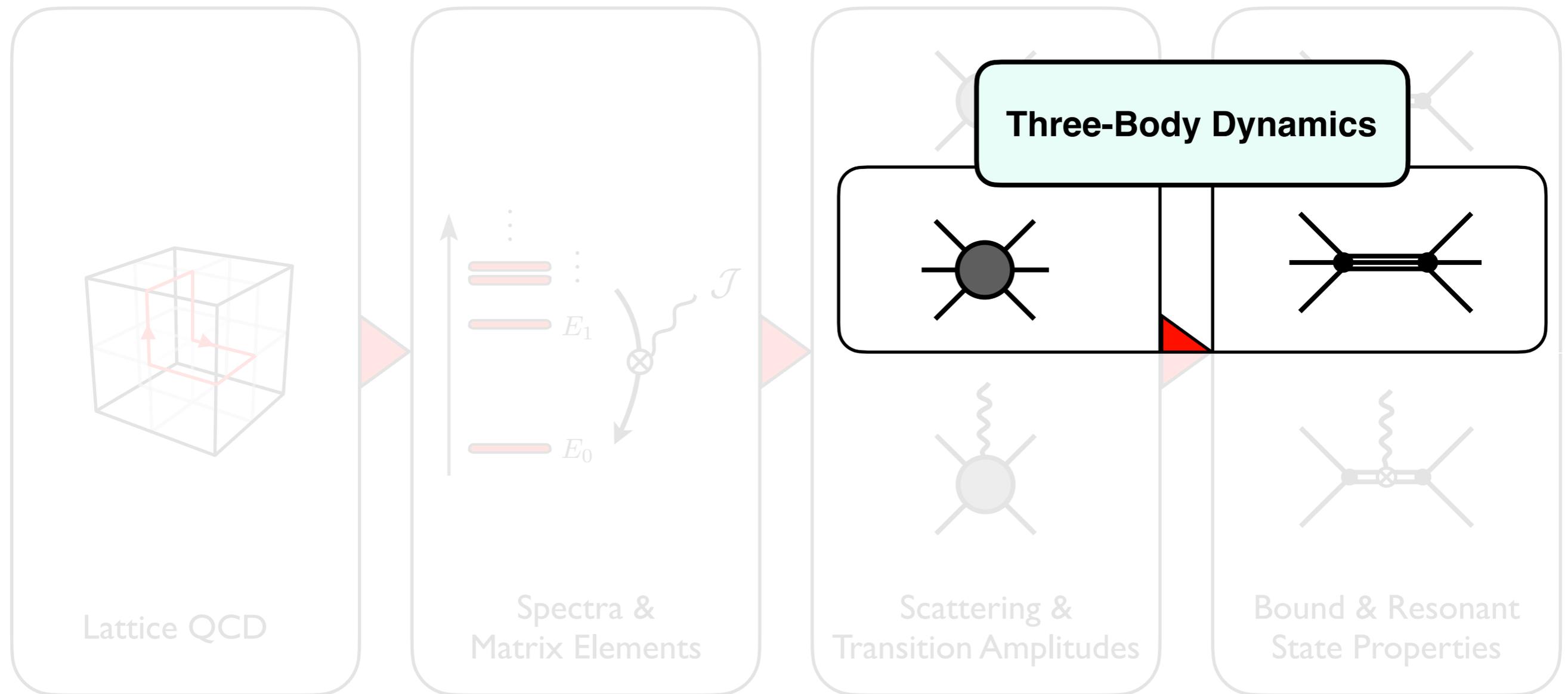
At physical point, 3, 4,.. body decays



# Few-Body Physics from QCD

Path to few-body physics from QCD

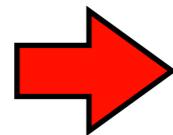
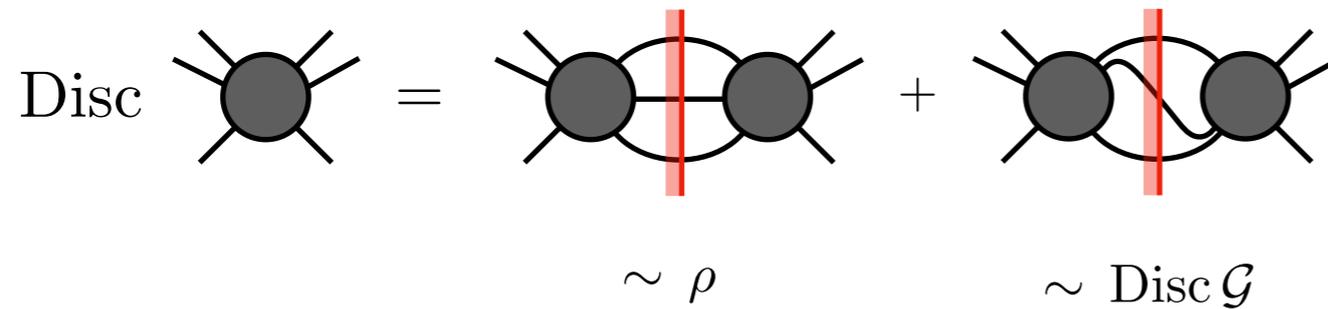
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Three-Body Dynamics

## On-shell scattering relations

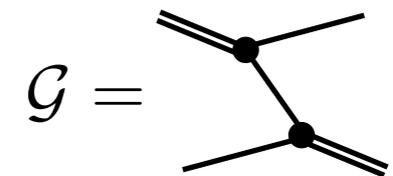
*Unitarity condition*



*On-shell scattering equation*

$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$

$\mathcal{K}_3$  Unknown!  
Obtained from Lattice QCD



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak  
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]  
Eur. Phys. J. C **79**, no. 1, 56 (2019)

AJ et al. [JPAC]  
Phys. Rev. D **100**, 034508 (2019)

AJ, arxiv:2208.10587

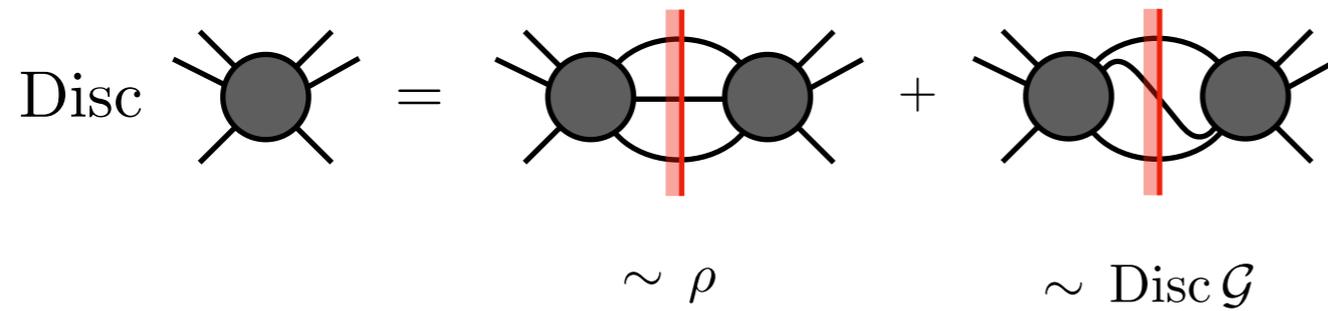
*cf. two-body case:*

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_2$$

# Three-Body Dynamics

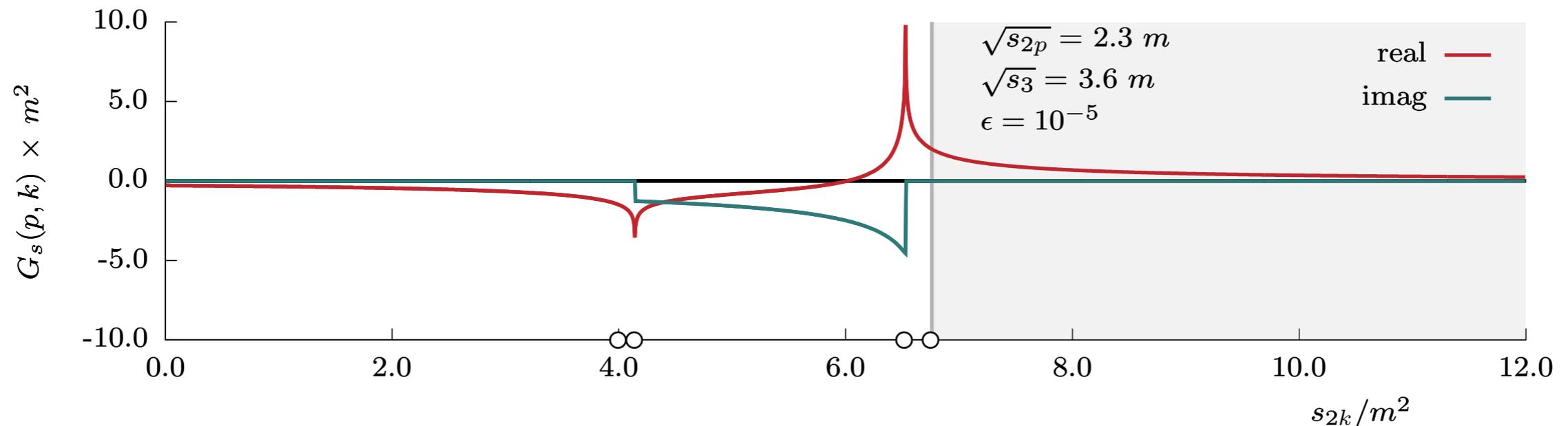
## On-shell scattering relations

Unitarity condition



On-shell scattering equation

$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$



M. M  
Eur. J  
AJ et  
Eur. J  
AJ et  
Phys

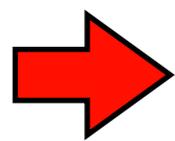
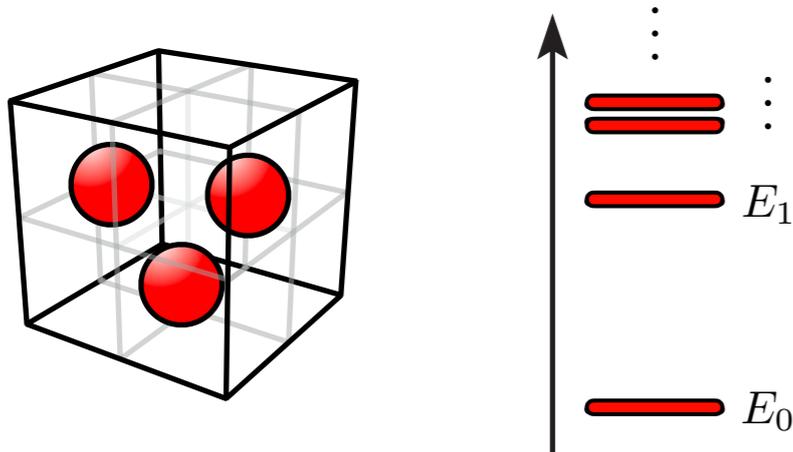
AJ, arxiv:2208.10587

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_2$$

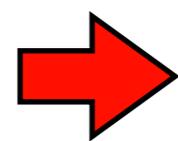
# Three-Body Dynamics

## Connecting to finite-volume spectra

*Finite-volume quantization condition*



$$\det \left( 1 + \mathcal{K}_3 ( \mathcal{F}_L + \mathcal{G}_L ) \right)_{E=E_n} = 0$$



$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$

M. Hansen and S. Sharpe  
Phys. Rev. D **90**, 116003 (2014), Phys. Rev. D **95**, 034501 (2017)

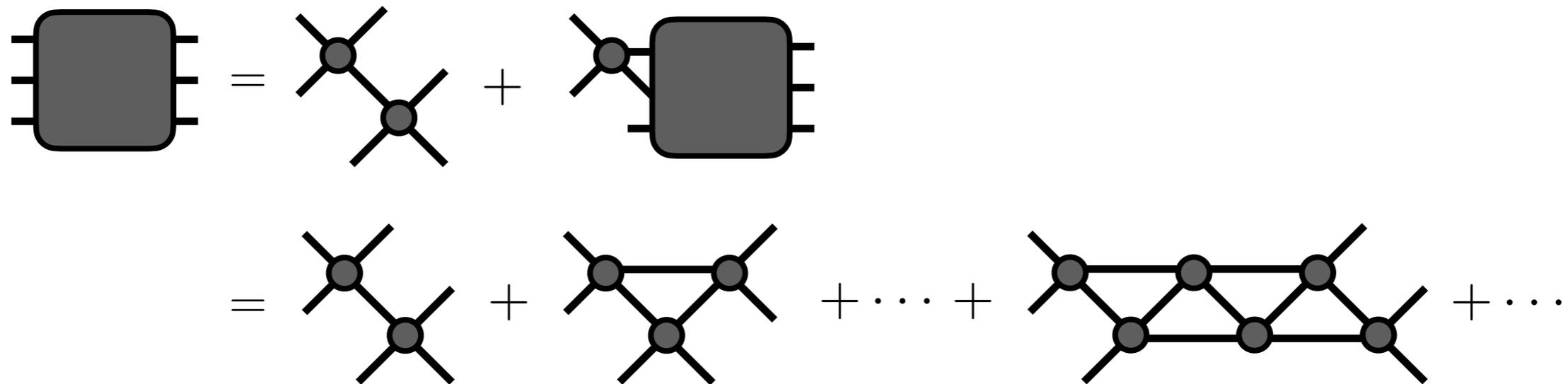
M. Mai and M. Döring  
Eur. Phys. J. A **53**, 240 (2017), Phys. Rev. Lett. **122**, 062503 (2019)

# Three-Body Dynamics

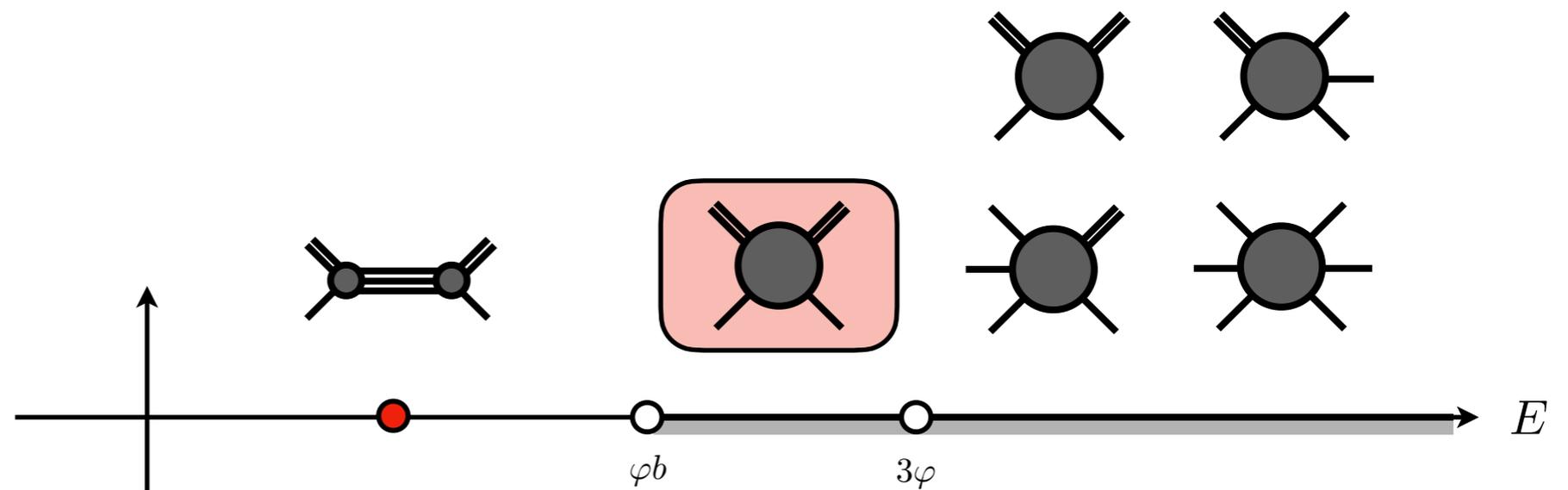
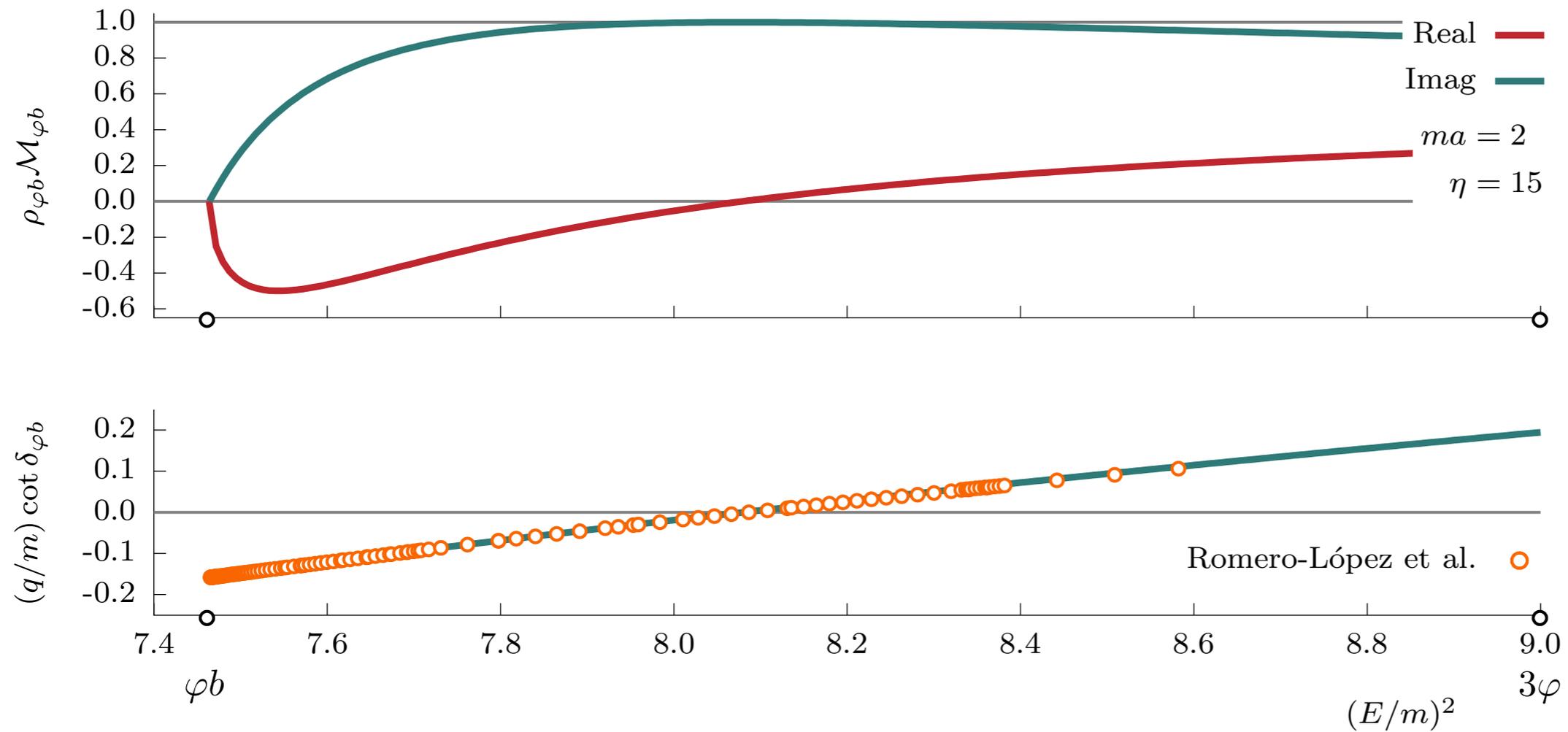
Examine toy-model —  $3\varphi \rightarrow 3\varphi$

- Assume exchange dominance — **No short-range three-body forces**
- Scalar system —  $J = 0$
- Two-hadron pair forms bound state —  $2\varphi \rightarrow b$

*Toy model version of  $3N \rightarrow 3N$  with  $2N \rightarrow d$*



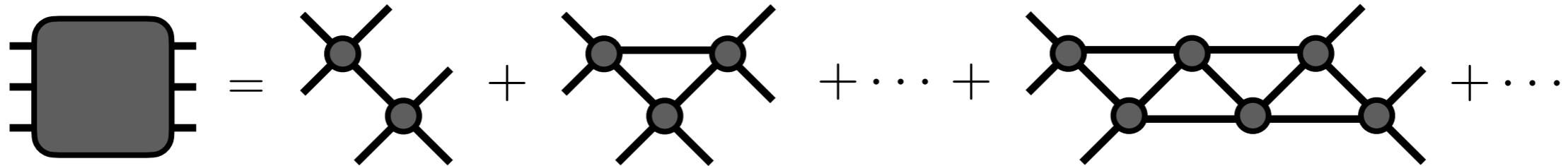
# Three-Body Dynamics



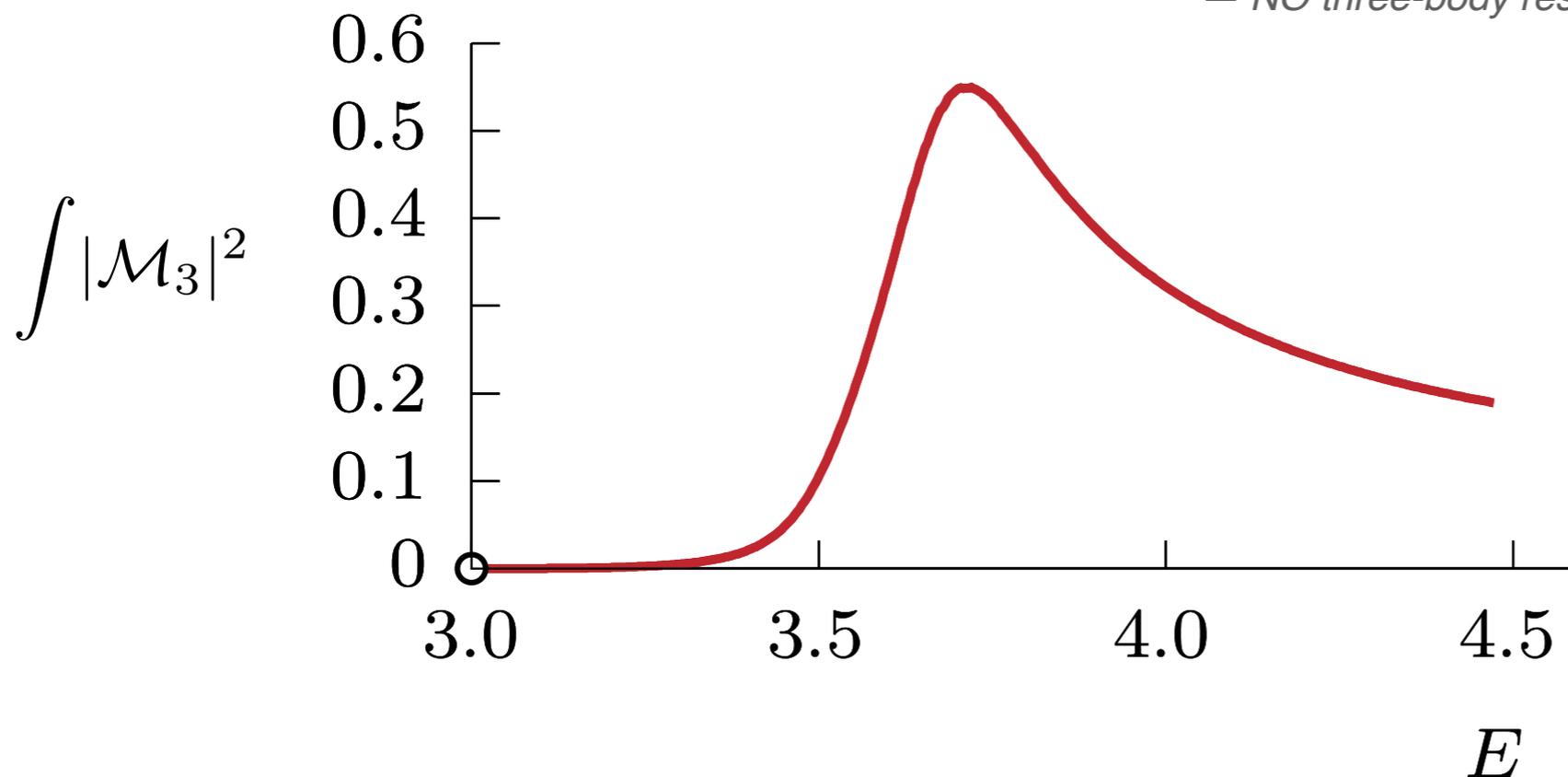
# Three-Body Dynamics

Three-body physics contains more degrees-of-freedom

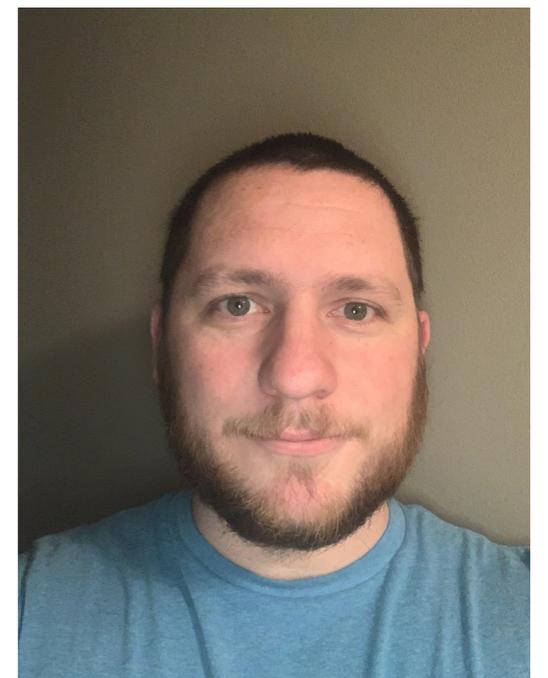
- Find new features not encountered in two-body systems



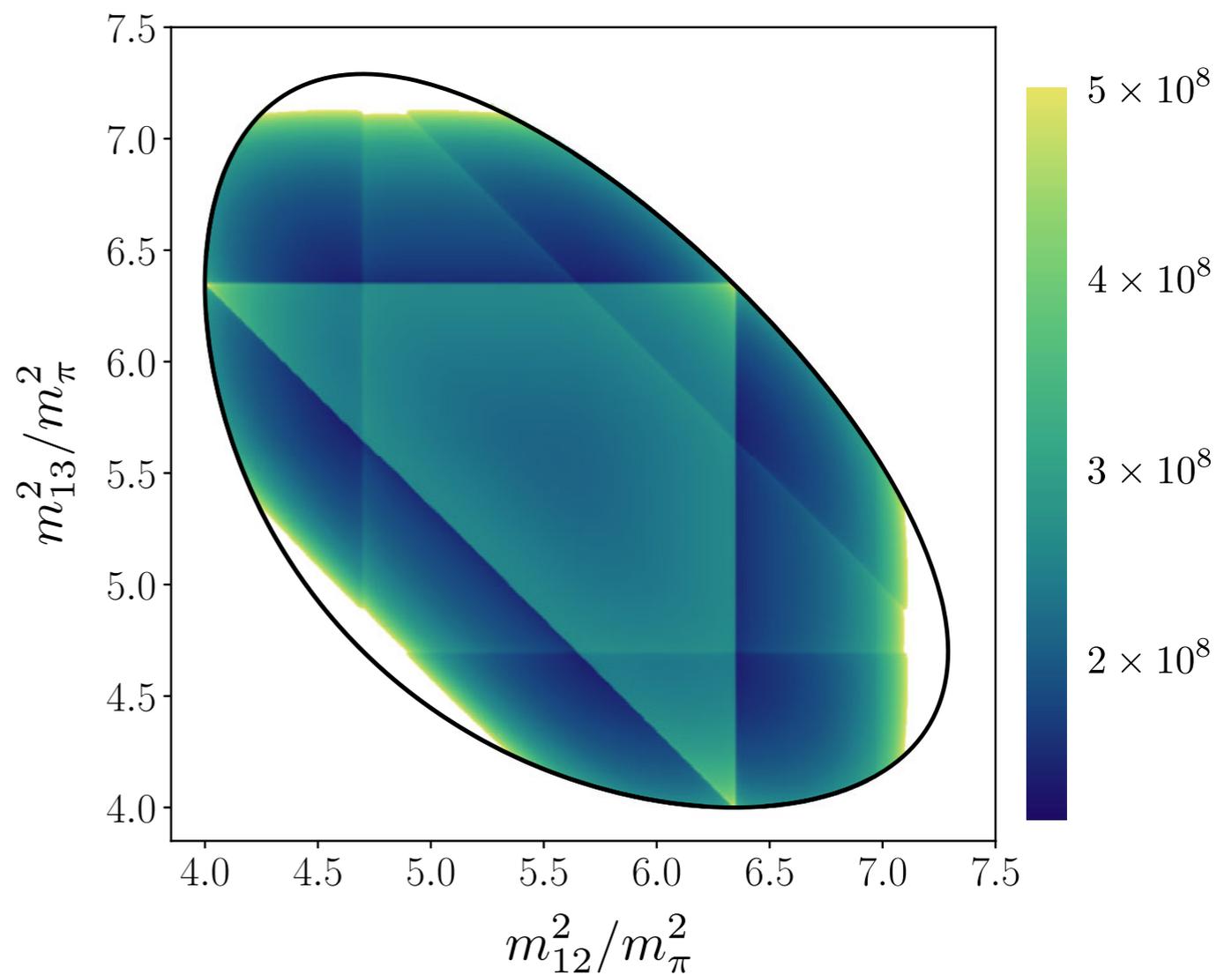
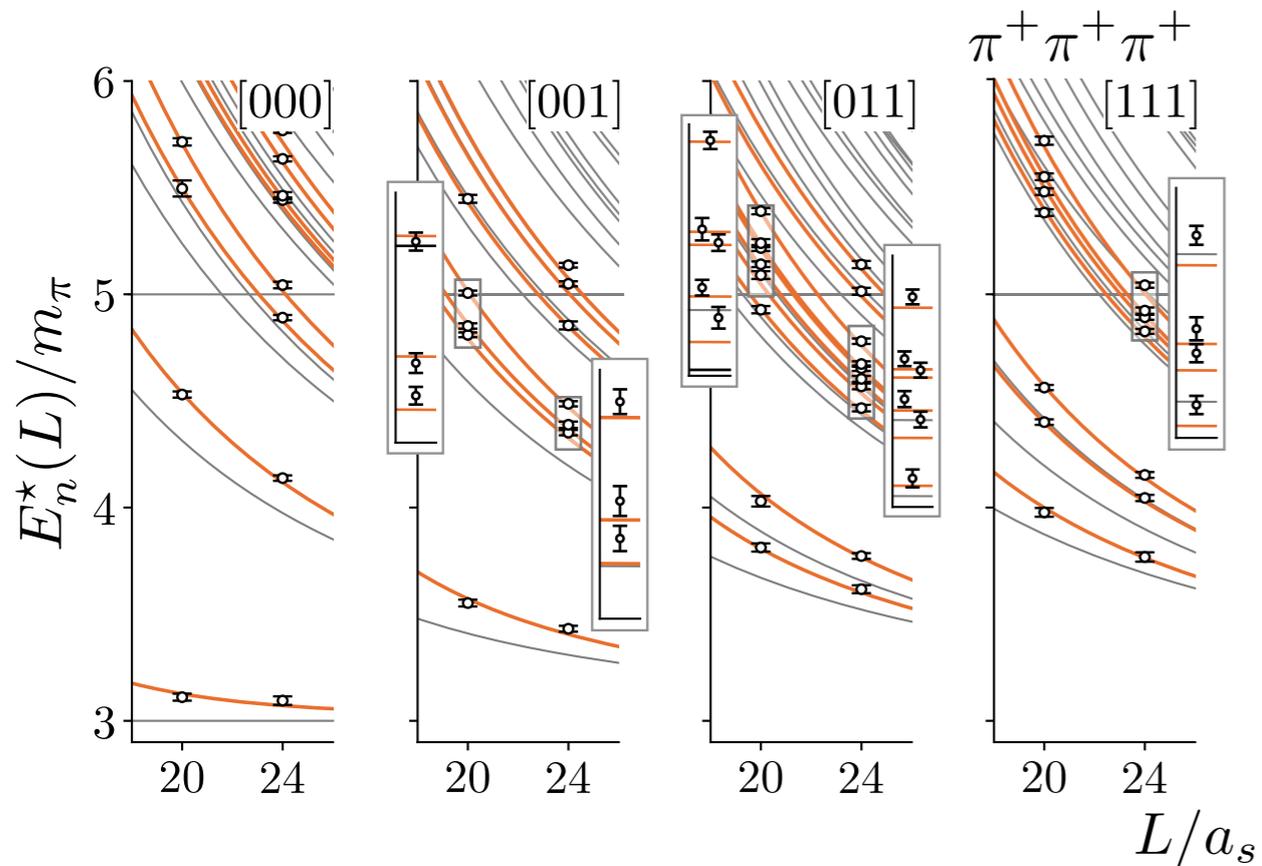
*Bump in the spectrum due to exchange  
— NO three-body resonance*



*Taylor Powell (W&M)*



# Applications to $3\pi^+$

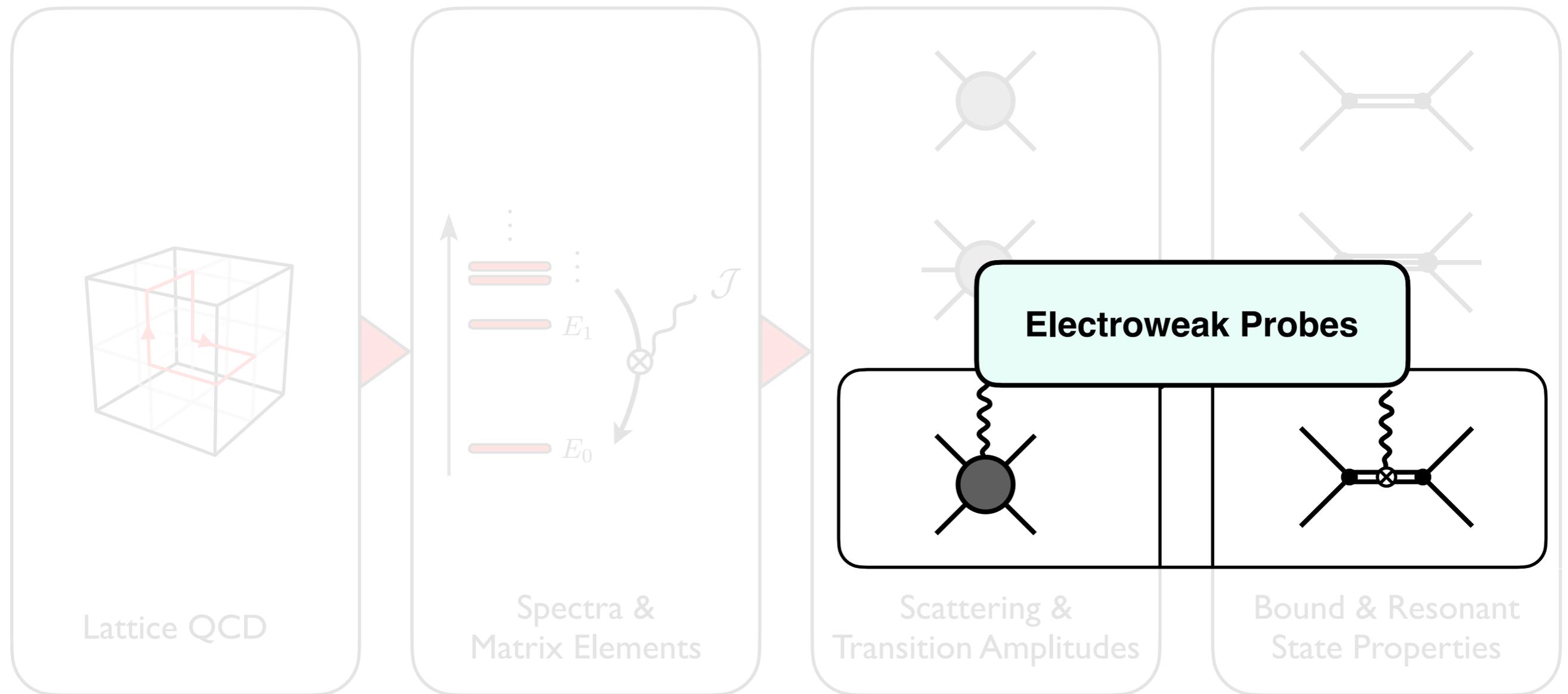


M. Hansen et al. [HadSpec]  
Phys. Rev. Lett. **126**, (2021) 012001

# Few-Body Physics from QCD

Path to few-body physics from QCD

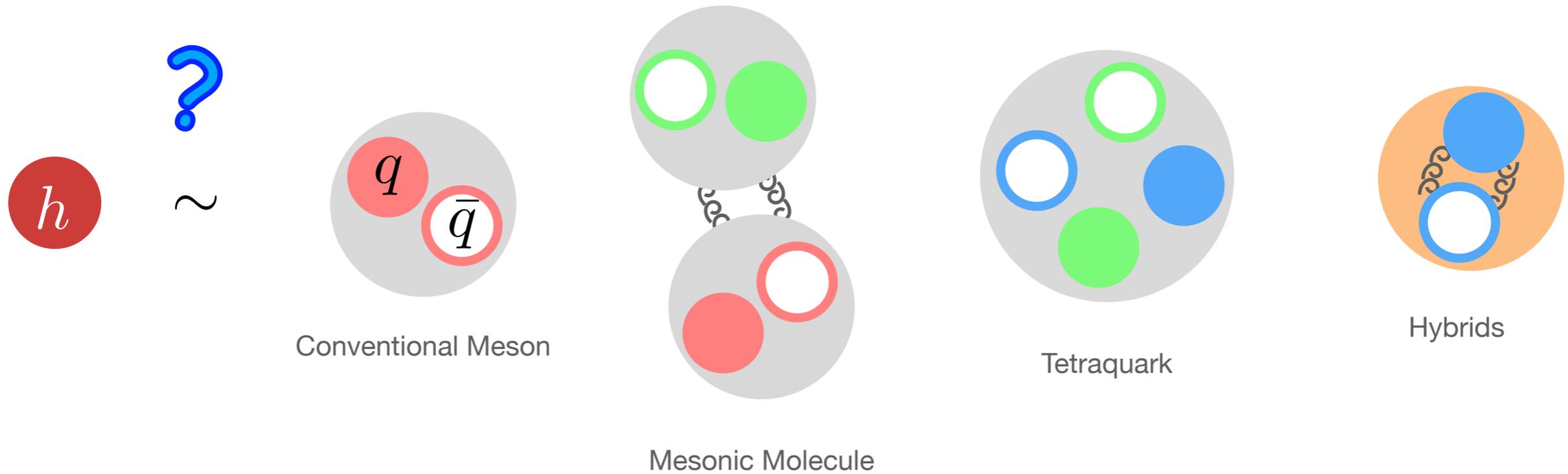
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Hadronic Structure & Electroweak Probes

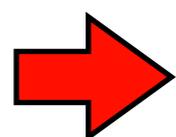
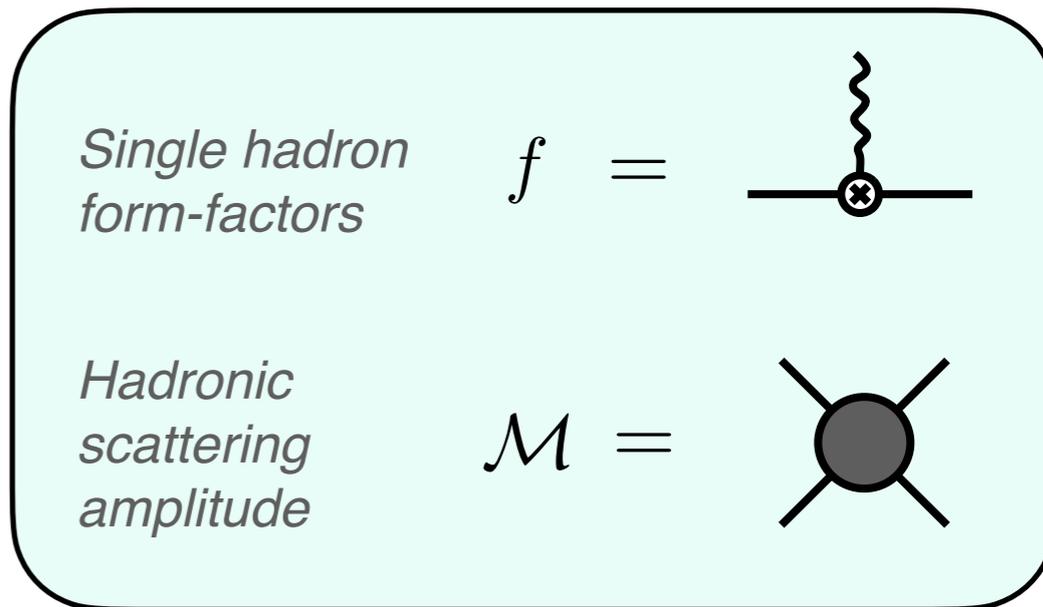
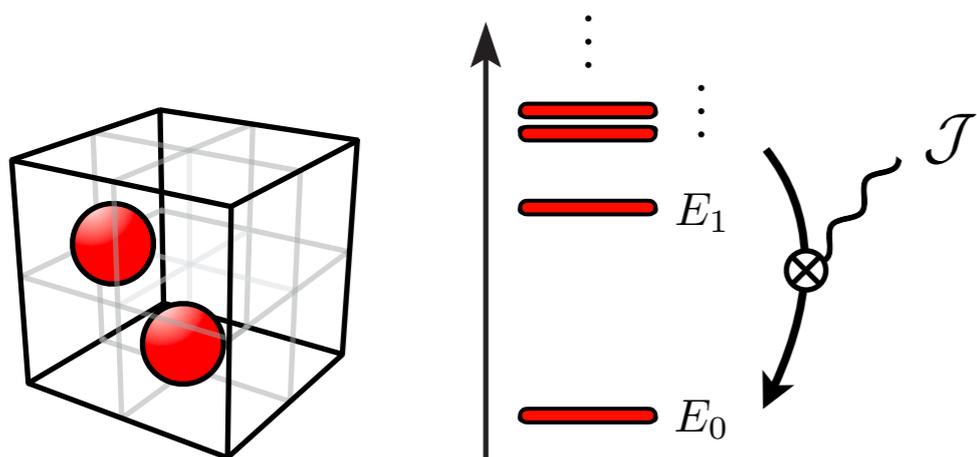
Can we get more than the spectrum?

