

Revisiting GPD evolution

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GPD definition

- Generalised parton distributions (GPDs) are a “byproduct” of factorisation of *amplitudes* for **exclusive** processes such as deeply-virtual Compton scattering.

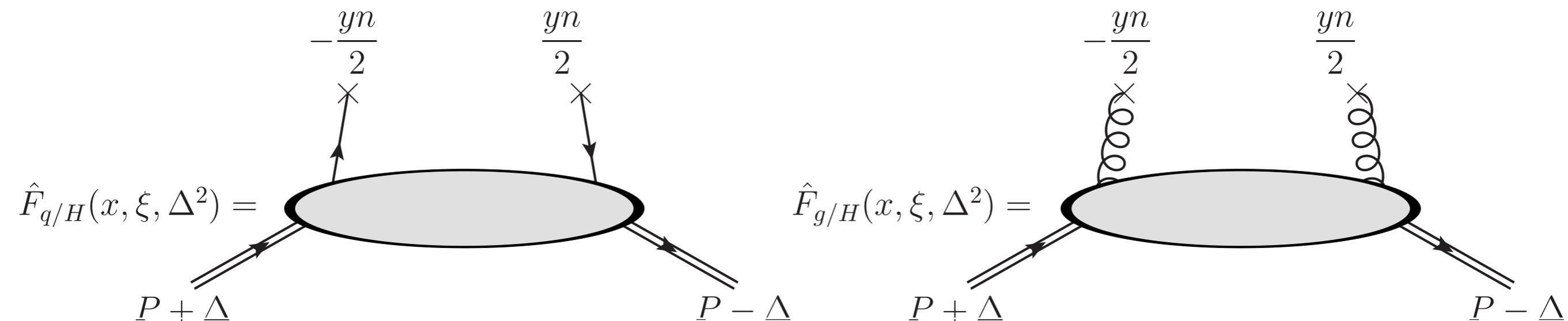
[Collins, Freund, *Phys.Rev.D* 59 (1999) 074009] [Ji, *Phys.Rev.D* 55 (1997) 7114-7125]

- An operator definition of the GPDs in the **light-cone gauge** ($n \cdot A = 0$) reads:

$$\hat{F}_{q/H}(x, \xi, \Delta^2) = \frac{1}{\sqrt{1-\xi^2}} \int \frac{dy}{2\pi} e^{-ix(n \cdot P)y} \left\langle P - \Delta \left| \bar{\psi}_q \left(\frac{yn}{2} \right) \frac{\not{n}}{2} \psi_q \left(-\frac{yn}{2} \right) \right| P + \Delta \right\rangle$$

$$\hat{F}_{g/H}(x, \xi, \Delta^2) = -x(n \cdot P) \int \frac{dy}{2\pi} e^{-ix(n \cdot P)y} \left\langle P - \Delta \left| A_a^\alpha \left(\frac{yn}{2} \right) A_{a\alpha} \left(-\frac{yn}{2} \right) \right| P + \Delta \right\rangle$$

$\xi = \frac{\Delta^+}{P^+}$



- No Wilson line, simpler gluon GPD, more complicated gluon propagator:

$$\mathcal{D}_{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \left(-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} \right)$$

- These definitions are affected by UV divergences that need to be renormalised.

GPD evolution

Using dim. reg., the renormalisation of GPDs can be implemented as follows:

$$F_{i/H}(x, \xi, \mu) = \sum_{j=q,g} \int_{-1}^1 \frac{dy}{|y|} Z_{ij} \left(\frac{x}{y}, \frac{\xi}{x}, \alpha_s(\mu), \varepsilon \right) \hat{F}_{j/H}(y, \xi, \varepsilon)$$

In the $\overline{\text{MS}}$ scheme renormalisation constants have the following structure:

$$Z_{ij}(z, \kappa, \alpha_s, \varepsilon) = \delta_{ij} \delta(1-z) + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^n \sum_{p=1}^n \frac{1}{\varepsilon^p} Z_{ij}^{[n,p]}(z, \kappa)$$

Exploiting the independence of the bare GPDs on μ , one can derive a **RGE**:

$$\frac{dF_{i/H}(x, \xi, \mu)}{d \ln \mu^2} = \sum_{k=q,g} \int_{-1}^1 \frac{dz}{|z|} \mathcal{P}_{i/k} \left(\frac{x}{z}, \frac{\xi}{x}, \alpha_s(\mu) \right) F_{k/H}(z, \xi, \mu)$$

The evolution kernel are finite quantities computable in perturbation theory:

$$\mathcal{P}_{i/k}(z, \kappa, \alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \mathcal{P}_{i/k}^{[n]}(z, \kappa)$$

They are related to the renormalisation constants Z_{ij} . At LO one finds:

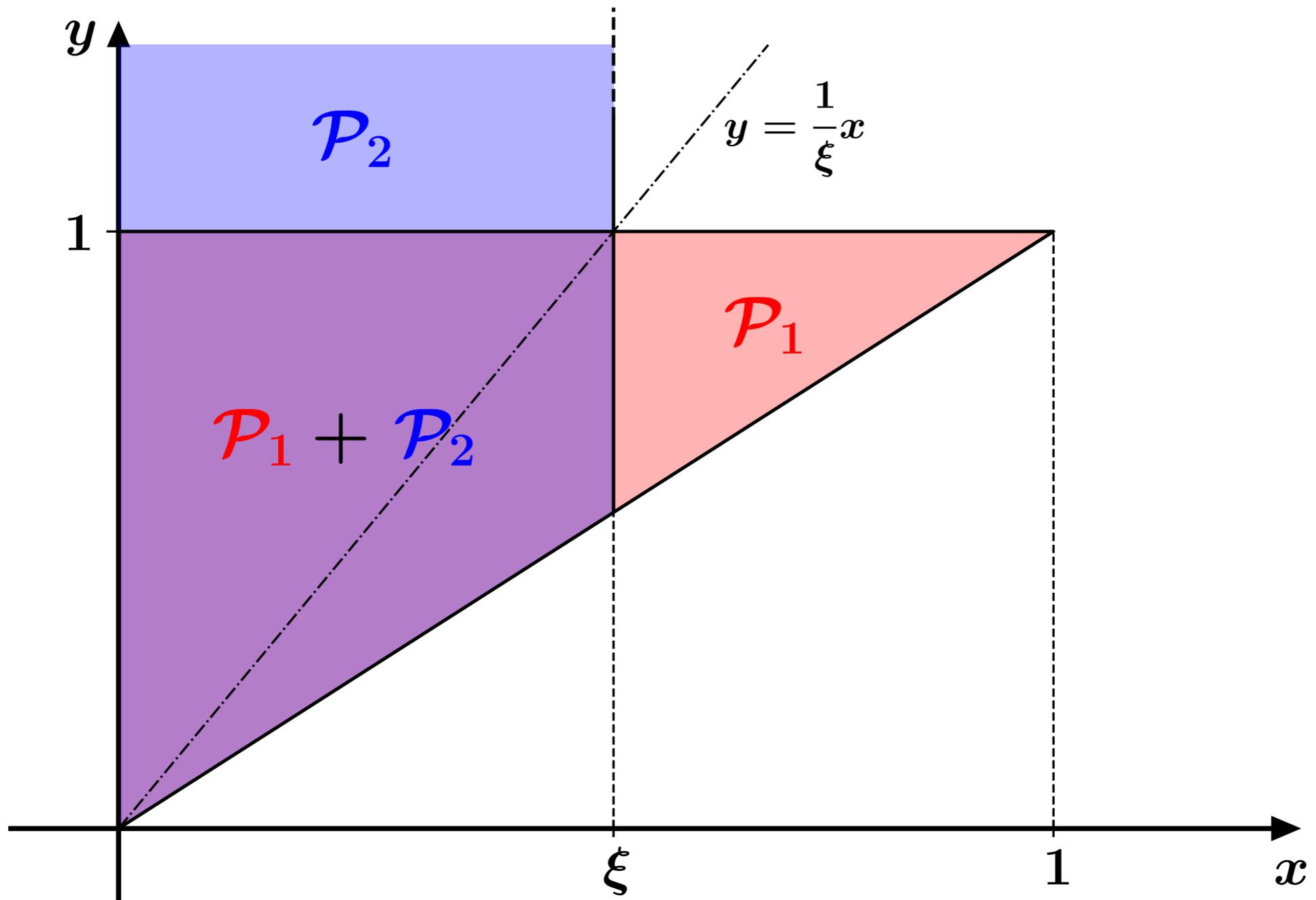
$$\mathcal{P}_{i/k}^{[0]}(z, \kappa) = -Z_{ik}^{[1,1]}(z, \kappa)$$

Bottomline: the computation of the **evolution kernels** boils down to computing the **GPD renormalisation constants**.

