GPD modeling with Continuum QCD

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Strong QCD from Hadron Structure Experiments; Newport News, 5-9 November 2019

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Overview
Timetable
Participant List

UNIVERSIDAD PABLO DE OLAVIDE SEVILLA

Universidad de Huelva

Universidad de Sevilla

This conference series began in 1970 at Duke University. The Seville meeting follows the successful events in Glasgow (2013) and Tallahassee (2016); and canvass similar themes. It will highlight the physics of baryons and related subjects in astro-, nuclear- and particle-physics, developing our understanding of the spectrum, structure and reactions of baryons using all available tools. Particular emphasis will be placed on elucidating the role of confinement and emergent mass, key non-perturbative phenomena within the Standard Model. Recent developments at existing facilities and those anticipated from the next-generation will also be showcased.

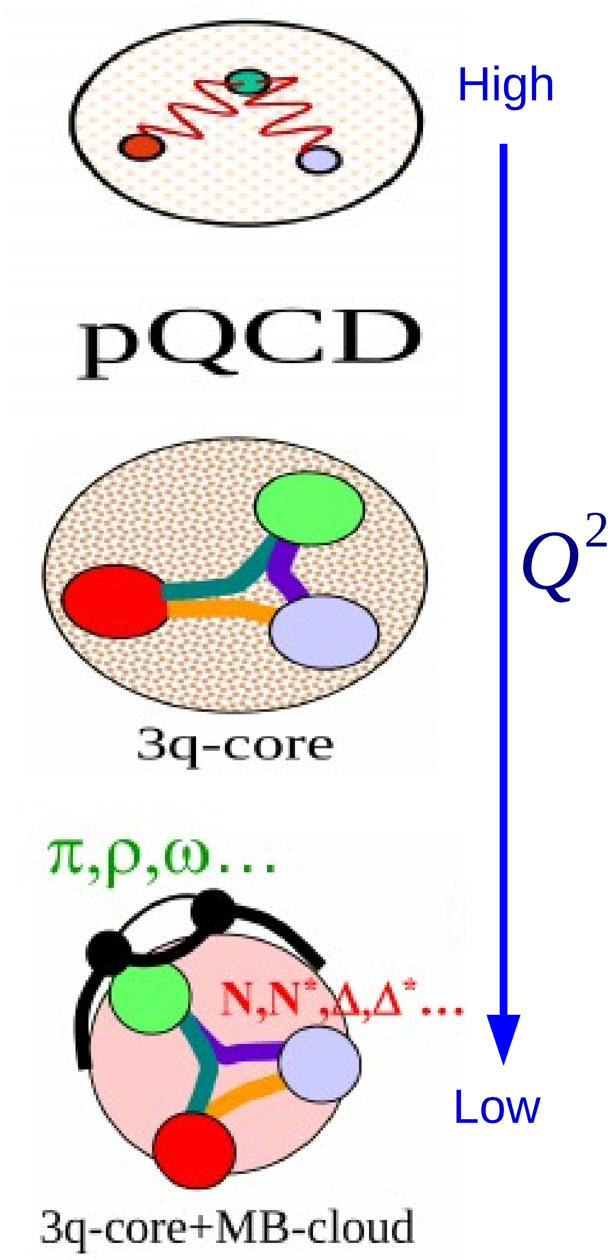
Main Topics Include:

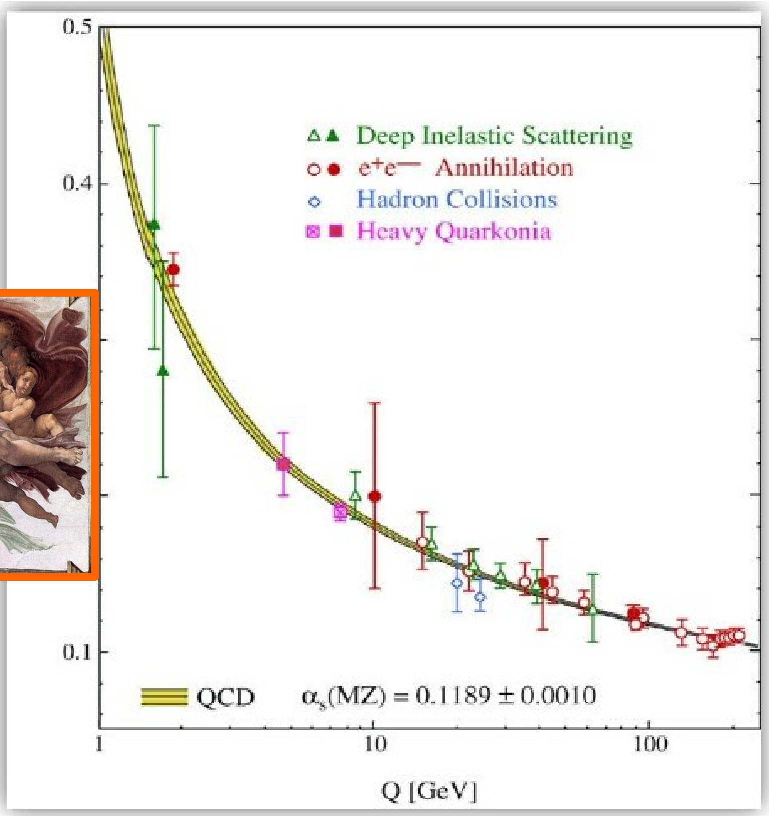
- Spectroscopy of Light / Heavy Flavor Hadrons.
- Structure of Hadrons.
- Recent Approaches to Non-Perturbative QCD.
- Exotic Baryons.
- Hadrons at Finite Density and Temperature.
- Electromagnetic and Weak Interactions.
- Hadron-Hadron Interactions.
- New Facilities and Instrumentation.