- Want to know the substructure of the excited hadrons



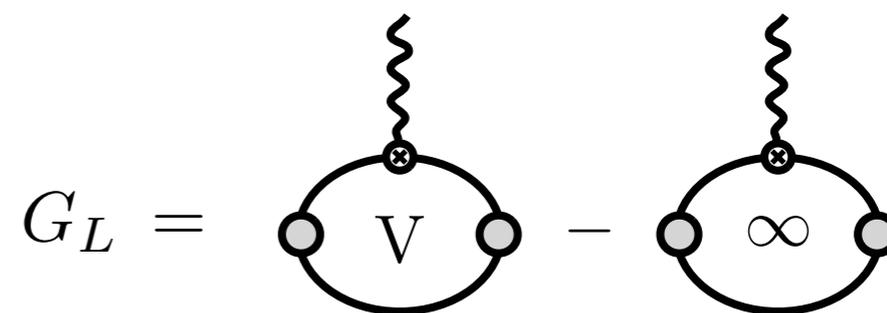
# Hadronic Structure & Electroweak Probes

Mapping between matrix elements and  $2 + \mathcal{J} \rightarrow 2$  amplitudes



$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L = \frac{1}{L^3} \mathcal{W}_{L,df} \cdot \sqrt{\mathcal{R}_{L,m} \cdot \mathcal{R}_{L,n}} \quad \leftarrow \text{FV conversion factors}$$

$$\mathcal{W}_{L,df} = \mathcal{W}_{df} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$



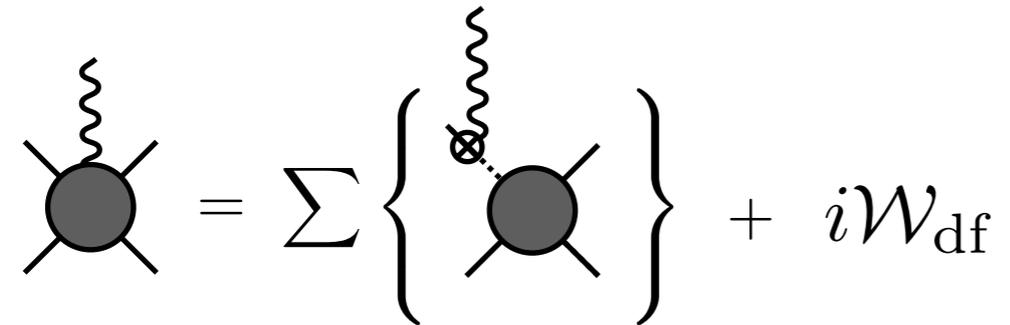
FV geometric function

R. Briceño, M. Hansen,  
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,  
Phys. Rev. D **100** 034511 (2019)

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes


$$\text{Diagram} = \sum \left\{ \text{Diagram} \right\} + i\mathcal{W}_{\text{df}}$$

*After considerable manipulations...*

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot G) \cdot \mathcal{M}$$

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes

$$\text{Diagram with wavy line} = \sum \left\{ \text{Diagram with wavy line and blob} \right\} + i\mathcal{W}_{\text{df}}$$

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot G) \cdot \mathcal{M}$$

Unknown short-distance function

- Constrain using Lattice QCD
- Constrained by Ward-Takahashi identity

Single hadron form-factors

$$f = \text{Diagram: wavy line to blob on a line}$$

Hadronic scattering amplitude

$$\mathcal{M} = \text{Diagram: blob with four external lines}$$

Triangle diagram

Contains normal and anomalous singularities from intermediate on-shell particles

$$G = \text{Diagram: triangle loop with wavy line}$$

$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L \sim \mathcal{W}_{\text{df}} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes

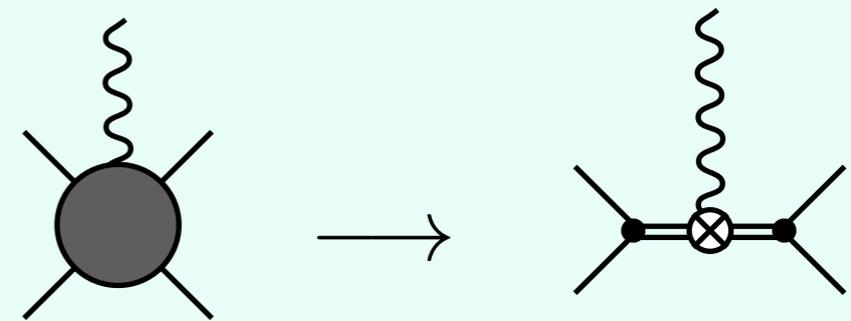
$$\text{Diagram with wavy line} = \sum \left\{ \text{Diagram with wavy line and blob} \right\} + i\mathcal{W}_{\text{df}}$$

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot G) \cdot$$

Rigorous definition for resonance form factors

$$\mathcal{W}_{\text{df}} \sim \frac{g}{s_f - s_p} \cdot f_p \cdot \frac{g}{s_i - s_p}$$



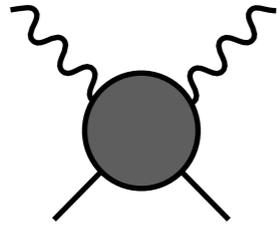
$$f_p = g^2 (\mathcal{A} + f \cdot G) \Big|_{s_f = s_i = s_p}$$

# Two-current systems

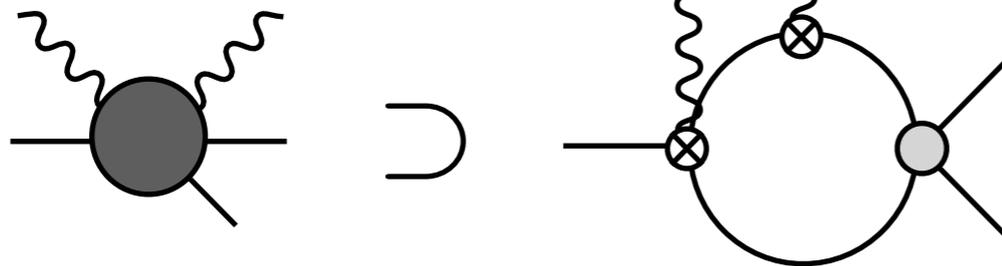
## Coupling two currents to hadronic systems – Compton-like processes

$1 + \mathcal{J} \rightarrow 1 + \mathcal{J}$

R. Briceño, Z. Davoudi, M. Hansen, M. Schindler, A. Baroni  
Phys. Rev. D **101** 014509 (2020)

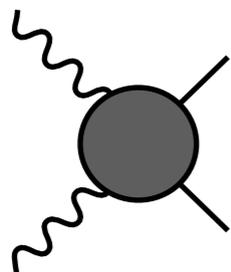


$1 + \mathcal{J} \rightarrow 2 + \mathcal{J}$



F. Ortega-Gama, K. Sherman, AJ, R. Briceño,  
Phys. Rev. D **105** (2022)

$\mathcal{J} + \mathcal{J} \rightarrow 2$



AJ, R. Briceño, A. Rodas, J. Guerrero  
*In preparation*

# Summary

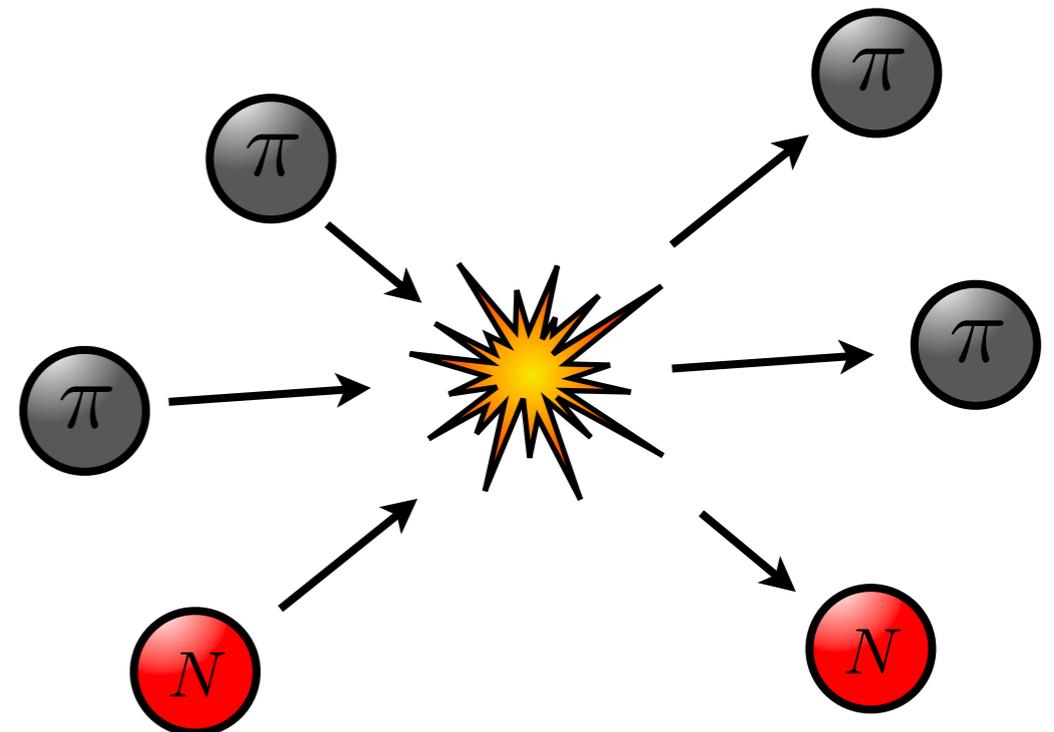
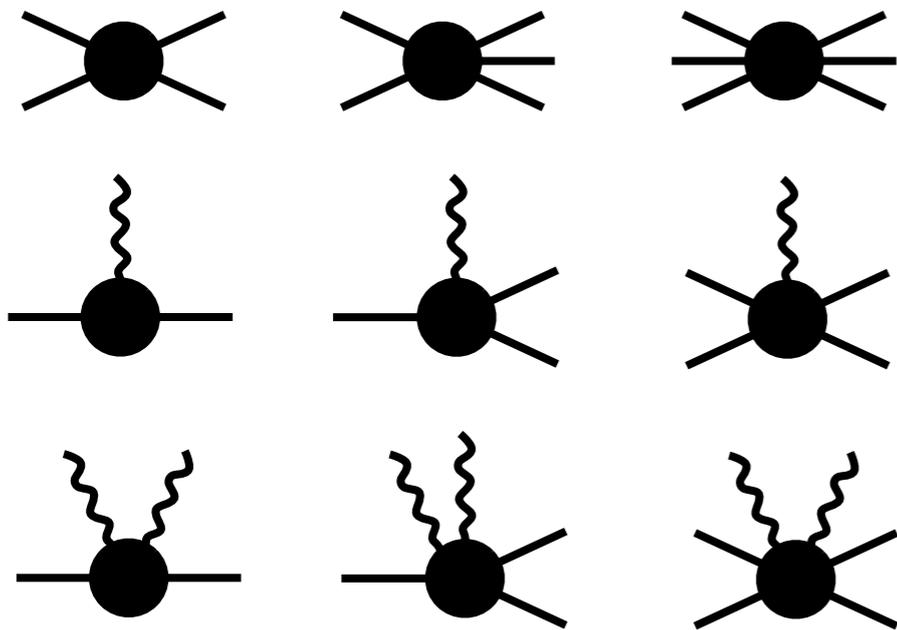
Few-body interactions play a key role in many outstanding problems in nuclear & hadron physics

Lattice QCD, EFTs, & Scattering theory combined provide useful tools to extract physics from QCD

- Rapid development in formalisms relating lattice QCD observables to amplitudes
- Scattering phenomenology is advancing in tandem

Latest developments in three-body scattering & two-body matrix elements

- First applications appearing in literature
- Can address increasingly complicated processes

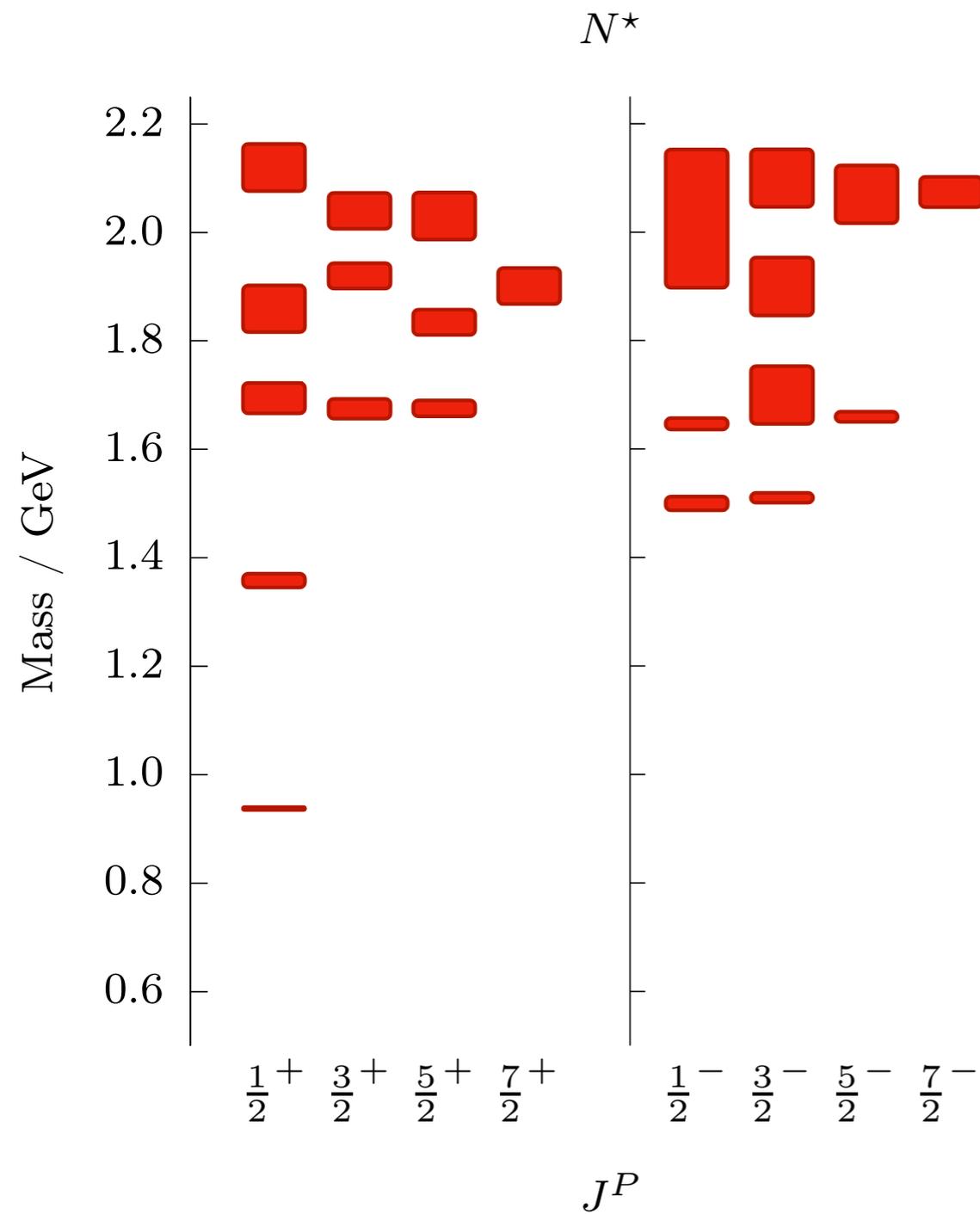


**Much more to come!**



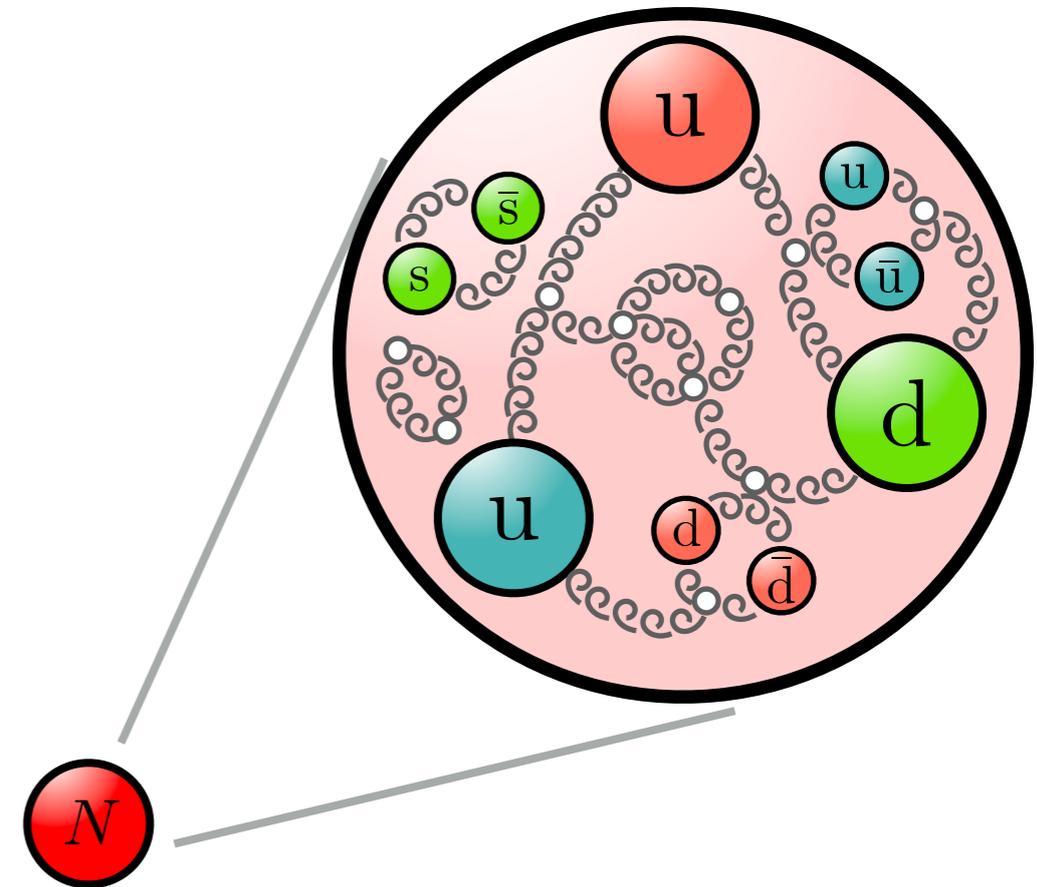
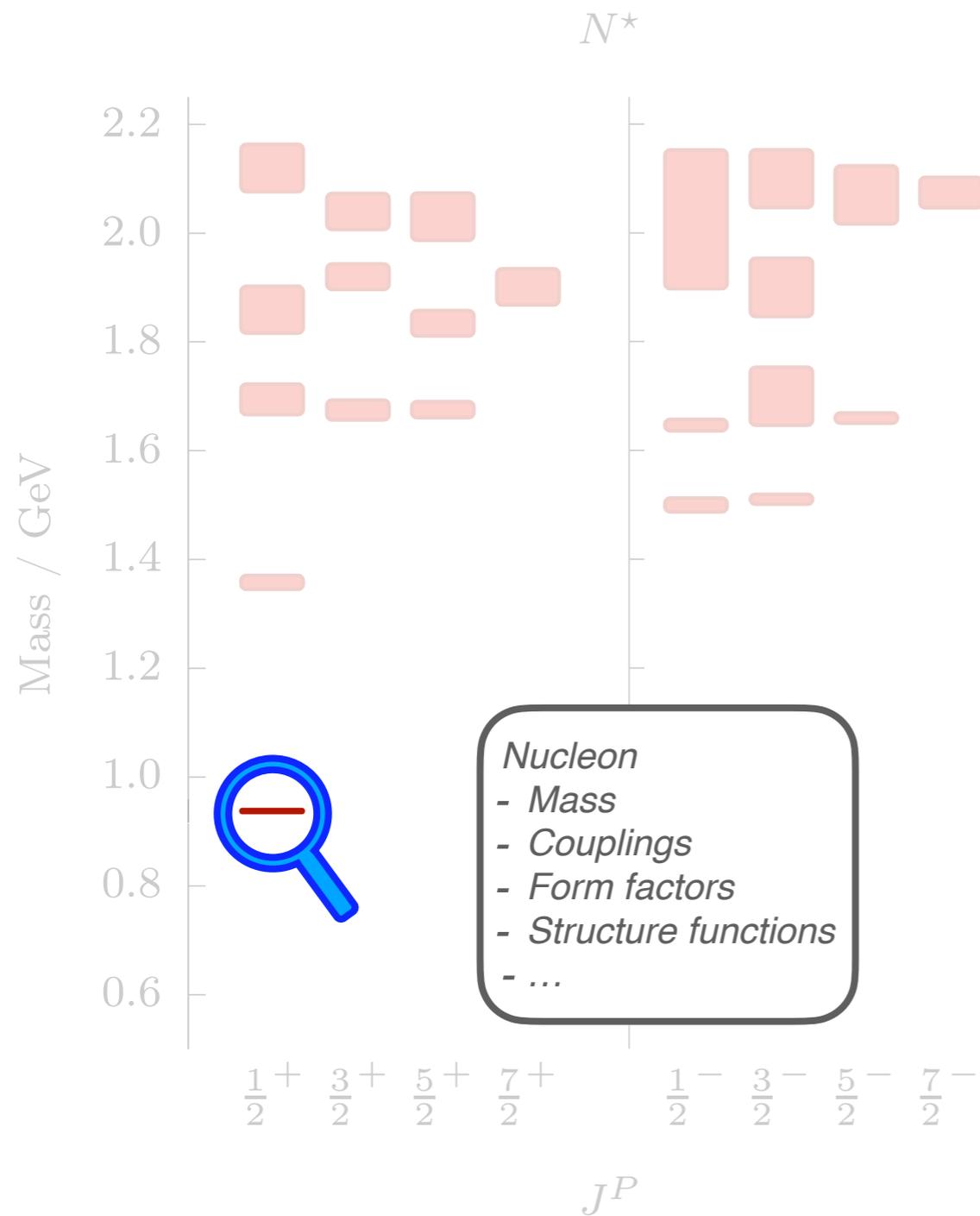
# Why Few-Body Physics? — Excited Nucleon Spectrum

Spectroscopy and structure of hadrons



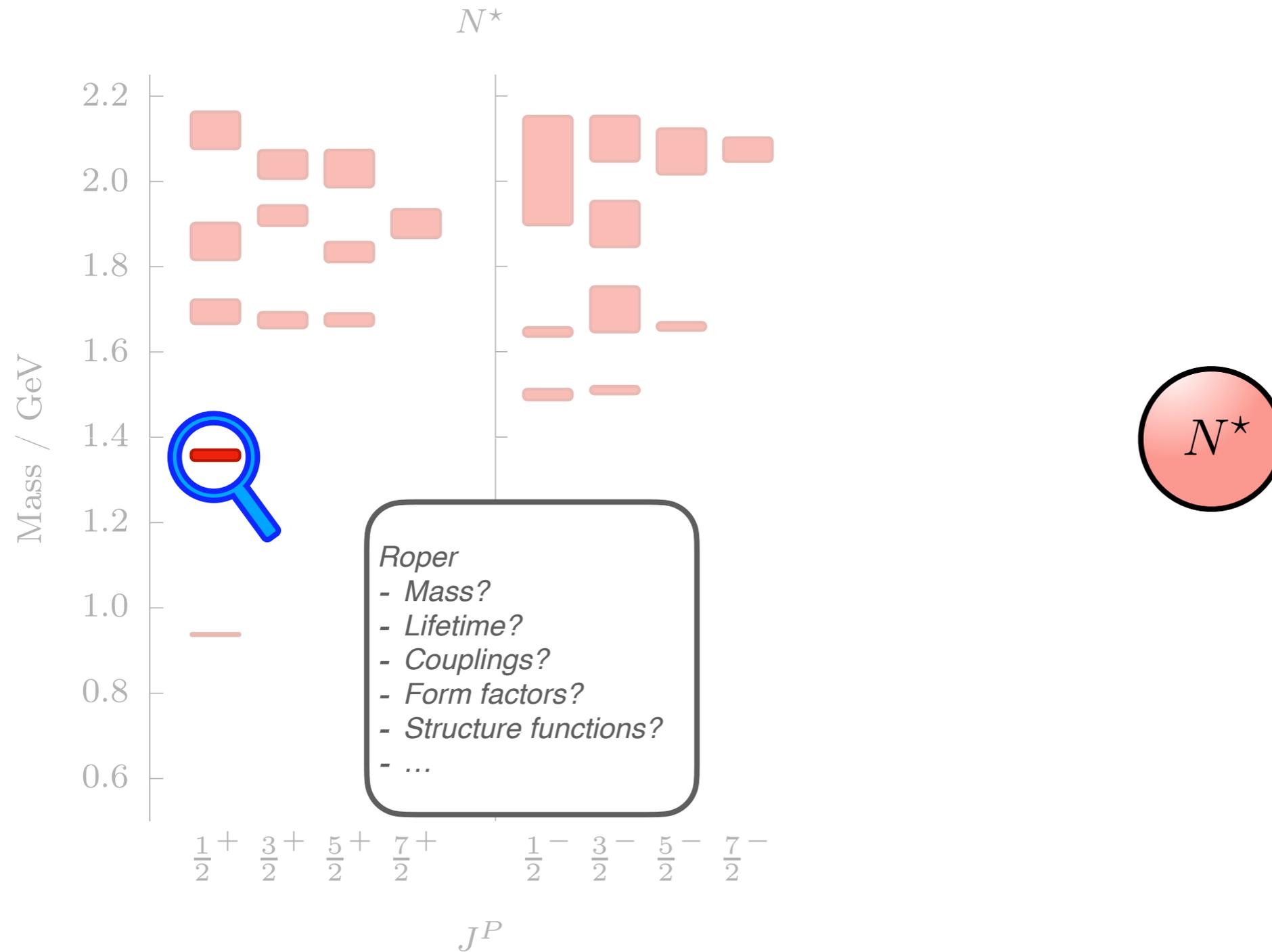
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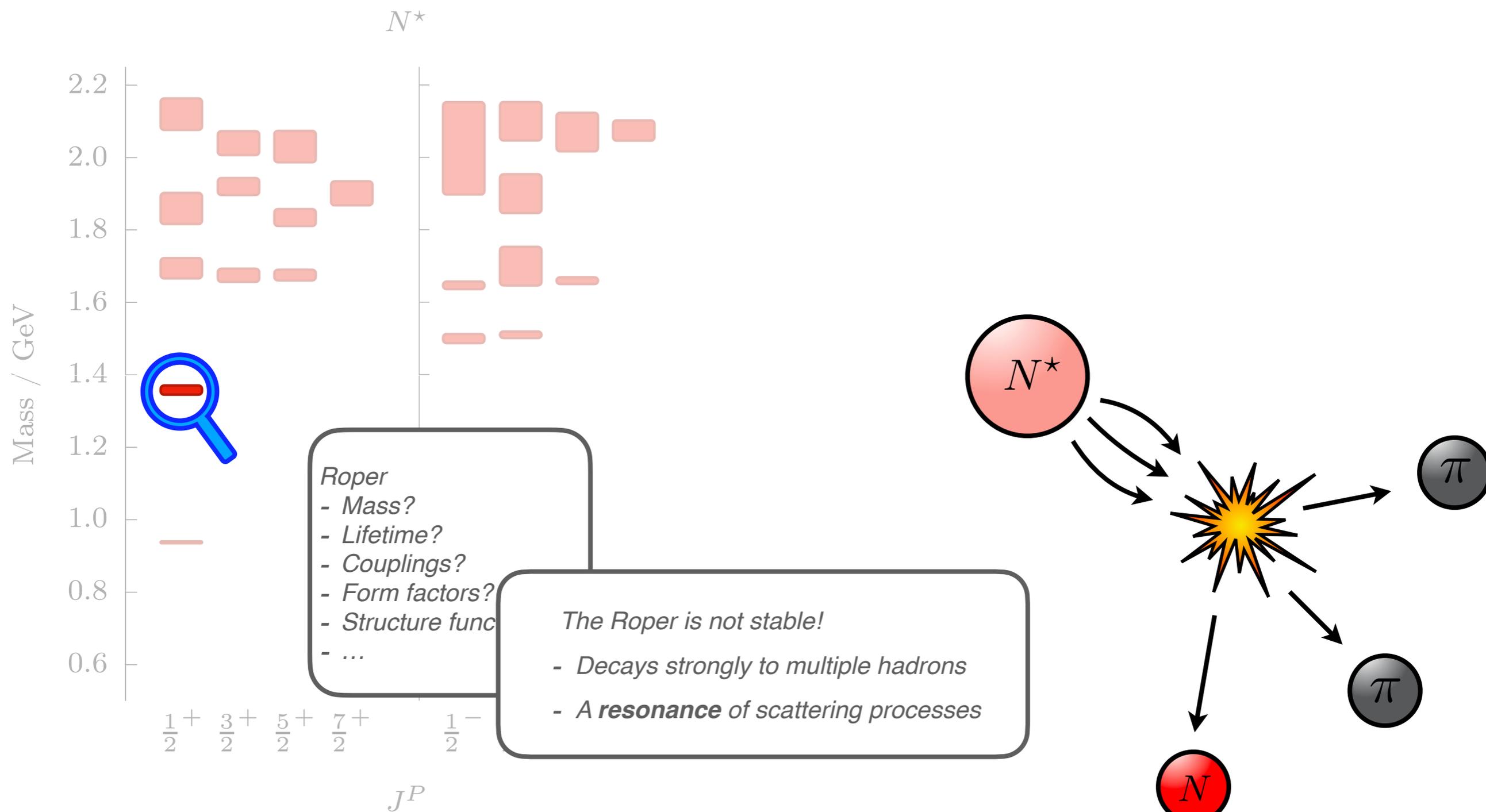
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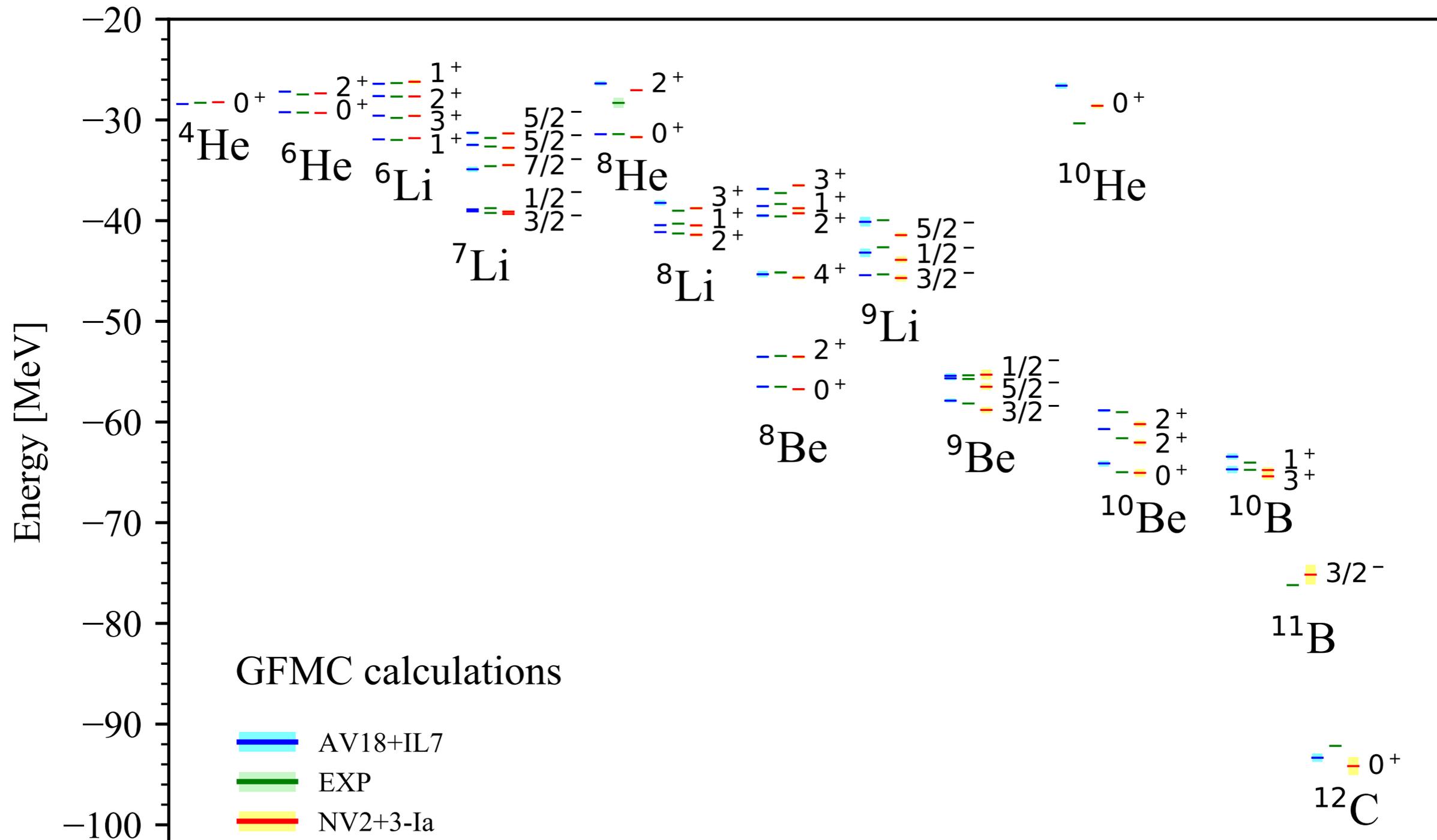
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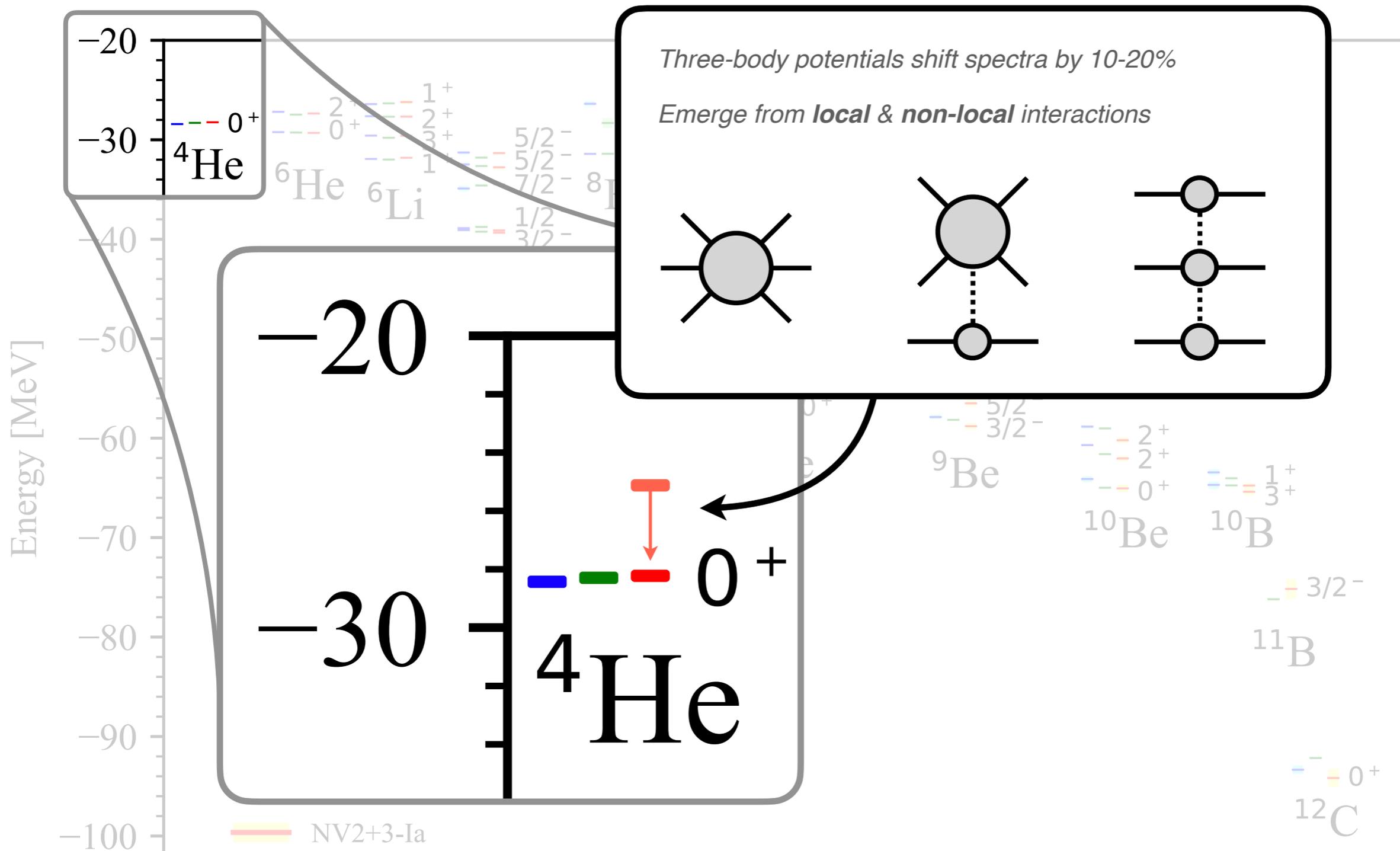
# Why Few-Body Physics? — Three-Body Nuclear Force

Three-body (or  $n$ -body) forces in nuclear structure



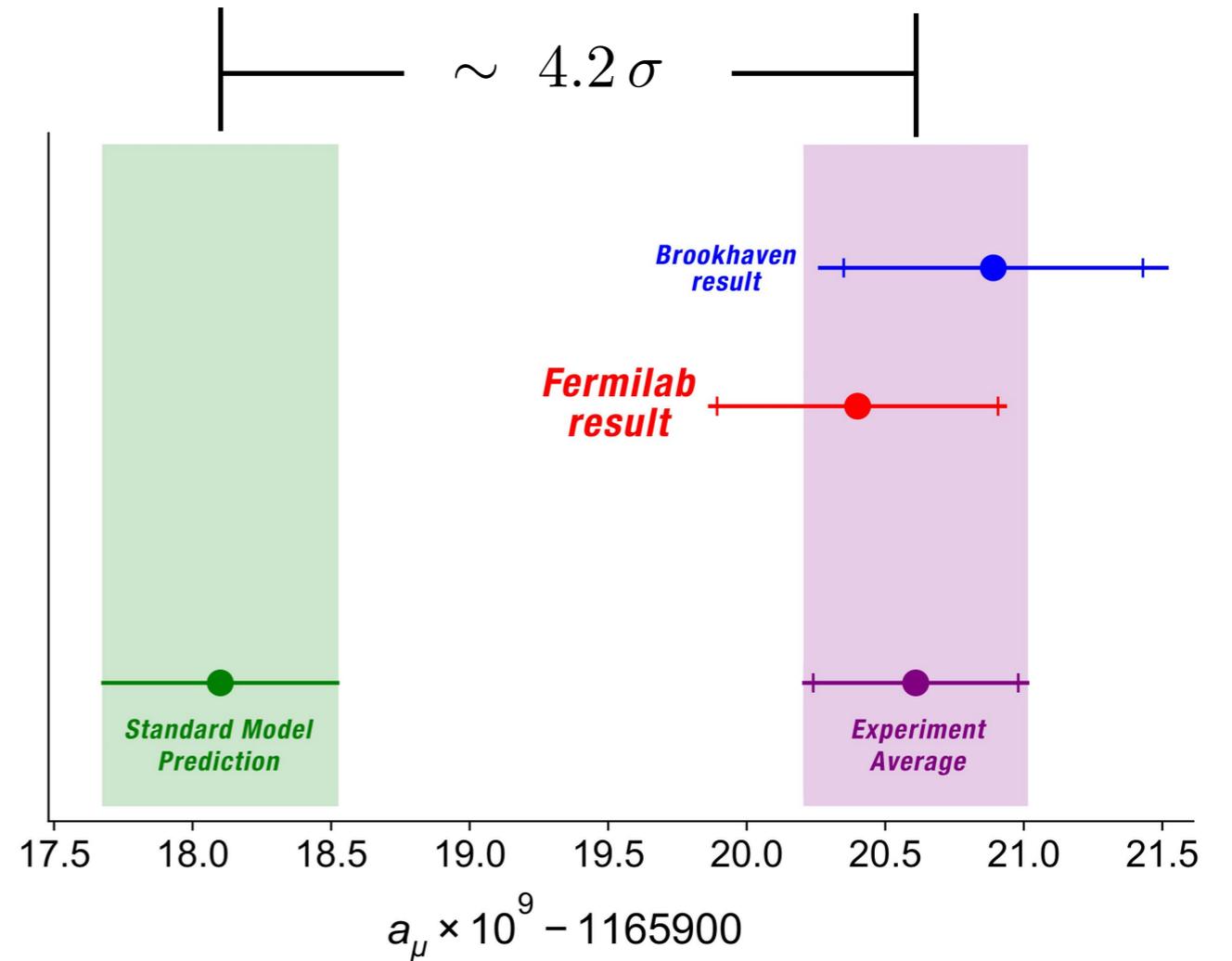
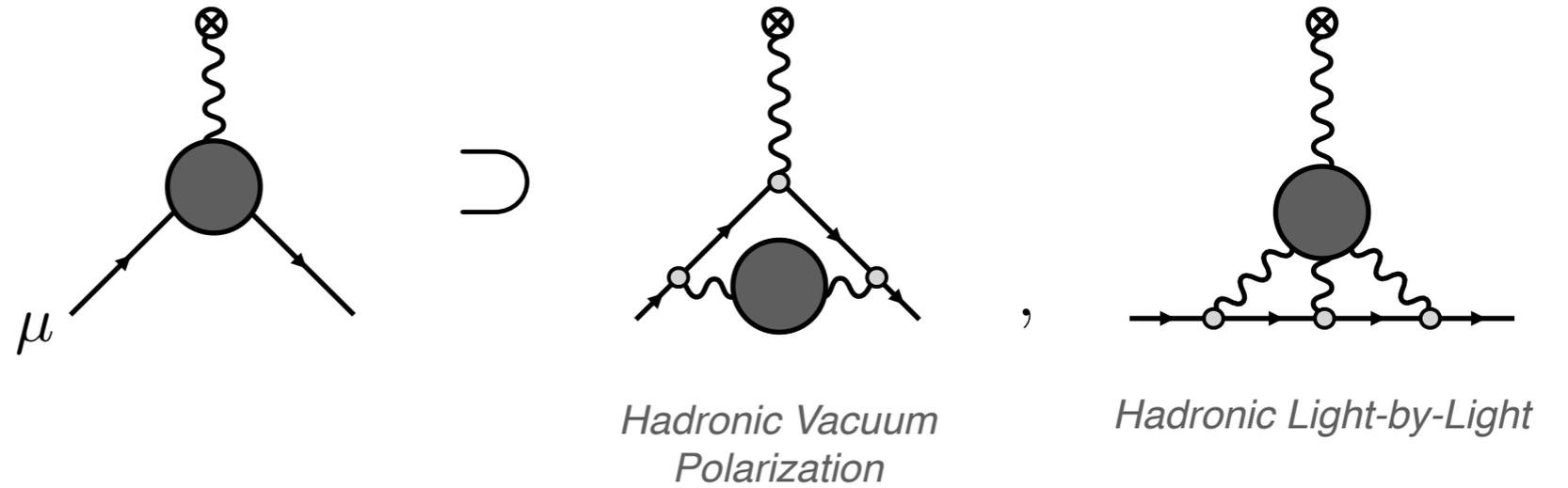
# Why Few-Body Physics? — Three-Body Nuclear Force