GPD evolution

$$\frac{dF^\pm(x, \xi, \mu)}{d \ln \mu^2} = \frac{\alpha_s(\mu)}{4\pi} \int_x^\infty \frac{dy}{y} \mathcal{P}^{\pm, [0]} \left(y, \frac{\xi}{x} \right) F^\pm \left(\frac{x}{y}, \xi, \mu \right)$$

$$\mathcal{P}^{\pm, [0]} \left(y, \frac{\xi}{x} \right) = \theta(1 - y) \mathcal{P}_1^{\pm, [0]} \left(y, \frac{\xi}{x} \right) + \theta(\xi - x) \mathcal{P}_2^{\pm, [0]} \left(y, \frac{\xi}{x} \right)$$



Properties of the kernels

$$\mathcal{P}^{\pm,[0]}(y, \kappa) = \theta(1-y) \mathcal{P}_1^{\pm,[0]}(y, \kappa) + \theta(\kappa-1) \mathcal{P}_2^{\pm,[0]}(y, \kappa) \quad \kappa = \frac{\xi}{x}$$

🍏 In the limit $\kappa \rightarrow 0$ the **DGLAP** splitting functions are recovered:

$$\lim_{\kappa \rightarrow 0} \mathcal{P}^{\pm,[0]}(y, \kappa) = \theta(1-y) P^{\pm,[0]}(y)$$

🍏 In the limit $\kappa \rightarrow 1/x$ the **ERBL** non-singlet kernel is recovered:

e.g. [Mikhailov, Radyushkin, *Nucl.Phys.B* 254 (1985) 89-126]

or [Blümlein, Geyer, Robaschik, *Phys.Lett.B* 406 (1997) 161-170]

$$\frac{1}{2u-1} \mathcal{P}^{-,[0]} \left(\frac{2t-1}{2u-1}, \frac{1}{2t-1} \right) = C_F \left[\theta(u-t) \left(\frac{t-1}{u} + \frac{1}{u-t} \right) - \theta(t-u) \left(\frac{t}{1-u} + \frac{1}{u-t} \right) \right]_+$$

$$\text{with } [f(t, u)]_+ \equiv f(t, u) - \delta(u-t) \int_0^1 du' f(t, u')$$

🍏 **Continuity** of GPDs at the crossover point $x = \xi$ ($\kappa = 1$) guaranteed:

$$\lim_{\kappa \rightarrow 1} \mathcal{P}_1^{\pm,[0]}(y, \kappa) = \text{finite} \quad \mathcal{P}_2^{\pm,[0]}(y, \kappa) \propto (1-\kappa)$$

🍏 Cancellation of **spurious divergencies** (stable numerical implementation)

$$\lim_{y \rightarrow \kappa^{-1}} (1 - \kappa^2 y^2)^\alpha \mathcal{P}_1^{\pm,[0]}(y, \kappa) = - \lim_{y \rightarrow \kappa^{-1}} (1 - \kappa^2 y^2)^\alpha \mathcal{P}_2^{\pm,[0]}(y, \kappa) = \text{finite}$$

Properties of the kernels

$$\mathcal{P}^{\pm,[0]}(y, \kappa) = \theta(1-y) \mathcal{P}_1^{\pm,[0]}(y, \kappa) + \theta(\kappa-1) \mathcal{P}_2^{\pm,[0]}(y, \kappa) \quad \kappa = \frac{\xi}{x}$$

🍏 Valence **sum rule** (polynomiality for the first moment of the non-singlet) conserved:

$$\int_0^1 dx F^-(x, \xi, \mu) = \text{FF} \quad \Rightarrow \quad \int_0^1 dz \mathcal{P}_1^{-,[0]} \left(z, \frac{\xi}{yz} \right) + \int_0^{\xi/y} dz \mathcal{P}_2^{-,[0]} \left(z, \frac{\xi}{yz} \right) = 0$$

🍏 As consequence of the **Ji's sum rule** one also finds: [\[Ji, Phys.Rev.Lett. 78 \(1997\) 610-613\]](#)

$$\int_0^1 dx x \left[F_q^+(x, \xi, \mu) + F_g^+(x, \xi, \mu) \right] = \text{constant}$$

🍏 that leads to:

$$\int_0^1 dz z \left[\mathcal{P}_{1,qq}^{+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{1,gq}^{+,[0]} \left(z, \frac{\xi}{yz} \right) \right] + \int_0^{\xi/y} dz z \left[\mathcal{P}_{2,qq}^{+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{2,gq}^{+,[0]} \left(z, \frac{\xi}{yz} \right) \right] = 0$$

$$\int_0^1 dz z \left[\mathcal{P}_{1,qg}^{+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{1,gg}^{+,[0]} \left(z, \frac{\xi}{yz} \right) \right] + \int_0^{\xi/y} dz z \left[\mathcal{P}_{2,qg}^{+,[0]} \left(z, \frac{\xi}{yz} \right) + \mathcal{P}_{2,gg}^{+,[0]} \left(z, \frac{\xi}{yz} \right) \right] = 0$$

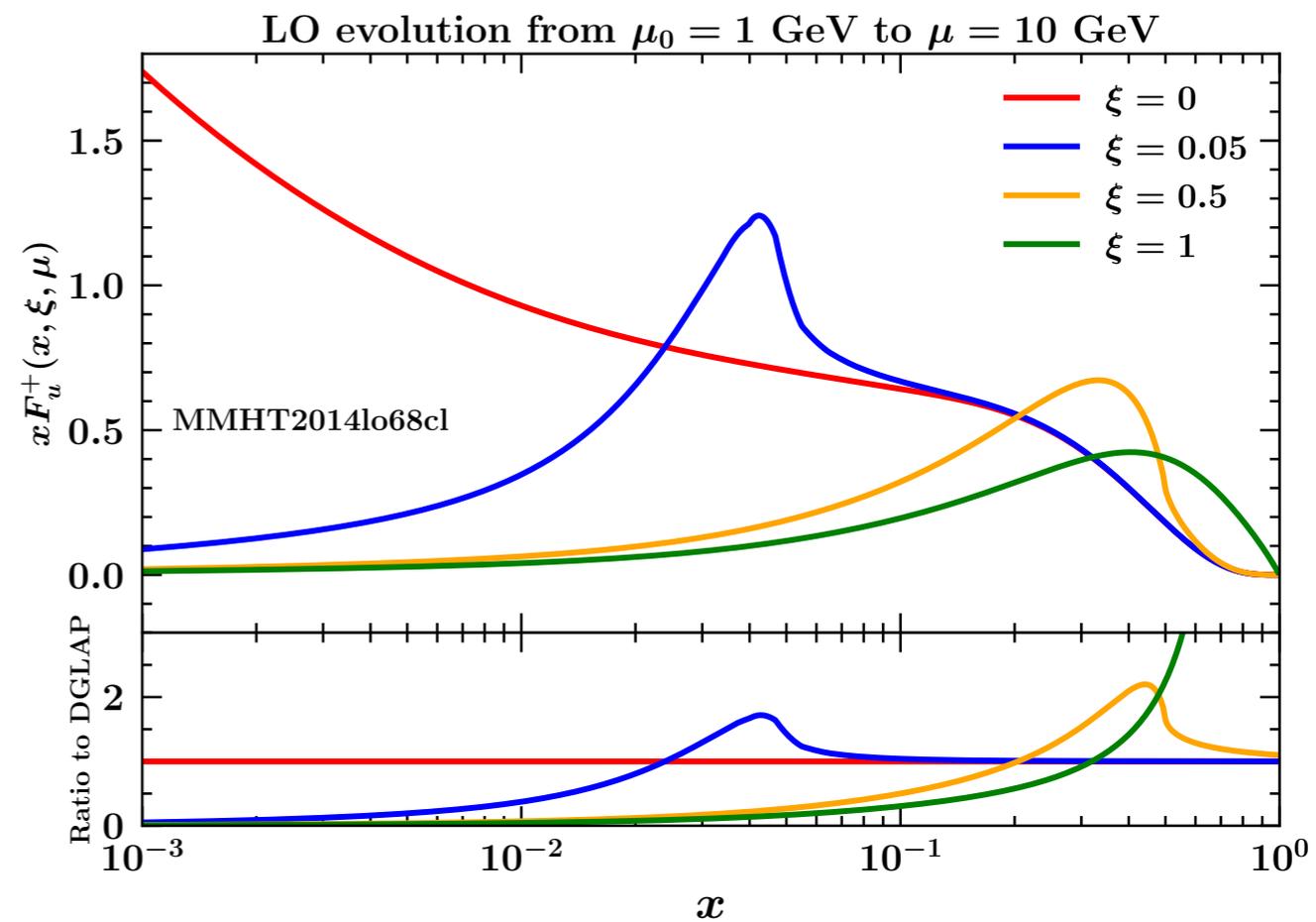
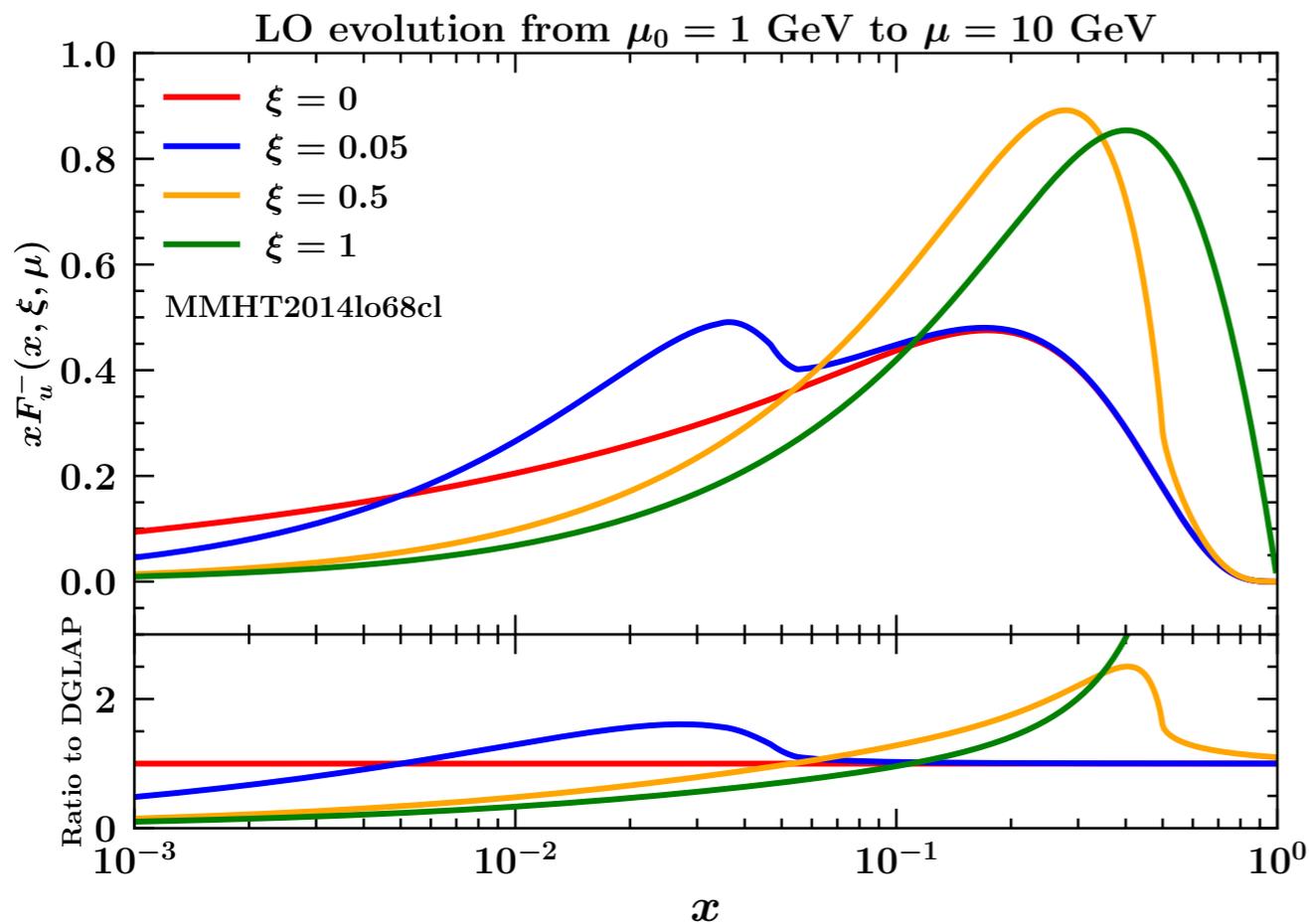
🍏 Explicit computation of conformal moments reveals that Gegenbauer polynomials of rank 3/2 diagonalise the LO non-singlet evolution kernels with **ξ -independent kernels**:

$$\int_{-1}^1 \frac{dx}{|\xi|} C_{2n}^{(3/2)} \left(\frac{x}{\xi} \right) \mathbb{V}^{-,[0]} \left(\frac{x}{\xi}, \frac{y}{\xi} \right) = V_{2n}^{[0]} C_n^{(3/2)} \left(\frac{y}{\xi} \right)$$