Three-body (or  $n$ -body) forces in nuclear structure



# Why Few-Body Physics? – Muon $g-2$

Multi-hadron states in physics Beyond the Standard Model searches



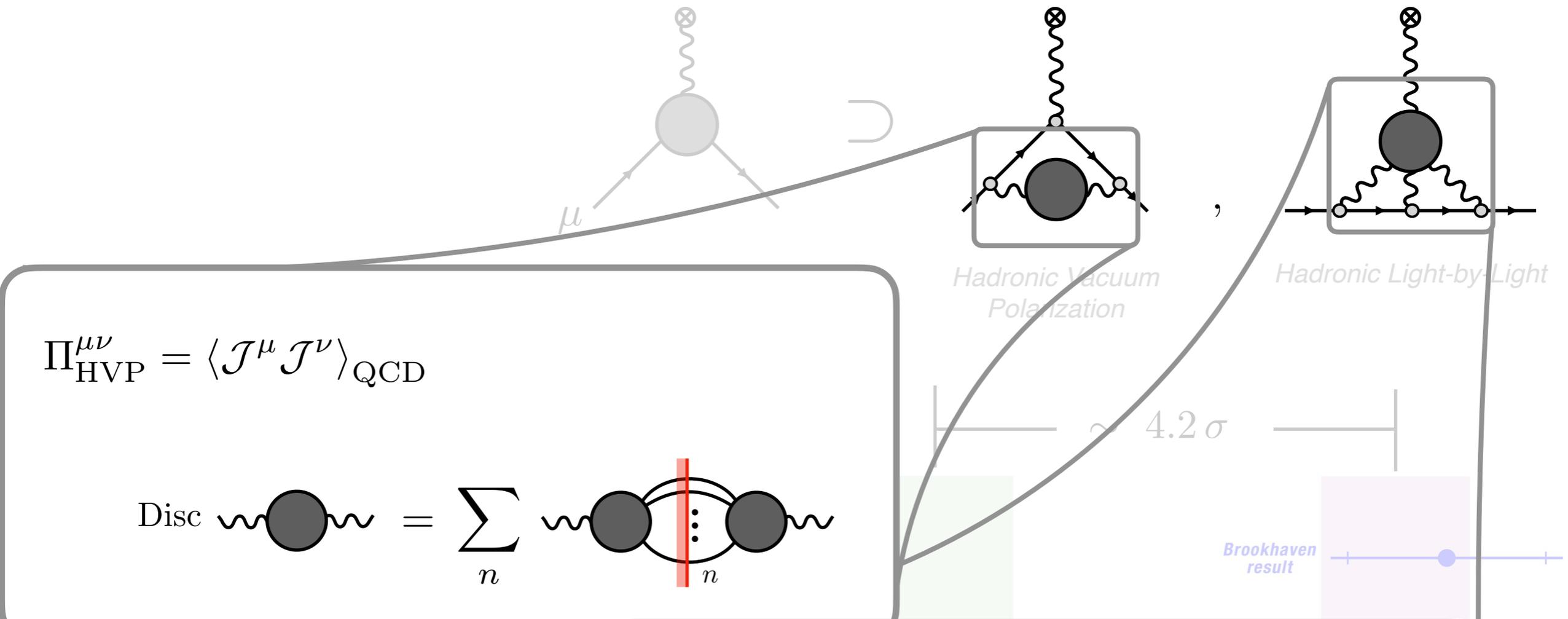
B. Abi et al. [Muon  $g-2$  Collaboration]  
Phys. Rev. Lett. **126**, 141801 (2021)

G.W. Bennett et al. [Muon  $g-2$  Collaboration]  
Phys. Rev. D. **73**, 072003 (2006)

T. Aoyama et al.  
Phys. Rep. **887**, 1 (2020)

# Why Few-Body Physics? – Muon $g-2$

Multi-hadron states in physics Beyond the Standard Model searches



$$\Pi_{\text{HVP}}^{\mu\nu} = \langle \mathcal{J}^\mu \mathcal{J}^\nu \rangle_{\text{QCD}}$$

$$\text{Disc} \text{ (loop)} = \sum_n \text{ (disc)}_n$$

$$\Pi_{\text{HLbL}}^{\mu\nu\rho\sigma} = \langle \mathcal{J}^\mu \mathcal{J}^\nu \mathcal{J}^\rho \mathcal{J}^\sigma \rangle_{\text{QCD}}$$

$$\text{Disc} \text{ (box)} = \sum_n \text{ (disc)}_n$$

B. Abi et al. [Muon  $g-2$  Collaboration]  
Phys. Rev. Lett. **126**, 141801 (2021)

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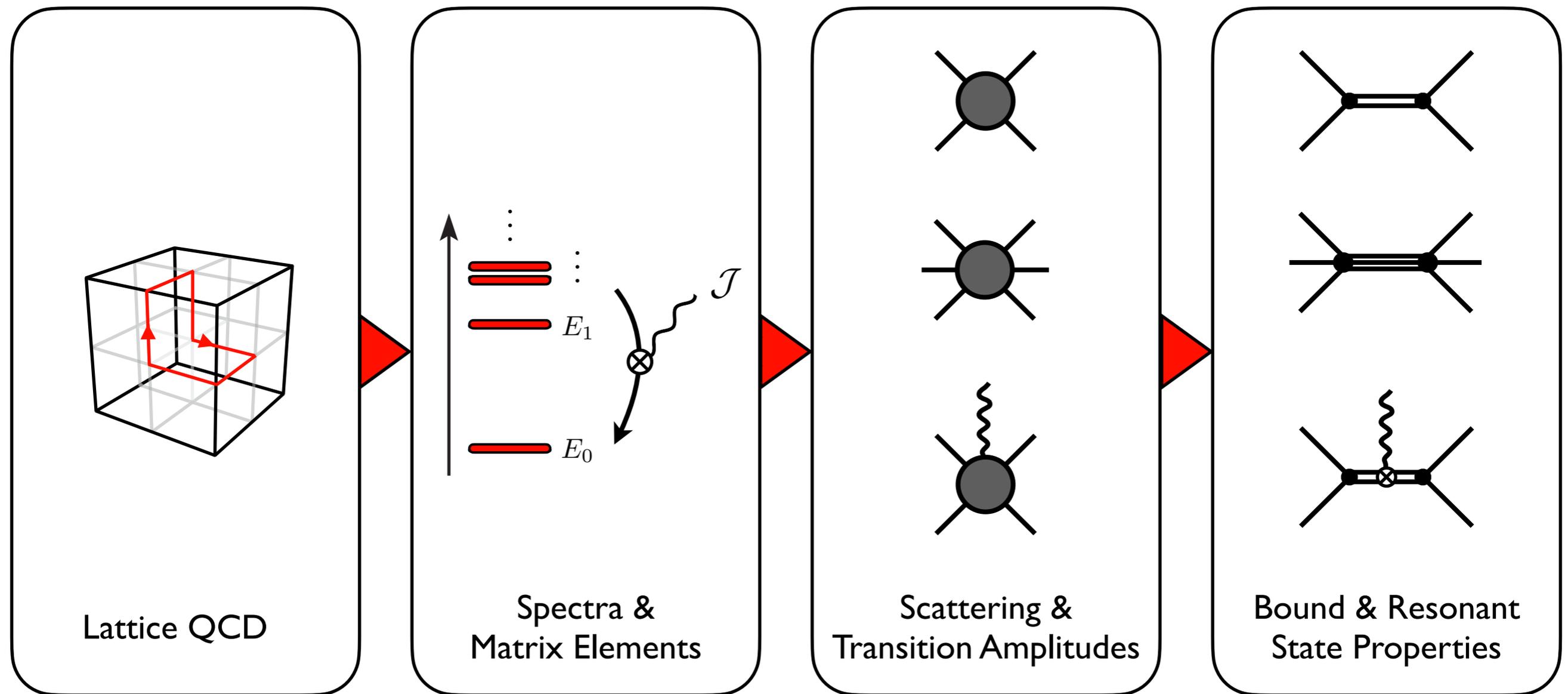
T. Aoyama et al.  
Phys. Rep. **887**, 1 (2020)

21.5

# Few-Body Physics & QCD

Path to few-body physics from QCD

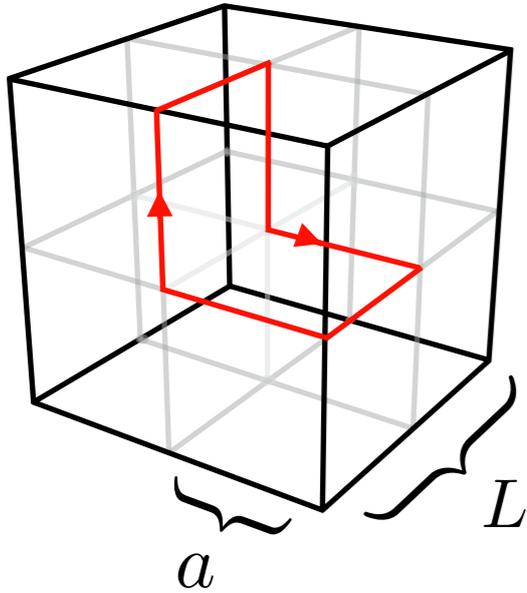
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Lattice QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



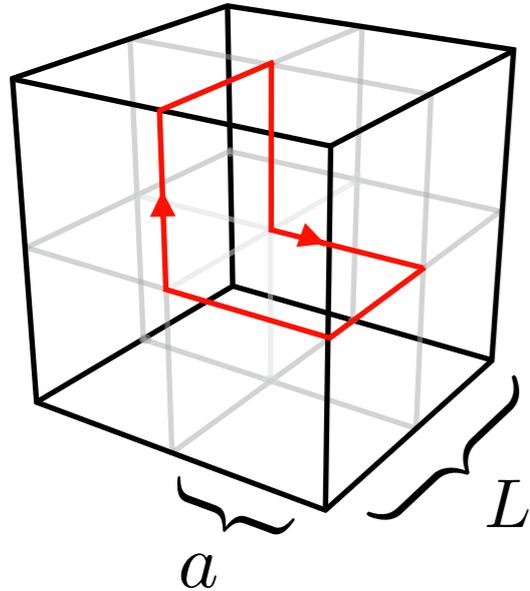
$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_{\mu} e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_{\mu})}$$

- Euclidean spacetime,  $t \rightarrow -i\tau$
- Finite volume,  $L$
- Discrete spacetime,  $a$
- Heavier than physical quark mass,  $m > m_{\text{phys}}$ .

# Lattice QCD

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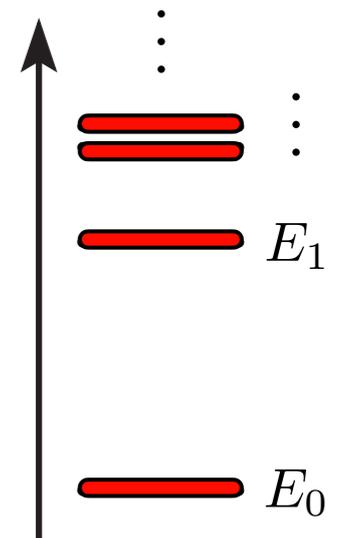


$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

- Euclidean spacetime,  $t \rightarrow -i\tau$
- Finite volume,  $L$
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- Heavier than physical quark mass,  $m$

Correlation functions yield discrete spectrum

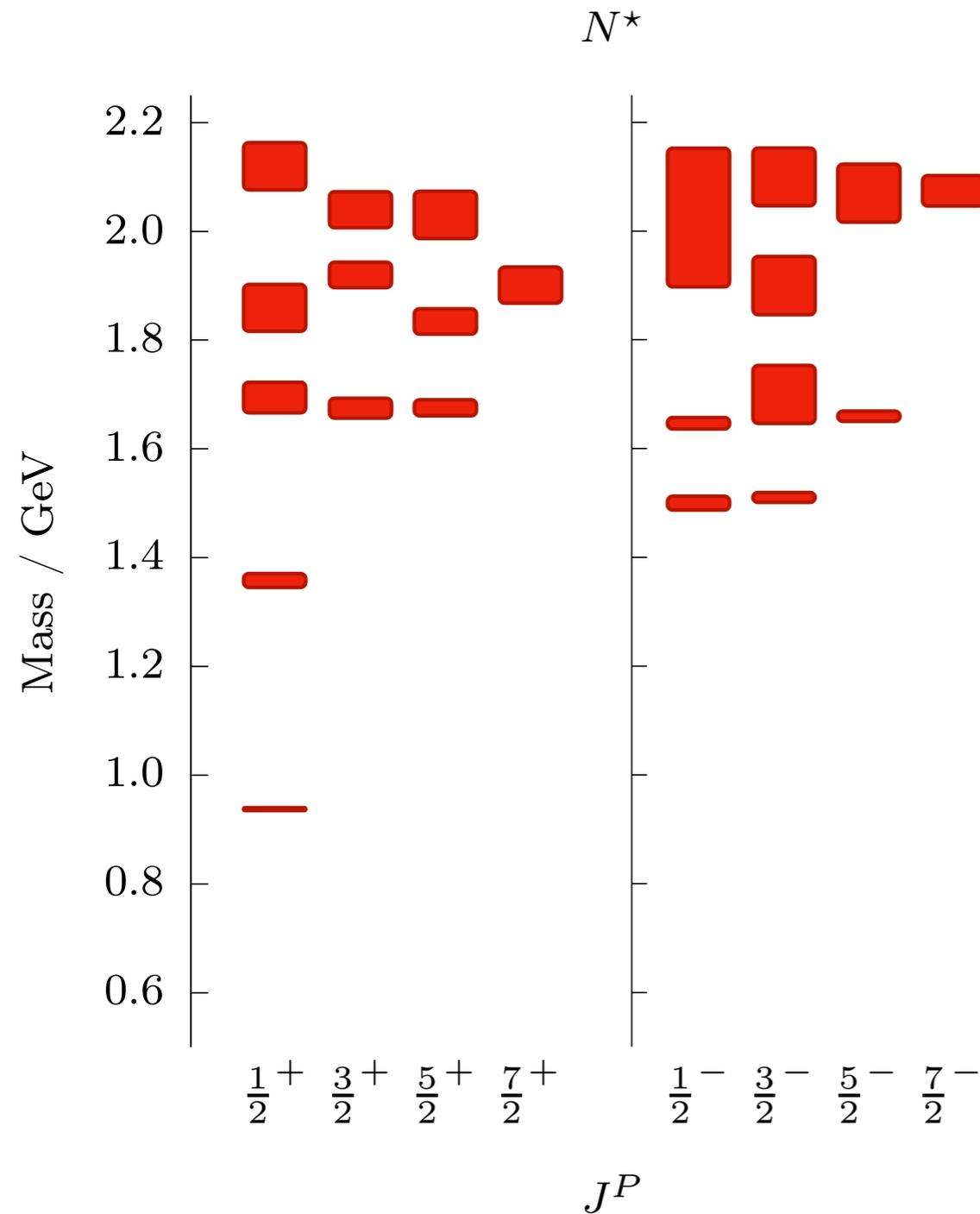
$$\langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle = \sum_{\mathbf{n}} |\langle 0 | \mathcal{O} | \mathbf{n} \rangle|^2 e^{-E_{\mathbf{n}} \tau}$$



# Few-Body Physics from QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

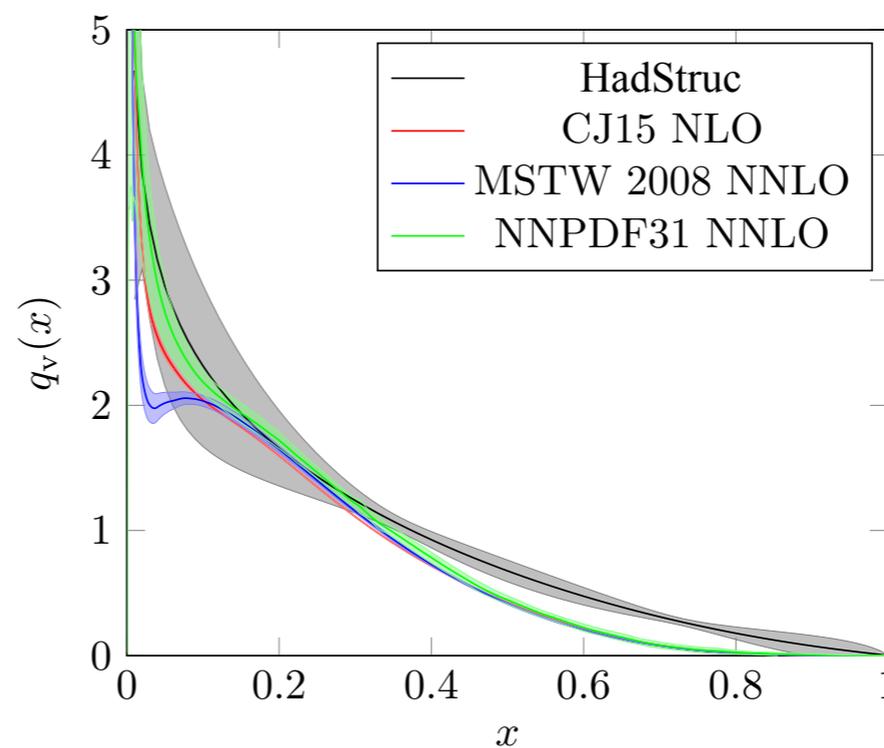
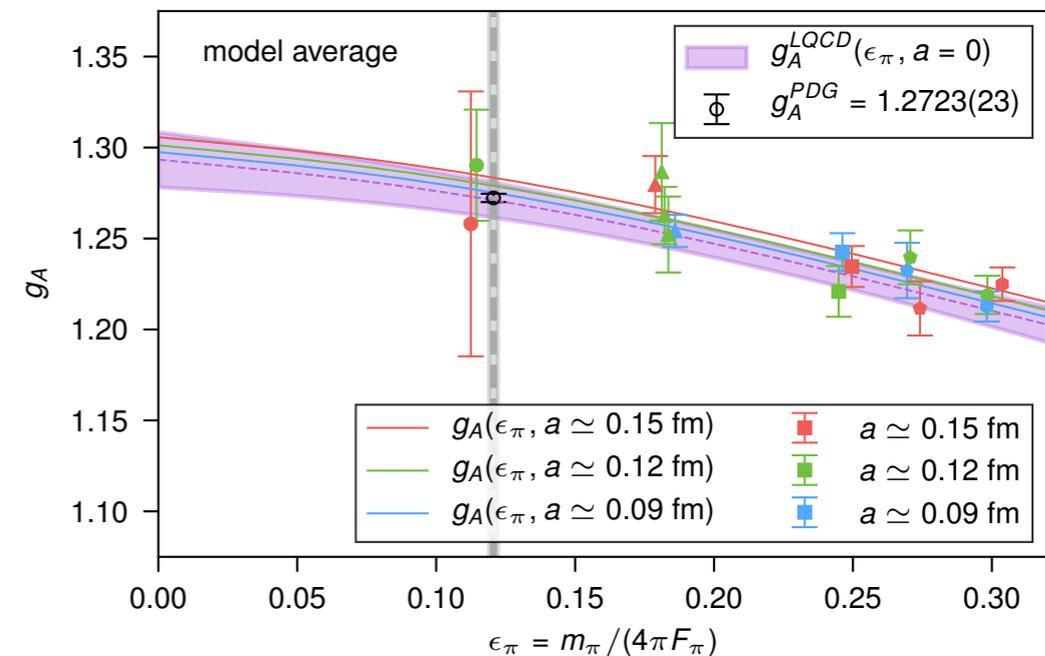
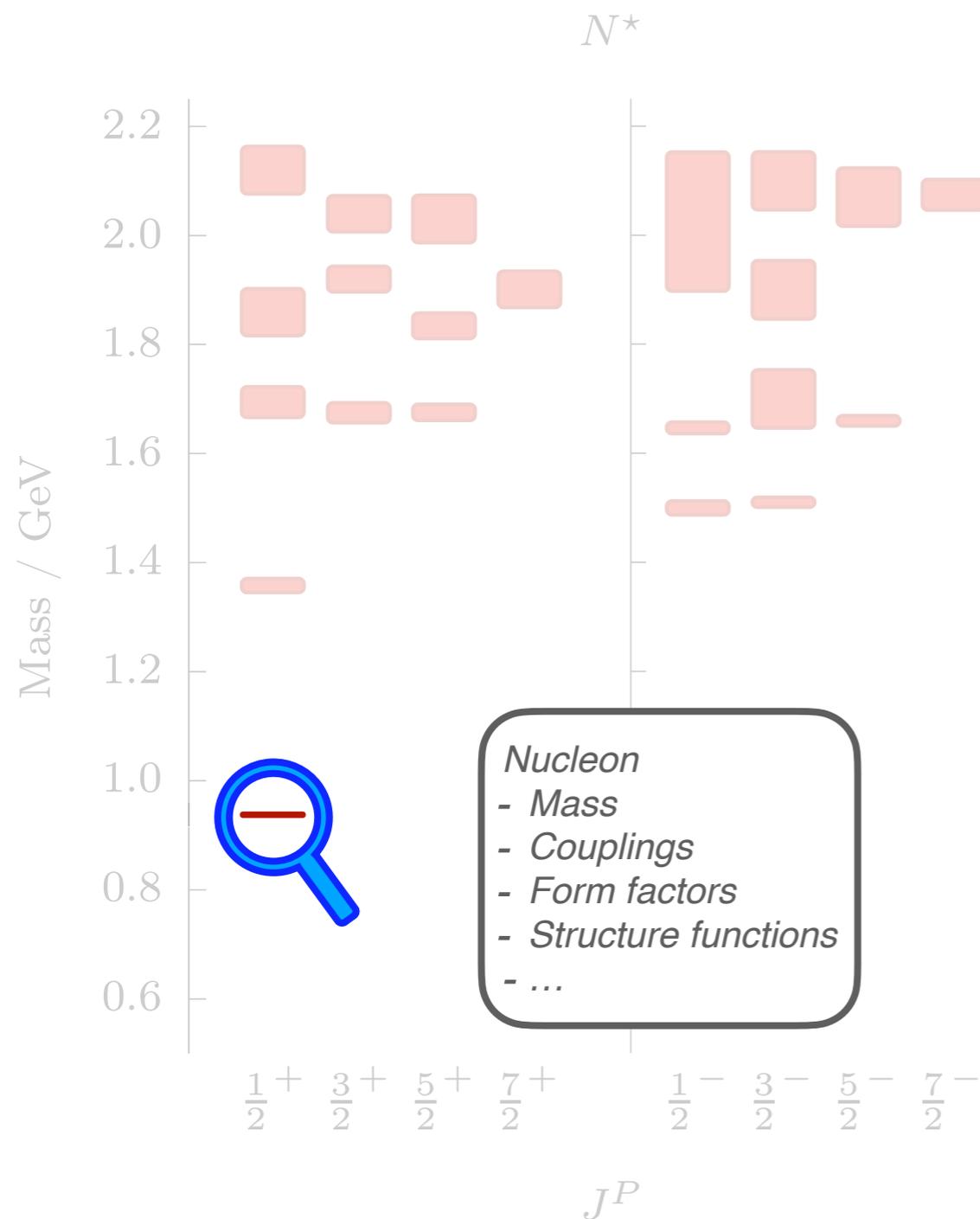
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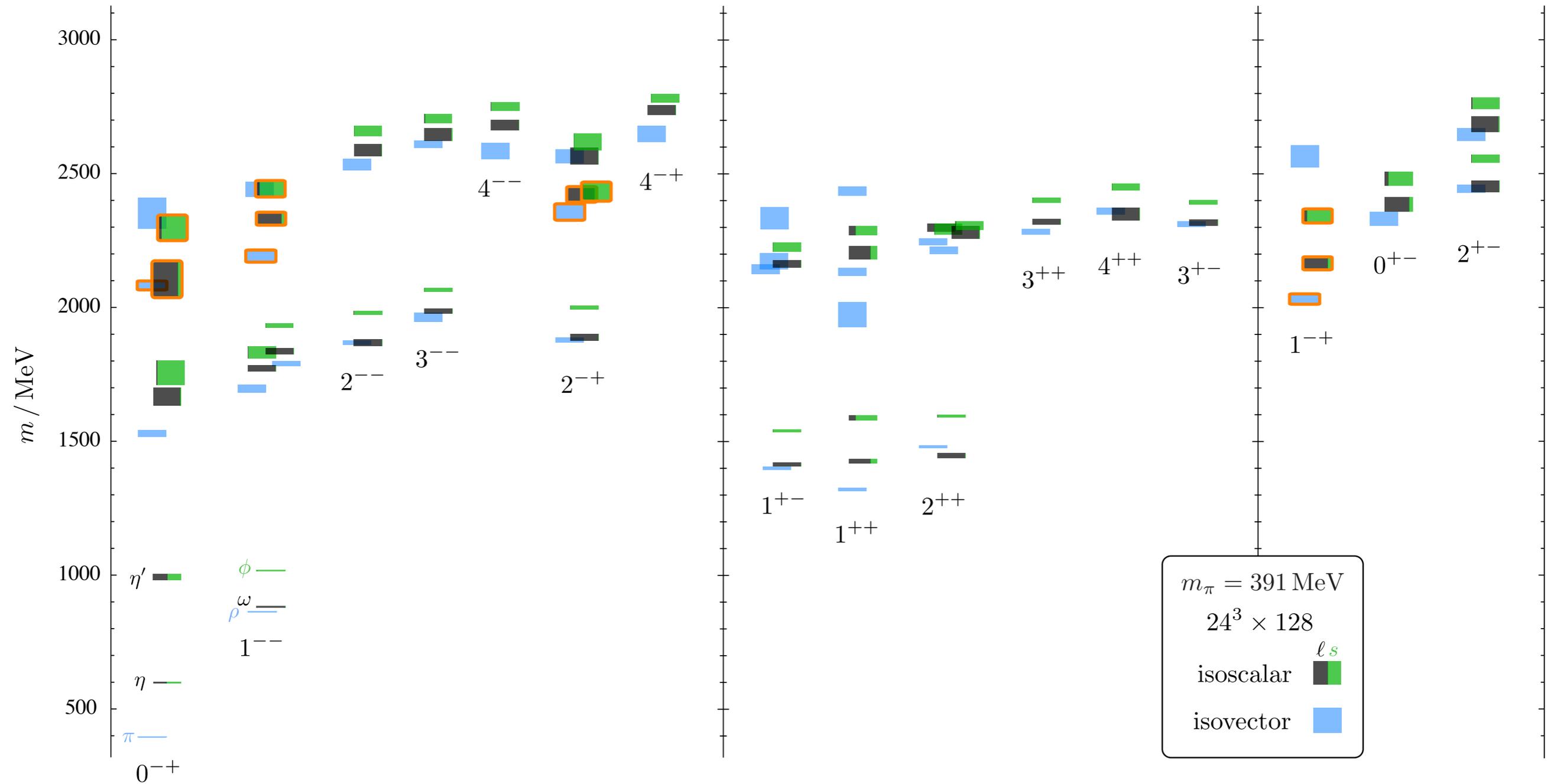
C. Chang et al.  
Nature **558**, (2018) 7708

B. Joó et al. [HadStruc]  
Phys. Rev. Lett. **125**, (2020) 232003

# Lattice QCD

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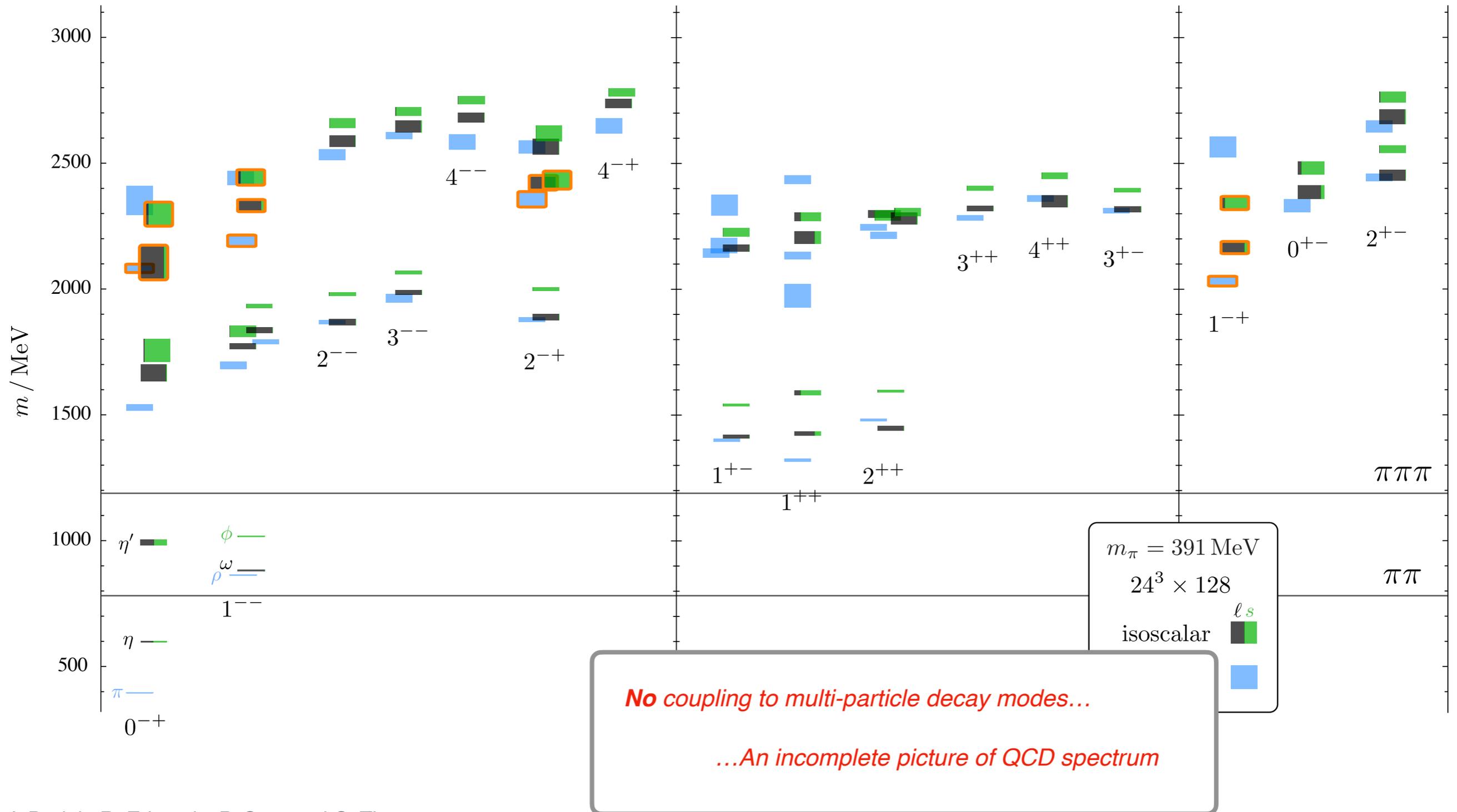
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# Scattering Theory & QCD Spectrum

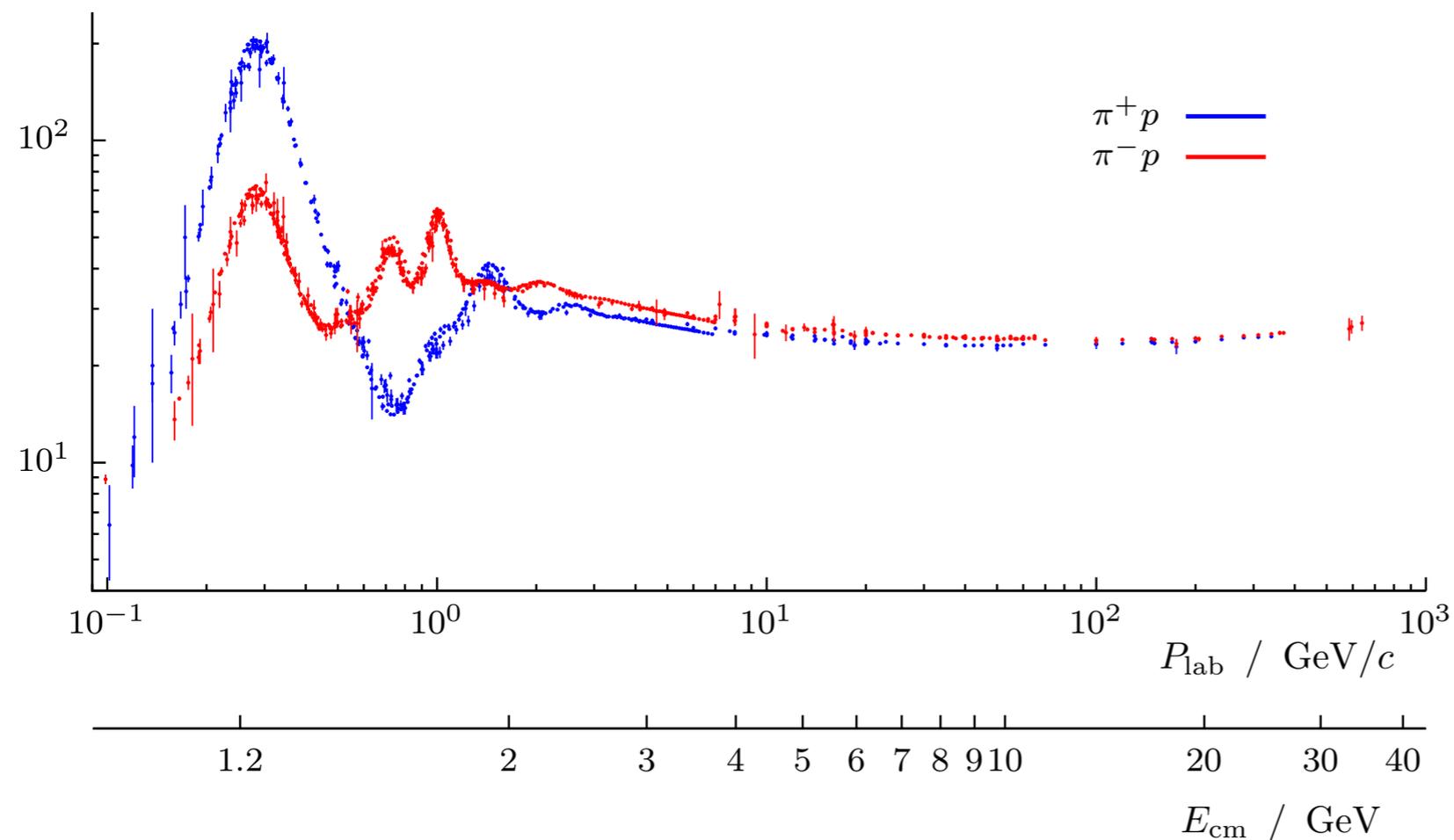
Excited hadronic states are not stable particles...

...but are **resonances** coupling to multi-particle decay channels

...need to understand scattering amplitudes and properties

*e.g.,  $\pi N$  scattering and the excited Nucleon/Delta spectrum*

$\sigma_{\text{tot}} / \text{mb}$



# Scattering Theory & QCD Spectrum

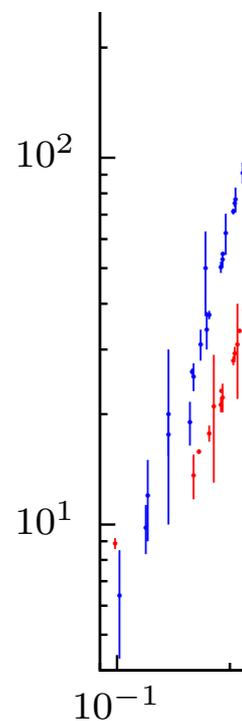
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$\sigma_{\text{tot}} / \text{mb}$



*Need to rigorously define bound & resonant states within **scattering theory***

## **Symmetry**

Lorentz invariance, CPT, Flavor, baryon number, ...