Numerical setup

- 🍏 The evolution kernels for *unpolarised* evolution that we have recomputed are now implemented in **APFEL++** and available through **PARTONS** allowing for LO GPD evolution in momentum space.
- 🍏 The remarkable properties of the evolution kernels allowed us to obtain for the first time a stable numerical implementation over the full range $\mathbf{0 \leq \xi \leq 1}$:
 - 🍏 first numerical check that both the **DGLAP** and **ERBL** limits are recovered,
 - 🍏 first numerical check of **polynomiality** conservation.
- 🍏 Numerical tests mostly use the MMHT14 PDF set at LO as an initial-scale set of distributions evolved from 1 to 10 GeV for the first time in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing.
- 🍏 Tests have also been performed using more realistic GPD models such as the Goloskokov-Kroll model [*Eur.Phys.J.C* 53 (2008) 367-384] based on the Radyushkin double-distribution ansatz [*Phys.Lett.B* 449 (1999) 81-88].

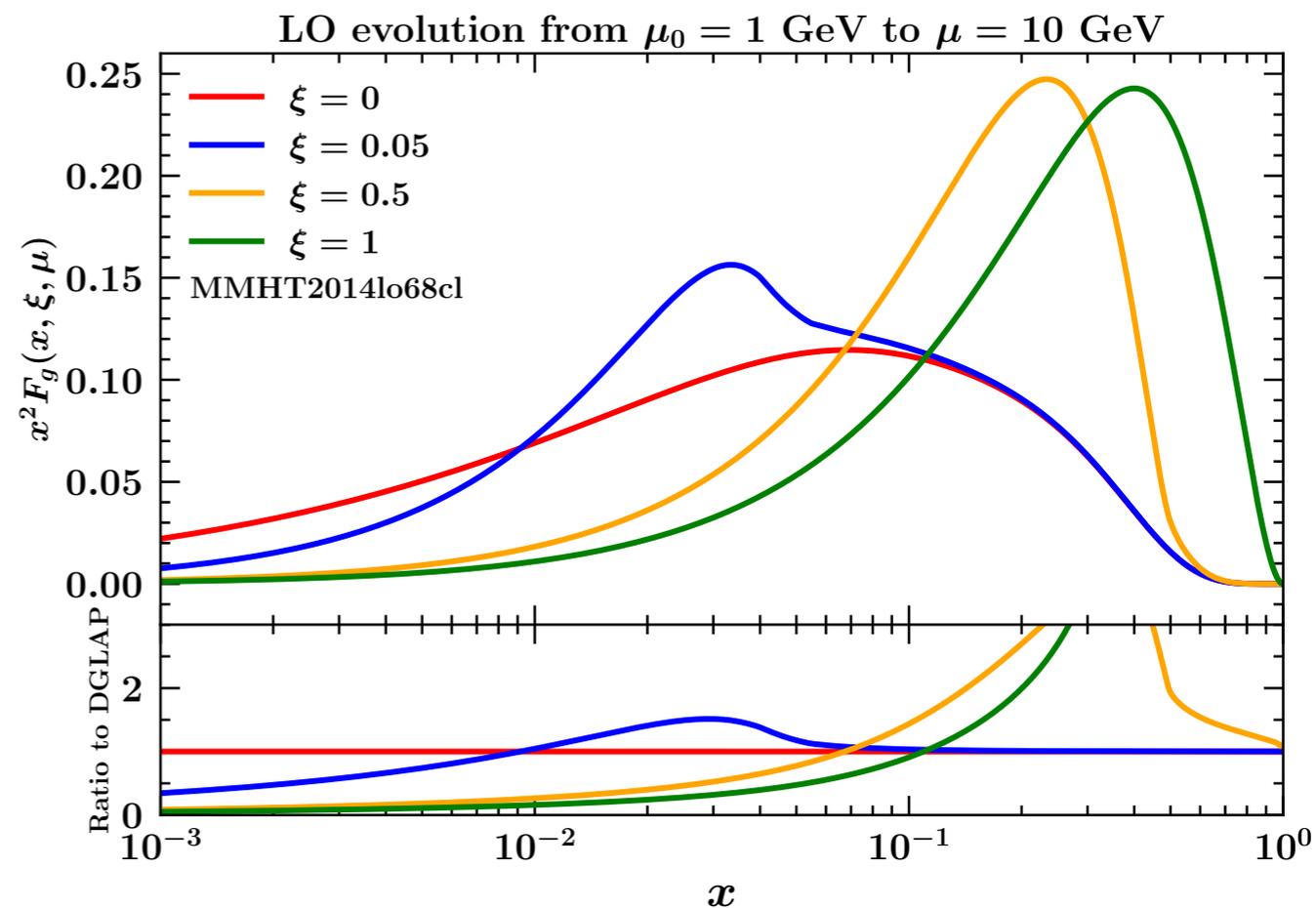
The DGLAP limit



 **DGLAP limit** reproduced within 10^{-5} relative accuracy.

 GPD evolution may significantly deviate from DGLAP evolution.

 The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.



The ERBL limit

🍏 The limit $\xi \rightarrow 1$ ($\kappa \rightarrow 1/x$) we should reproduce the **ERBL equation**.

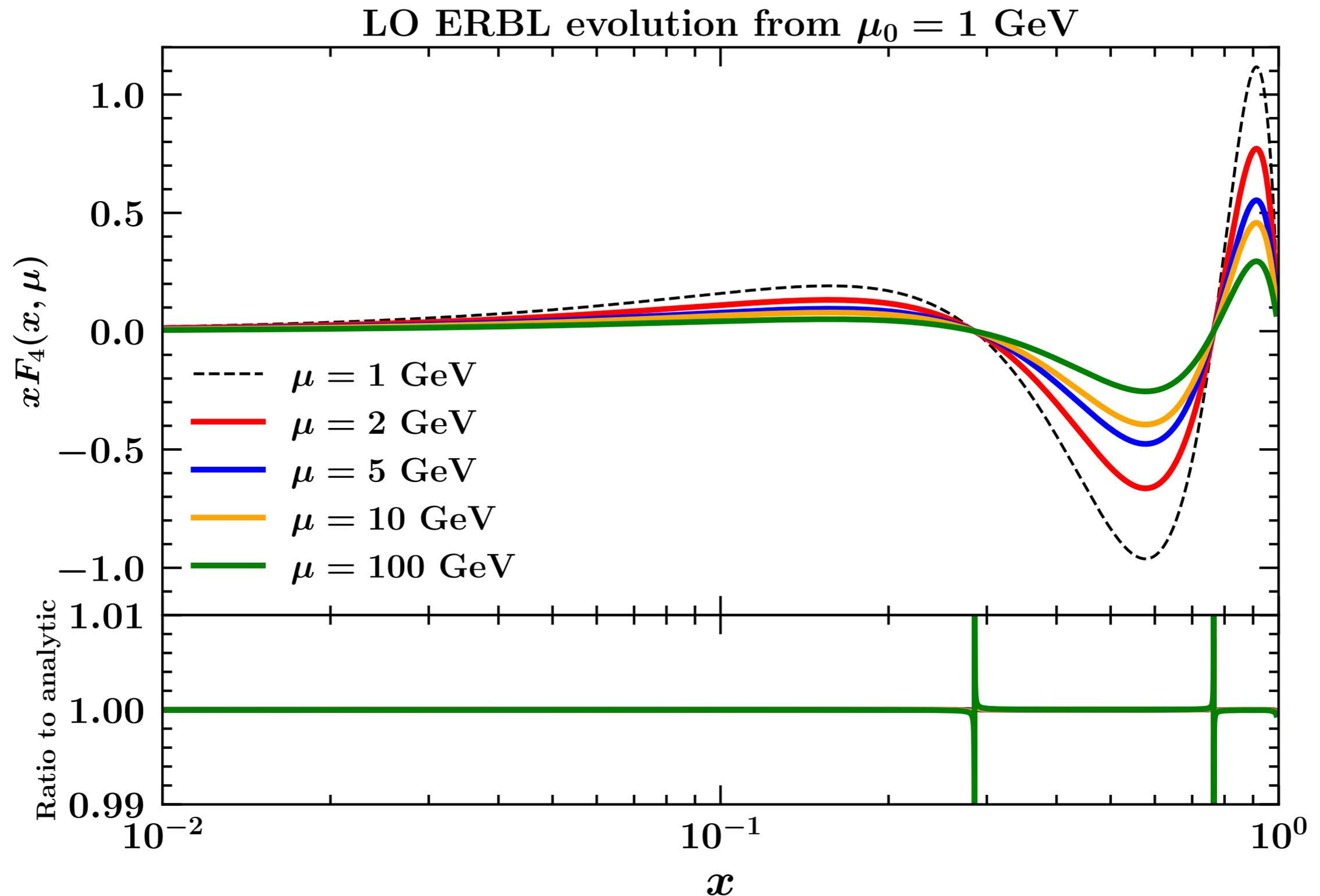
🍏 It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x, \mu_0) = (1 - x^2) C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x, \mu) = \exp \left[\frac{V_{2n}^{[0]}}{4\pi} \int_{\mu_0}^{\mu} d \ln \mu^2 \alpha_s(\mu) \right] F_{2n}(x, \mu_0)$$

🍏 where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

🍏 We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

The ERBL limit



🍏 **ERBL limit** reproduced within less than 10^{-9} relative accuracy,

🍏 Same accuracy for **higher-degree** Gegenbauer polynomials.

Conformal-space evolution

🍏 In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x, \xi, \mu_0) = \int_{\Omega} d\beta d\alpha \delta(x - \beta - \xi\alpha) q(|\beta|) \pi(\beta, \alpha)$$

with:

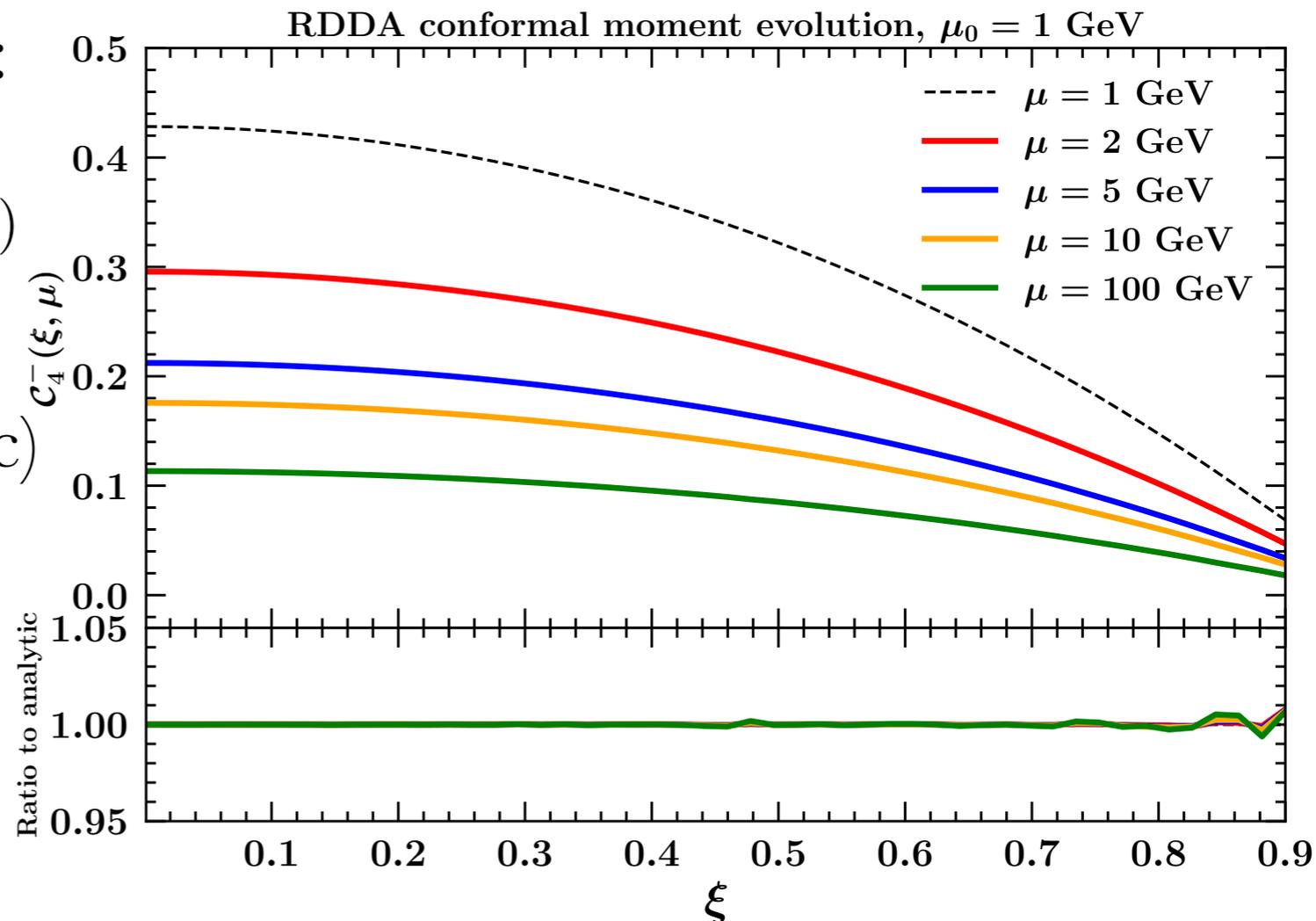
$$q(x) = \frac{35}{32} x^{-1/2} (1-x)^3, \quad \pi(\beta, \alpha) = \frac{3}{4} \frac{((1-|\beta|)^2 - \alpha^2)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

$$C_4^-(\xi, \mu) = \xi^4 \int_{-1}^1 dx C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x, \xi, \mu)$$

from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

we found excellent agreement.



Polynomiality

🍏 GPD evolution should preserve **polynomiality**.

[Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]

🍏 The following relations for the Mellin moments must hold at **all scales**:

$$\int_0^1 dx x^{2n} F_q^-(x, \xi, \mu) = \sum_{k=0}^n A_k(\mu) \xi^{2k}$$

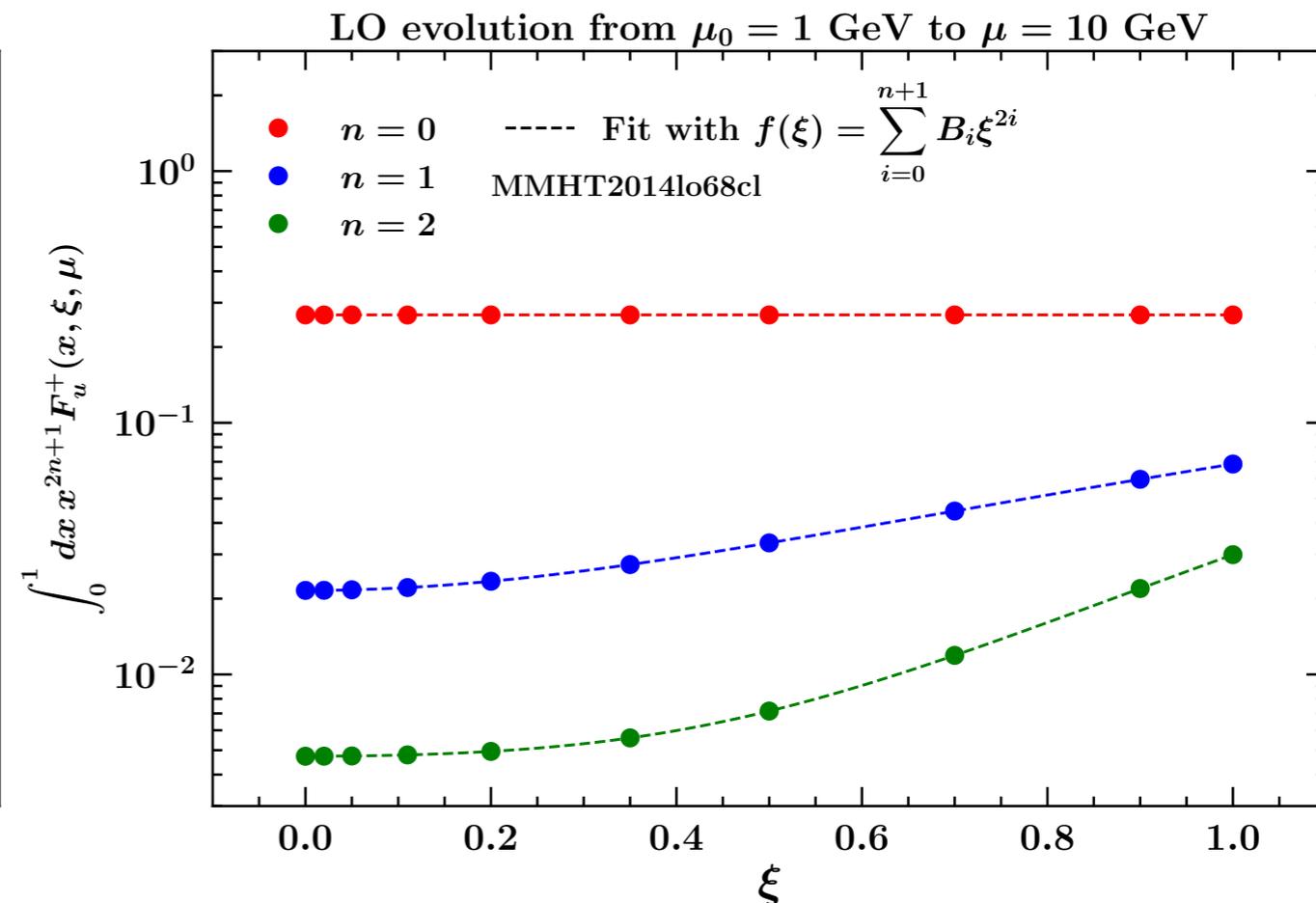
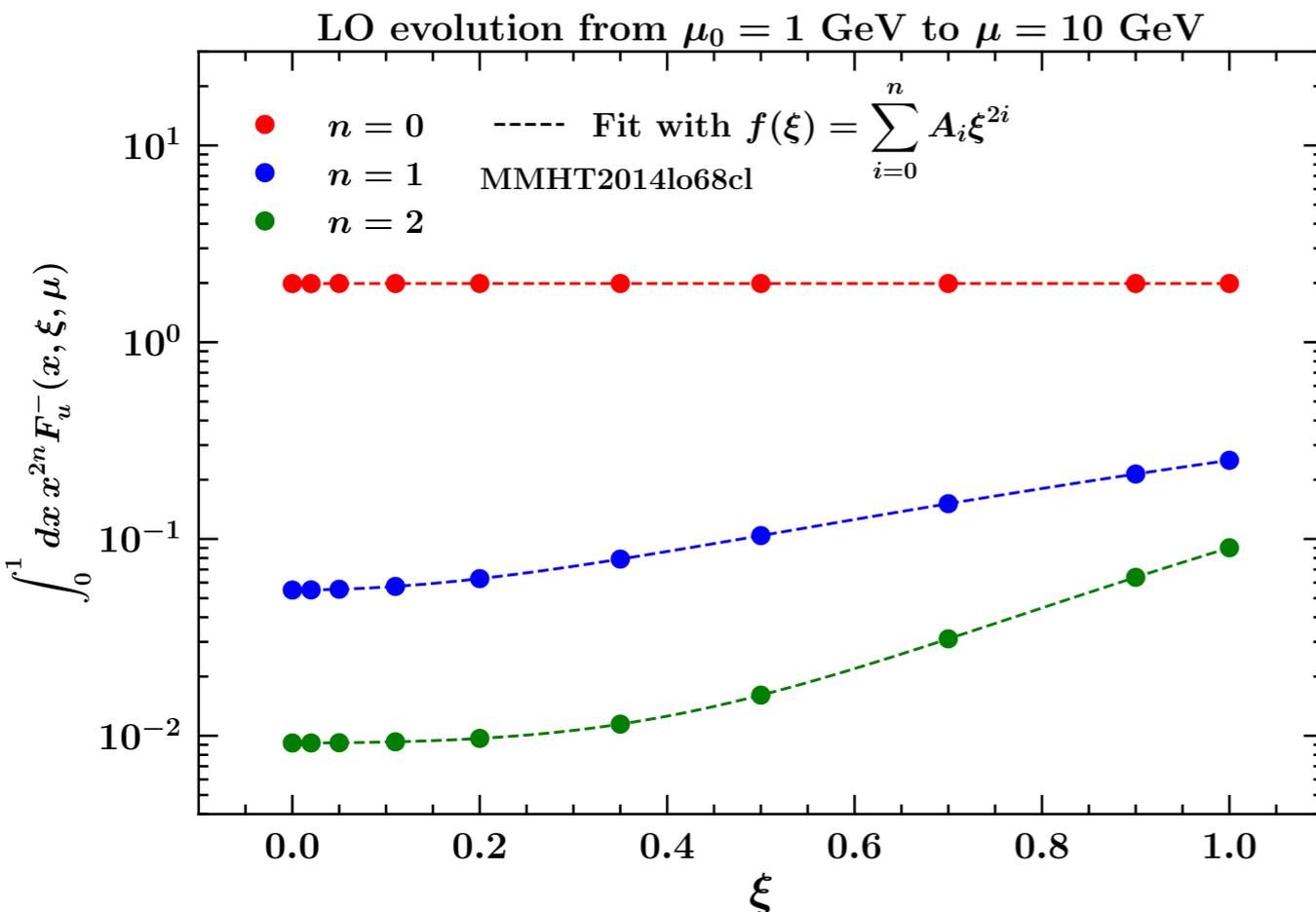
$$\int_0^1 dx x^{2n+1} F_q^+(x, \xi, \mu) = \sum_{k=0}^{n+1} B_k(\mu) \xi^{2k}$$