## **Unitarity**

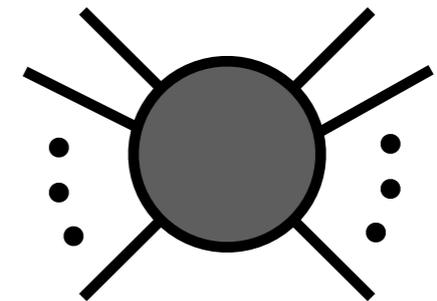
*Probability conservation*  $\implies$  The  $S$  matrix is a unitary operator

## **Analyticity**

*Causality*  $\implies$  Amplitudes are boundary values of analytic functions in complex energy plane

## **Crossing**

*CPT symmetry*  $\implies$  Relates particle–anti-particles in scattering processes



$E_{\text{cm}} / \text{GeV}$

# Scattering Theory & QCD Spectrum

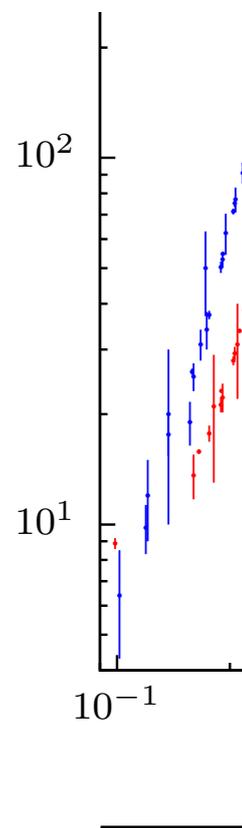
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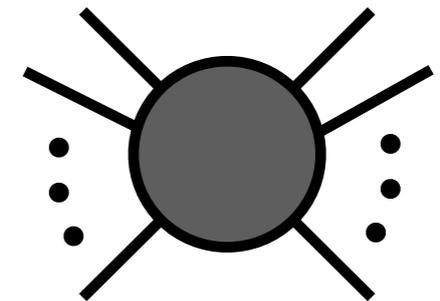
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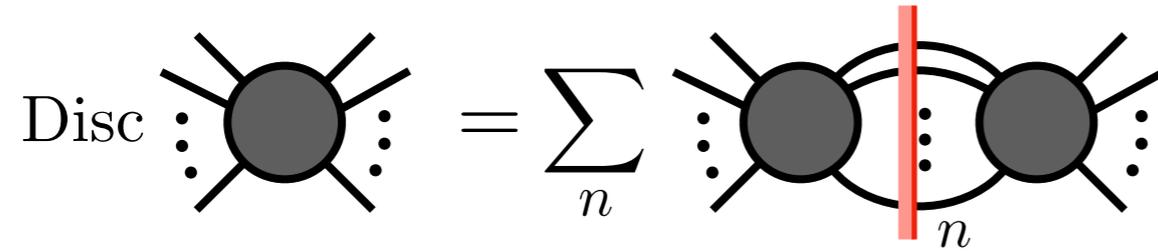
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$E_{\text{cm}} / \text{GeV}$

# Scattering Theory & QCD Spectrum

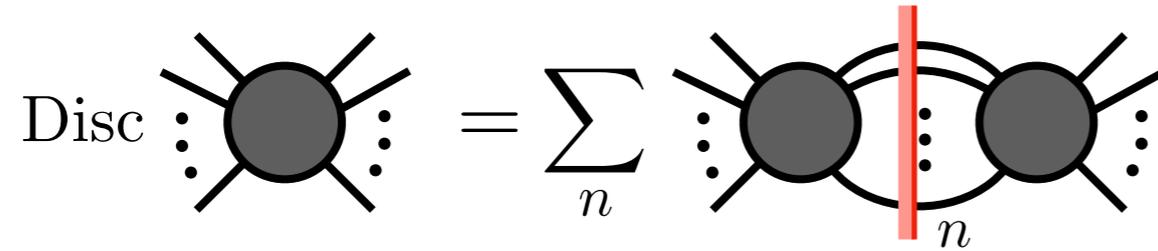
S matrix unitarity ( $S^\dagger S = 1$ ) enforces restricted analytic structure



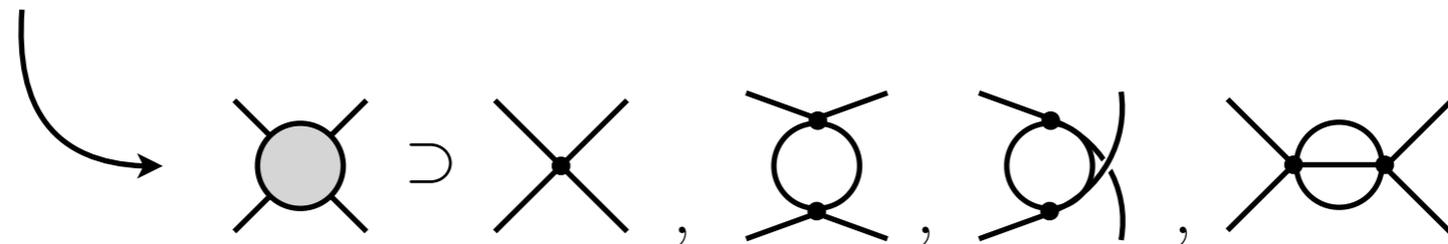
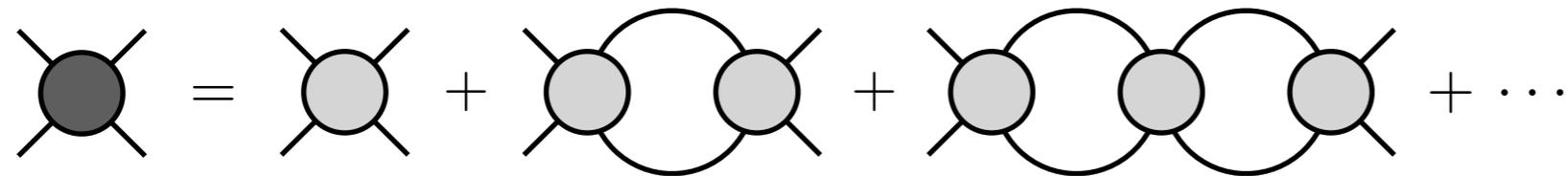


# Scattering Theory & QCD Spectrum

S matrix unitarity ( $S^\dagger S = 1$ ) enforces restricted analytic structure



e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT



*Bethe-Salpeter kernels*

*All 2PI diagrams - left hand cuts & higher multi-particle thresholds*

# Scattering Theory & QCD Spectrum

S matrix unitarity ( $S^\dagger S = 1$ ) enforces restricted analytic structure

$$\text{Disc} \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array} = \sum_n \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array}$$

e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT

$$\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \circ \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \circ \quad \circ \quad \circ \\ \diagdown \quad \diagup \end{array} + \dots$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \circ \text{ PV } \circ \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \circ \quad \circ \\ \diagdown \quad \diagup \end{array}$$

$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\text{th}}}$$

# Scattering Theory & QCD Spectrum

S matrix unitarity ( $S^\dagger S = 1$ ) enforces restricted analytic structure

$$\text{Disc} \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array} = \sum_n \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \\ \vdots \quad \vdots \end{array}$$

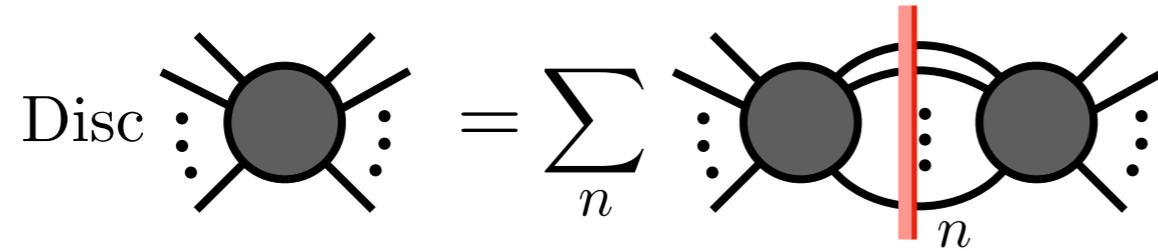
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*K-matrix* — All short-distance physics which cannot go on-shell  
 — Unknown! — theory specific

# Scattering Theory & QCD Spectrum

S matrix unitarity ( $S^\dagger S = 1$ ) enforces restricted analytic structure



e.g.,  $2 \rightarrow 2$  scattering within some generalized EFT

On-shell representation of scattering amplitude

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$

$2m$

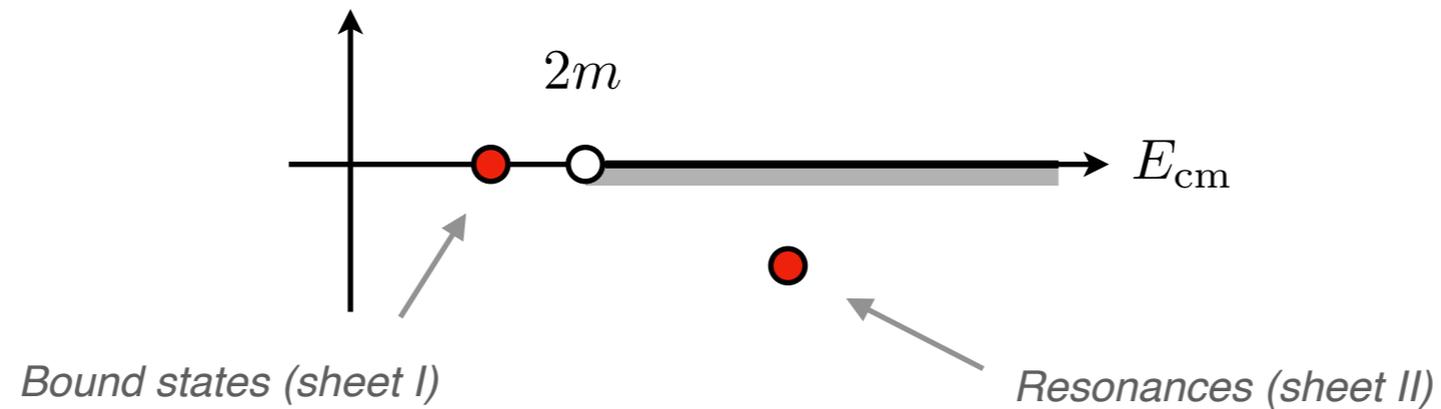
$E_{\text{cm}}$

Scattering Region

# Scattering Theory & QCD Spectrum

Resonances & Bound states are pole singularities of scattering amplitudes

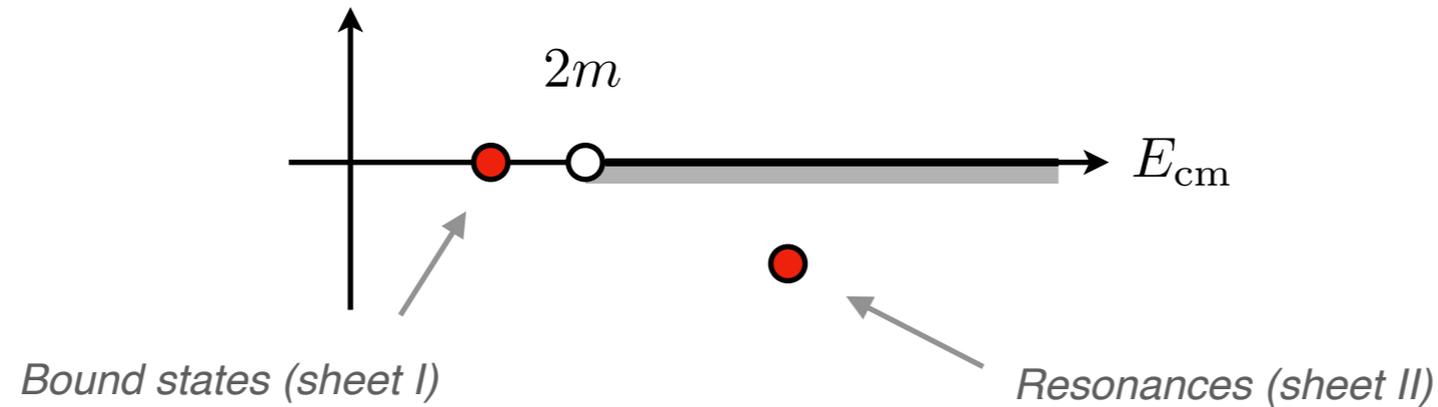
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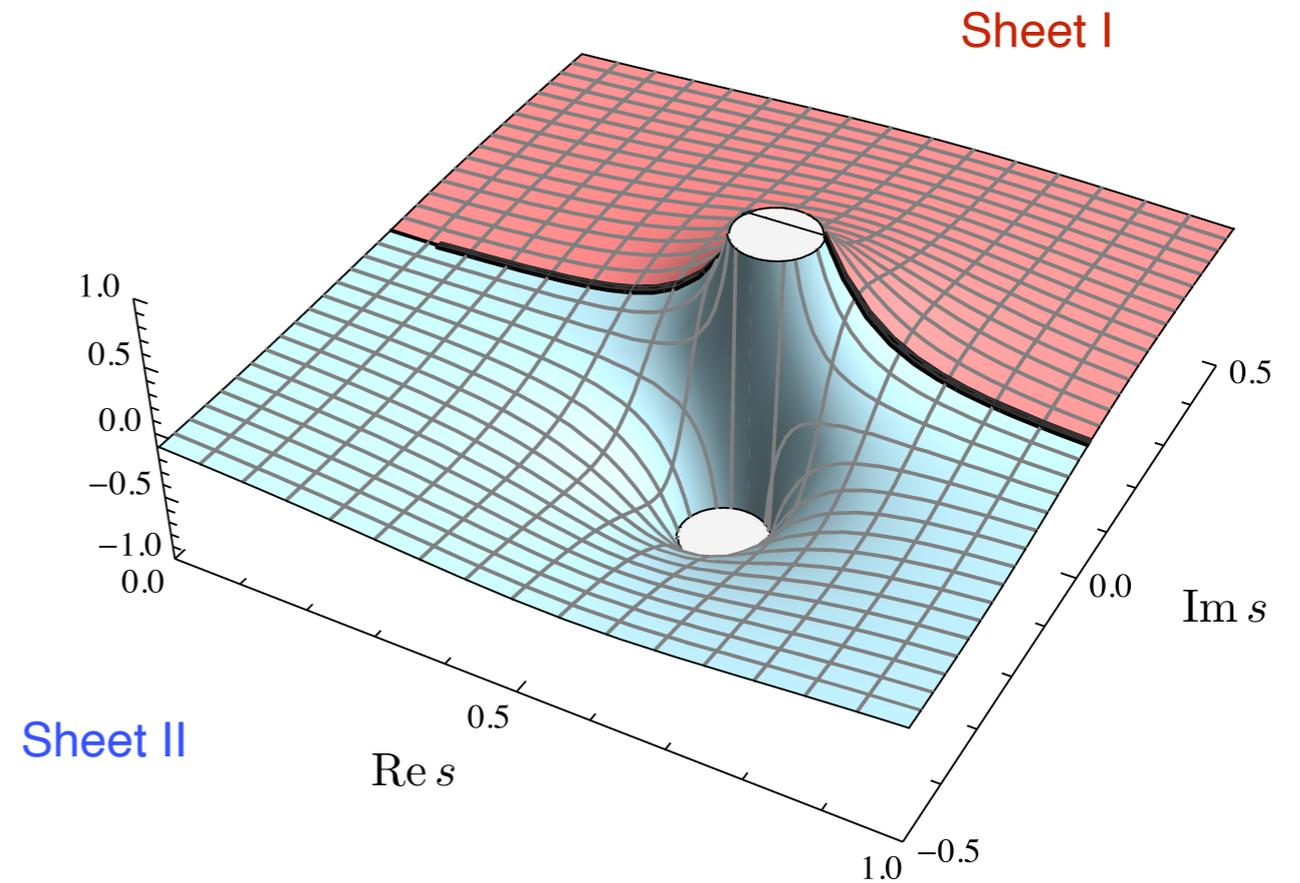
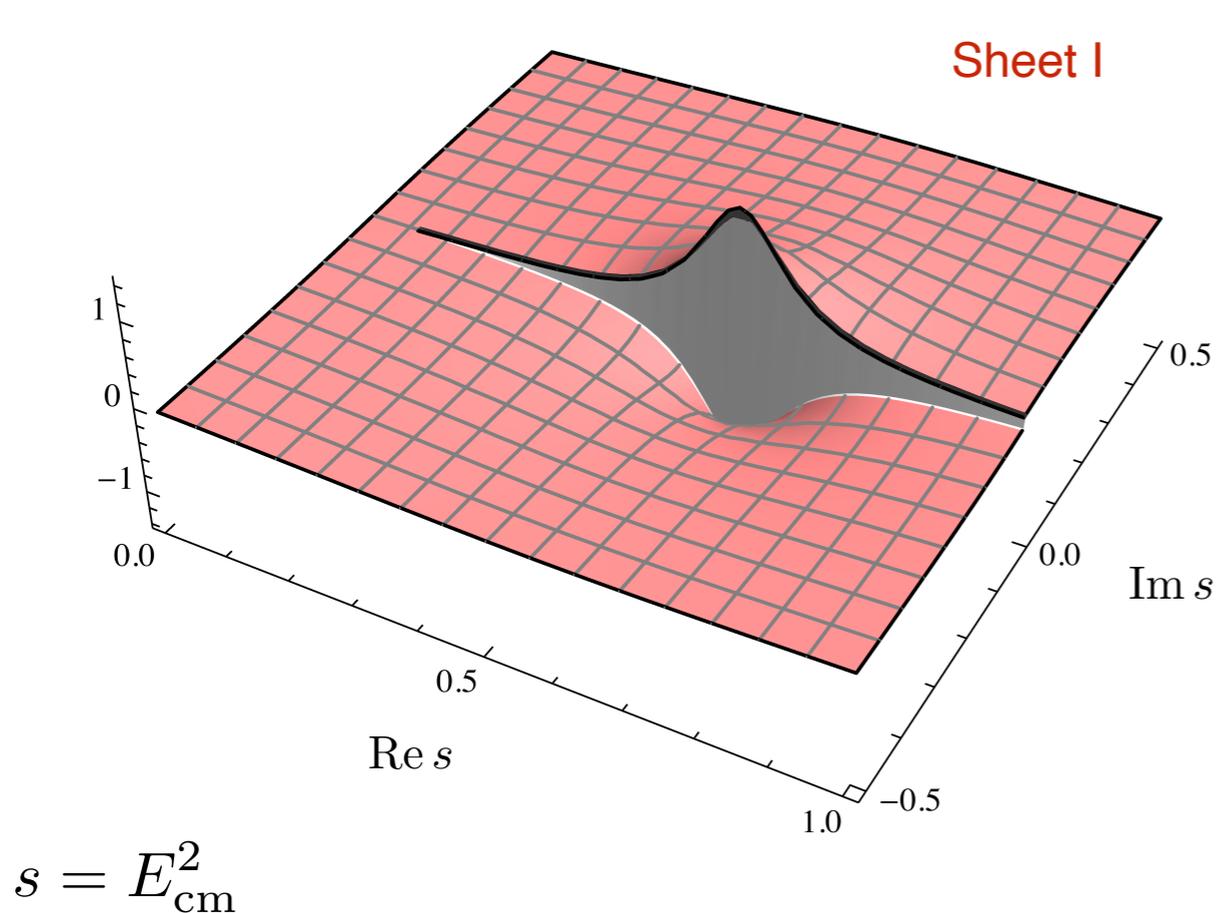
# Scattering Theory & QCD Spectrum

Resonances & Bound states are pole singularities of scattering amplitudes

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$



e.g., Narrow resonance

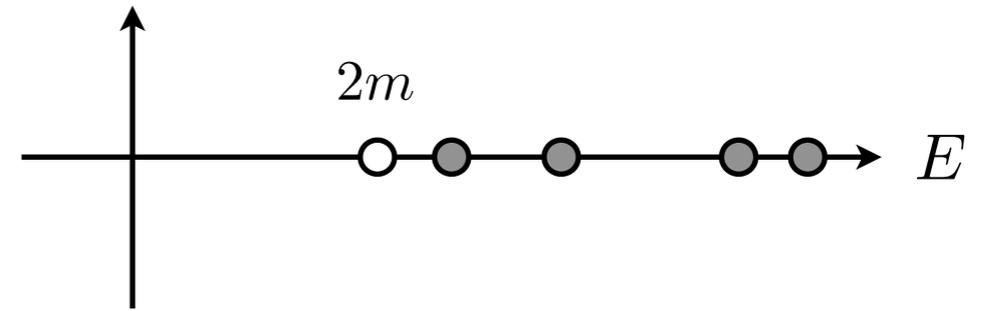


$$s = E_{\text{cm}}^2$$

# Connecting Scattering Physics to QCD

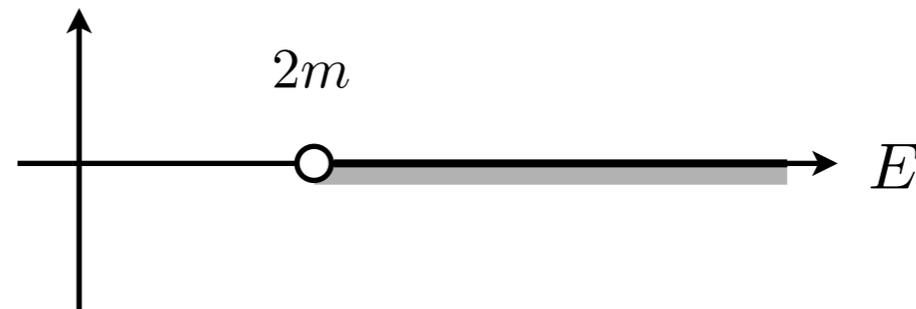
Q: How do we connect a finite-volume spectrum computed from QCD...

$$\int_L d^4x e^{iP \cdot x} \langle \mathcal{O}(t) \mathcal{O}^\dagger(0) \rangle = \sum_n \frac{i |\langle 0 | \mathcal{O} | \mathbf{n} \rangle|^2}{E - E_n}$$



...to infinite-volume scattering amplitudes?

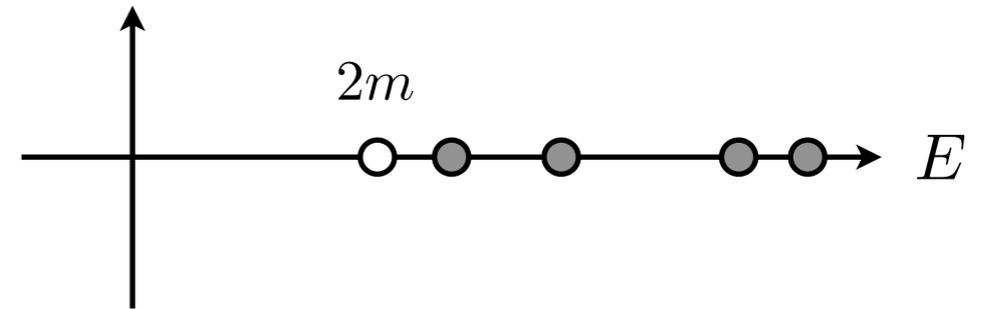
$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$



# Connecting Scattering Physics to QCD

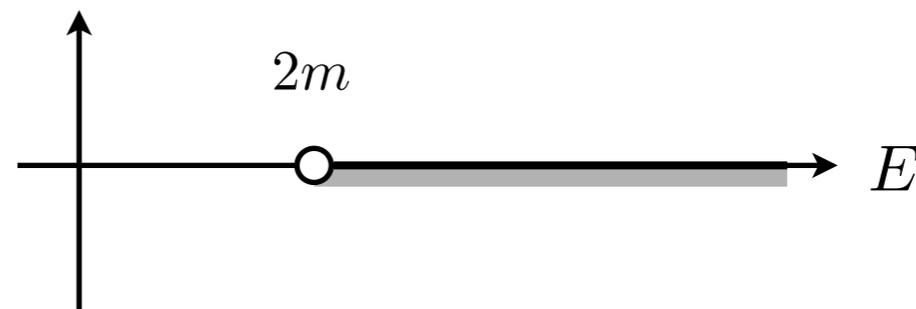
**Q:** How do we connect a finite-volume spectrum computed from QCD...

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...to infinite-volume scattering amplitudes?

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$



**A:** Correct analytic structure of finite-volume correlators

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows the two-point correlator  $C_L(P)$  as a sum of Feynman diagrams. The first diagram is a single loop with two external legs, labeled 'V'. The second diagram is a two-loop diagram with two external legs, labeled 'V', and a central shaded vertex. The third diagram is a three-loop diagram with two external legs, labeled 'V', and two central shaded vertices. The series continues with an ellipsis.

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$C_L(P) = \text{[Diagram: circle with V]} + \text{[Diagram: two circles with V]} + \text{[Diagram: three circles with V]} + \dots$$

$$\text{[Diagram: circle with V]} = \text{[Diagram: circle with } \infty \text{]} + \left[ \text{[Diagram: circle with V]} - \text{[Diagram: circle with } \infty \text{]} \right]$$

$$= \text{[Diagram: circle with } \infty \text{]} + \text{[Diagram: circle with V and a red vertical line through it]}$$

$i\rho + F_L$

$$F_L(P) = \left[ \frac{1}{L^3} \sum_{\mathbf{k}} -\text{PV} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right] \frac{1}{2\omega_k} \frac{1}{(P - k)^2 - m^2 + i\epsilon} \Big|_{k^0 = \omega_k}$$

*Geometric function — characterizes finite-volume distortions*

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned}
 C_L(P) &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 &= C(P) + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A circle with two external grey circles, labeled 'V'.
- Diagram 2: Two circles connected at a central grey circle, with two external grey circles.
- Diagram 3: Three circles connected in a chain at two central grey circles, with two external grey circles.
- Diagram 4: A circle with two external black circles, with a vertical red line through the center, labeled 'V'.
- Diagram 5: Two circles connected at a central black circle, with two external black circles, and two vertical red lines through the circles, labeled 'V'.
- Diagram 6: Three circles connected in a chain at two central black circles, with two external black circles, and three vertical red lines through the circles, labeled 'V'.

$$\text{Diagram 7} = \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \dots$$

The diagrams are as follows:

- Diagram 7: A central black circle with four external lines.
- Diagram 8: A central grey circle with four external lines.
- Diagram 9: A central grey circle with two external lines, connected to another grey circle via a loop with an infinity symbol.
- Diagram 10: A central grey circle with two external lines, connected to two grey circles in a chain via two loops with infinity symbols.

$$\text{Diagram 11} = \text{Diagram 12} + \text{Diagram 13} + \text{Diagram 14} + \dots$$

The diagrams are as follows:

- Diagram 11: A central black circle with three external lines.
- Diagram 12: A central grey circle with three external lines.
- Diagram 13: A central grey circle with one external line, connected to another grey circle via a loop with an infinity symbol.
- Diagram 14: A central grey circle with one external line, connected to two grey circles in a chain via two loops with infinity symbols.

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned}
 C_L(P) &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 &= C(P) + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 &= C(P) + i\mathcal{A} \frac{i}{\mathcal{M}_2^{-1} + i\rho + F_L} (i\rho + F_L) i\mathcal{A}
 \end{aligned}$$

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned}
 C_L(P) &= \text{[Diagram: circle with V and two grey dots]} + \text{[Diagram: two circles with V, V and two grey dots]} + \text{[Diagram: three circles with V, V, V and two grey dots]} + \dots \\
 &= C(P) + \text{[Diagram: circle with V and two black dots, red vertical lines]} + \text{[Diagram: two circles with V, V and two black dots, red vertical lines]} + \text{[Diagram: three circles with V, V, V and two black dots, red vertical lines]} + \dots \\
 &= C(P) + i\mathcal{A} \frac{i}{\mathcal{M}_2^{-1} + i\rho + F_L} (i\rho + F_L) i\mathcal{A}
 \end{aligned}$$

Finite-Volume poles must match!

$$\det \left( \mathcal{M}_2^{-1} + i\rho + F_L \right)_{E=E_n} = 0$$

Lüscher quantization condition

# Connecting Scattering Physics to QCD

Two-point correlator to all-orders

$$\begin{aligned}
 C_L(P) &= \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 &= C(P) + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 &= C(P) + i\mathcal{A} \frac{i}{\mathcal{M}_2^{-1} + i\rho + F_L} (i\rho + F_L) i\mathcal{A}
 \end{aligned}$$

Finite-Volume poles must match!

$$\det \left( \mathcal{M}_2^{-1} + i\rho + F_L \right)_{E=E_n} = 0$$

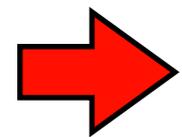
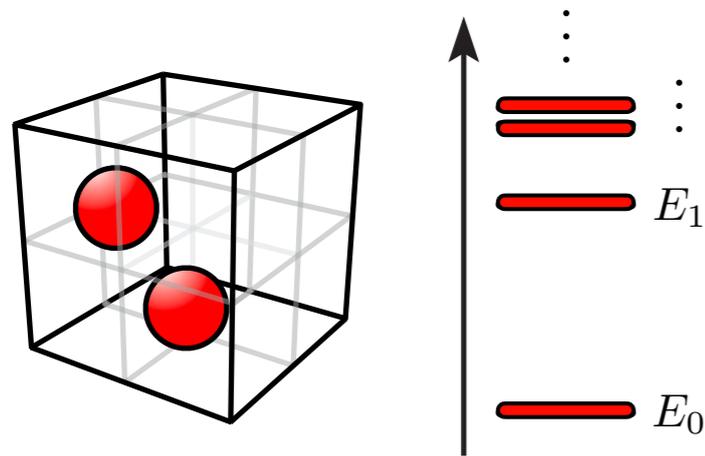
Lüscher quantization condition

$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$

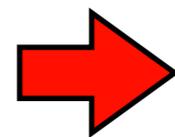
$$\det \left( 1 + \mathcal{K}_2 F_L \right)_{E=E_n} = 0$$

# Few-Body Physics & QCD

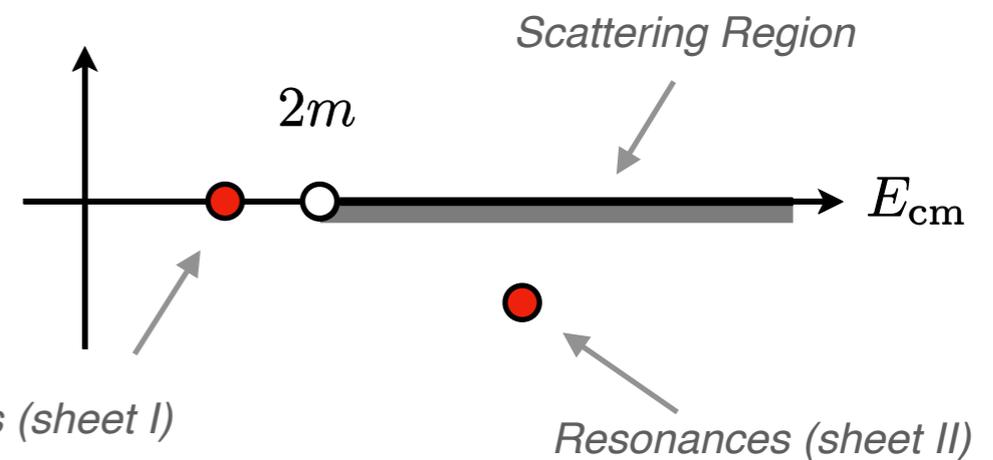
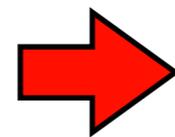
Employing scattering theory and EFTs to all-orders connects lattice QCD to scattering observables



$$\det ( 1 + \mathcal{K}_2 F_L )_{E=E_n} = 0$$



$$\mathcal{M}_2 = \mathcal{K}_2 \frac{1}{1 - i\rho \mathcal{K}_2}$$



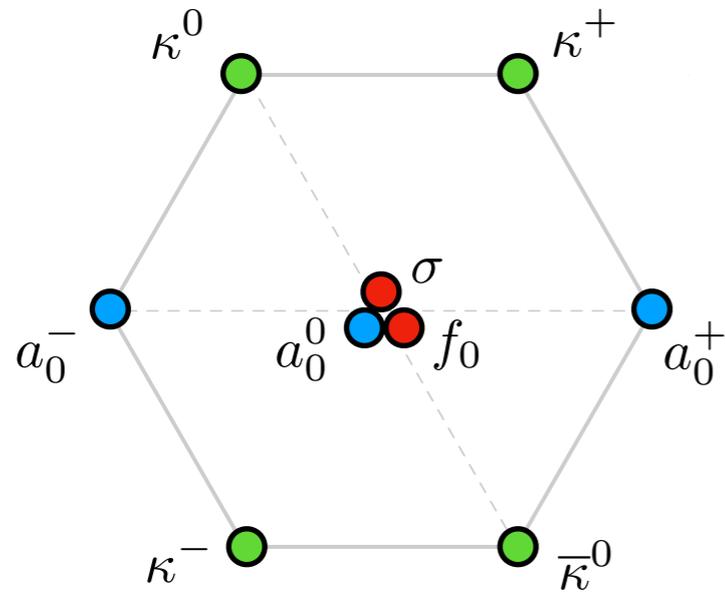
M. Lüscher  
Commun.Math.Phys. **105**, 153 (1986)  
Nucl.Phys. **B354**, 531 (1991)

Many others...

# Connecting Scattering Physics to QCD

Much success in two-body sector

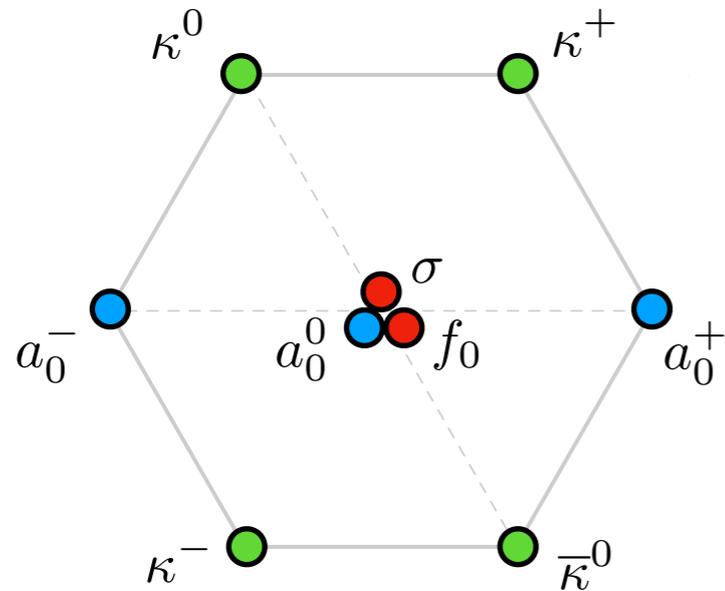
- e.g., *the Scalar nonet*



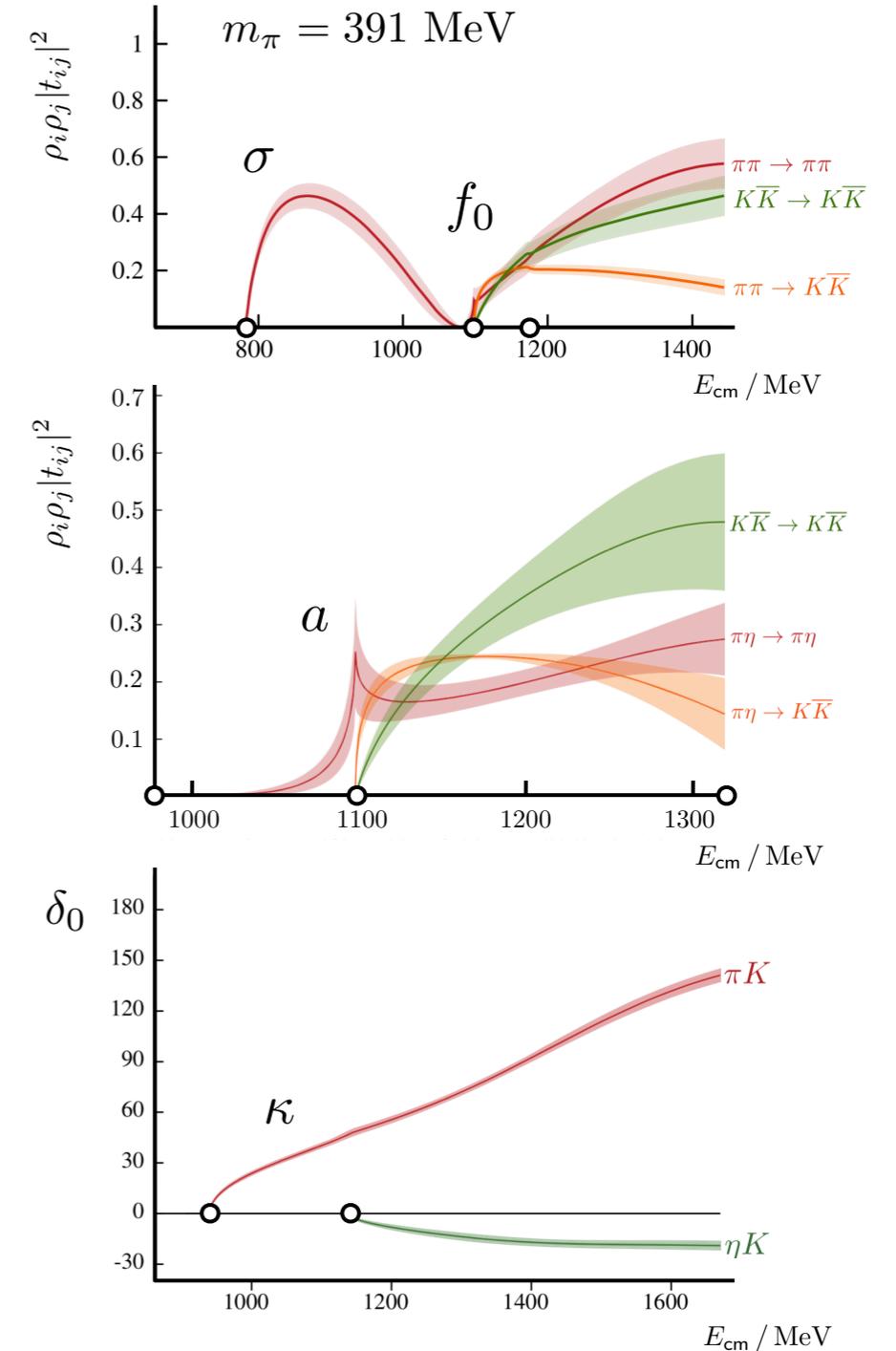
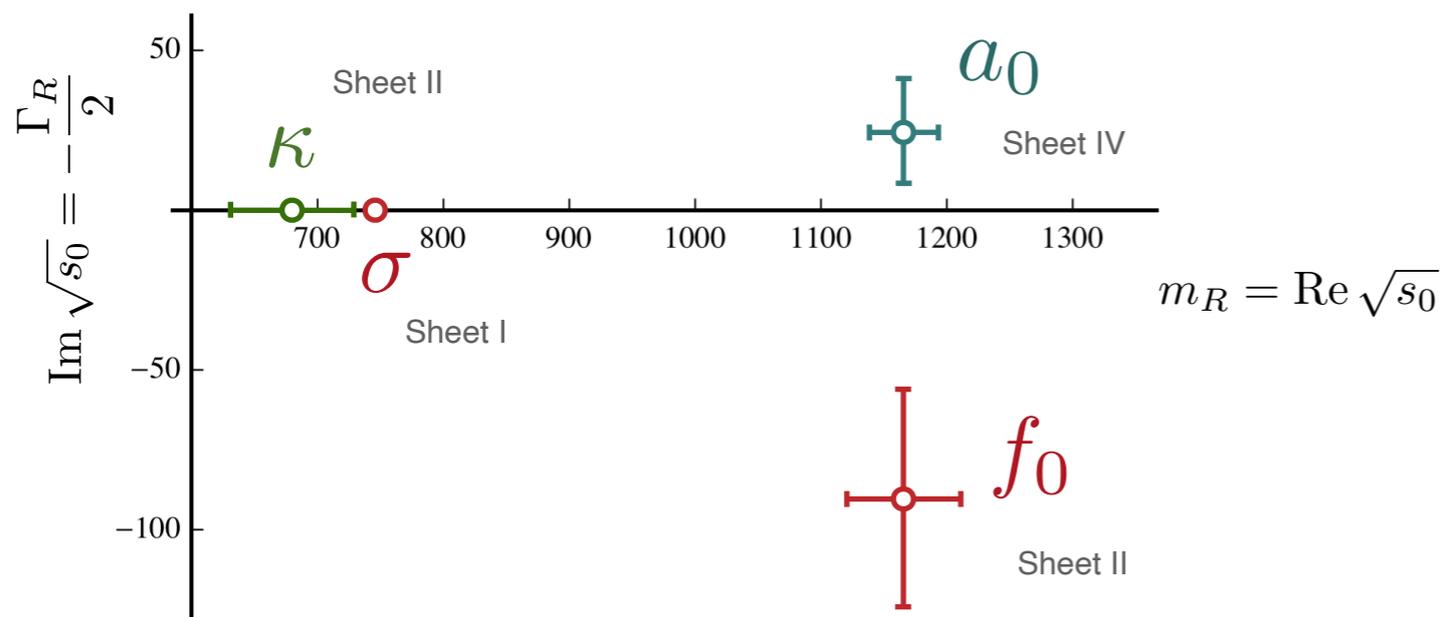
# Connecting Scattering Physics to QCD

Much success in two-body sector

- e.g., *the Scalar nonet*



$m_\pi = 391$  MeV



R.A. Briceño et al. [HadSpec]  
Phys. Rev. **D97**, 054513 (2018)

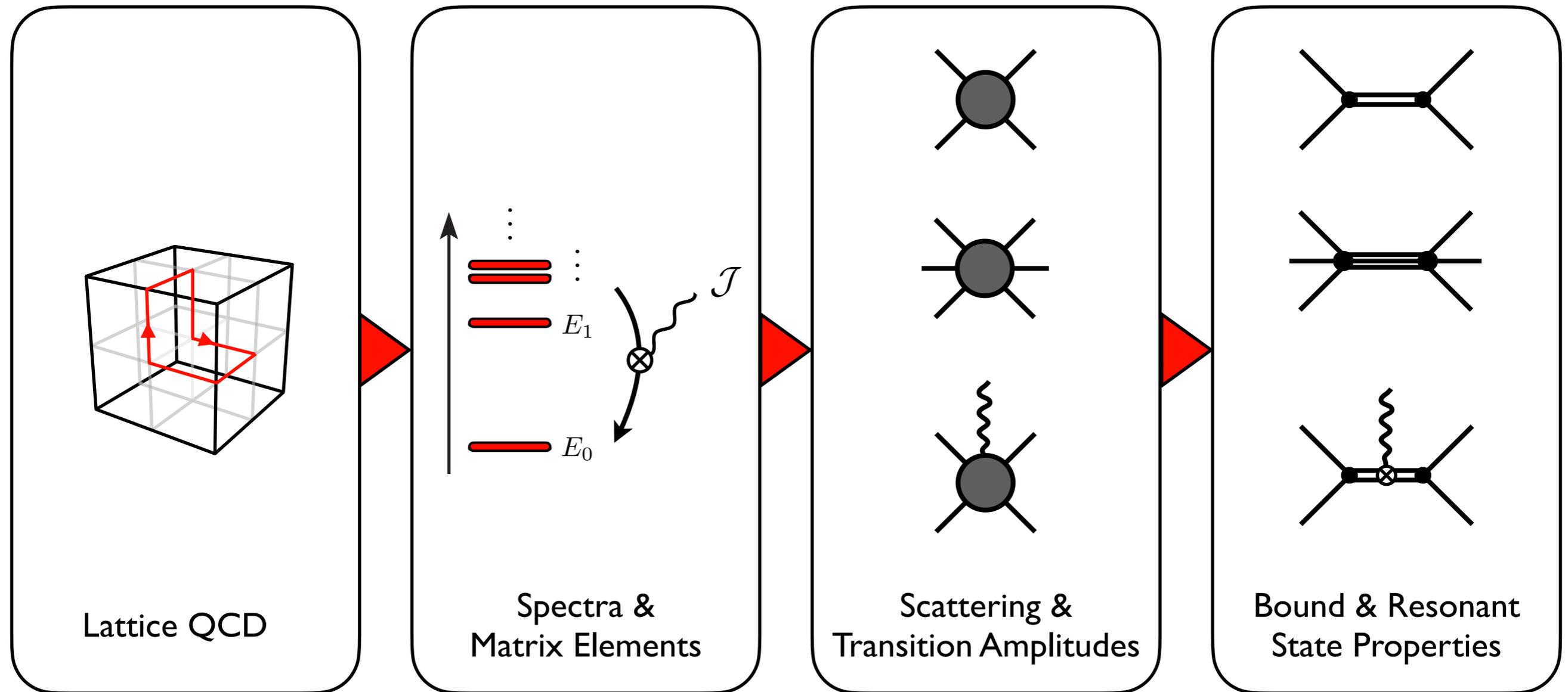
J.J. Dudek et al. [HadSpec]  
Phys. Rev. **D93**, 094506 (2016)

J.J. Dudek et al. [HadSpec]  
Phys. Rev. Lett. **113**, 182001 (2014)

# Few-Body Physics & QCD

Path to few-body physics from QCD

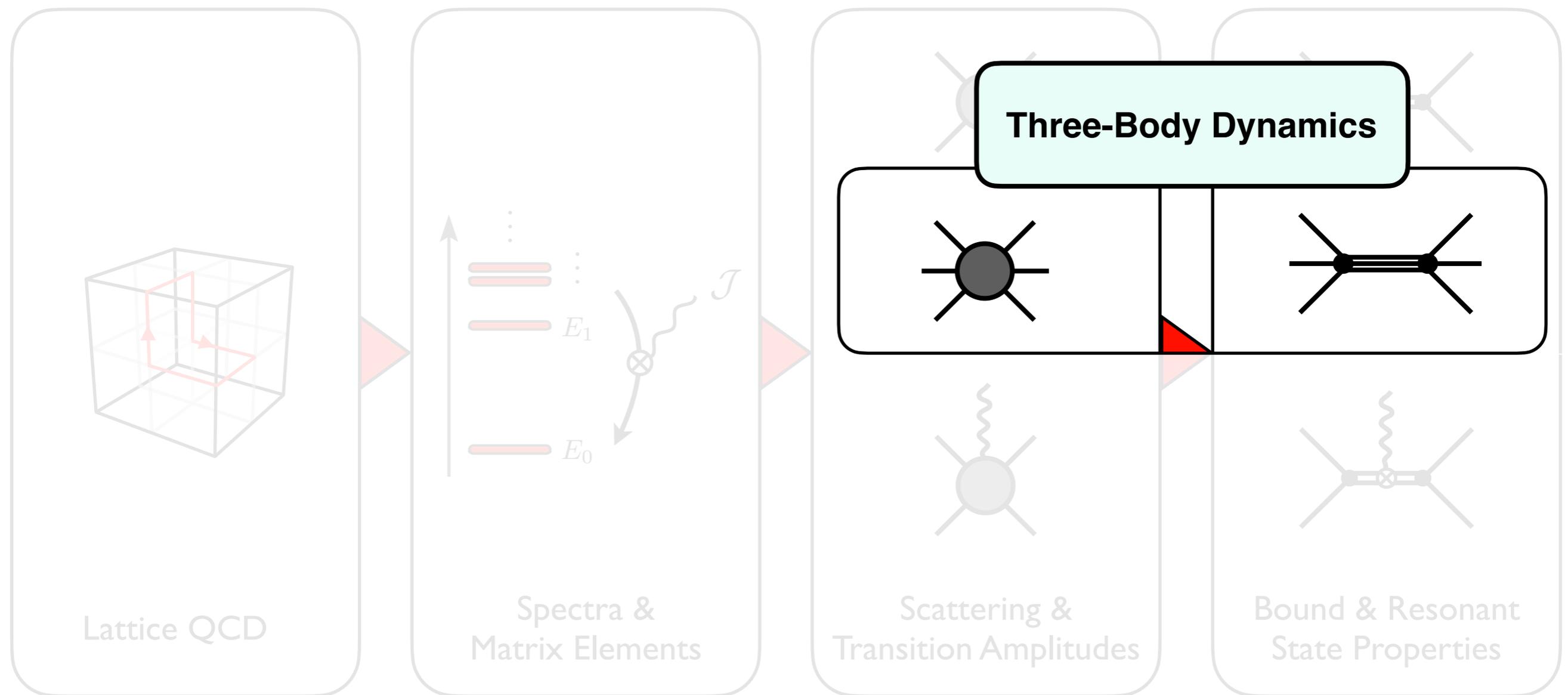
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Few-Body Physics & QCD