🍏 Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in ξ .

🍏 The coefficient of the ξ^{2n+2} term of the *singlet* (D-term) is absent in our initial conditions and it is *not* generated by evolution, so that also the first moment of the singlet is expected to be **constant** in ξ .

🍏 For the other values of n one can just **fit** the behaviour in ξ and check that it follows the **expected power law**.

Polynomiality



🍏 **First moment** for both singlet and non-singlet is indeed **constant** in ξ :

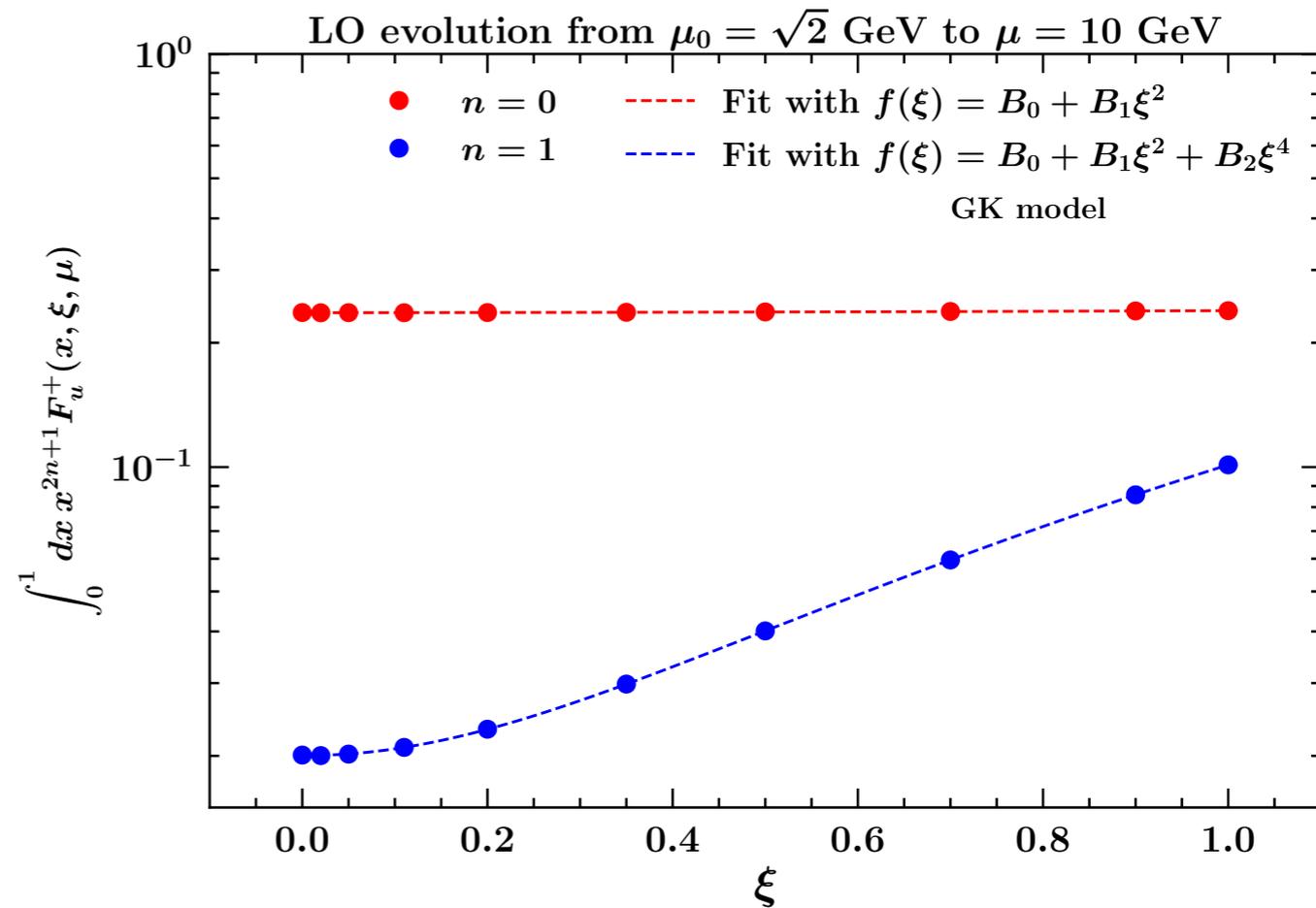
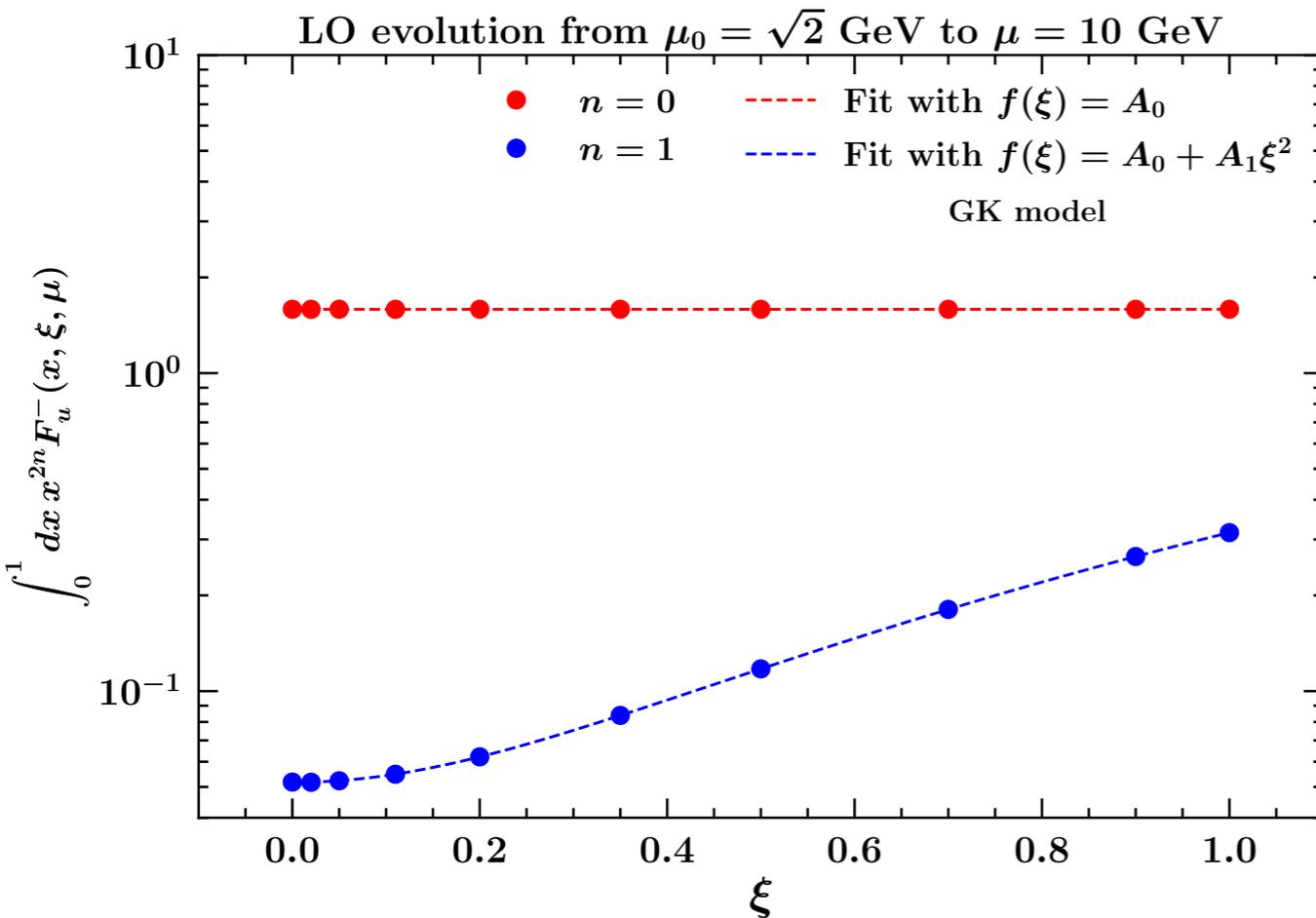
🍏 this was expected and the expectation is very nicely fulfilled.

🍏 **Second and third moments** follow the expected law:

🍏 including odd-power terms in the fit gives coefficients very close to zero.

🍏 B_{n+1} in the singlet is consistently found to be compatible with zero (no D-term).

Polynomiality



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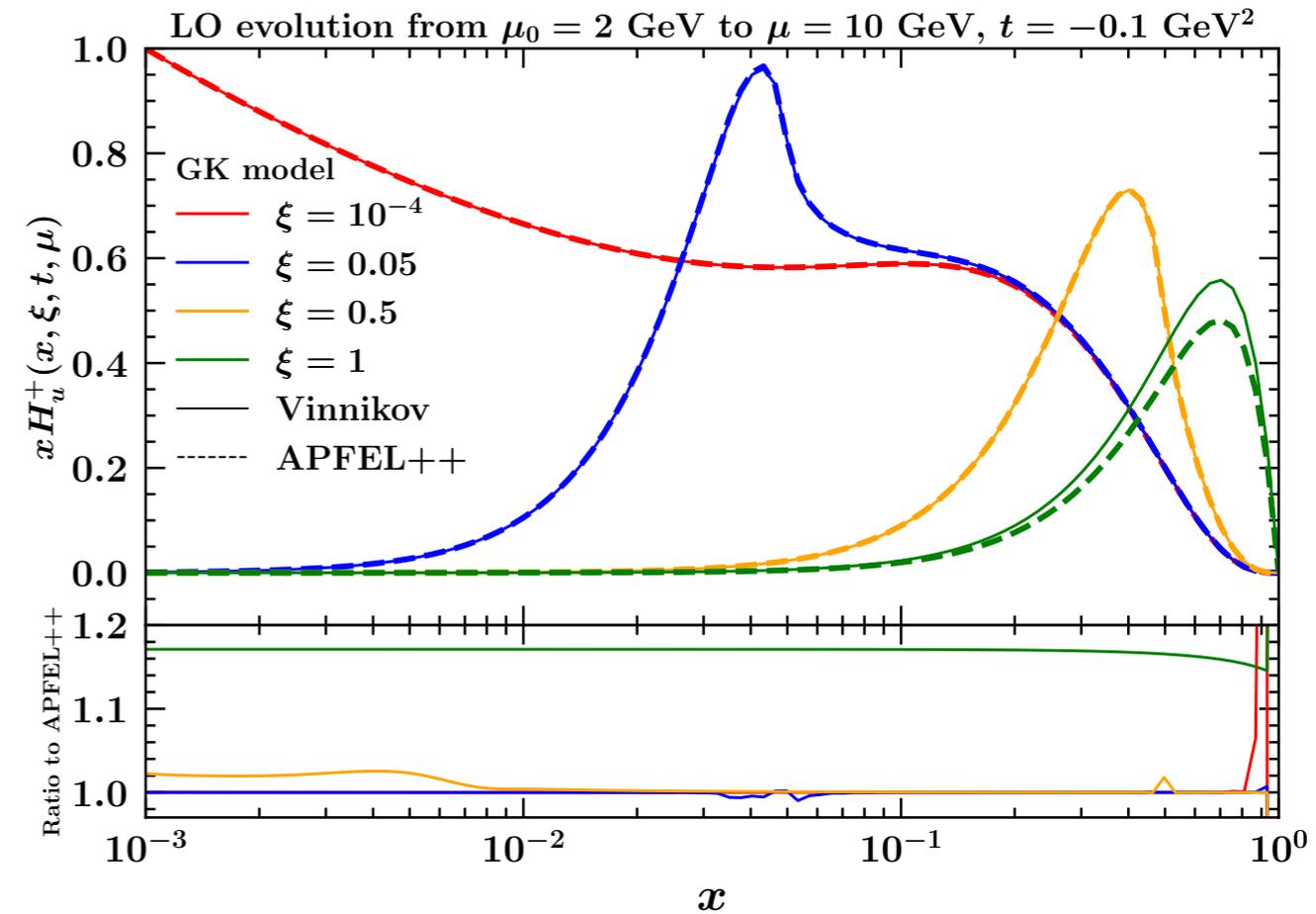
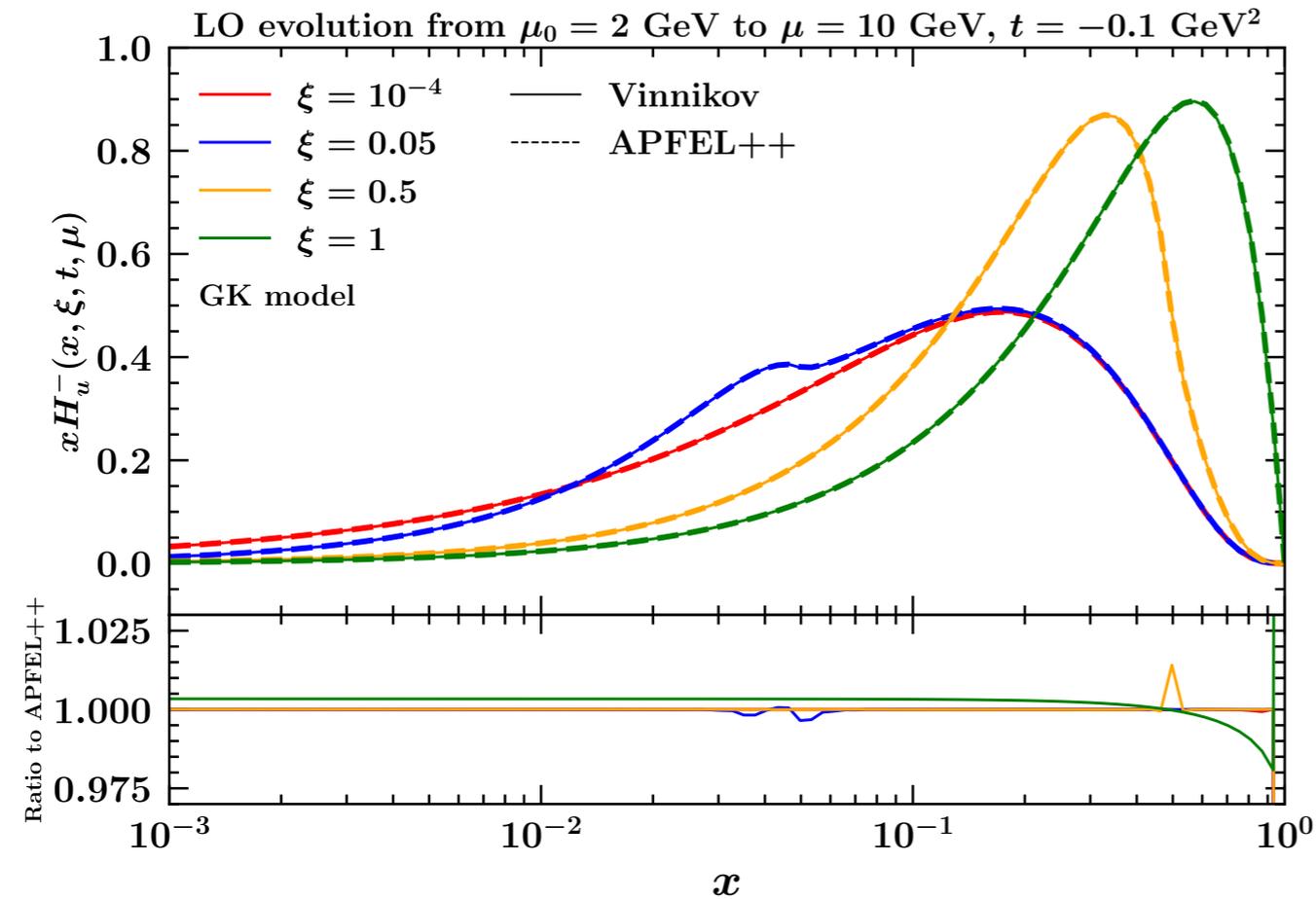
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APFEL vs. Vinnikov's code