Path to few-body physics from QCD

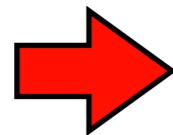
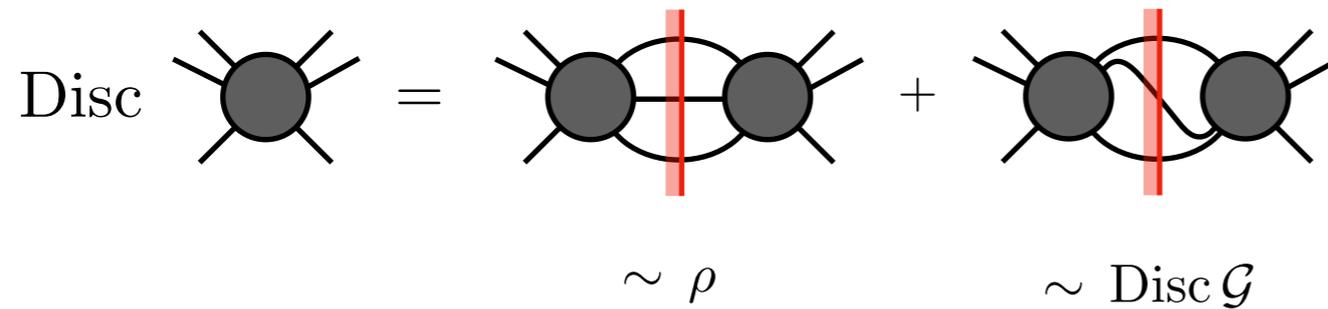
- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



# Three-Body Dynamics

## On-shell scattering relations

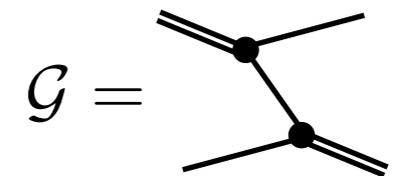
*Unitarity condition*



*On-shell scattering equation*

$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$

$\mathcal{K}_3$  Unknown!  
Obtained from Lattice QCD



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak  
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]  
Eur. Phys. J. C **79**, no. 1, 56 (2019)

AJ et al. [JPAC]  
Phys. Rev. D **100**, 034508 (2019)

AJ, *in preparation*

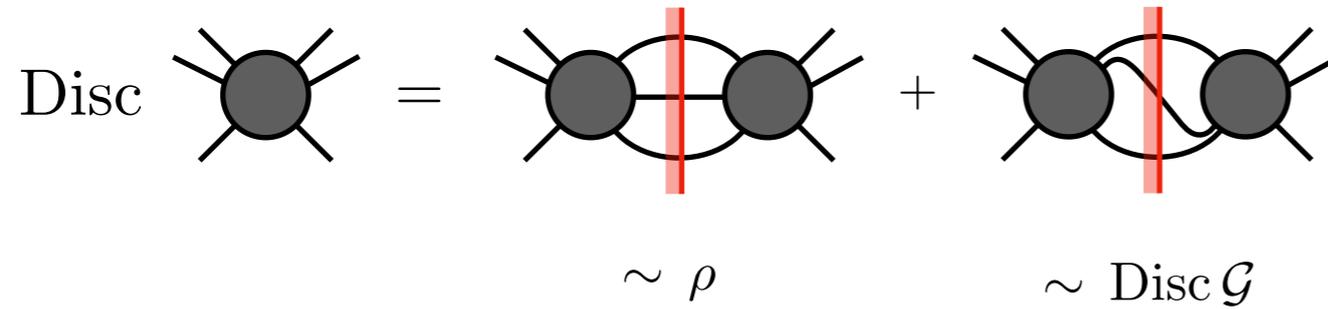
*cf. two-body case:*

$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_2$$

# Three-Body Dynamics

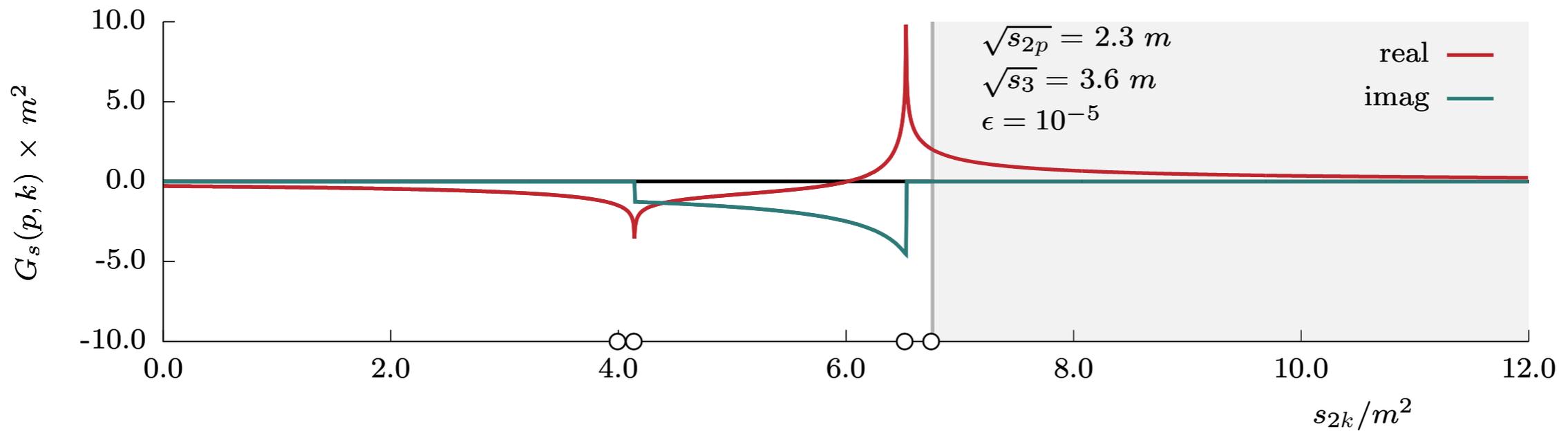
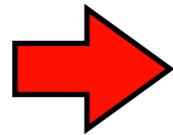
## On-shell scattering relations

Unitarity condition



On-shell scattering equation

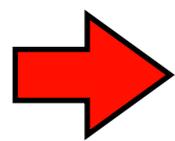
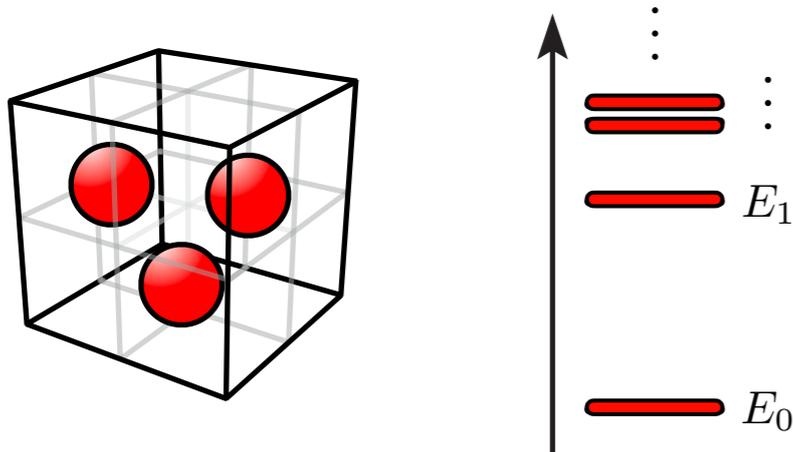
$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$



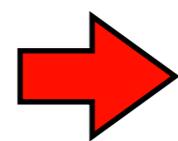
# Three-Body Dynamics

## Connecting to finite-volume spectra

*Finite-volume quantization condition*



$$\det \left( 1 + \mathcal{K}_3 ( \mathcal{F}_L + \mathcal{G}_L ) \right)_{E=E_n} = 0$$



$$\mathcal{M}_3 = \mathcal{K}_3 + \int \mathcal{K}_2 \cdot i\rho \cdot \mathcal{M}_3 + \iint \mathcal{K}_3 \cdot \mathcal{G} \cdot \mathcal{M}_3$$

M. Hansen and S. Sharpe  
Phys. Rev. D **90**, 116003 (2014), Phys. Rev. D **95**, 034501 (2017)

M. Mai and M. Döring  
Eur. Phys. J. A **53**, 240 (2017), Phys. Rev. Lett. **122**, 062503 (2019)

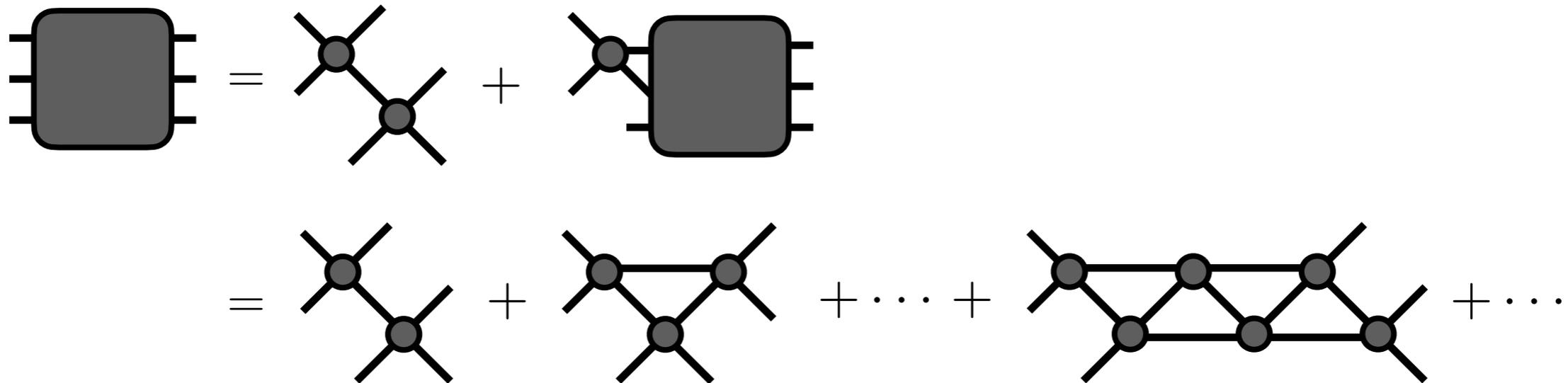
AJ, *in preparation*

# Three-Body Dynamics

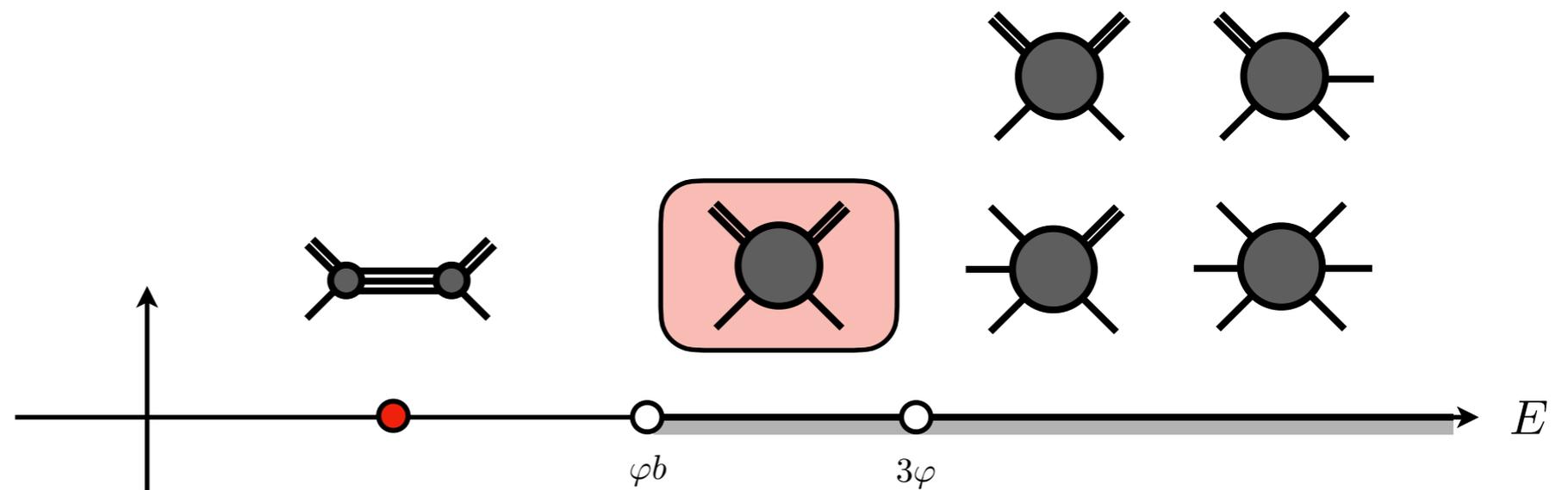
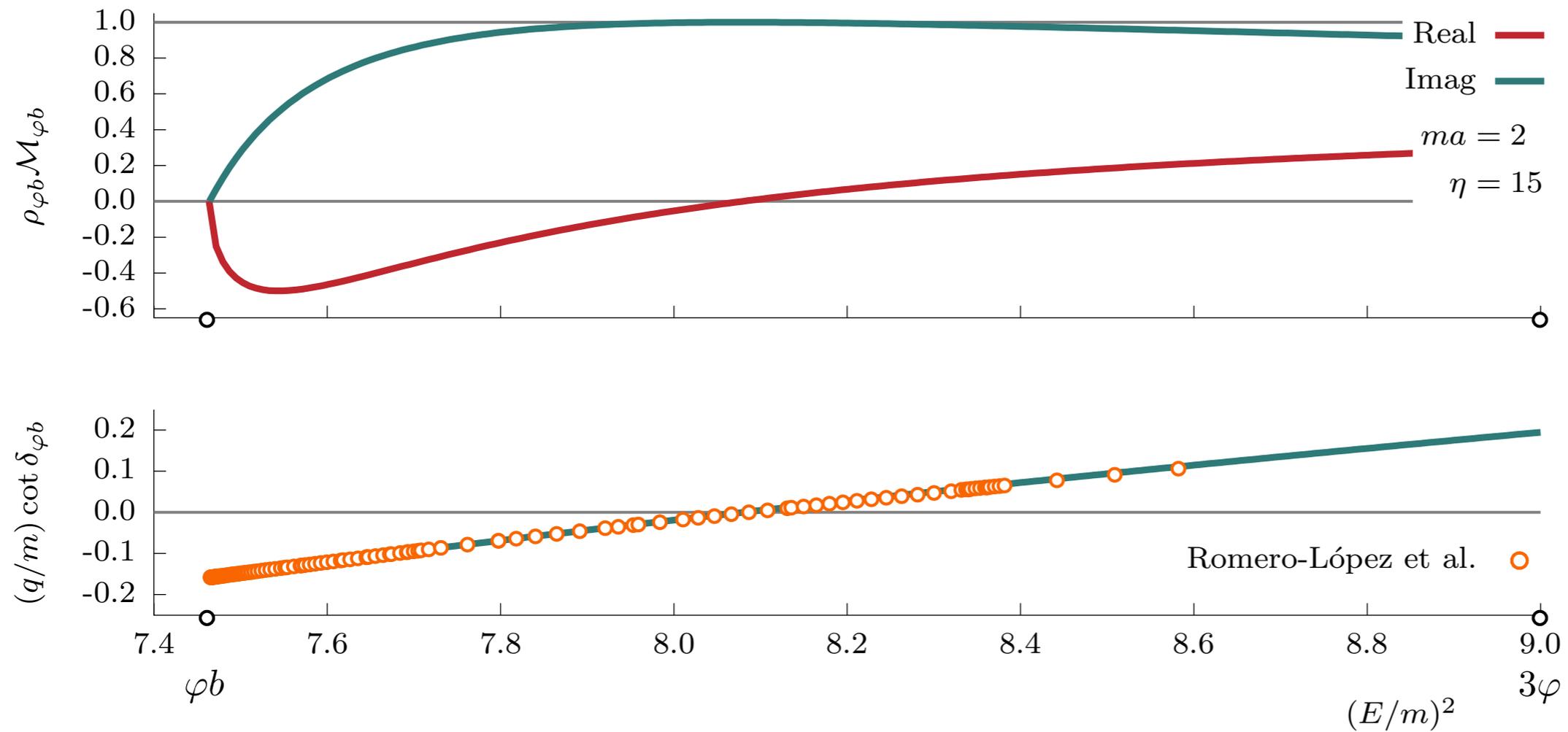
Examine toy-model —  $3\varphi \rightarrow 3\varphi$

- Assume exchange dominance — **No short-range three-body forces**
- Scalar system —  $J = 0$
- Two-hadron pair forms bound state —  $2\varphi \rightarrow b$

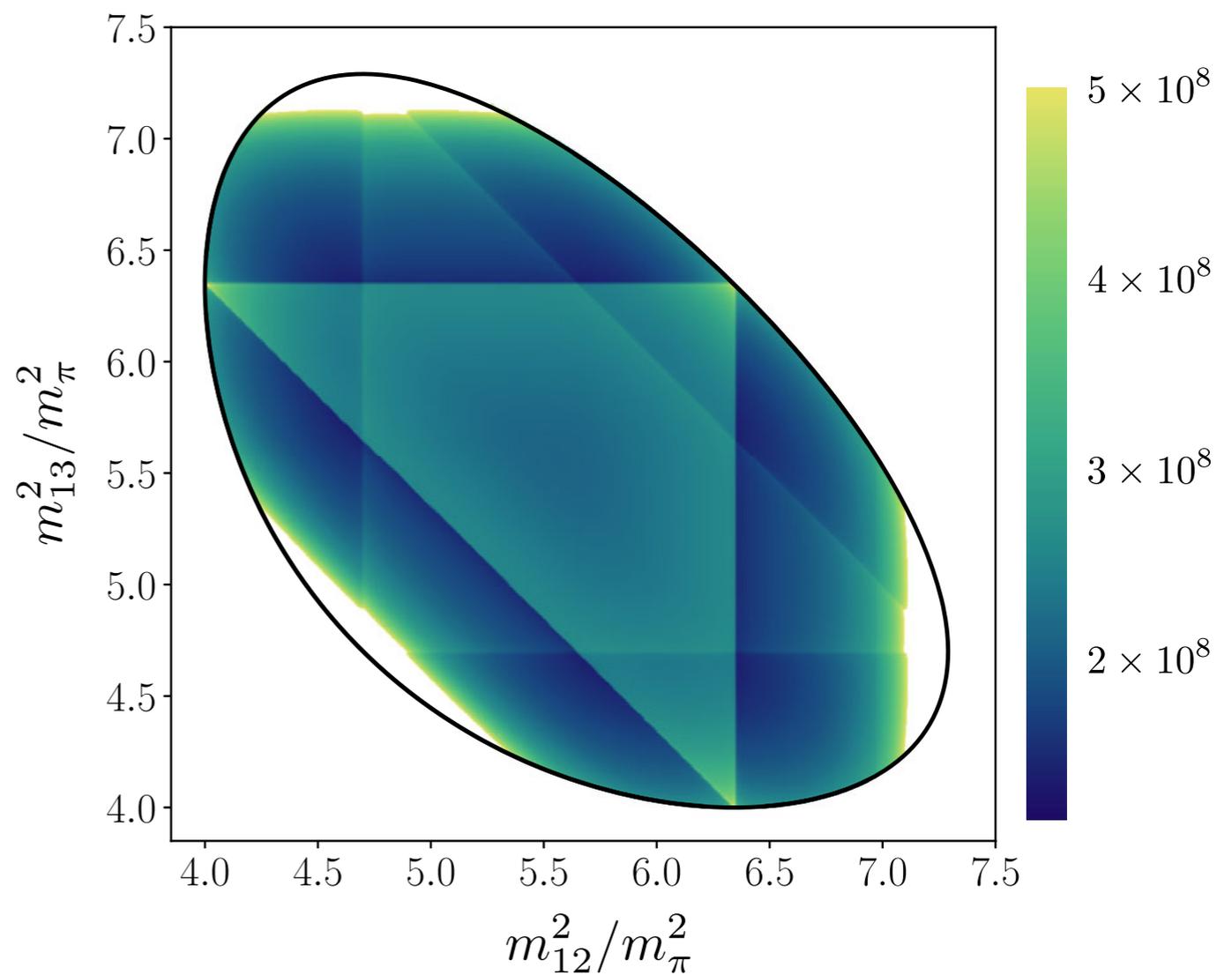
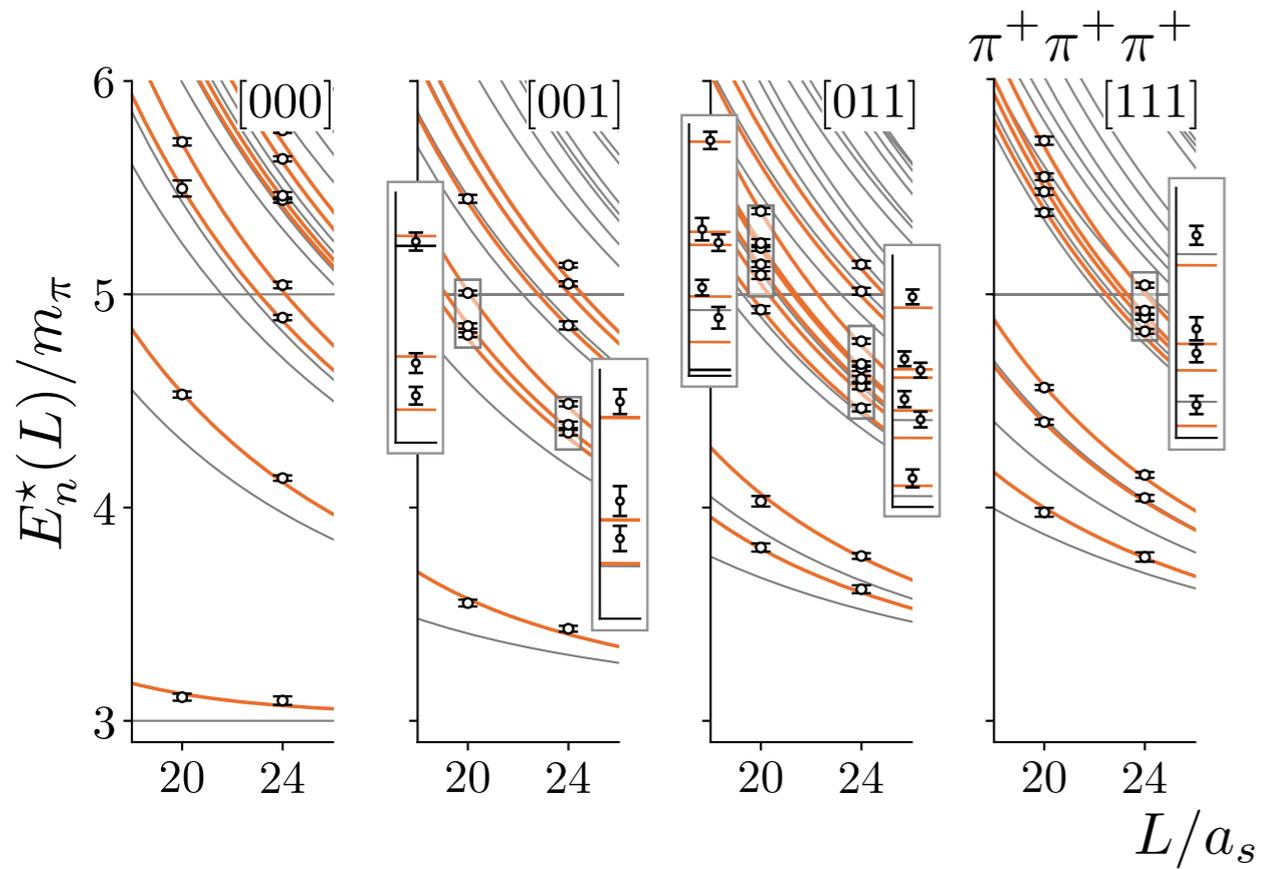
*Toy model version of  $3N \rightarrow 3N$  with  $2N \rightarrow d$*



# Three-Body Dynamics



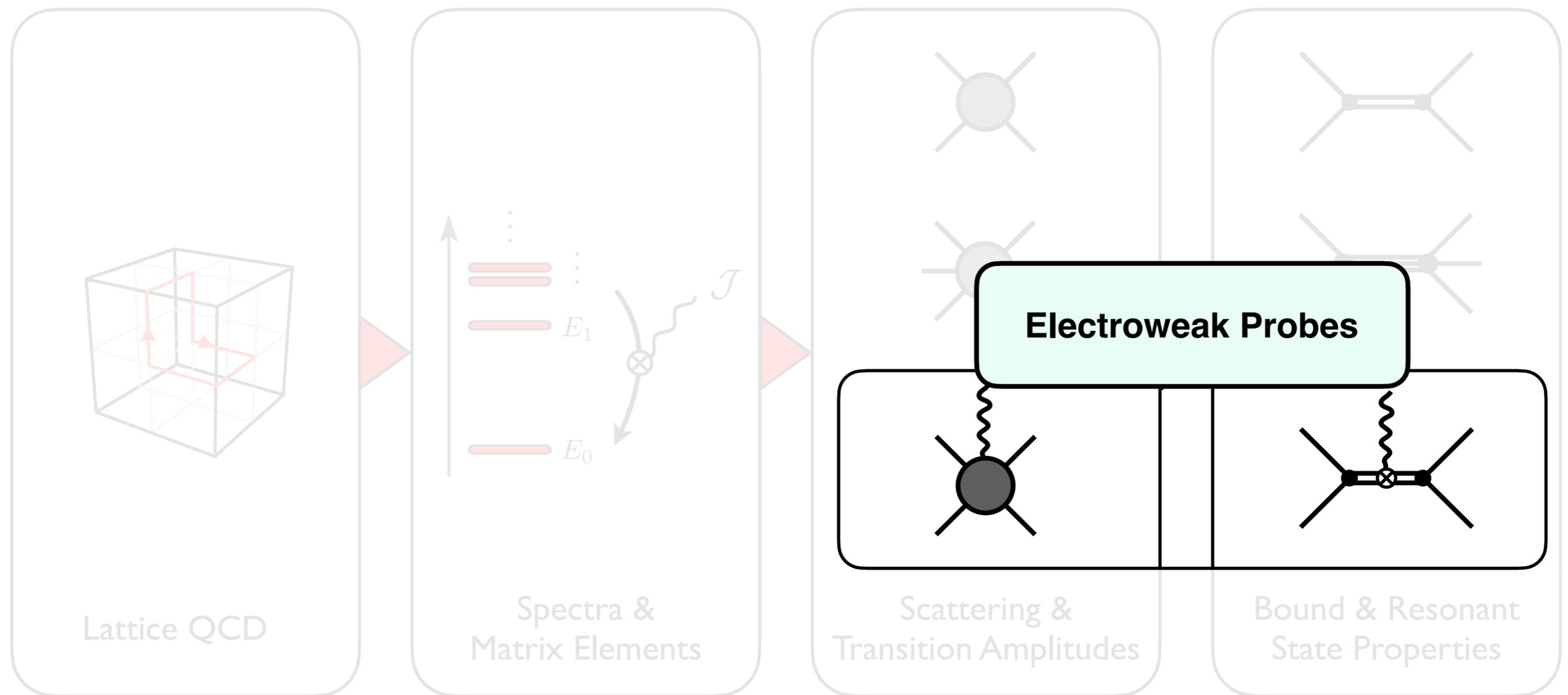
# Applications to $3\pi^+$



# Few-Body Physics & QCD

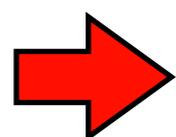
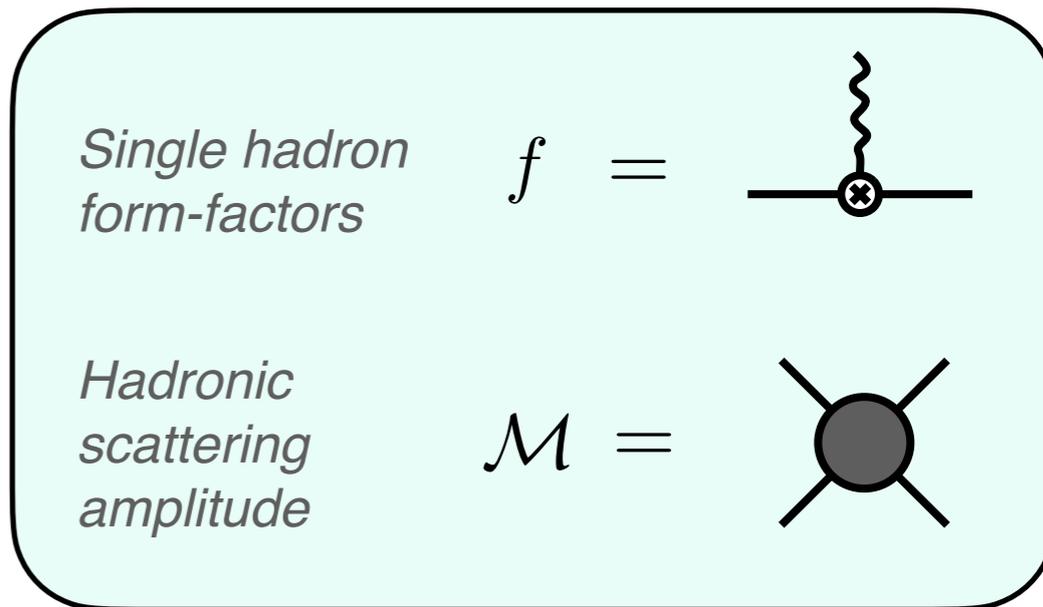
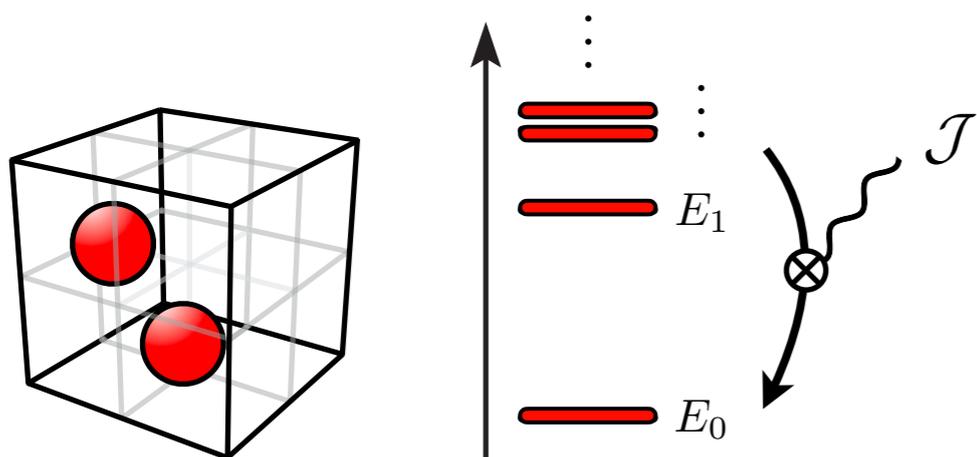
Path to few-body physics from QCD

- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



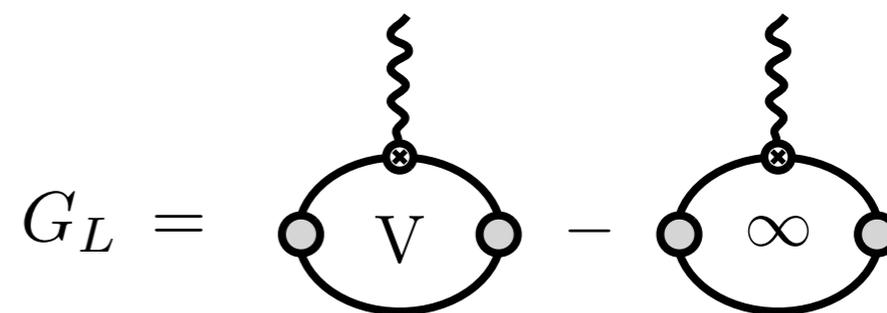
# Hadronic Structure & Electroweak Probes

Mapping between matrix elements and  $2 + \mathcal{J} \rightarrow 2$  amplitudes



$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L = \frac{1}{L^3} \mathcal{W}_{L,df} \cdot \sqrt{\mathcal{R}_{L,m} \cdot \mathcal{R}_{L,n}} \quad \leftarrow \text{FV conversion factors}$$

$$\mathcal{W}_{L,df} = \mathcal{W}_{df} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$



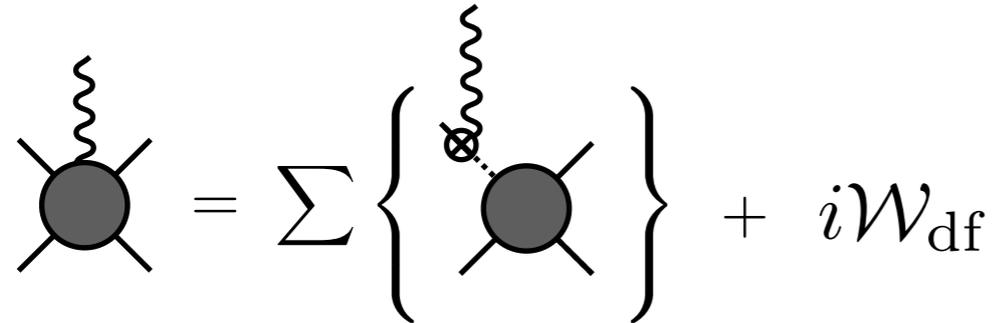
FV geometric function

R. Briceño, M. Hansen,  
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,  
Phys. Rev. D **100** 034511 (2019)

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes


$$\text{Diagram} = \sum \left\{ \text{Diagram} \right\} + i\mathcal{W}_{\text{df}}$$

*After considerable manipulations...*

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot G) \cdot \mathcal{M}$$

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes

$$\text{Diagram with wavy line} = \sum \left\{ \text{Diagram with wavy line and dashed line} \right\} + i\mathcal{W}_{\text{df}}$$

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\boxed{A} + f \cdot G) \cdot \mathcal{M}$$

Unknown short-distance function

- Constrain using Lattice QCD
- Constrained by Ward-Takahashi identity

Single hadron form-factors

$$f = \text{Diagram: wavy line to a vertex on a line}$$

Hadronic scattering amplitude

$$\mathcal{M} = \text{Diagram: a circle with four external lines}$$

Triangle diagram

Contains normal and anomalous singularities from intermediate on-shell particles

$$G = \text{Diagram: triangle loop with wavy line and two shaded vertices}$$

$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L \sim \mathcal{W}_{\text{df}} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$

# Hadronic Structure & Electroweak Probes

On-shell representation of  $2 + \mathcal{J} \rightarrow 2$  amplitudes

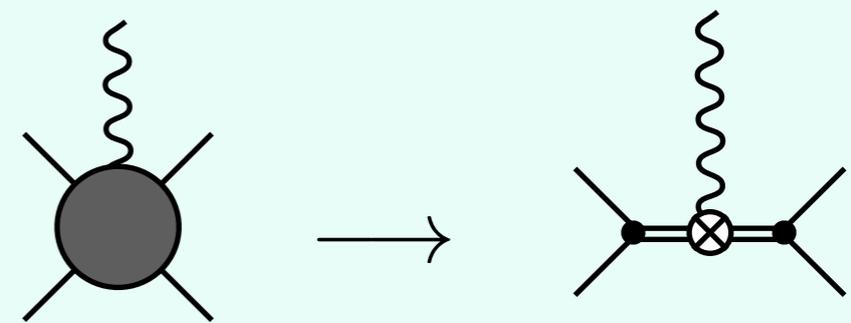
$$\text{Diagram with wavy line} = \sum \left\{ \text{Diagram with wavy line and blob} \right\} + i\mathcal{W}_{\text{df}}$$

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot G) \cdot$$

Rigorous definition for resonance form factors

$$\mathcal{W}_{\text{df}} \sim \frac{g}{s_f - s_p} \cdot f_p \cdot \frac{g}{s_i - s_p}$$



$$f_p = g^2 (\mathcal{A} + f \cdot G) \Big|_{s_f = s_i = s_p}$$

# Hadronic Structure & Electroweak Probes

---

Investigations into features of the framework



Conserved vector currents



Bound States

- Infinite-volume limit
- Leading order FV corrections



Non-Relativistic Behavior

- Expansion near threshold, leading FV corrections
- Numerical cross-check
- Time-ordered perturbation theory

AJ, R. Briceño, M. Hansen,  
Phys. Rev. D **100** 114505 (2019)

AJ, R. Briceño, M. Hansen,  
Phys. Rev. D **101** 094508 (2020)

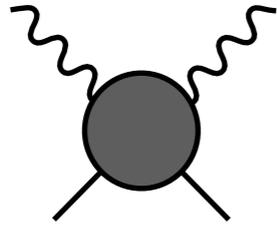
# Two-current systems

Coupling two currents to hadronic systems – Compton-like processes

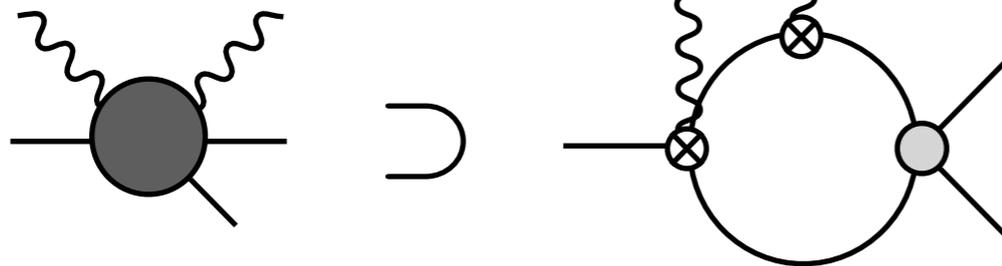


$$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{1} + \mathcal{J}$$

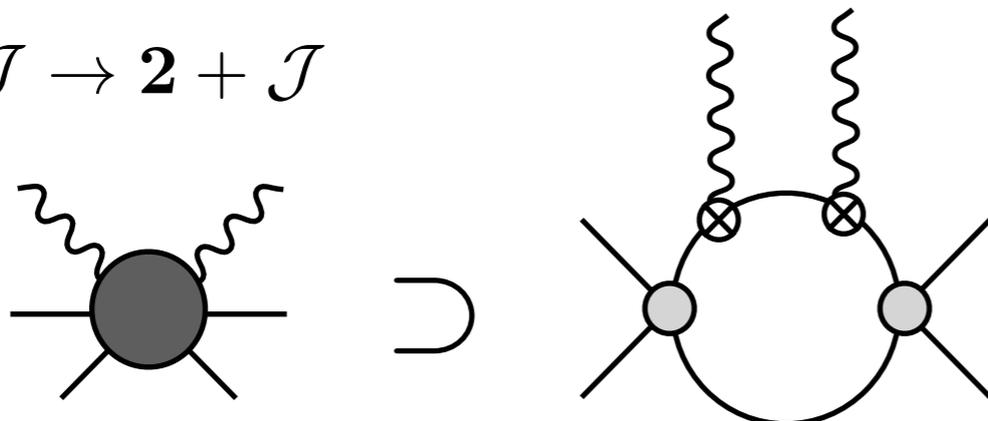
R. Briceño, Z. Davoudi, M. Hansen, M. Schindler, A. Baroni  
Phys. Rev. D **101** 014509 (2020)



$$\square \quad \mathbf{1} + \mathcal{J} \rightarrow \mathbf{2} + \mathcal{J}$$



$$\square \quad \mathbf{2} + \mathcal{J} \rightarrow \mathbf{2} + \mathcal{J}$$



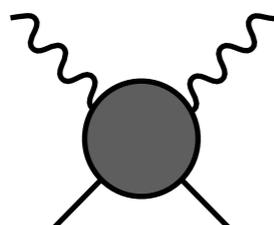
# Two-current systems

## Coupling two currents to hadronic systems – Compton-like processes



$$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{1} + \mathcal{J}$$

R. Briceño, Z. Davoudi, M. Hansen, M. Schindler, A. Baroni  
Phys. Rev. D **101** 014509 (2020)



$$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{2} + \mathcal{J}$$



$$\mathbf{2} + \mathcal{J}$$



**Coming Soon!**

JLAB-THY-22-3552

### Two-current transition amplitudes with two-body final states

Keegan H. Sherman,<sup>1,\*</sup> Felipe G. Ortega-Gama,<sup>2,3,†</sup> Raúl A. Briceño,<sup>2,4,‡</sup> and Andrew W. Jackura<sup>2,4,§</sup>

<sup>1</sup>*Department of Physics, Old Dominion University, Norfolk, Virginia 23529, USA*

<sup>2</sup>*Thomas Jefferson National Accelerator Facility,*

*12000 Jefferson Avenue, Newport News, Virginia 23606, USA*

<sup>3</sup>*Department of Physics, William & Mary, Williamsburg, Virginia 23187, USA*

<sup>4</sup>*Department of Physics, Old Dominion University, Norfolk, Virginia 23529, USA*

(Dated: February 2, 2022)