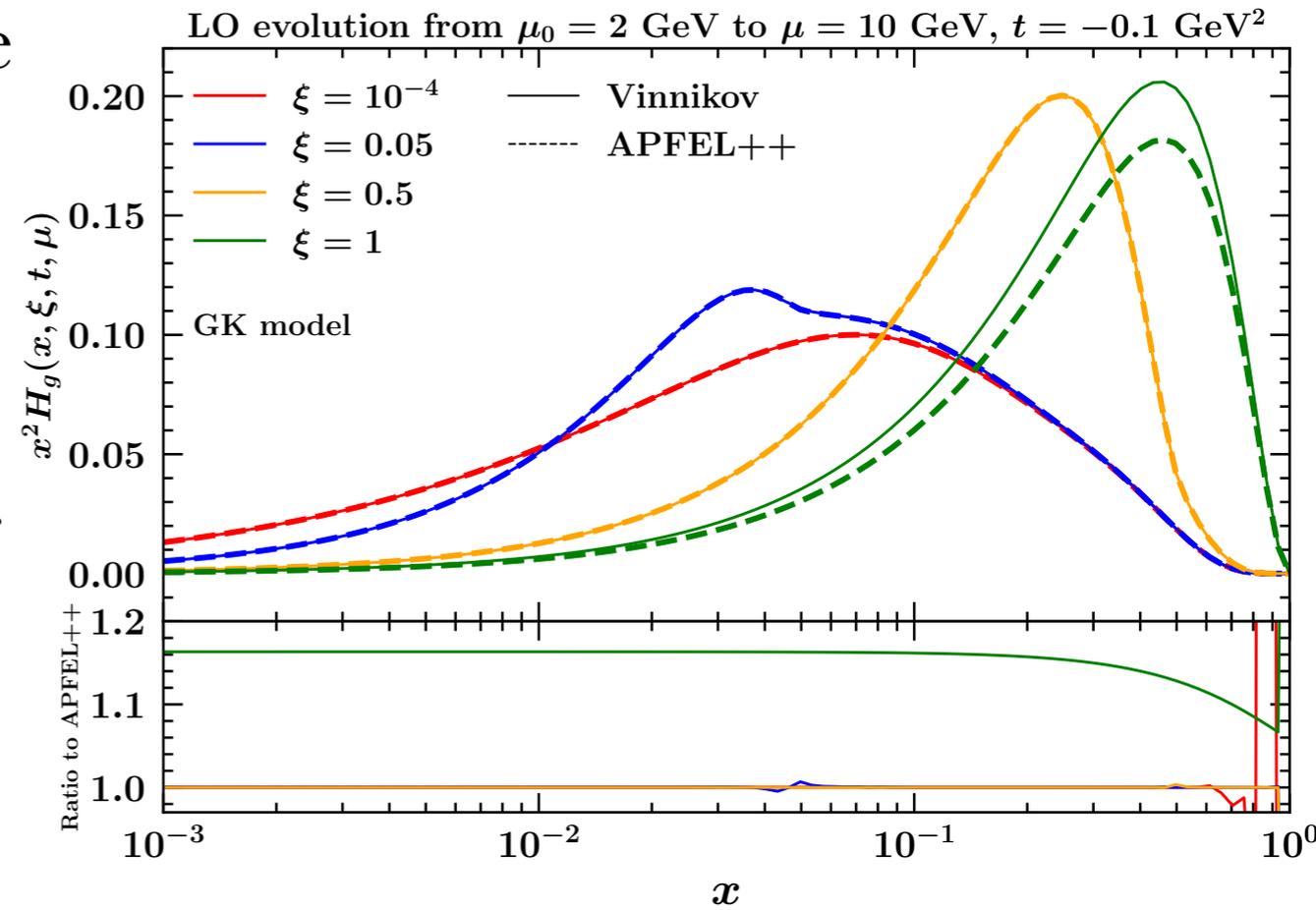
🍏 **Excellent agreement** between the two code for $\xi \lesssim 0.6$.

🍏 Agreement deteriorates for $\xi \gtrsim 0.6$:

🍏 discrepancy larger for the singlets ($\sim 20\%$) than for the non-singlet ($\sim 1\%$).

🍏 possible numerical instabilities of Vinnikov's code?

🍏 Inability to check the ERBL limit.



Conclusions and outlook

🍏 We have **revisited LO GPD evolution** in momentum space:

- 🍏 *Ab-initio* calculation of the LO unpolarised splitting kernels based on Feynman diagrams in light-cone gauge.
- 🍏 GPD evolution equations recasted in a DGLAP-like form convenient for implementation.
- 🍏 Various analytical properties of the kernels highlighted and numerically checked.
- 🍏 DGLAP and ERBL limits correctly recovered within excellent accuracy.
- 🍏 Evolution conserves polynomiality and agrees with conformal-space evolution.
- 🍏 the code (**APFEL++**) is public and available within  **PARTONS**
<https://github.com/vbertone/apfelxx>

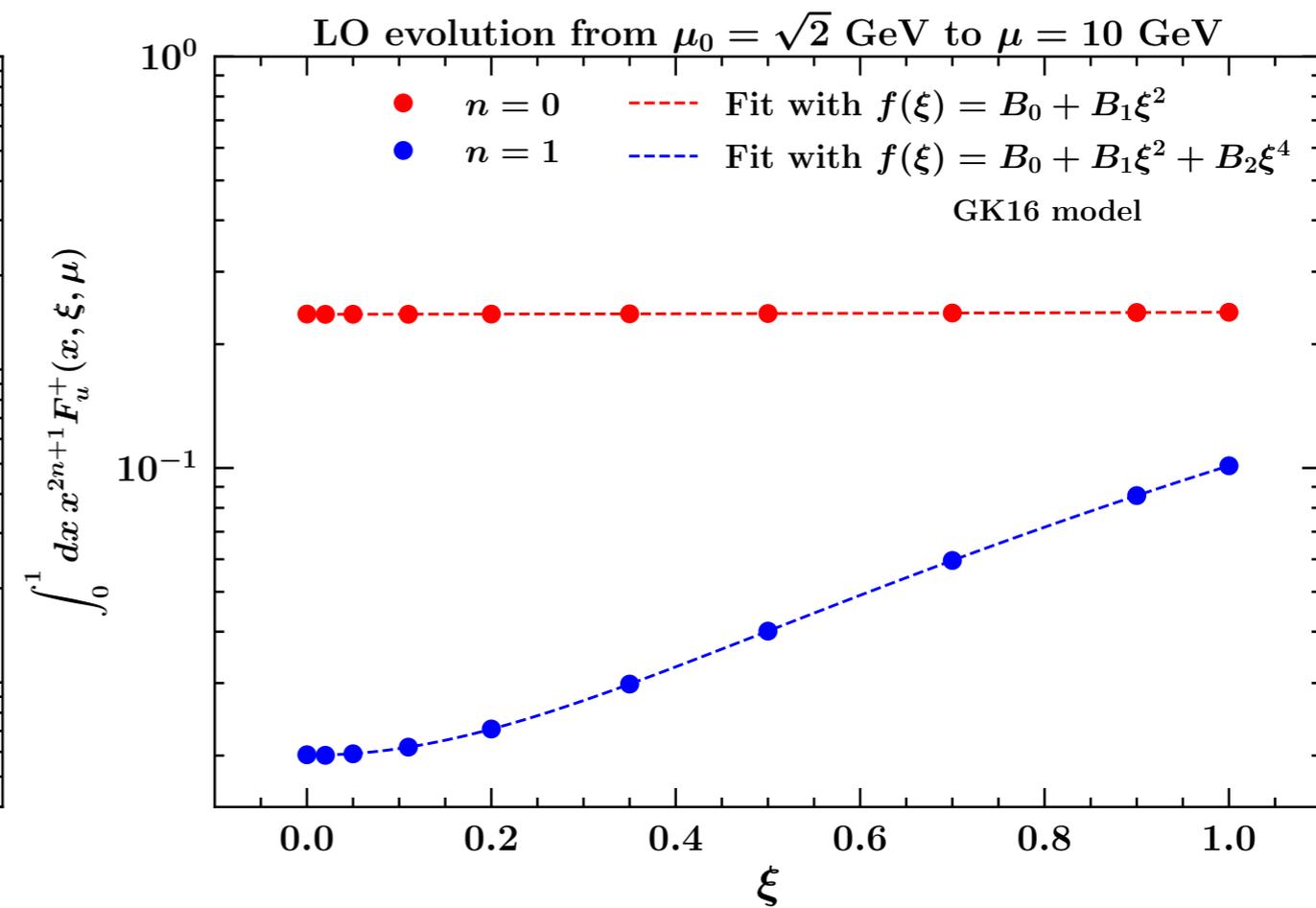
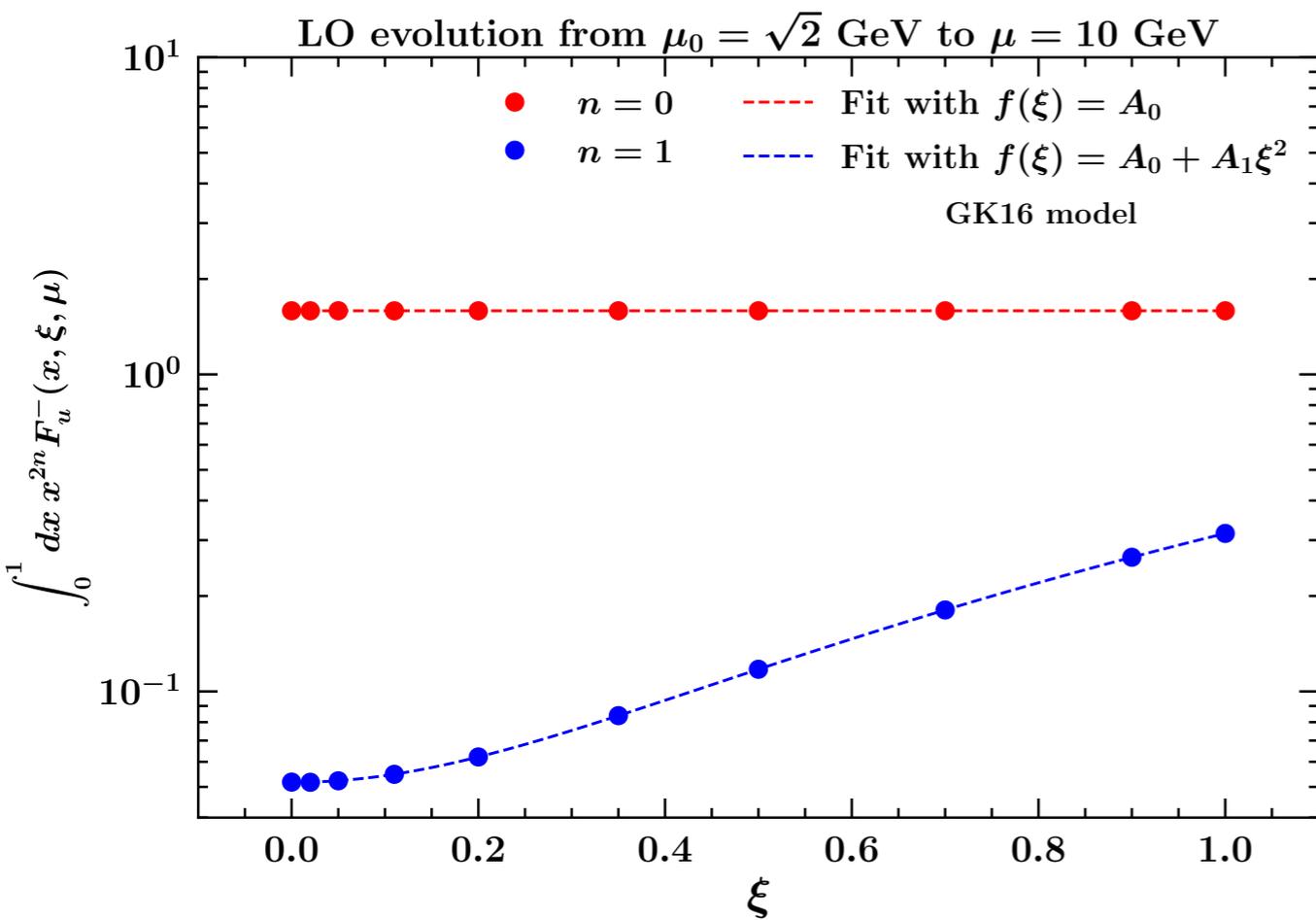
<http://partons.cea.fr/partons/doc/html/index.html>

🍏 **Next steps:**

- 🍏 **short term:** calculation/implementation of polarised (long. and trans. (?)) evolutions,
- 🍏 **middle term:** benchmark of the public evolution codes (discussion already started),
- 🍏 **longer term:** calculation and implementation of the NLO corrections.