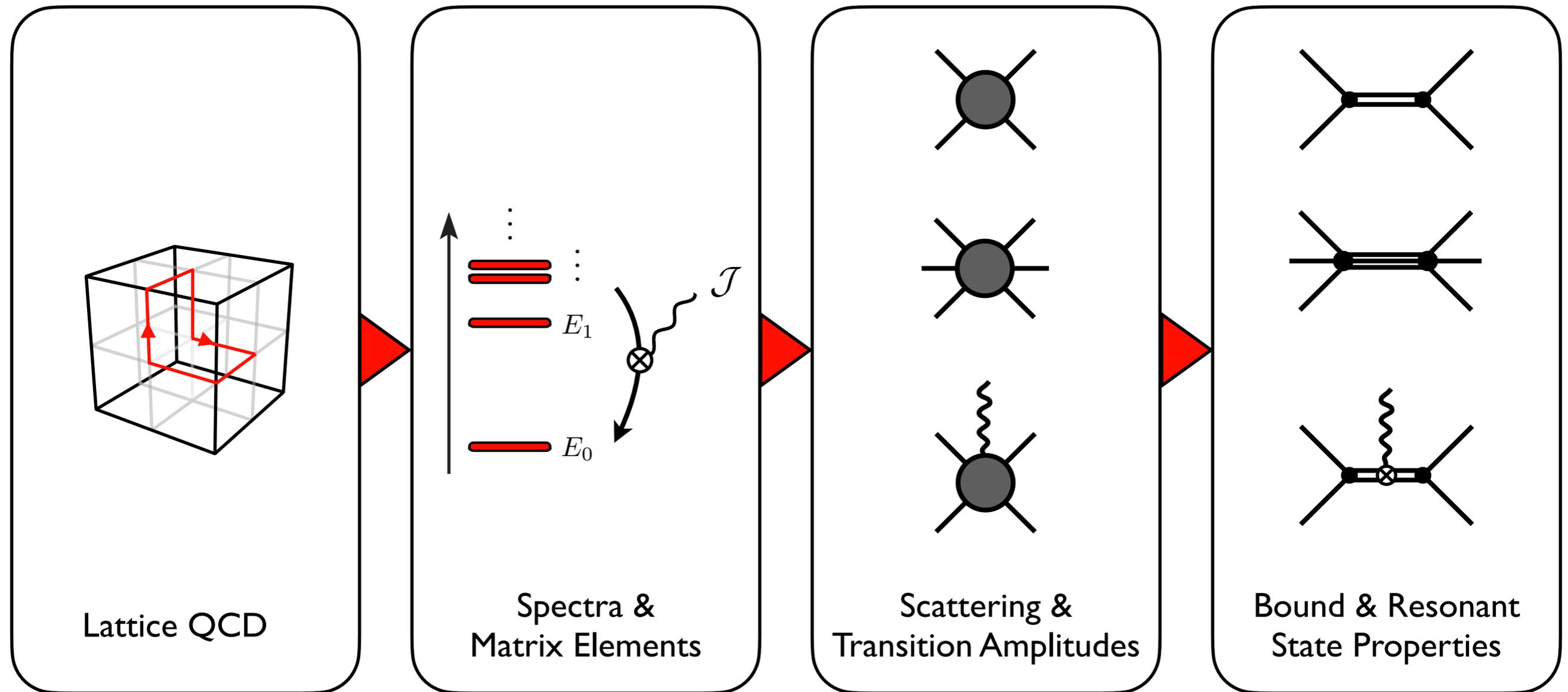
We derive the on-shell form of amplitudes containing two external currents with a single hadron in the initial state and two hadrons in the final state, denoted as  $1 + \mathcal{J} \rightarrow 2 + \mathcal{J}$ . This class of amplitude is relevant in precision tests of the Standard Model as well as for exploring the structure of excited states in the QCD spectrum. We present a model-independent description of the amplitudes where we sum to all orders in the strong interaction. From this analytic form we are able to extract transition and elastic resonance form factors consistent with previous work as well as a

# Few-Body Physics & QCD

Path to few-body physics from QCD

- Link finite-volume spectra and matrix elements to scattering amplitudes
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*

**Applications of formalisms beginning, stay tuned for results...**



# Summary

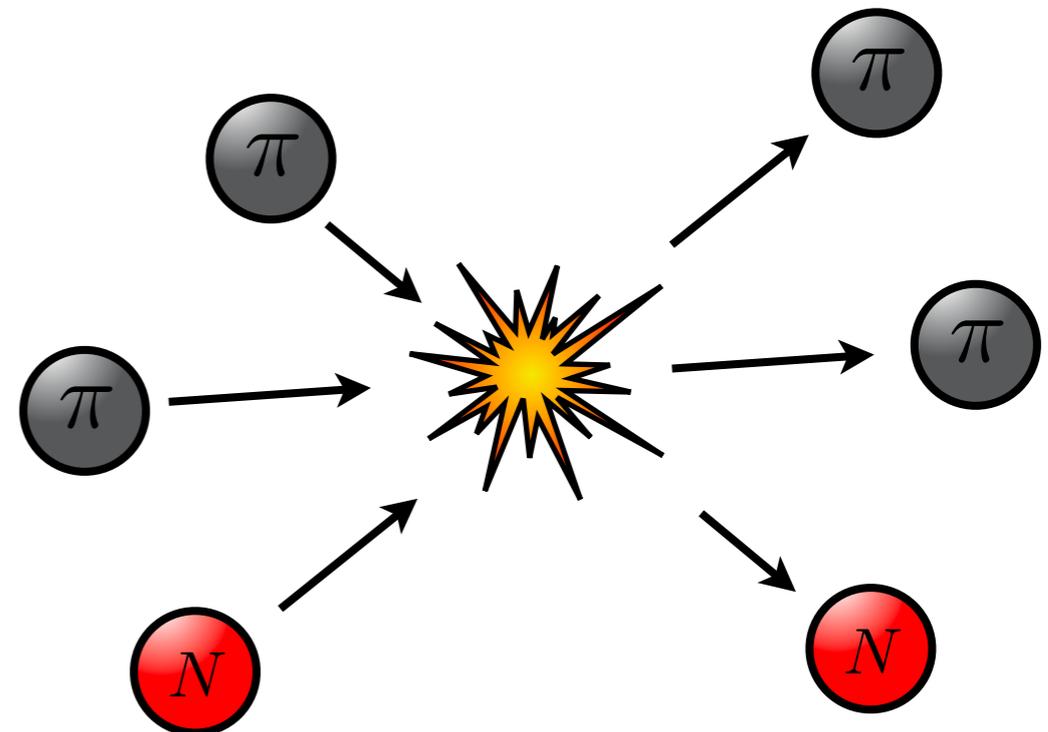
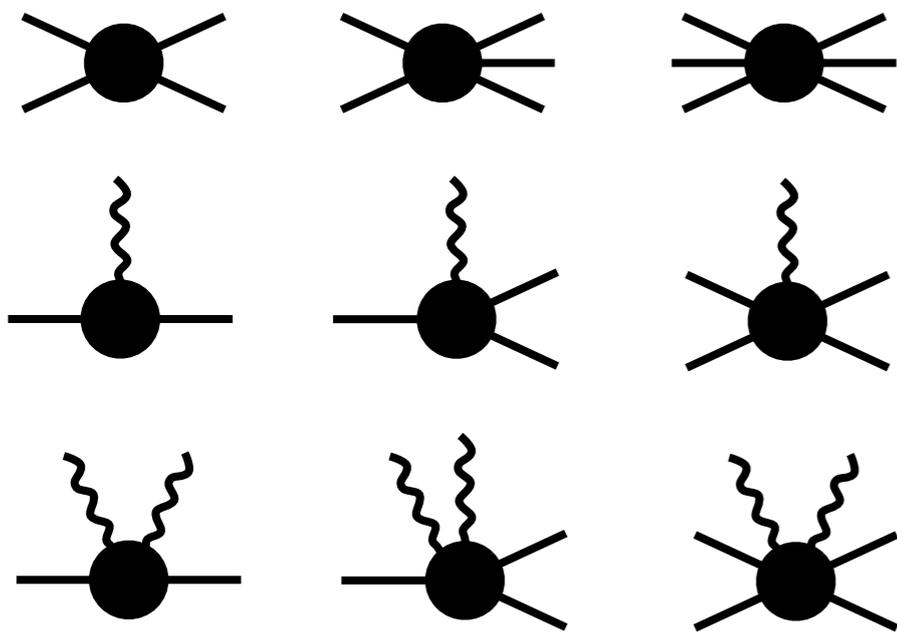
Few-body interactions play a key role in many outstanding problems in nuclear & hadron physics

Lattice QCD, EFTs, & Scattering theory combined provide useful tools to extract physics from QCD

- Rapid development in formalisms relating lattice QCD observables to amplitudes
- Scattering phenomenology is advancing in tandem

Latest developments in three-body scattering & two-body matrix elements

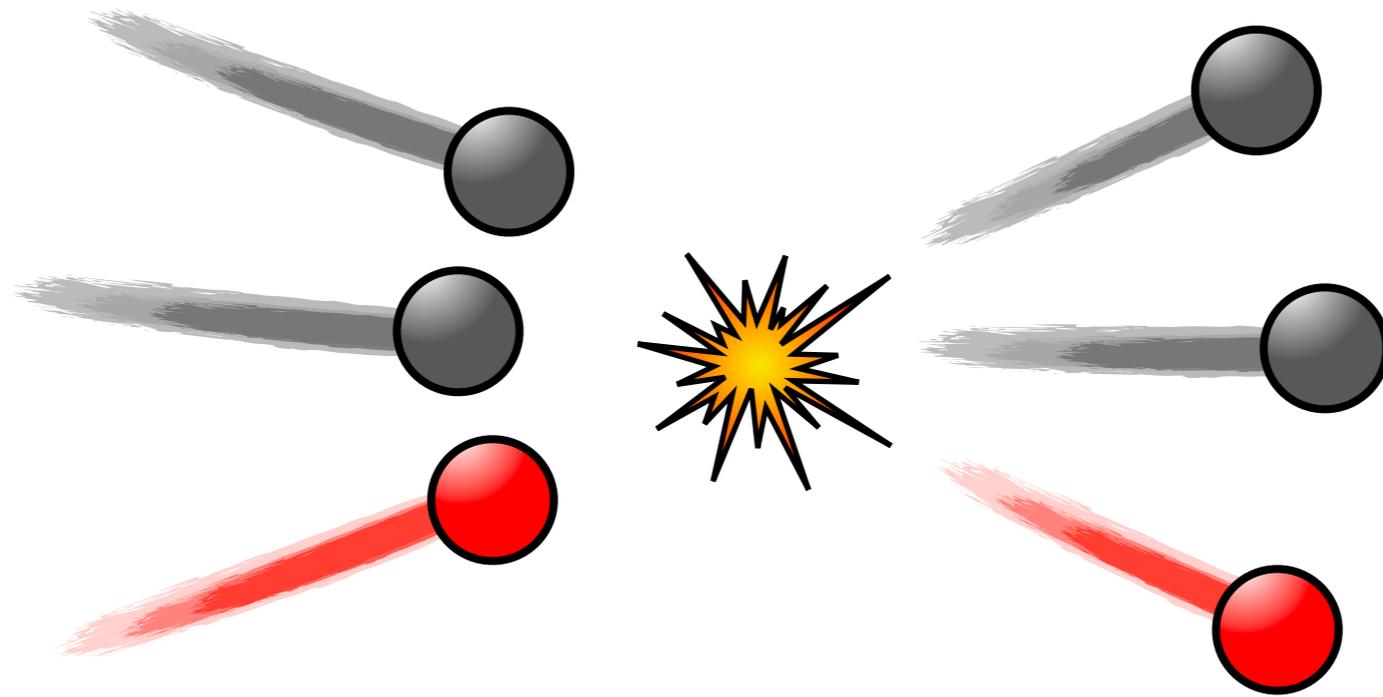
- First applications appearing in literature
- Can address increasingly complicated processes



**Much more to come!**



# Extra Slides

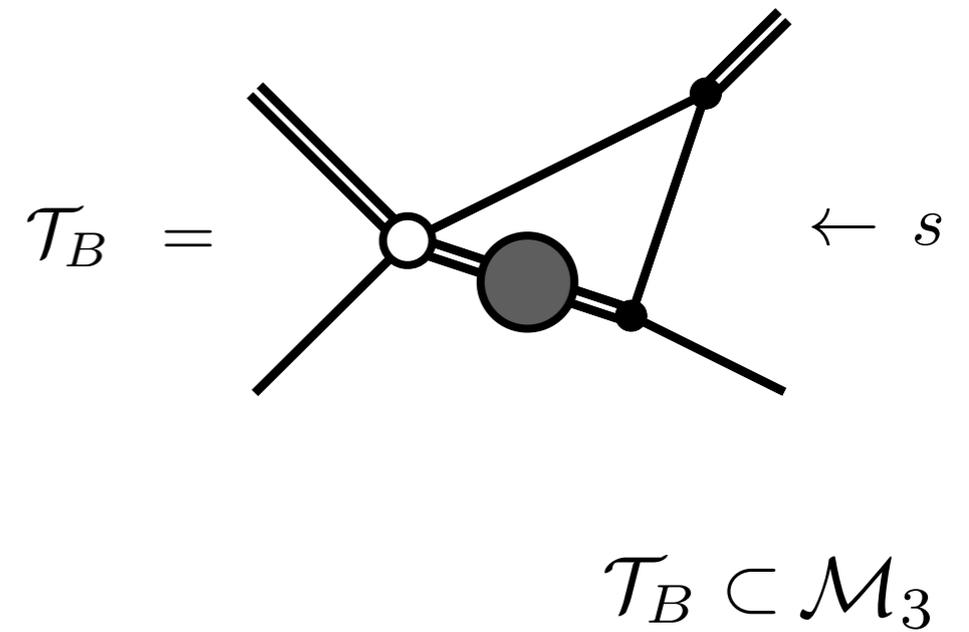
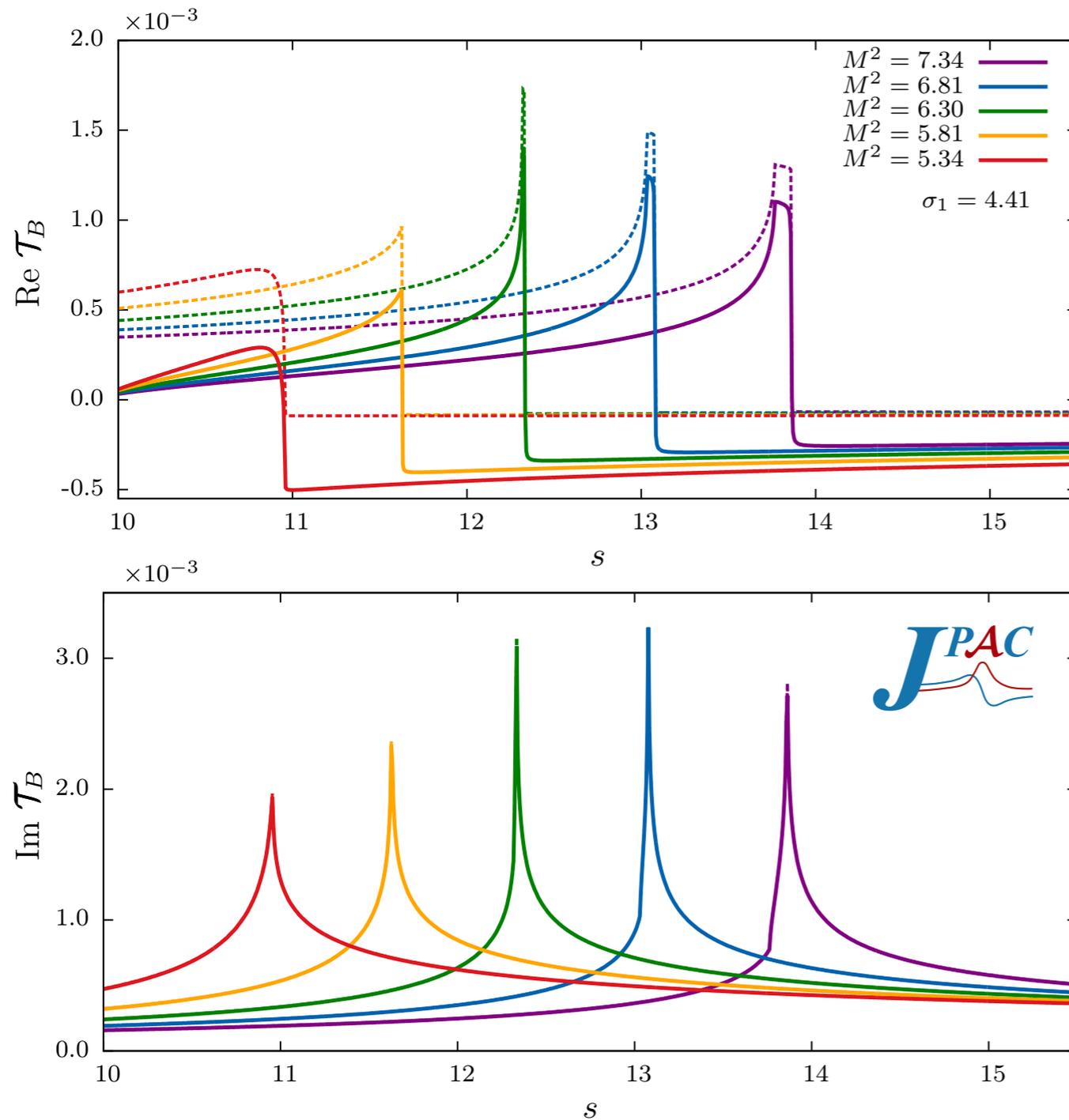


# Three-Body Dynamics

On-shell scattering relations

Scattering amplitudes have more complicated singularities

*e.g. effects of triangle singularities*



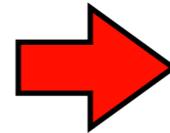
AJ et al. [JPAC]  
 Eur. Phys. J. C **79**, no. 1, 56 (2019)

# Hadronic Structure & Electroweak Probes

Investigations into features of the framework

*e.g. Conserved vector current*

$$\langle \mathbf{n} | Q | \mathbf{n} \rangle_L = L^3 \langle \mathbf{n} | \mathcal{J}^0 | \mathbf{n} \rangle_L = Q_0$$



*Ward-Takahashi identity*

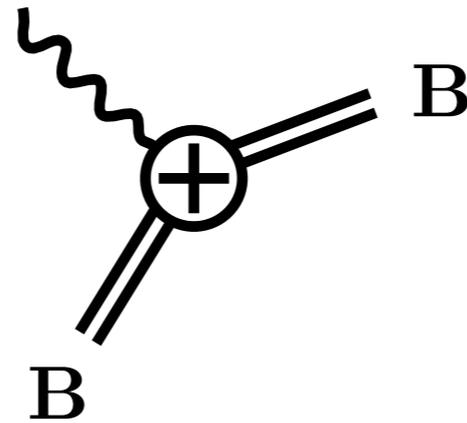
$$\lim_{P_i \rightarrow P_f} \mathcal{W}_{\text{df}}^\mu = Q_0 \frac{\partial}{\partial P_\mu} \mathcal{M}$$

*Leads to constraint for short-distance function*

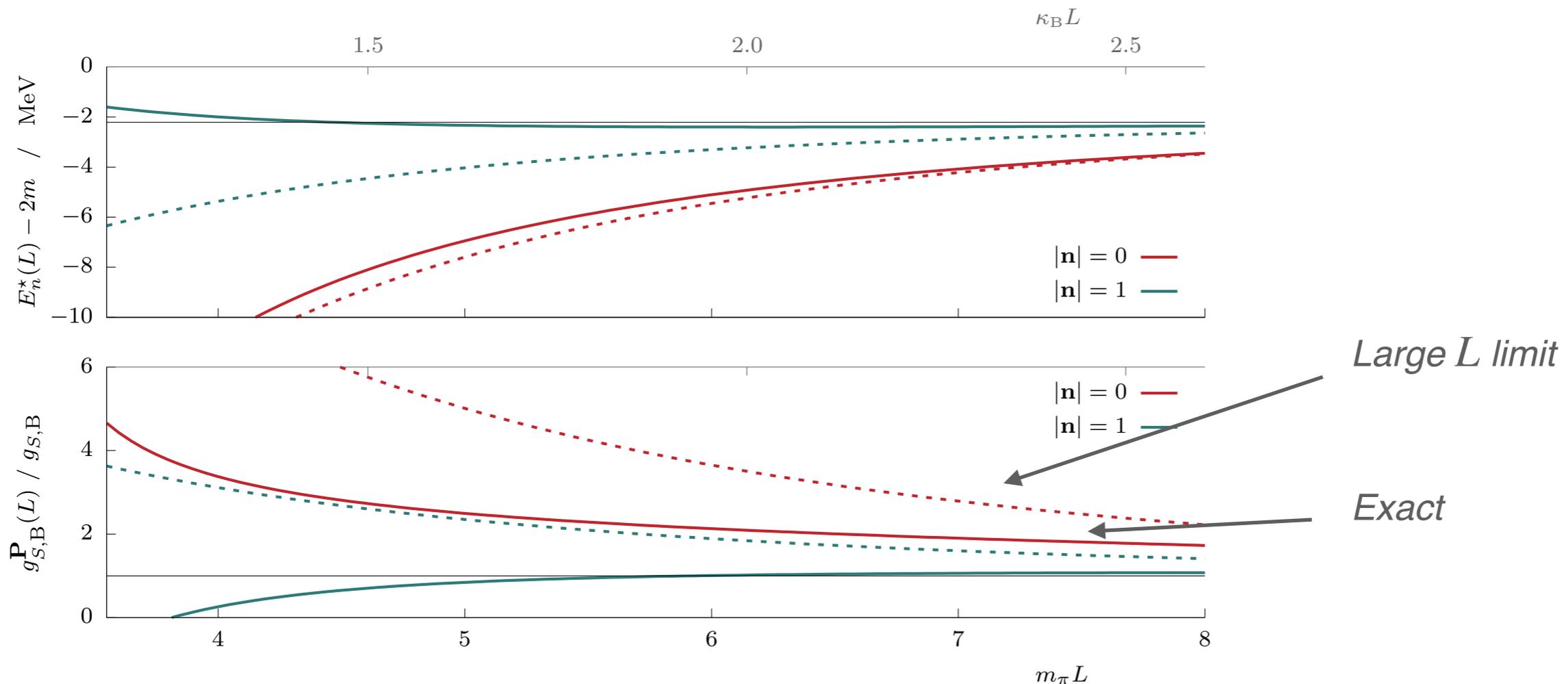
# Hadronic Structure & Electroweak Probes

Investigations into features of the framework

e.g. Bound systems



$$\frac{g_B(L)}{g_B} = 1 + \mathcal{O}(e^{-\kappa L})$$



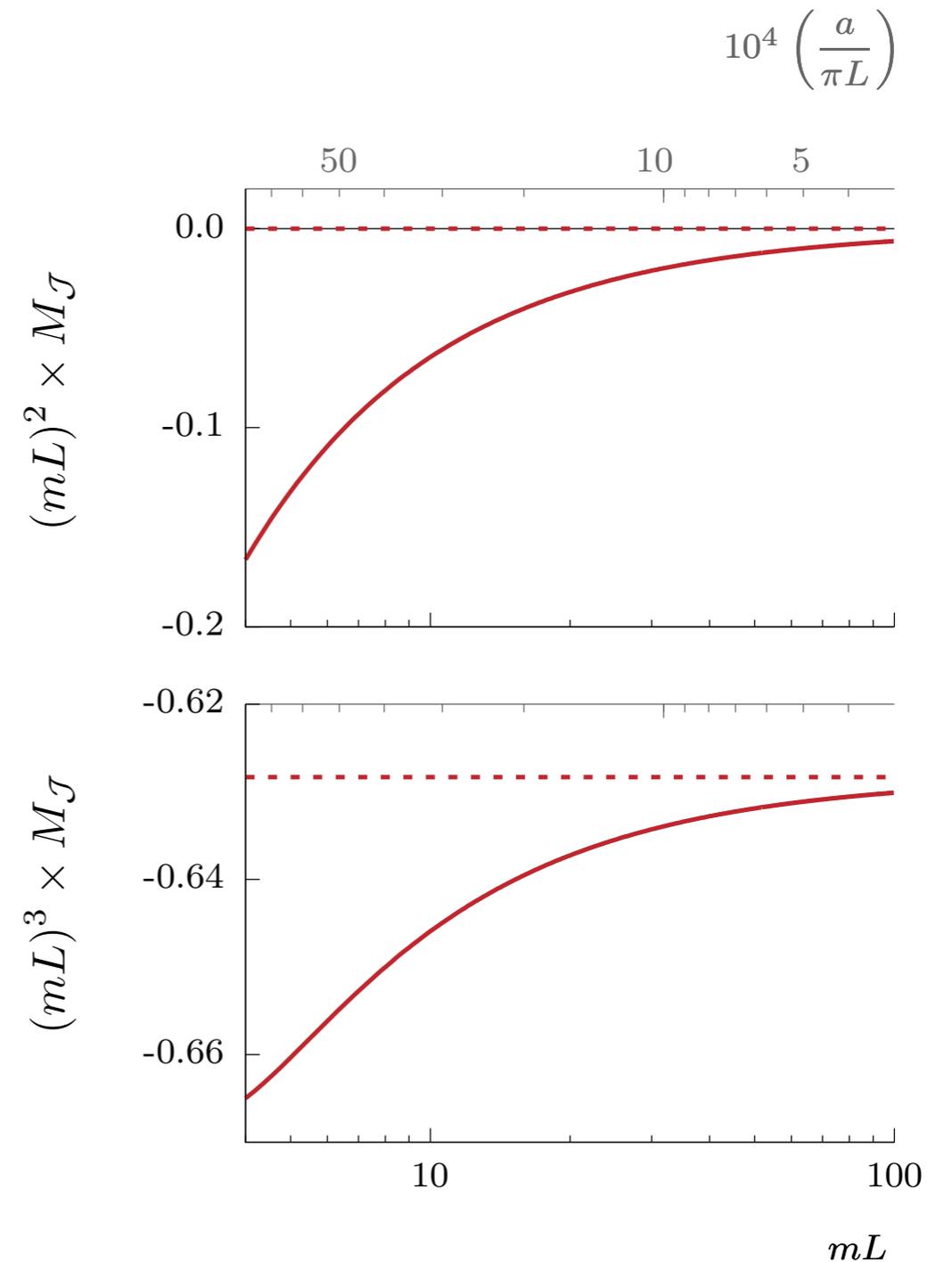
# Hadronic Structure & Electroweak Probes

Investigations into features of the framework

*e.g. Large  $L$  behavior*

$$E_0(L) = 2m + \frac{4\pi a}{mL^3} + \mathcal{O}(1/L^4)$$

$$L^3 \langle 0 | \mathcal{J} | 0 \rangle = \frac{g}{m} \left( 1 - \frac{2\pi a}{m^2 L^3} + \mathcal{O}(1/L^4) \right)$$

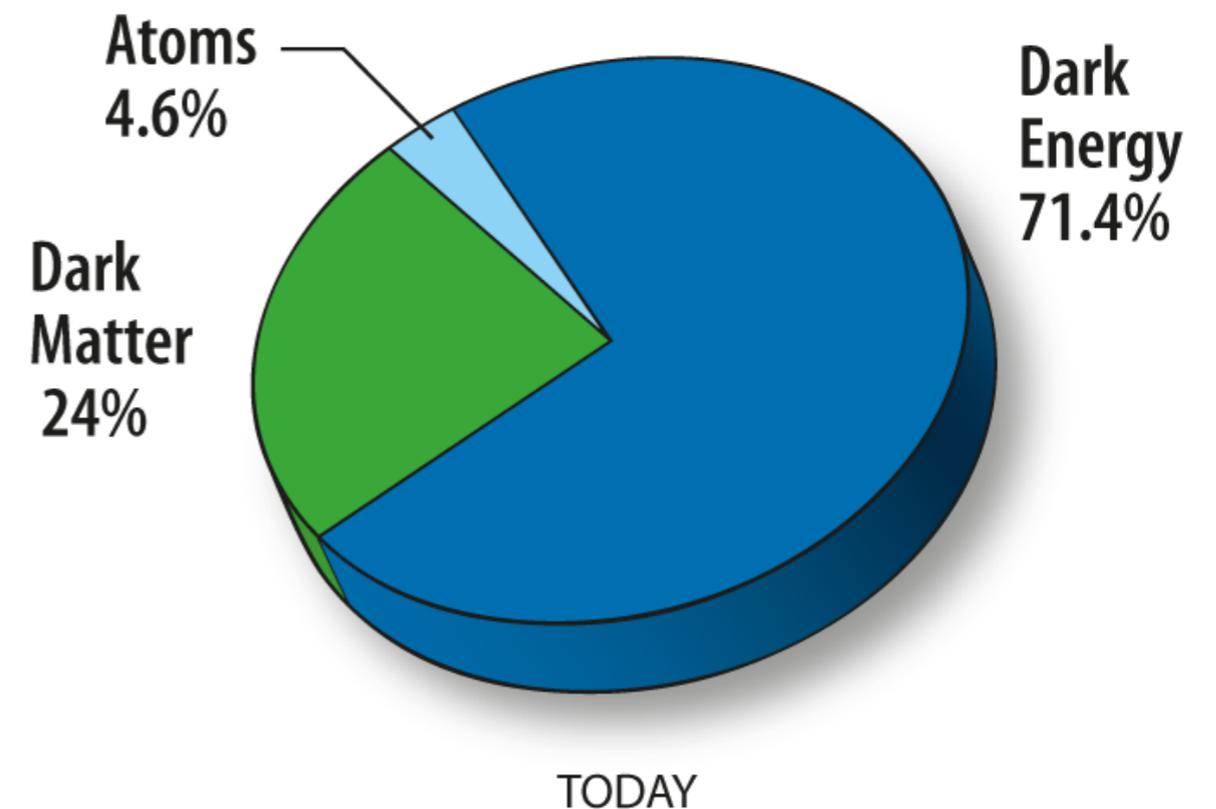
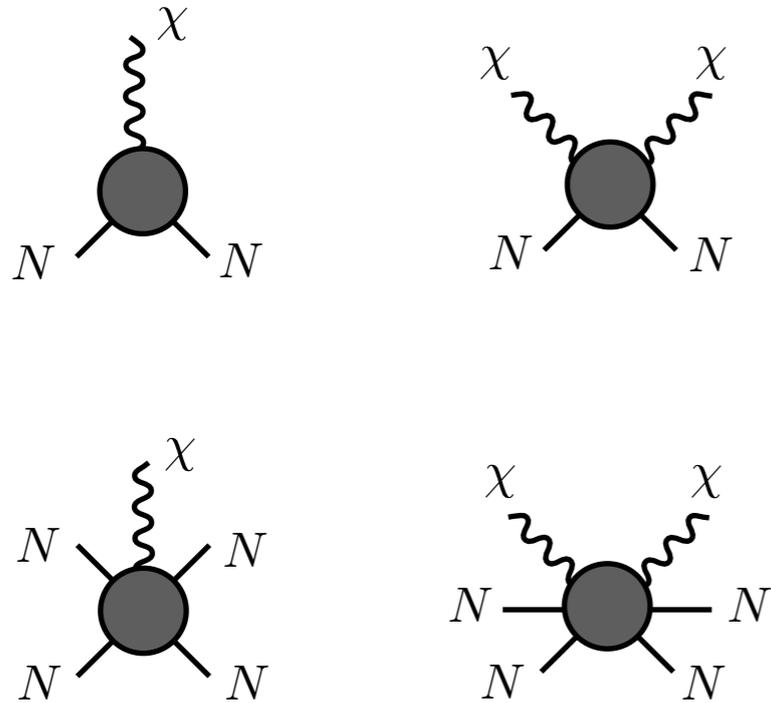


# Beyond Hadron Structure

Our framework has drawn the attention of people outside hadron spectroscopy/ structure

- Searches for new physics Beyond the Standard Model require nuclear matrix elements

*e.g. Dark Matter Searches*



# On-shell scattering equations

S-matrix unitarity fixes on-shell structure of scattering amplitudes

$2 \rightarrow 2$  scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

*On-shell two-particle rescattering*

$$\text{Im } \mathcal{I} = -\rho \sim \text{Im} \left( \text{circle with vertical dashed line and two dots} \right)$$

*K-matrices*

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
- *Scheme dependent (unphysical)*

# On-shell scattering equations

S-matrix unitarity fixes on-shell structure of scattering amplitudes

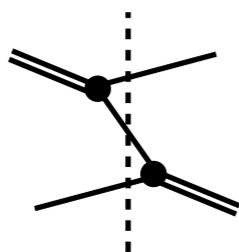
**2**  $\rightarrow$  **2** scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

**3**  $\rightarrow$  **3** scattering

$$\begin{aligned} \mathcal{M}_3 = & \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3 \\ & - \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \iint \mathcal{K}_3 G \mathcal{M}_3 \end{aligned}$$

*On-shell exchange*

$$\text{Im } G = -\Delta \sim \text{diagram}$$


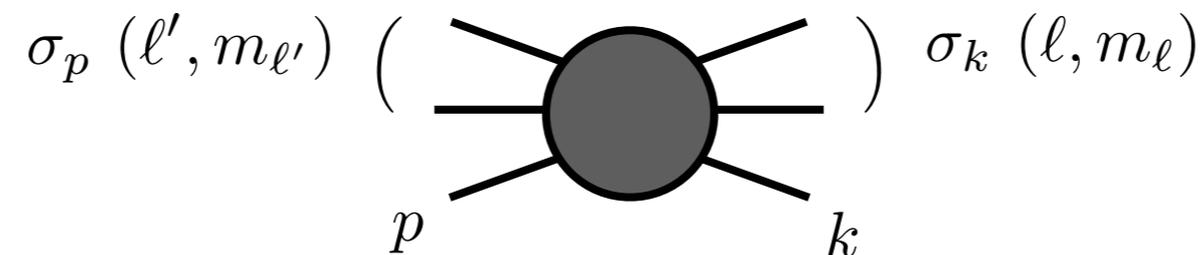
*K-matrices*

- Unknown real function characterizing short-distance physics
- Parameterize with analytic function and determine from lattice QCD
- Scheme dependent (unphysical)

# Solving three-body scattering equations

Given  $\mathcal{K}_2$ , and  $\mathcal{K}_3$ , solve integral equation for  $\mathcal{M}_3$

$$\begin{aligned} \mathcal{M}_3 = & \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3 \\ & - \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \iint \mathcal{K}_3 G \mathcal{M}_3 \end{aligned}$$



- Many equivalent forms (Hansen-Sharpe, Blanton-Sharpe, Döring-Mai, JPAC-AJ, Mikhasenko)
- Matrix equation in pair angular momenta
- Integral equation in spectator momenta
- Singular kernels
- Scheme dependent K-matrices (unphysical)

## Solving three-body scattering equations

Given  $\mathcal{K}_2$ , and  $\mathcal{K}_3$ , solve integral equation for  $\mathcal{M}_3$

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Consider toy model:  $3\varphi \rightarrow 3\varphi$  such that  $2\varphi \rightarrow b$

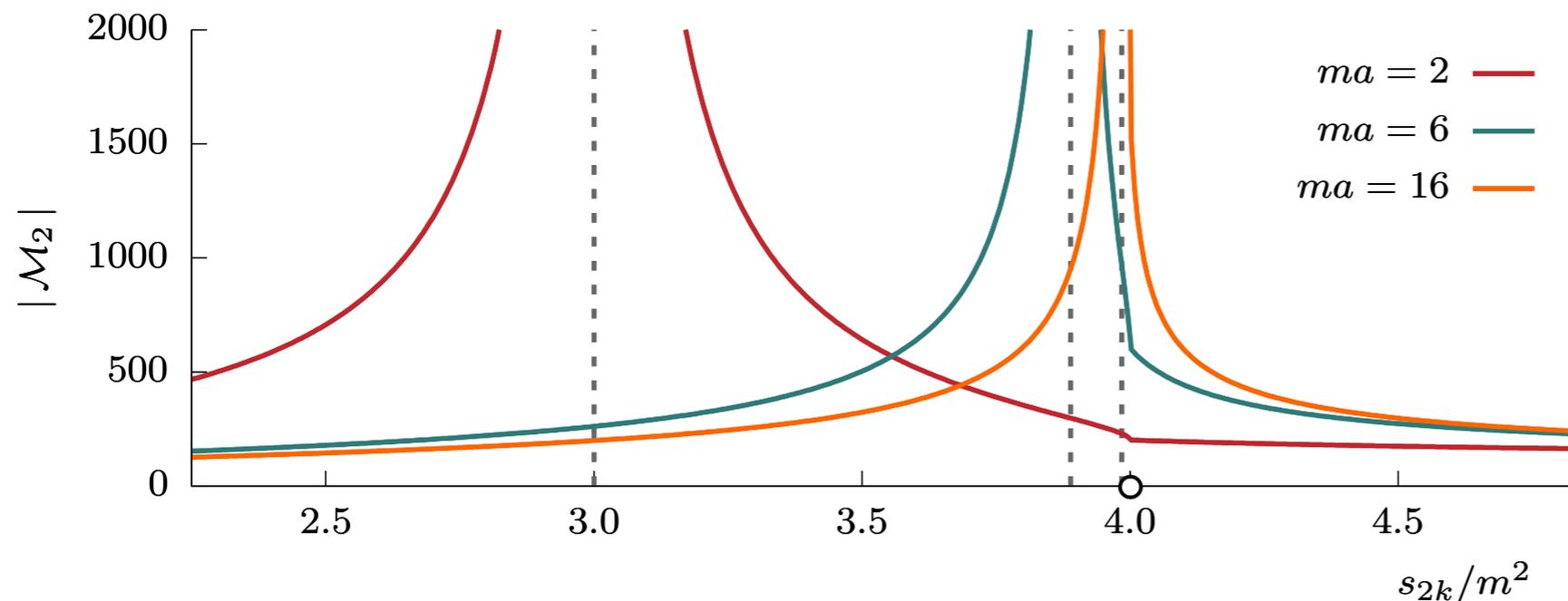
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Consider toy model:  $3\varphi \rightarrow 3\varphi$  such that  $2\varphi \rightarrow b$

$$\mathcal{K}_2^{-1} \sim -\frac{1}{a}$$



## Solving three-body scattering equations

Given  $\mathcal{K}_2$ , and  $\mathcal{K}_3$ , solve integral equation for  $\mathcal{M}_3$

$$\begin{aligned} \mathcal{M}_3 = & \cancel{\mathcal{K}_3} - \cancel{\mathcal{K}_3} \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \cancel{\mathcal{K}_3} \mathcal{I} \mathcal{M}_3 \\ & - \mathcal{K}_2 G \mathcal{M}_2 - \int \cancel{\mathcal{K}_3} G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int \int \cancel{\mathcal{K}_3} G \mathcal{M}_3 \end{aligned}$$

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Assume  $\mathcal{K}_3 = 0$

$$\mathcal{M}_3|_{\mathcal{K}_3=0} \equiv \mathcal{D} \quad \Longrightarrow \quad \mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int G \mathcal{D}$$

# Solving three-body scattering equations

Given  $\mathcal{K}_2$ , and  $\mathcal{K}_3$ , solve integral equation for  $\mathcal{M}_3$

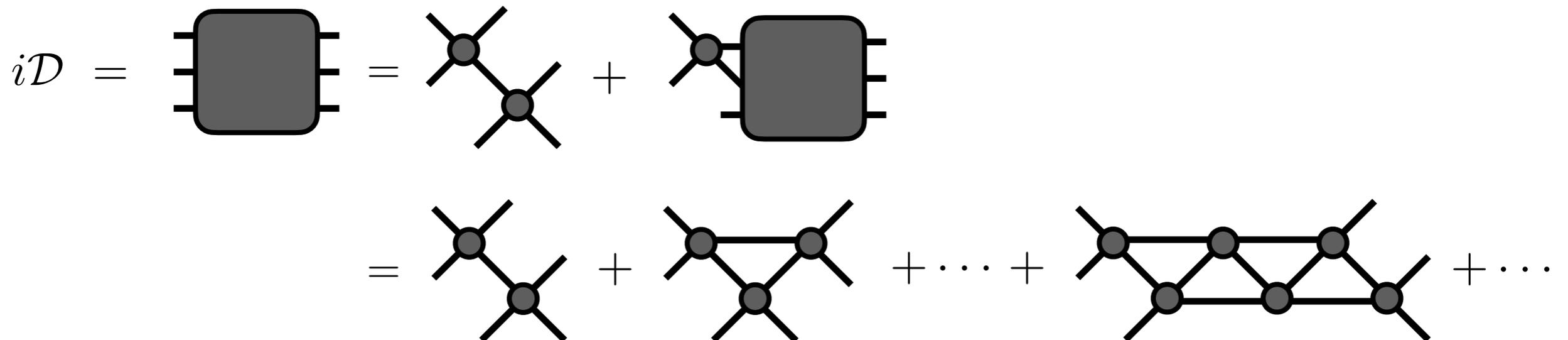
$$\mathcal{M}_3 = \cancel{\mathcal{K}_3} - \cancel{\mathcal{K}_3} \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \cancel{\mathcal{K}_3} \mathcal{I} \mathcal{M}_3$$

$$- \mathcal{K}_2 \mathcal{G} \mathcal{M}_2 - \int \cancel{\mathcal{K}_3} \mathcal{G} \mathcal{M}_2 - \mathcal{K}_2 \int \mathcal{G} \mathcal{M}_3 - \int \int \cancel{\mathcal{K}_3} \mathcal{G} \mathcal{M}_3$$

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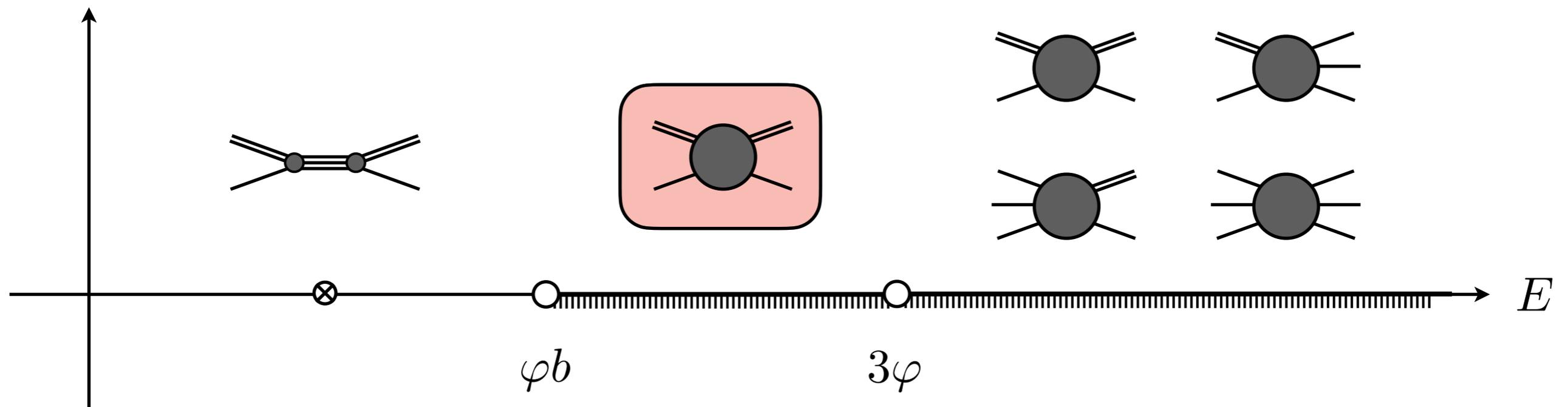


# Solving three-body scattering equations

Focus on case where 2-body systems forms bound state

- Consider energies below three particle threshold

$$\lim_{\sigma_p, \sigma_k \rightarrow \sigma_b} i\mathcal{M}_3 = ig_b \frac{i}{\sigma_p - \sigma_b} i\mathcal{M}_{\varphi b} \frac{i}{\sigma_k - \sigma_b} ig_b$$



# Solving three-body scattering equations

---

Convert integral equation to linear equation

- Introduce regulators  $N$  (matrix size) and  $\epsilon$  (pole shift)
- Recover amplitude in  $N \rightarrow \infty$ ,  $\epsilon \rightarrow 0^+$  limit

$$\mathcal{M}_{\varphi b} = \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \mathcal{M}_{\varphi b}(N, \epsilon)$$

*Several methods*

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys. **A305** (1982) 217*

# Solving three-body scattering equations

Convert integral equation to linear equation

- Introduce regulators  $N$  (matrix size) and  $\epsilon$  (pole shift)
- Recover amplitude in  $N \rightarrow \infty$ ,  $\epsilon \rightarrow 0^+$  limit

$$\mathcal{M}_{\varphi b} = \lim_{N \rightarrow \infty} \lim_{\epsilon \rightarrow 0^+} \mathcal{M}_{\varphi b}(N, \epsilon)$$

S-matrix unitarity provides a way to check quality of solutions

- Deviation from unitarity guides quality of solution

$$\text{Im } \mathcal{M}_{\varphi b}^{-1}(E) = -\rho_{\varphi b}(E)$$

$$\Delta\rho_{\varphi b}(E; N) \equiv \left| \frac{\text{Im} \left[ \mathcal{M}_{\varphi b}^{-1}(E; N) \right] + \rho_{\varphi b}(E)}{\rho_{\varphi b}(E)} \right| \times 100$$

*Several methods*

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys. **A305** (1982) 217*

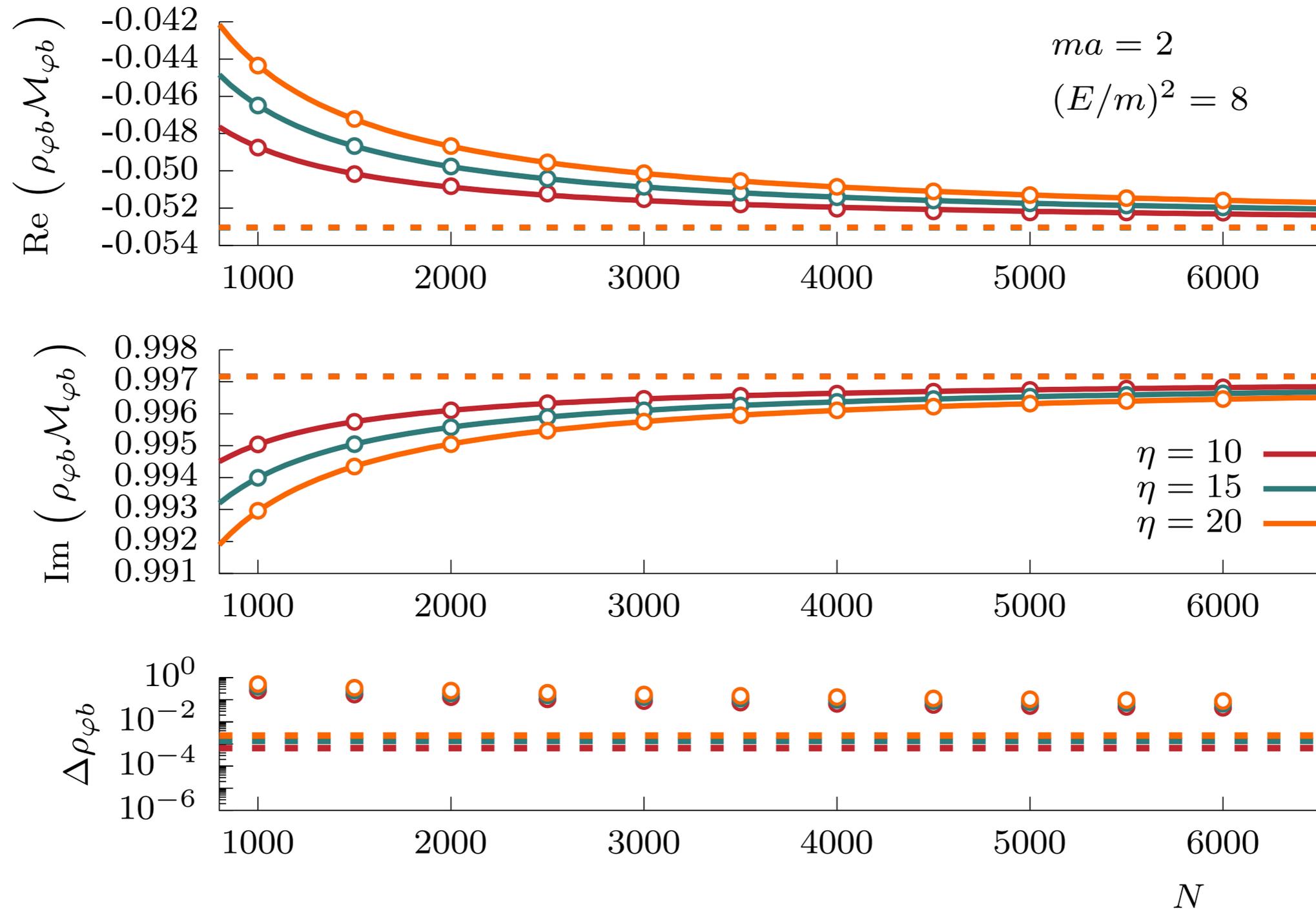
# $N \rightarrow \infty$ Extrapolations

Compute multiple  $N$  solutions — extrapolate to  $N \rightarrow \infty$  limit

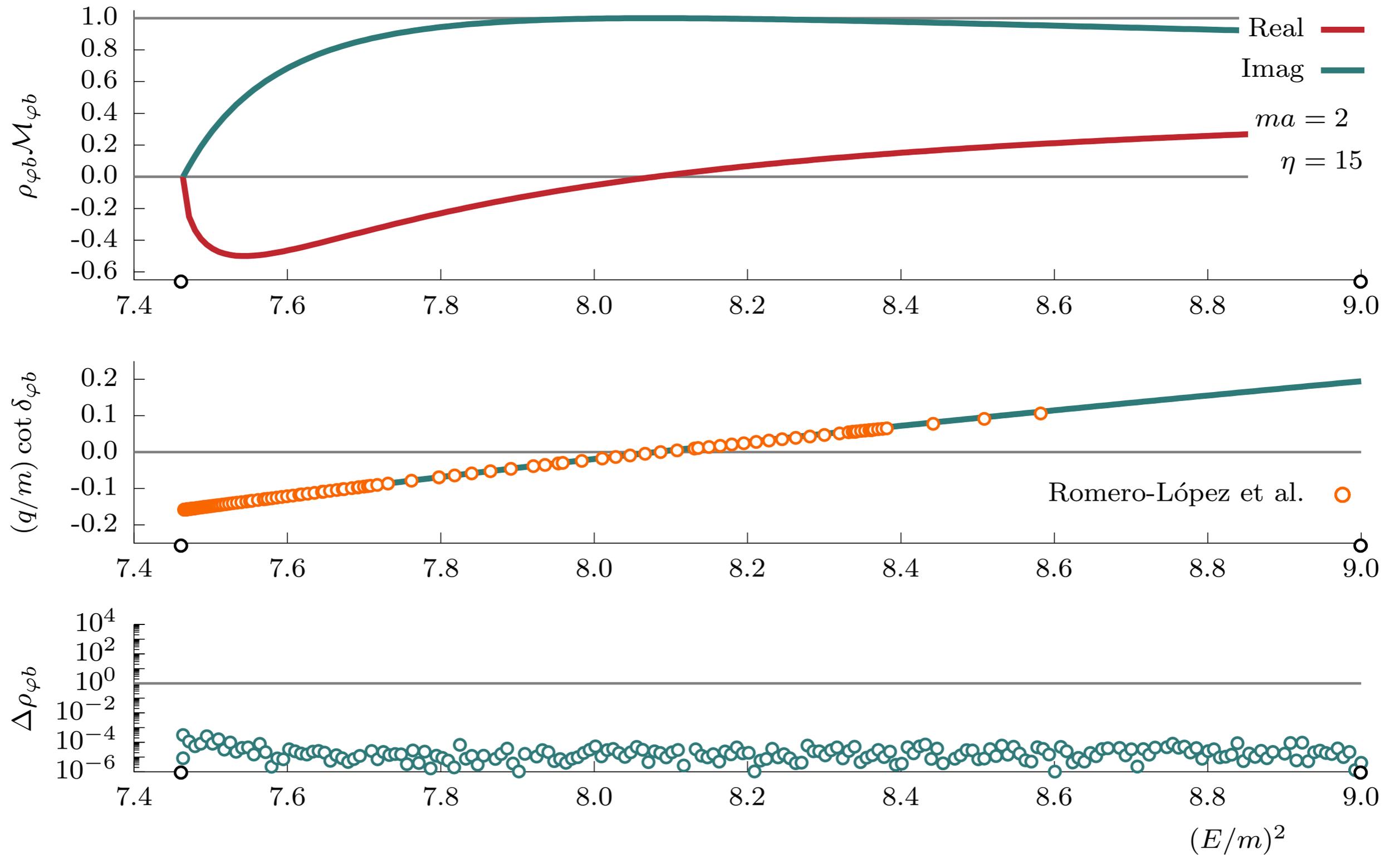
- Unitarity deviation greatly improved

$$\mathcal{M}_{\varphi b}(E; N) \approx \mathcal{M}_{\varphi b}(E) + \frac{\alpha}{N}$$

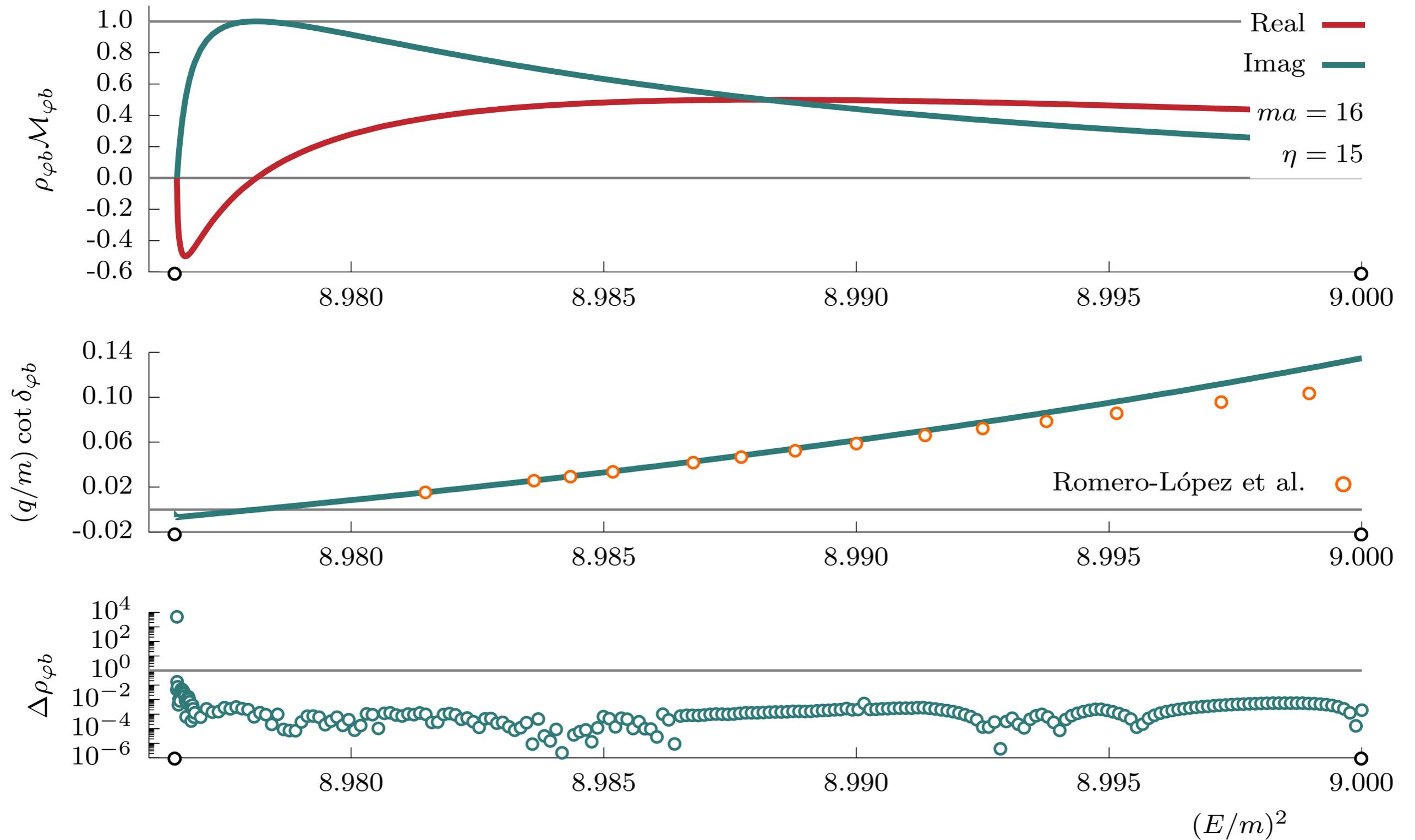
$$\epsilon \propto \frac{\eta}{N}$$



# Example of solution



# Example of solution

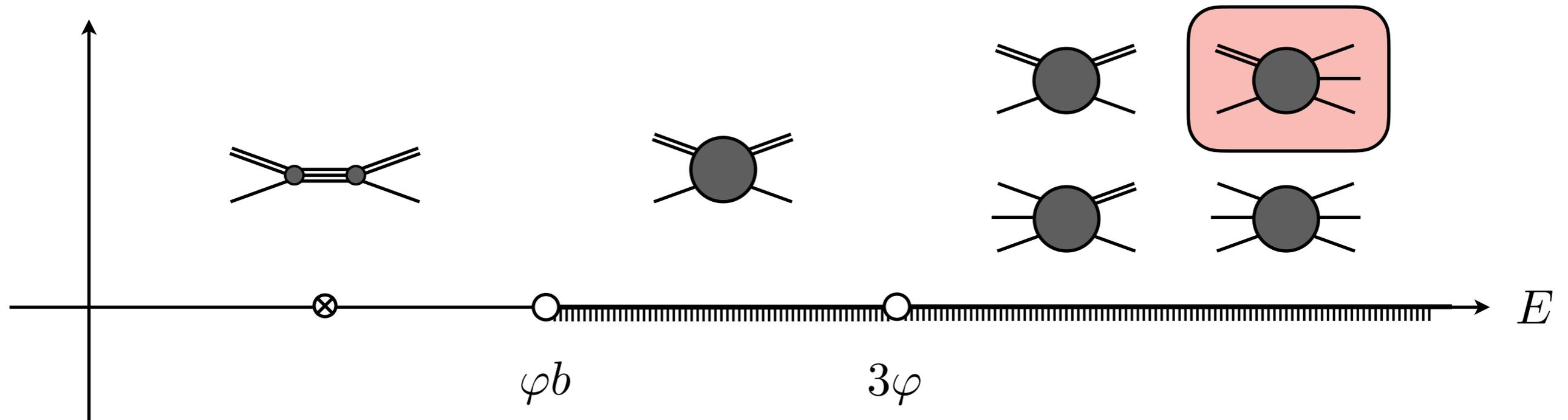


## Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude

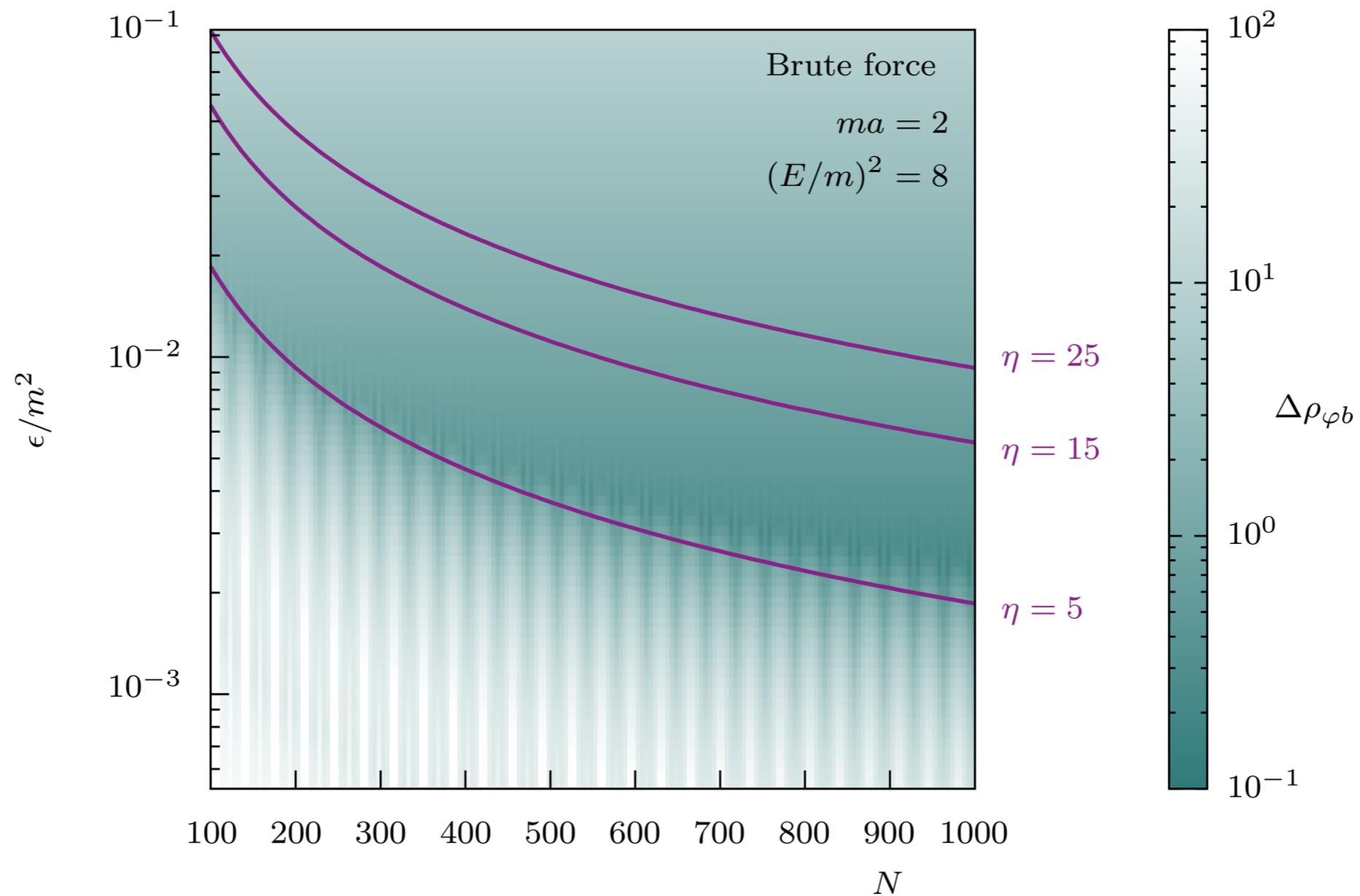
$$\mathcal{M}_{\varphi \rightarrow 3\varphi}(\sigma_p) = - \lim_{\sigma_k \rightarrow \sigma_b} \frac{\sigma_k - \sigma_b}{g} \mathcal{M}_3(\sigma_p, \sigma_k)$$



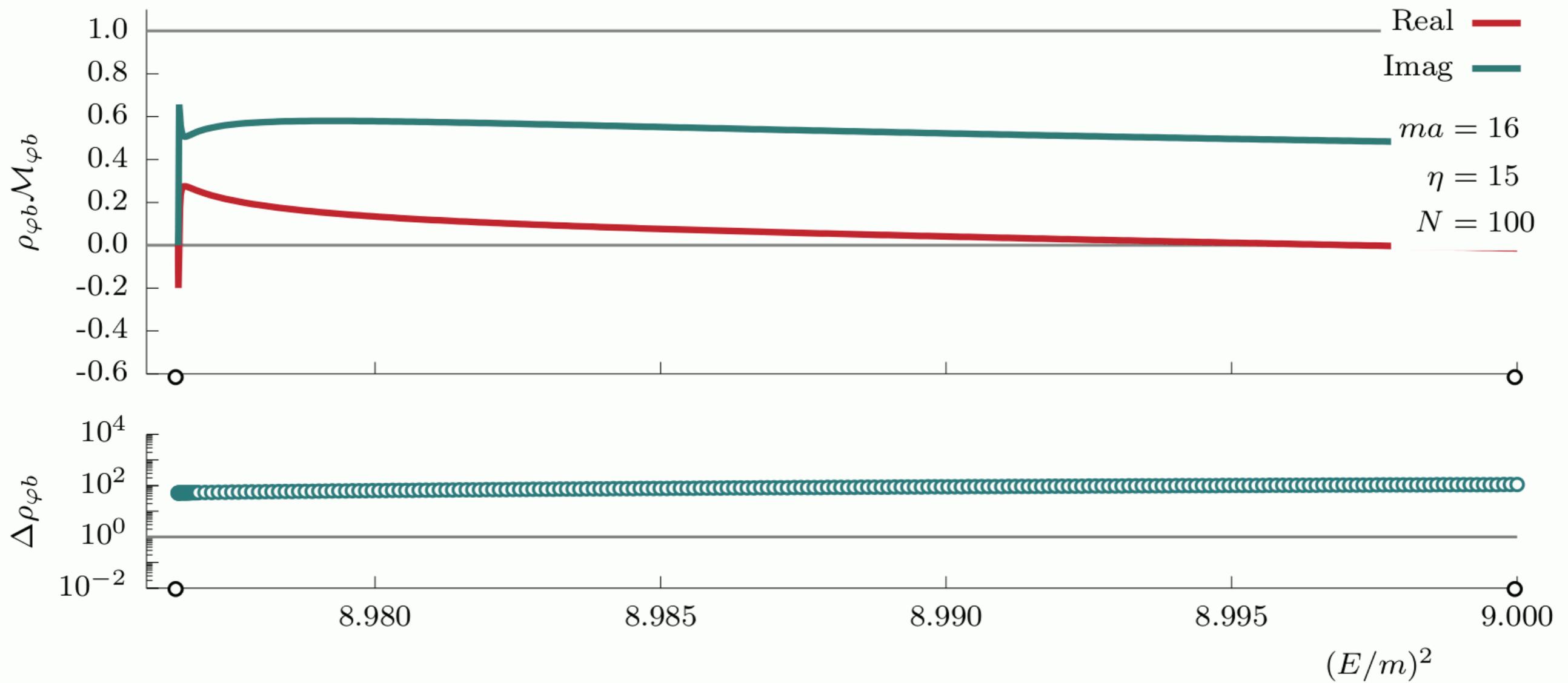
# $\epsilon \rightarrow 0$ limit

Ensure  $\epsilon \rightarrow 0$  through  $N \rightarrow \infty$  limit

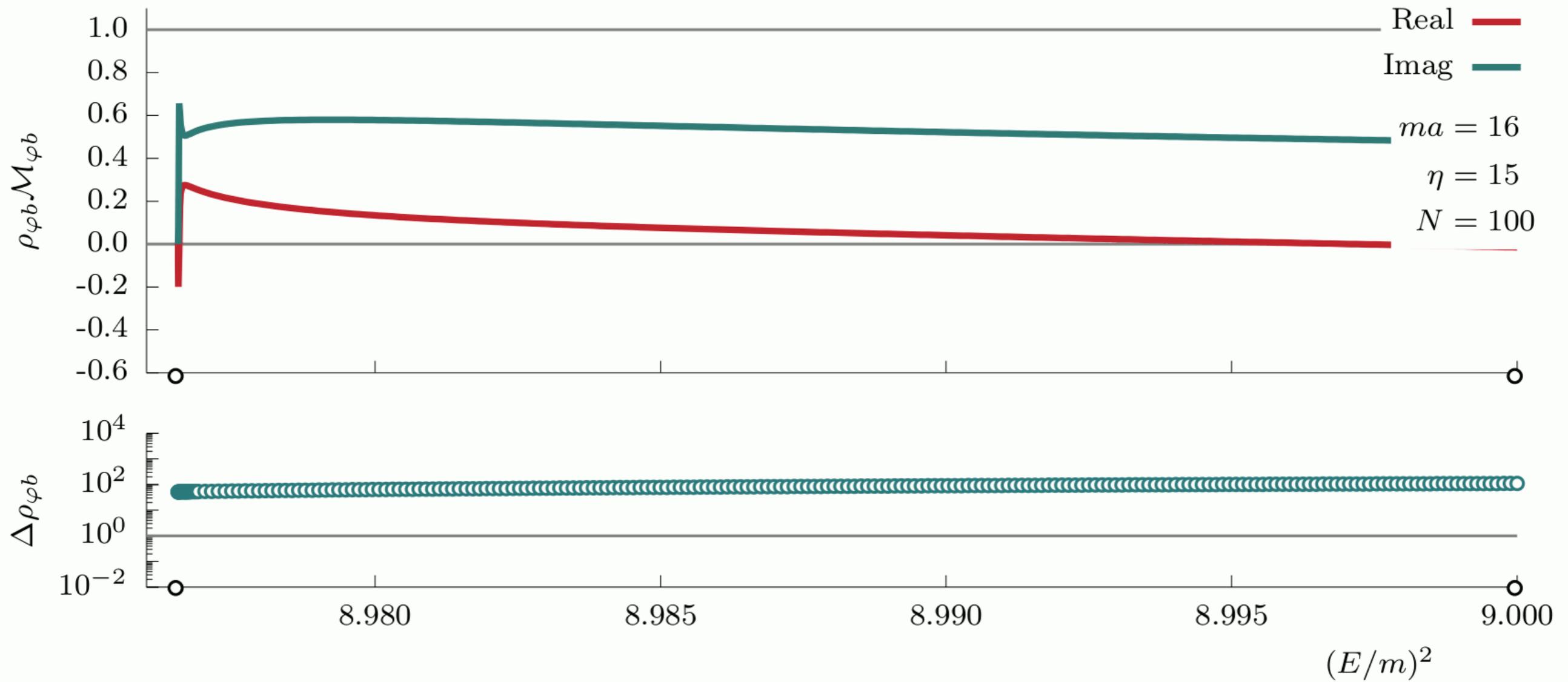
$$\left[ \sum_x \Delta x - \int dx \right] \frac{1}{x^2 - x_0^2 + i\epsilon} \sim e^{-2\pi\epsilon/\Delta x} \implies \epsilon \propto \frac{\eta}{N}$$



# Evolution of solutions



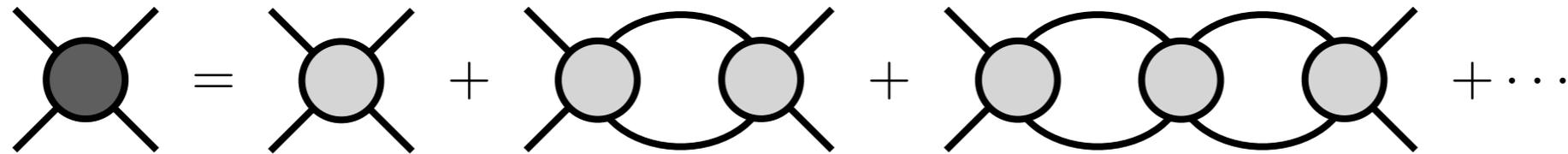
# Evolution of solutions



# On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

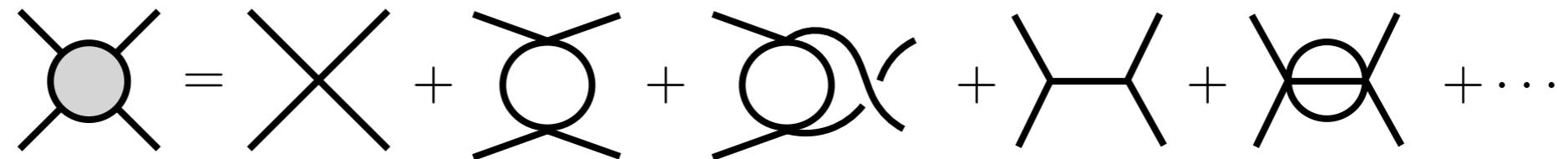
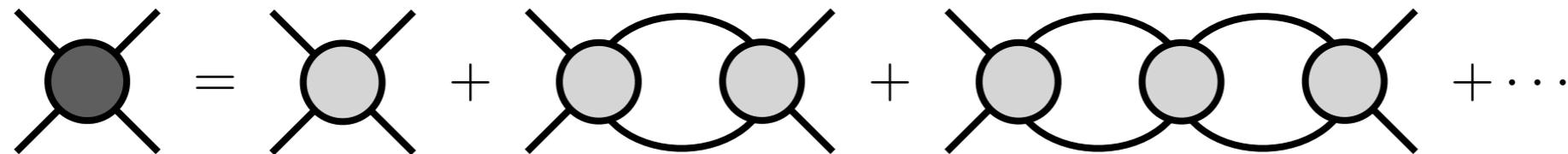
- e.g.  $2 \rightarrow 2$



# On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

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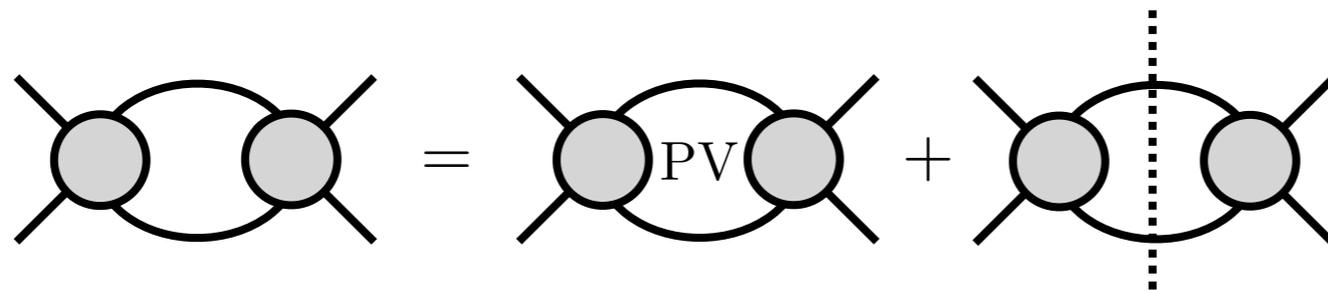
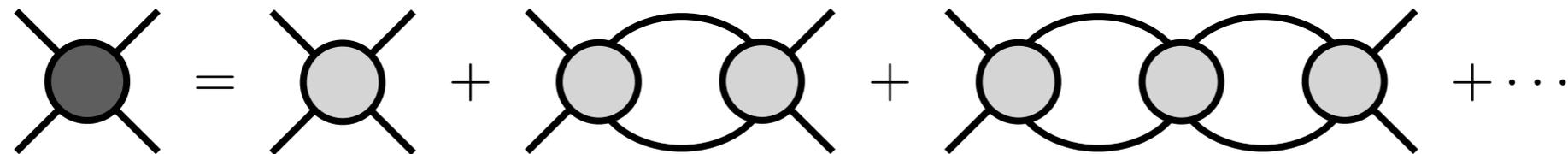


*All 2PI diagrams - left hand cuts and higher multi-particle thresholds*

# On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g.  $2 \rightarrow 2$

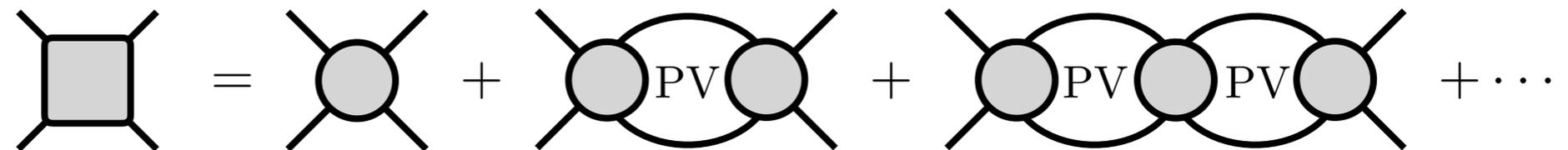
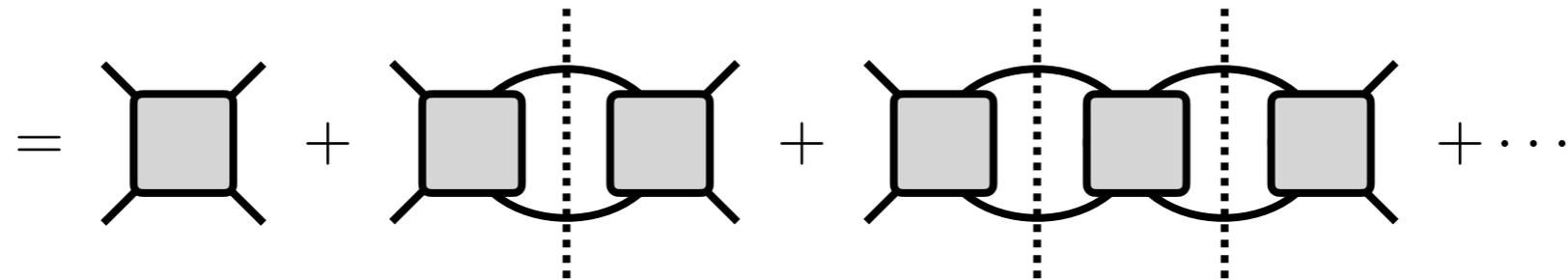
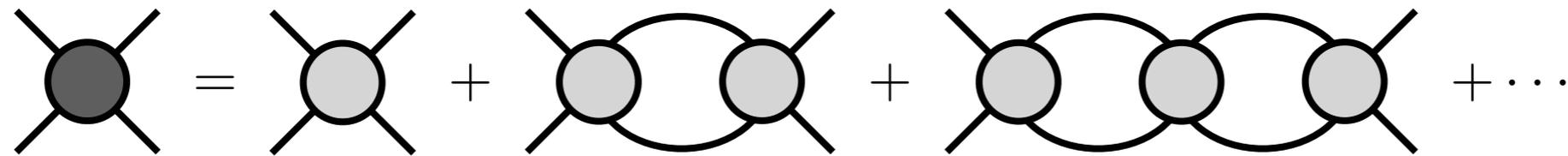


$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\text{th}}}$$

# On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g.  $2 \rightarrow 2$

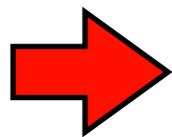
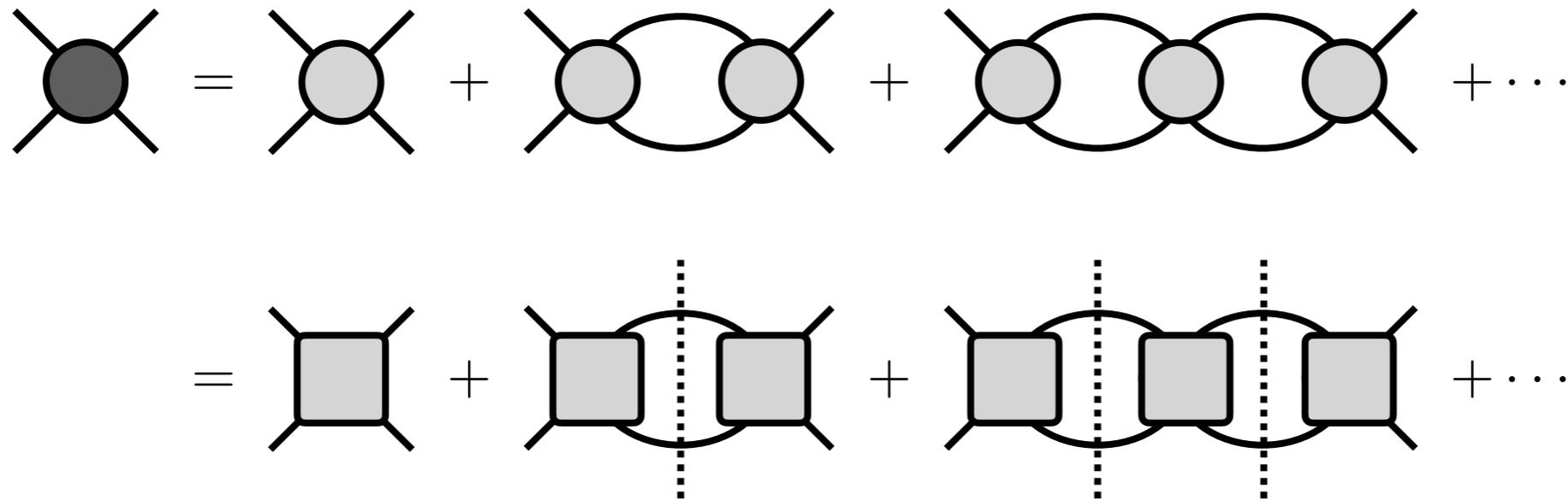


*K matrix — unknown dynamical function unconstrained by unitarity*

# On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g.  $2 \rightarrow 2$



$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho \mathcal{M}_2$$

*For given K matrix, obtain on-shell solution for amplitude*

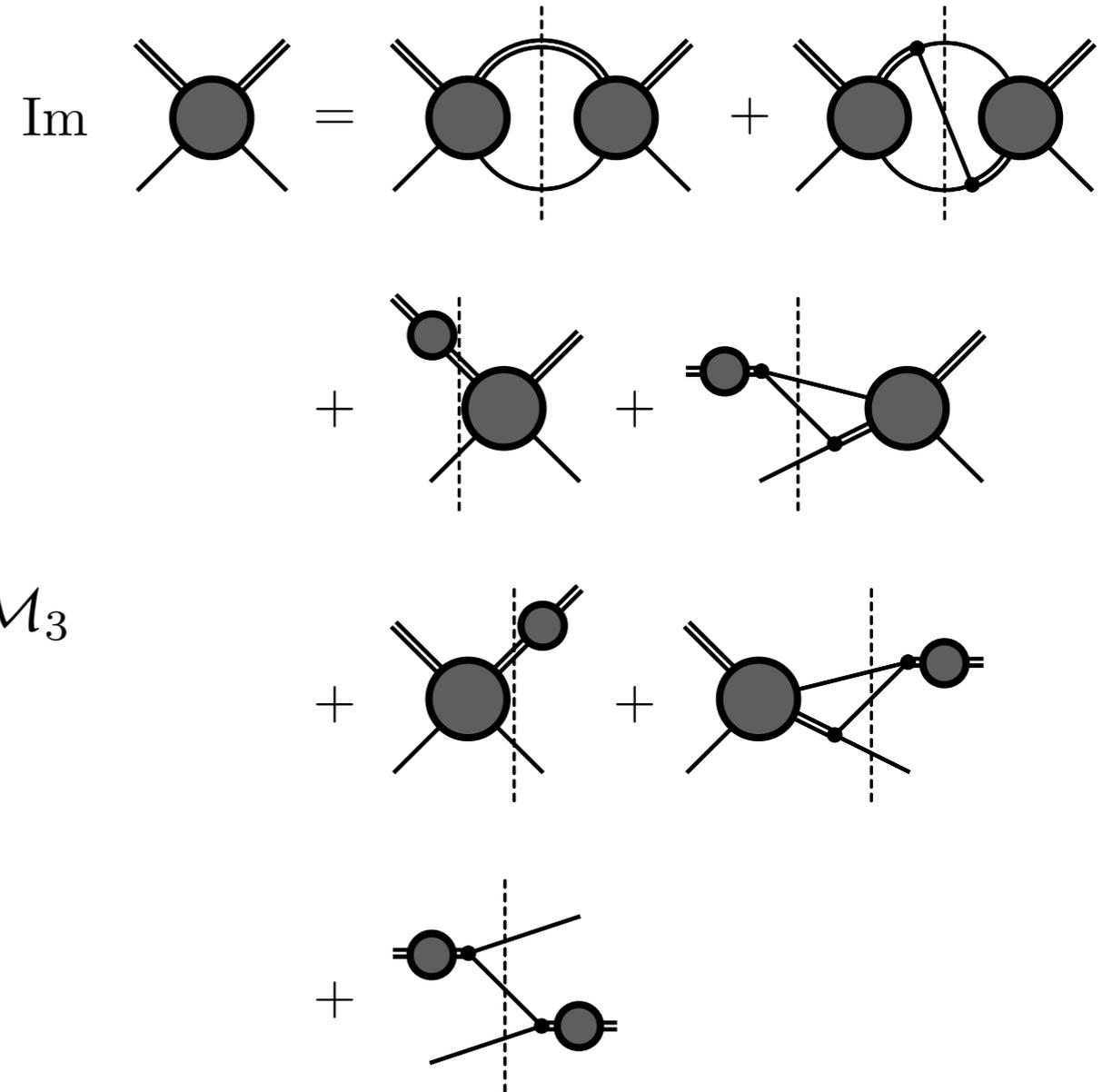
# On-shell scattering amplitudes from unitarity

Start with  $S$  matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- On-shell amplitude satisfies linear equation — check unitarity constraint

$$\mathcal{M}_3 = \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_2 + \int \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_3$$

$\mathcal{R}$  is a different short-distance function



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak  
 Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]  
 Eur. Phys. J. C **79**, no. 1, 56 (2019)

M. Mikhasenko, AJ et al. [JPAC]  
 Phys. Rev. D **98**, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC]  
 JHEP **08**, 080 (2019)