Back up

Numerics: polynomiality



🍏 **First moment** for both singlet and non-singlet is **constant** in ξ :

🍏 this was expected and the expectation is very nicely fulfilled.

🍏 **Second moments** follow the expected law:

🍏 including odd-power terms in the fit gives coefficients very close to zero.

🍏 B_2 in the singlet is consistently found to be compatible with zero (no D-term).

On the calculation of P_{qq} at LO

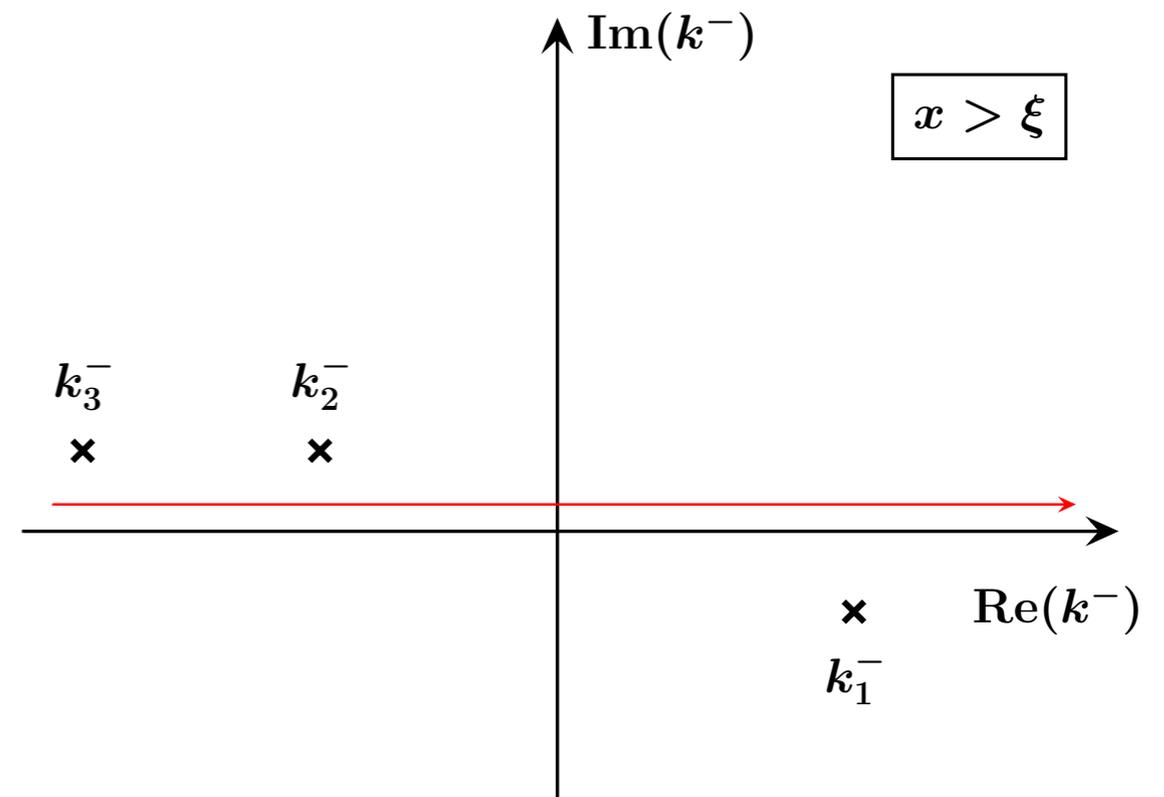
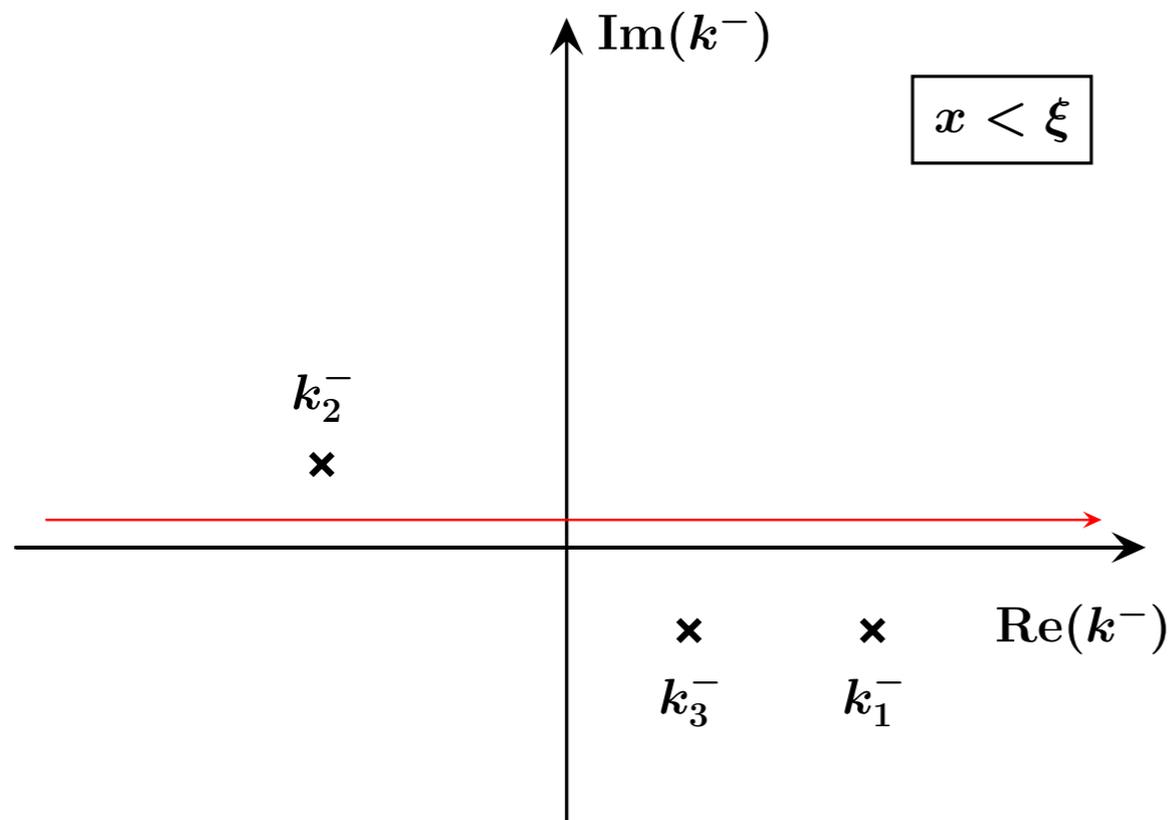
🍏 In light-cone gauge, there is one single real diagram:

$$\hat{F}_{(0),q/q}^{[1],(g^{\mu\nu})}(x, \xi) = \sqrt{1 - \xi^2} \frac{i}{2} C_F \frac{1}{(p^+)^2 (1-x)(x^2 - \xi^2)} \mu^{2\epsilon} \int \frac{d^{2-2\epsilon} \mathbf{k}_T}{(2\pi)^{2-2\epsilon}} \mathbf{k}_T^2$$

$$\times \int_{-\infty}^{+\infty} \frac{dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)},$$

$$k_1^- = \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i\epsilon, \quad k_2^- = -\frac{\mathbf{k}_T^2}{2(x+\xi)p^+} + i(x+\xi)\epsilon, \quad k_3^- = -\frac{\mathbf{k}_T^2}{2(x-\xi)p^+} + i(x-\xi)\epsilon,$$

🍏 Pole structure:



On the calculation of P_{qq} at LO

🍏 The real diagram gives:

$$\begin{aligned}\hat{F}_{(0),q/q}^{[1]}(x, \xi) &= \hat{F}_{(0),q/q}^{[1],(n^\mu)}(x, \xi) + \hat{F}_{(0),q/q}^{[1],(g^{\mu\nu})}(x, \xi) \\ &= C_F \frac{\sqrt{1-\xi^2}}{\xi(1-x)} \left[\frac{(x+\xi)(1-x+2\xi)}{1+\xi} - \theta(x-\xi) \frac{(x-\xi)(1-x-2\xi)}{1-\xi} \right] \mu^{2\epsilon} S_\epsilon \int \frac{dk_T^2}{k_T^{2+2\epsilon}}\end{aligned}$$

🍏 The virtual contribution can be computed using the sum rule.

🍏 Including the virtual diagram and isolating the UV divergence gives:

$$\begin{aligned}P_{qq}^{[1]}(x, \xi) &= 2C_F \left\{ \frac{1}{2\xi(1-x)} \left[\frac{(x+\xi)(1-x+2\xi)}{1+\xi} - \theta(x-\xi) \frac{(x-\xi)(1-x-2\xi)}{1-\xi} \right] \right. \\ &\quad \left. - \delta(1-x) \left[\int_0^1 dz \frac{1+z^2}{1-z} + \ln(|1-\xi^2|) \right] \right\}.\end{aligned}$$