*Extension to FV*

M. Mai and M. Döring  
 Eur. Phys. J. A **53**, 240 (2017)

# Equivalence of relativistic methods

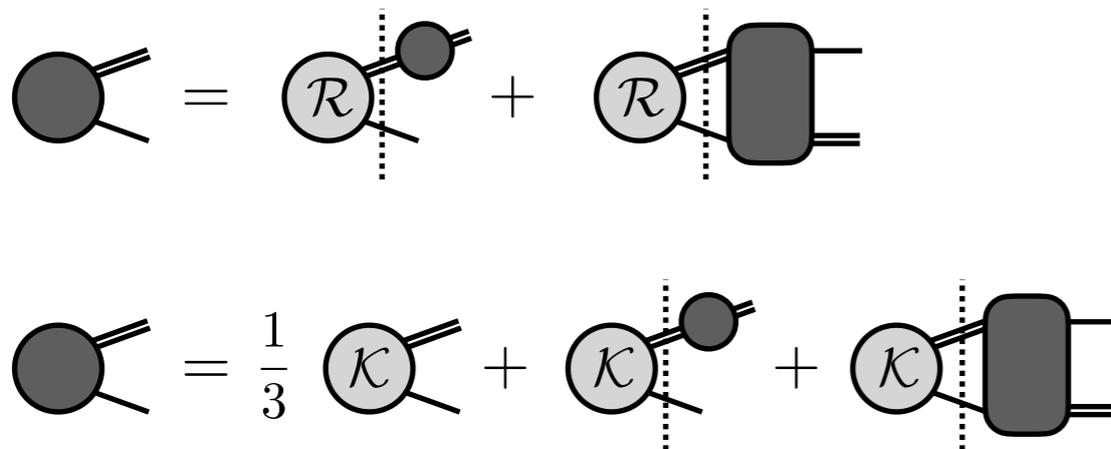
The RFT vs FVU methods can be summarized as “Bottom up” vs “Top down”

On-shell scattering equations are equivalent

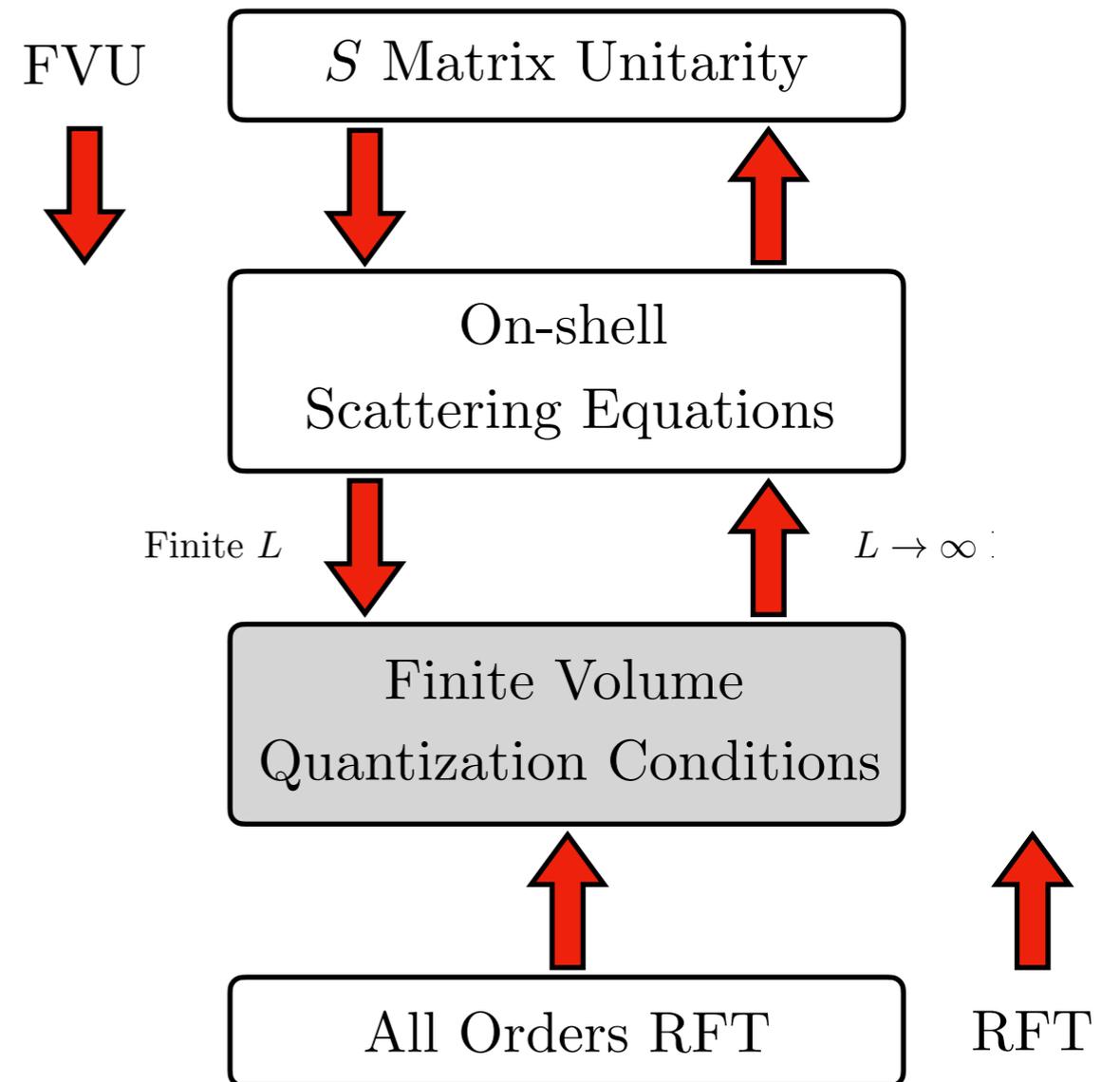
AJ et al.  
Phys. Rev. D **100**, 034508 (2019)

Quantization conditions are equivalent

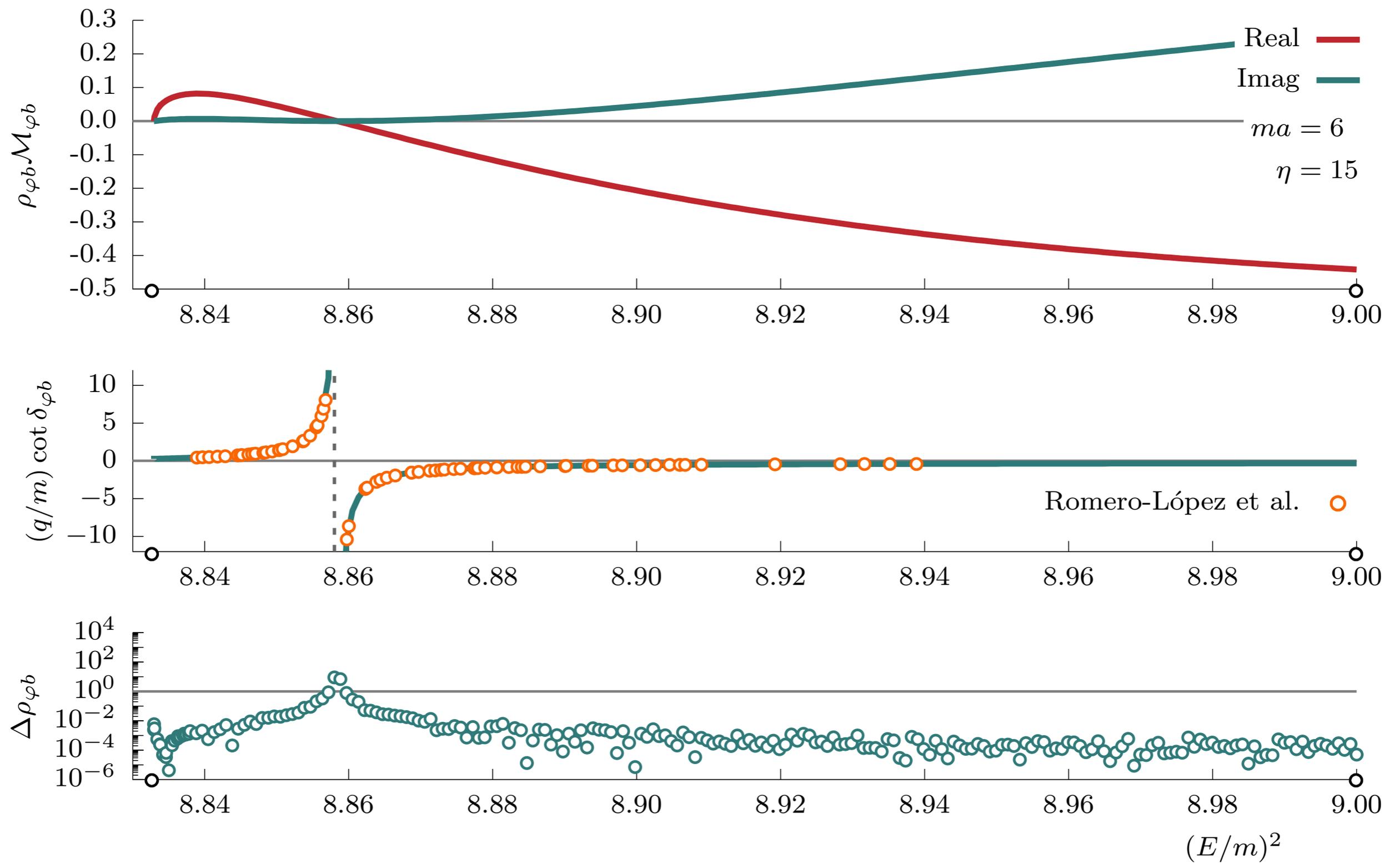
T. Blanton and S. Sharpe  
arXiv:2007.16190 (2020)



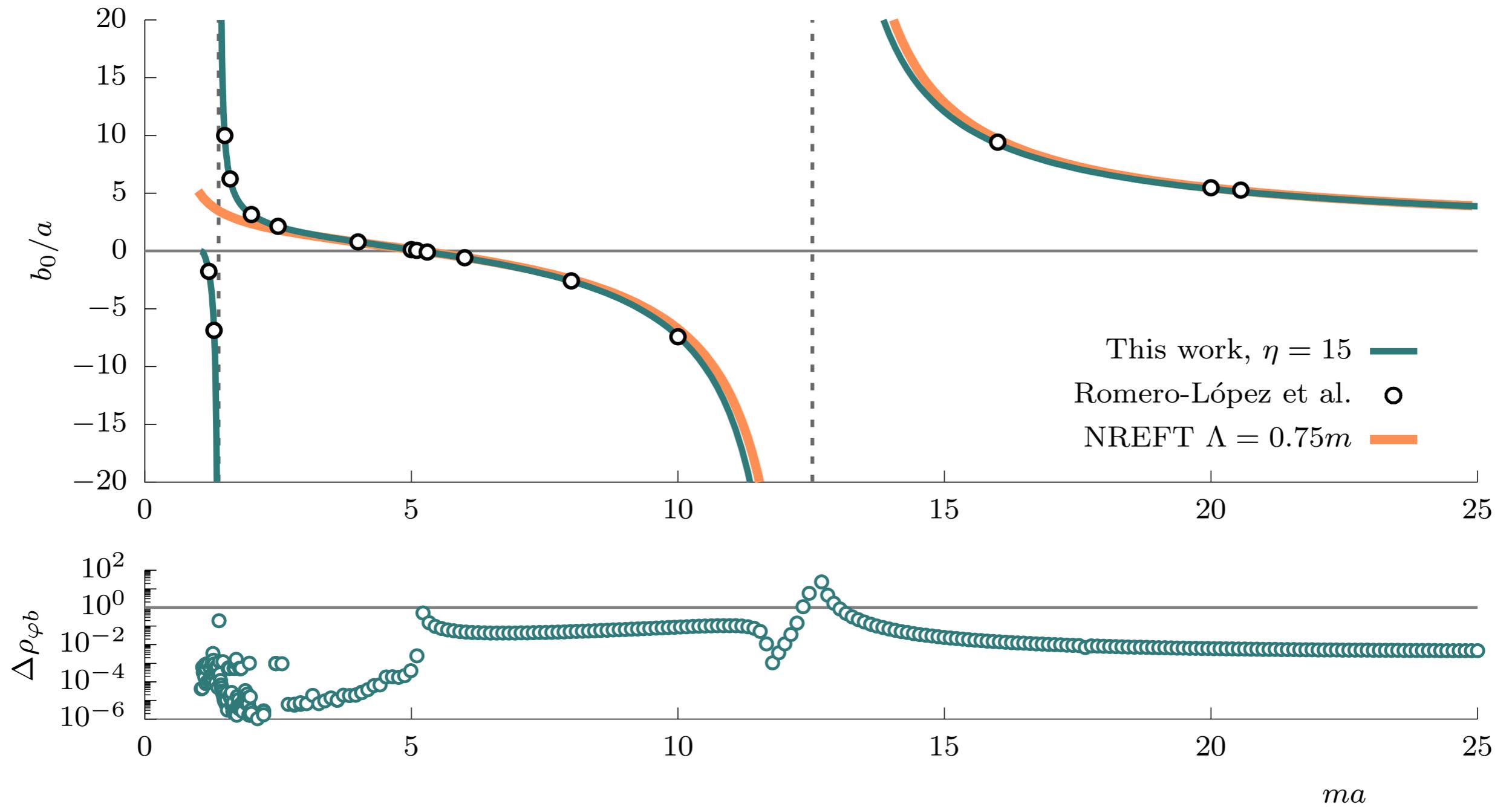
*Small details, e.g. aspects of symmetrization  
— see literature for information*



# Example of solution — Extrapolated result



# Example of solution — Extrapolated result

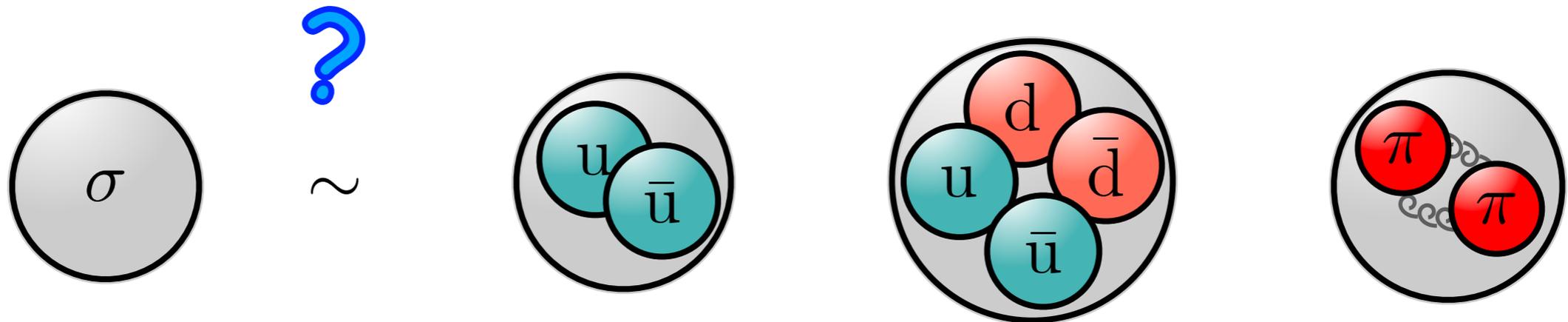


# Structural composition of hadrons

How do we determine structure of hadrons from QCD?

- Probe stable hadrons via external currents
- Gives access to form-factors, parton distribution functions, etc.

What about unstable particles, i.e. resonances? — e.g., the  $\sigma$  meson



*Standard meson*

*Tetraquark*

*Mesonic molecule*

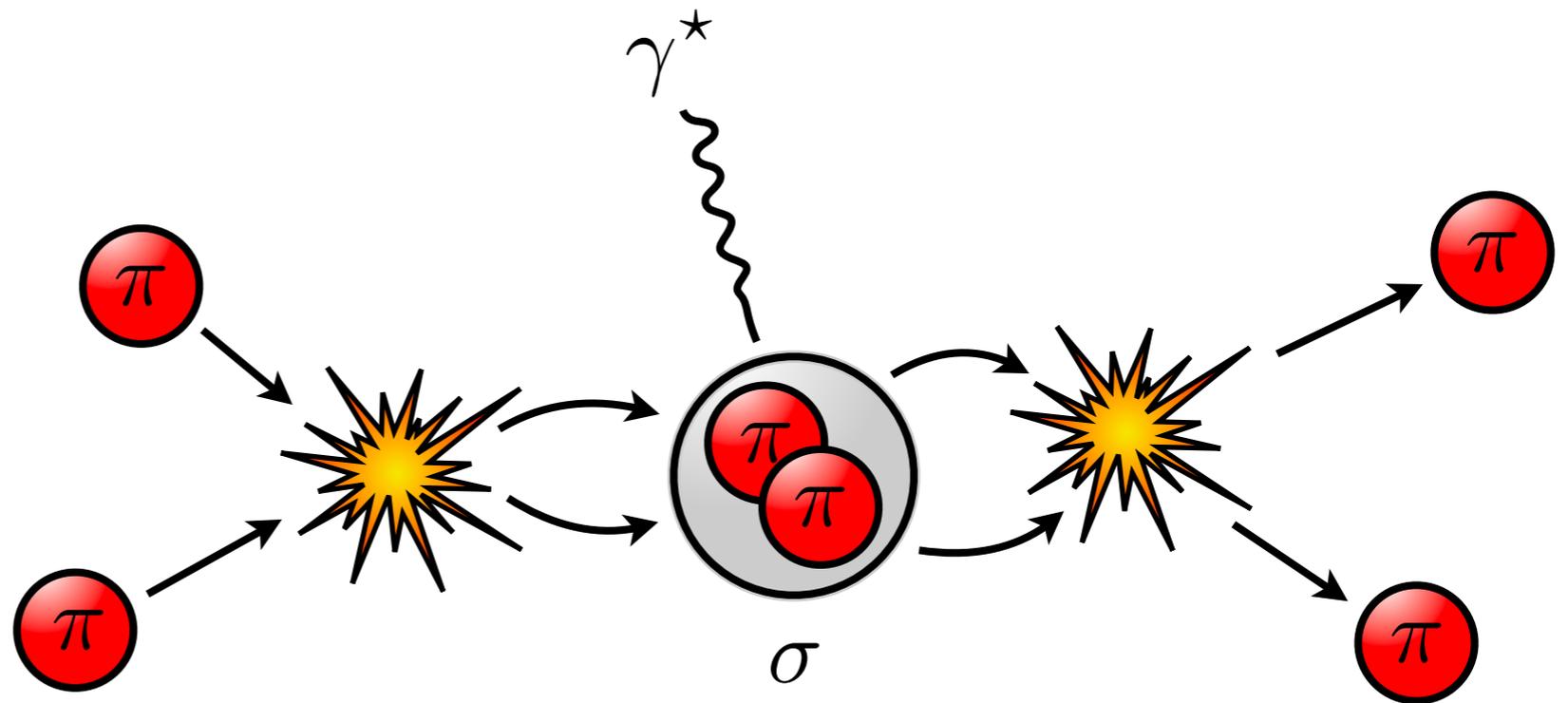
# Structural composition of hadrons

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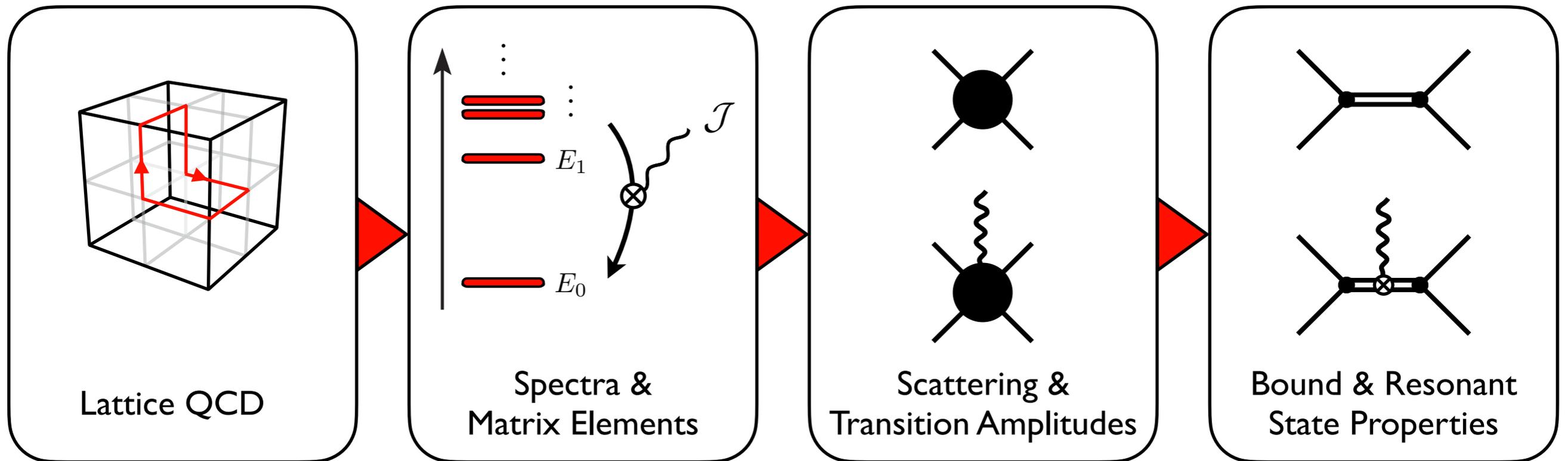
- Couple to multi-hadron states — need transition amplitudes evaluated at poles

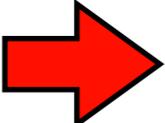
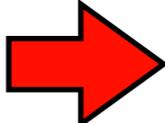


# Path to Hadronic Properties from QCD

Lattice QCD offers a systematic avenue to compute multi-hadron transition amplitudes

- Convert spectra and matrix elements to transition amplitudes via Lüscher methodology



Spectral Analysis  Finite-Volume Mappings  Analytic Continuation

# Finite volume mappings - Two hadron matrix elements

Mapping between matrix elements and  $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$  amplitudes

Features

- Relativistic
- Model independent
- Arbitrary current structure

Assumptions

- Spinless particles
- Below three-particle thresholds

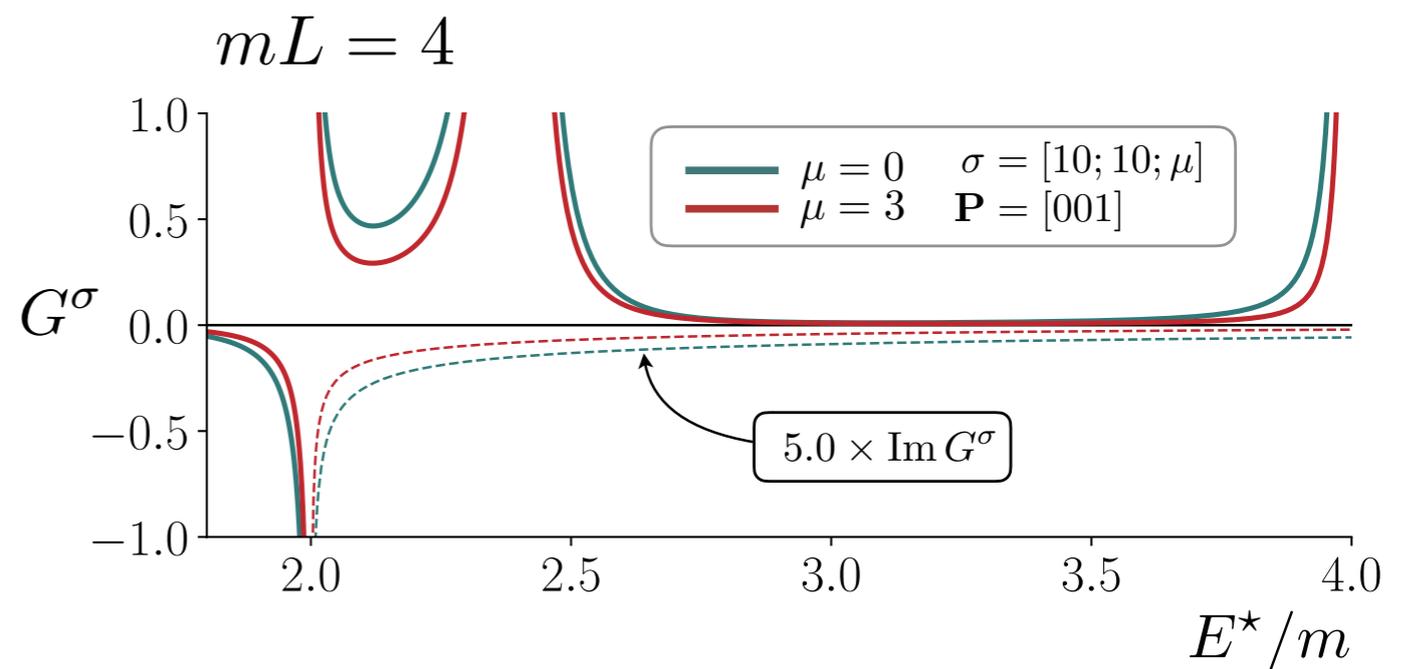
$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L = \frac{1}{L^3} \mathcal{W}_{L,\text{df}} \cdot \sqrt{\mathcal{R}_{L,\mathbf{m}} \cdot \mathcal{R}_{L,\mathbf{n}}}$$

$$\mathcal{W}_{L,\text{df}} = \mathcal{W}_{\text{df}} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$

Finite volume geometric function

$$G_L = \text{Diagram with } V \text{ and } \infty$$

The diagram shows the finite volume geometric function  $G_L$  as the difference between two diagrams. The first diagram is a circle with two external legs (grey circles) and a wavy line (representing a meson) attached to the top vertex, with a cross symbol  $\otimes$  at the vertex. The second diagram is identical but with an infinity symbol  $\infty$  inside the circle.



R. Briceño, M. Hansen,  
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,  
Phys. Rev. D **100** 034511 (2019)

# Finite volume mappings - Two hadron matrix elements

Mapping between matrix elements and  $2 + \mathcal{J} \rightarrow 2$  amplitudes

## Features

- Relativistic
- Model independent
- Arbitrary current structure

## Assumptions

- Spinless particles
- Below three-particle thresholds

$$\langle \mathbf{m} | \mathcal{J} | \mathbf{n} \rangle_L = \frac{1}{L^3} \mathcal{W}_{L,df} \cdot \sqrt{\mathcal{R}_{L,m} \cdot \mathcal{R}_{L,n}}$$

$$\mathcal{W}_{L,df} = \mathcal{W}_{df} + \mathcal{M} \cdot f \cdot G_L \cdot \mathcal{M}$$

Previously determined functions

Single hadron  
form-factors

$$f = \text{---} \otimes \text{---}$$

Hadronic  
scattering  
amplitude

$$\mathcal{M} = \text{---} \bullet \text{---}$$

R. Briceño, M. Hansen,  
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,  
Phys. Rev. D **100** 034511 (2019)

# Consistency Checks

---

Need to ensure formalism recovers known results in various limits

- Volume scaling of charge associated with conserved vector current
- Exponential suppression for two-body bound states
- Series expansion for perturbative systems

AJ, R. Briceño, M. Hansen,  
Phys. Rev. D **100** 114505 (2019)

AJ, R. Briceño, M. Hansen,  
Phys. Rev. D **101** 094508 (2020)

# Consistency Checks (I)

Expect conserved vector charge to remain independent of finite volume corrections

$$\begin{aligned}\langle \mathbf{n} | Q | \mathbf{n} \rangle_L &= L^3 \langle \mathbf{n} | \mathcal{J}^0 | \mathbf{n} \rangle_L \\ &= \mathcal{W}_{L,\text{df}} \cdot \mathcal{R}_L \\ &= \left( \mathcal{W}_{\text{df}}^0 + Q_0 \mathcal{M} \cdot G_L^0 \cdot \mathcal{M} \right) \left( \frac{\partial}{\partial E} \mathcal{M} + \mathcal{M} \cdot G_L^0 \cdot \mathcal{M} \right)^{-1}\end{aligned}$$

*Ward-Takahashi identity*

$$\lim_{P_i \rightarrow P_f} \mathcal{W}_{\text{df}}^\mu = Q_0 \frac{\partial}{\partial P_\mu} \mathcal{M}$$

$$\langle \mathbf{n} | Q | \mathbf{n} \rangle_L = Q_0$$

# Consistency Checks (II)

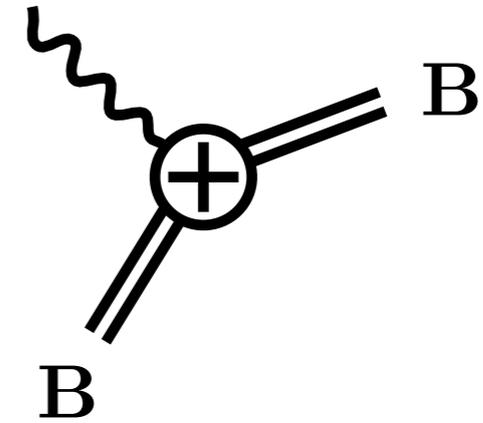
Bound state limit — corrections exponentially suppressed

$$m_B^2(L) = m_B^2 + \mathcal{O}(e^{-\kappa L})$$

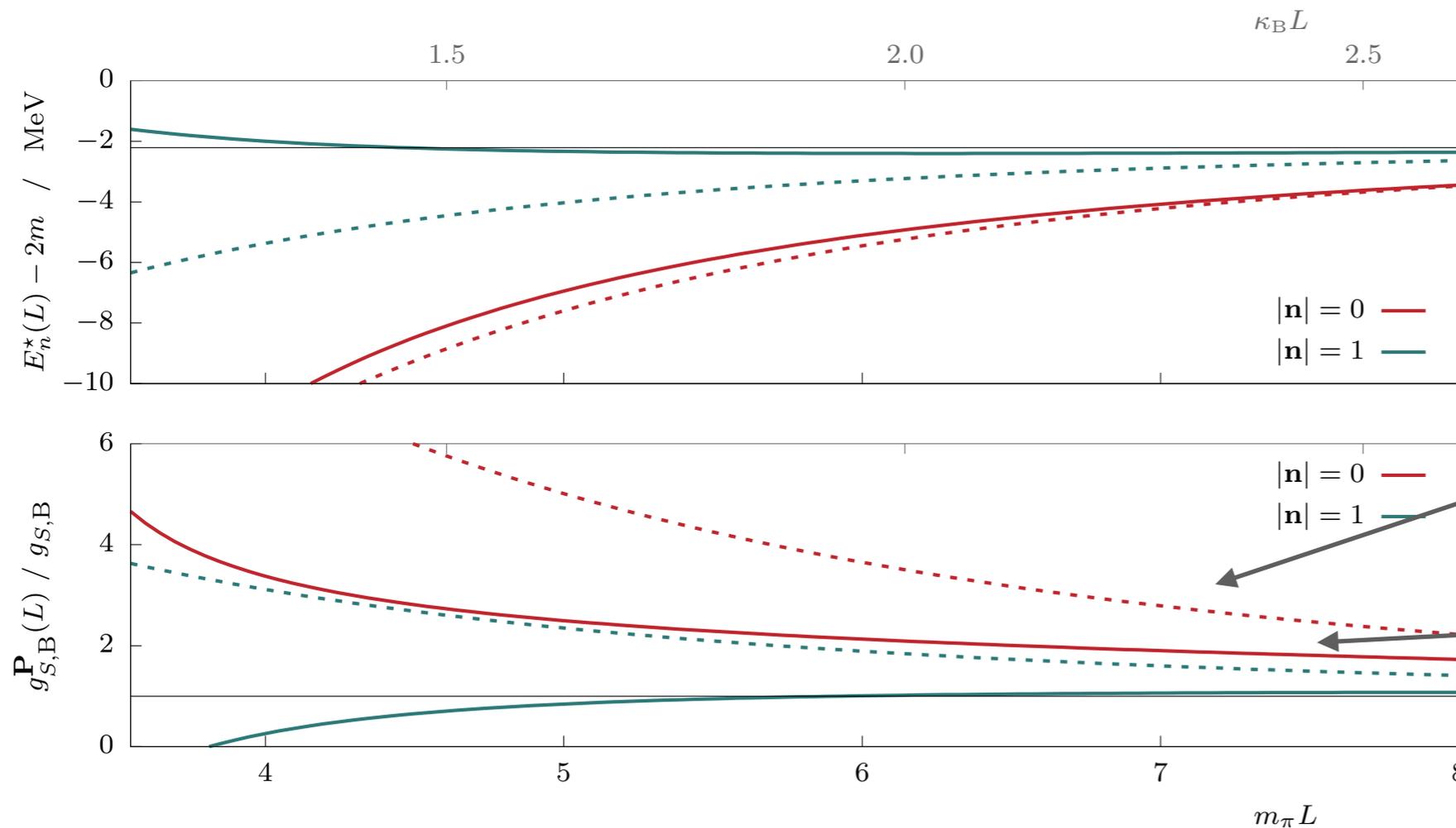
Z. Davoudi, M. Savage,  
Phys. Rev. D **84** 114502 (2011)

$$\frac{g_B(L)}{g_B} = 1 + \mathcal{O}(e^{-\kappa L})$$

AJ, R. Briceño, M. Hansen,  
Phys. Rev. D **100** 114505 (2019)



*Low-order large  $L$  behavior can have significant deviations*



# Consistency Checks (III)

Systems near threshold – Large  $L$  expansion

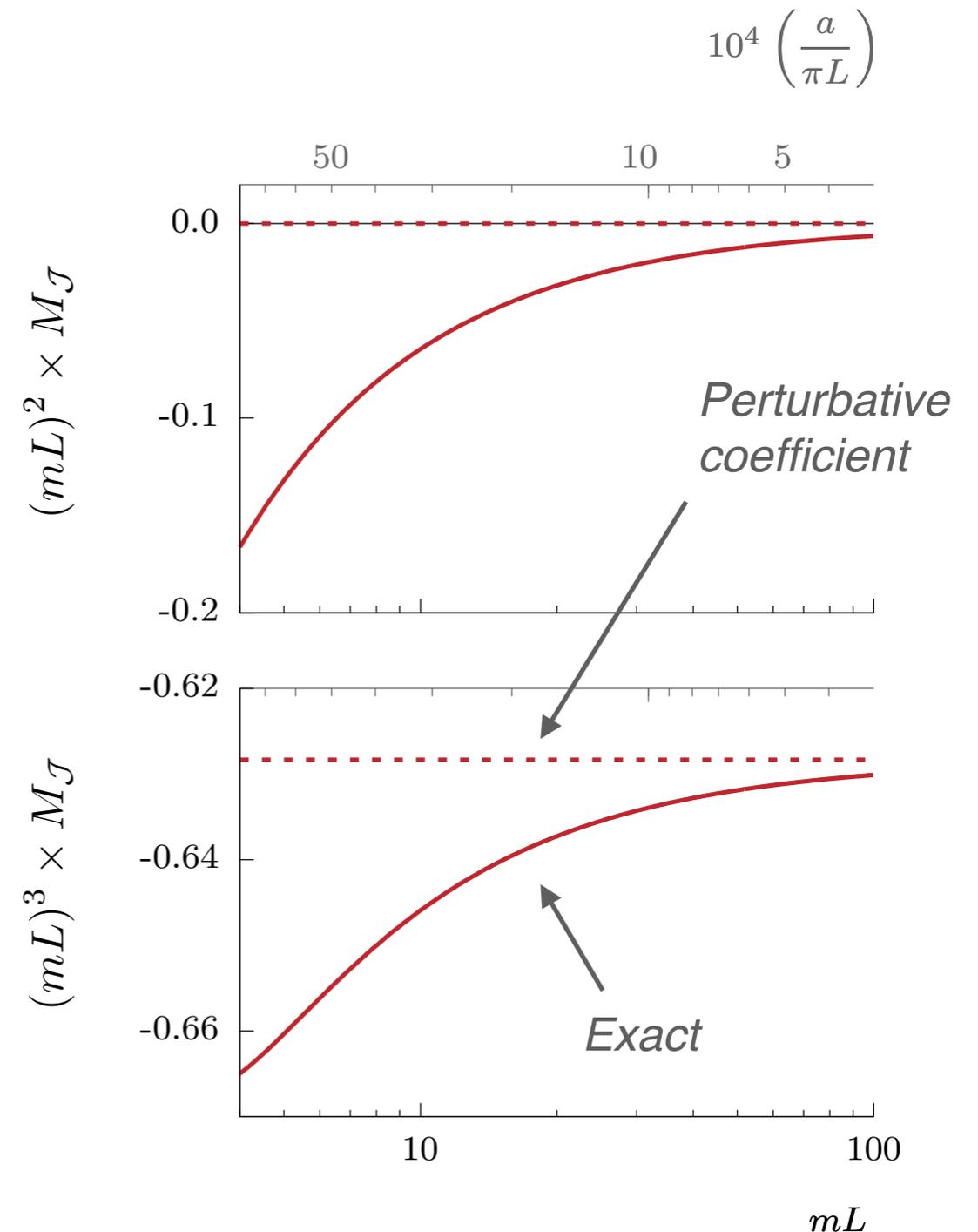
$$E_0(L) = 2m + \frac{4\pi a}{mL^3} + \mathcal{O}(1/L^4) \quad \text{Lüscher (1986), Huang and Yang (1957), many others...}$$

$$L^3 \langle 0 | \mathcal{J} | 0 \rangle = \frac{g}{m} \left( 1 - \frac{2\pi a}{m^2 L^3} + \mathcal{O}(1/L^4) \right)$$

Confirmation via

- Threshold expansion
- Perturbation theory
- Numerical verification
- Agrees with Feynman-Hellman theory

$$L^3 \langle 0 | \mathcal{J} | 0 \rangle = g \frac{dE_0(L)}{dm^2}$$



# Finite-Volume Formalism

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Have framework relating matrix elements to amplitudes



- Formalism is relativistic, model independent, valid for arbitrary local current
- Systematically controlled below three-hadron channels

Passes cross-checks in various limits



- Charge conservations ensures volume independence
- Bound states lead to exponential corrections
- Perturbative systems have consistent large  $L$  expansion

Need understanding of analytic behavior of infinite volume amplitudes

# Two hadron transition amplitudes

Sum to all orders in strong interaction — interested in two-particle kinematic energies

$$i\mathcal{W} = \text{[diagram: a black circle with four external lines and a wavy line pointing up]} \\ = \mathcal{S} \left\{ \text{[diagram: a black circle with four external lines, a wavy line pointing up, and a dashed line with a cross pointing down]} \right\} + i\mathcal{W}_{\text{df}}$$

*Object we can determine from FV formalism*

## Features

- *Relativistic*
- *Model independent*
- *Arbitrary current structure*

## Assumptions

- *Spinless particles*
- *Below three-particle thresholds*

## Two hadron transition amplitudes

Sum to all orders in strong interaction — interested in two-particle kinematic energies

$$\begin{aligned} i\mathcal{W} &= \text{Diagram 1} \\ &= \mathcal{S} \left\{ \text{Diagram 2} \right\} + i\mathcal{W}_{\text{df}} \end{aligned}$$

The diagram on the right is a black circle with four external lines. A wavy line enters from the top, and a dashed line with a cross in a circle at its end enters from the top-left. The other two lines are solid and extend downwards and outwards.

Projecting equations to on-shell form

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

# Two hadron transition amplitudes

Sum to all orders in strong interaction — interested in two-particle kinematic energies

$$i\mathcal{W} = \text{[Diagram: a black circle with four external lines and a wavy line entering from the top]} \\ = \mathcal{S} \left\{ \text{[Diagram: a black circle with four external lines, a wavy line entering from the top, and a dashed line connecting the wavy line to the circle]} \right\} + i\mathcal{W}_{\text{df}}$$

Projecting equations to on-shell form

$$\mathcal{W}_{\text{df}} = \boxed{\mathcal{M}} \cdot (\mathcal{A} + \boxed{f \cdot \mathcal{G}}) \cdot \mathcal{M}$$

*Known functions*

*Single hadron form-factors*

$$f = \text{[Diagram: a horizontal line with a wavy line entering from the top and a small circle with an 'x' on the line]} \\ = \text{[Diagram: a horizontal line with a wavy line entering from the top and a small circle with an 'x' on the line]}$$

*Hadronic scattering amplitude*

$$\mathcal{M} = \text{[Diagram: a black circle with four external lines]} \\ = \text{[Diagram: a black circle with four external lines]}$$

*Triangle diagram*

*Contains normal and anomalous singularities from intermediate on-shell particles*

$$\mathcal{G} = \text{[Diagram: a loop with two vertices (grey circles) and a wavy line entering from the top]} \\ = \text{[Diagram: a loop with two vertices (grey circles) and a wavy line entering from the top]}$$

# Two hadron transition amplitudes

Sum to all orders in strong interaction — interested in two-particle kinematic energies

$$i\mathcal{W} = \text{[Diagram: a black circle with four external lines and a wavy line entering from the top]} \\ = \mathcal{S} \left\{ \text{[Diagram: a black circle with four external lines, a wavy line entering from the top, and a dashed line with a cross entering from the left]} \right\} + i\mathcal{W}_{\text{df}}$$

Projecting equations to on-shell form

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot \left( \boxed{A} + f \cdot \mathcal{G} \right) \cdot \mathcal{M}$$

*Unknown function*

*Real and Smooth function  
characterizing short-distance physics*

*Parameterize in terms of energy-  
dependent form-factors*

*Only unknown function - determined  
from lattice QCD and FV formalism*

*Can be related to LEC's of EFT's*

# Two hadron transition amplitudes

Sum to all orders in strong interaction — interested in two-particle kinematic energies

$$i\mathcal{W} = \text{Diagram} = \mathcal{S} \left\{ \text{Diagram} \right\} + i\mathcal{W}_{\text{df}}$$

The diagram on the left is a black circle with four external lines: two solid lines and two wavy lines. The diagram in the curly braces is a black circle with four external lines: two solid lines and two wavy lines. A dashed line with a cross in a circle at its end connects the top wavy line to the top of the circle.

Projecting equations to on-shell form

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

*Rigorous definition for resonance form factors*

$$i\mathcal{W} \sim \frac{-g}{s_f - s_p} i f_p \frac{-g}{s_i - s_p}$$

$$i f_p = g^2 (\mathcal{A} + f \cdot \mathcal{G}) \Big|_{s_f = s_i = s_p}$$

$$= \text{Diagram}$$

The diagram shows a central circle with a plus sign inside. It has four external lines: one wavy line pointing up and left, and two double solid lines pointing down and right.

# Summary

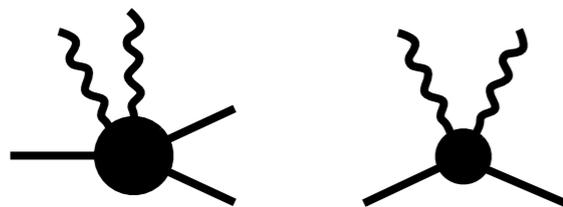
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Model-independent method to determine  $2 + \mathcal{J} \rightarrow 2$  processes

- Relates finite volume matrix elements to transition amplitudes
- Many cross-checks — confidence builder of formalism Briceño, Hansen, AJ (2019 & 2020)

Global infinite volume amplitude studies

- On-shell physics fixes amplitude — single unknown real function remains
- Have forms for one current, two hadron amplitudes
- Informs structures that will appear in future FV formalisms
- Extensions to two currents



*See R. Briceño, this session*