https://www.upo.es/baryons2020/
https://indico.cern.ch/event/853858/

Hadron Physics. General Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!





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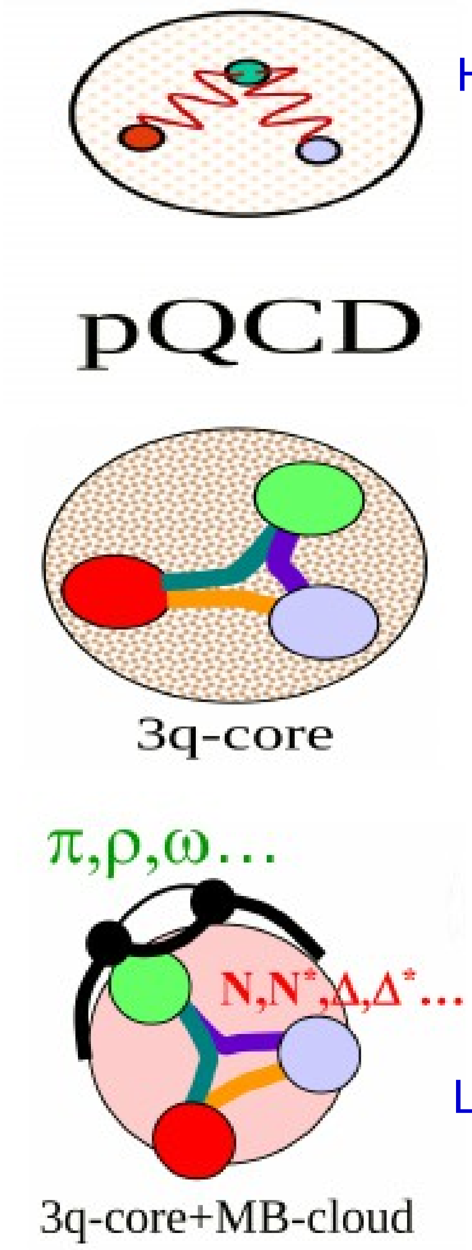


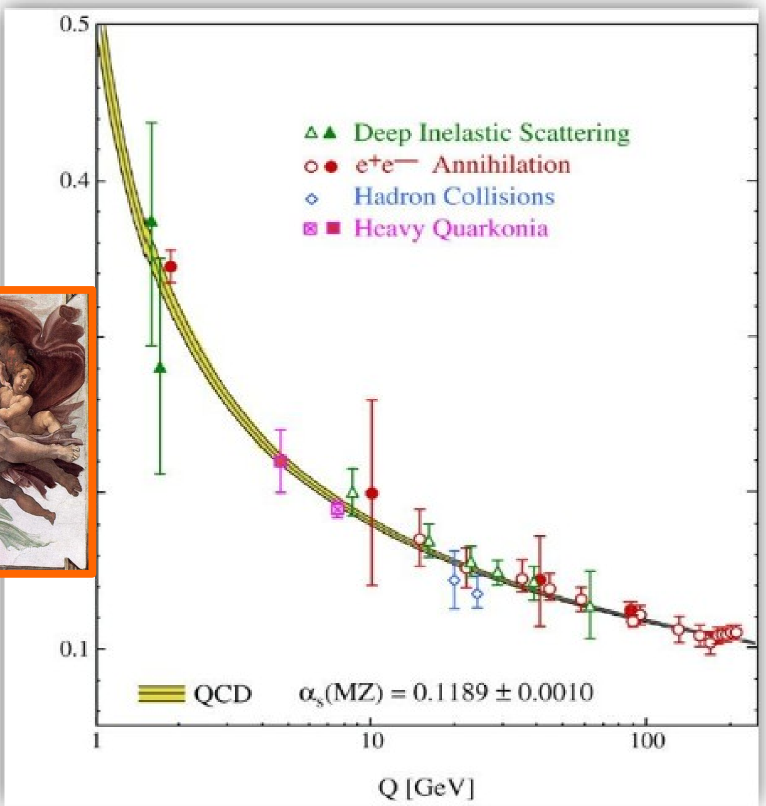
High

Q^2

Low

Confinement

Colored bound states have never been seen to exist as particles in nature



DCSB

Chiral symmetry appears dynamically violated in the Hadron spectrum

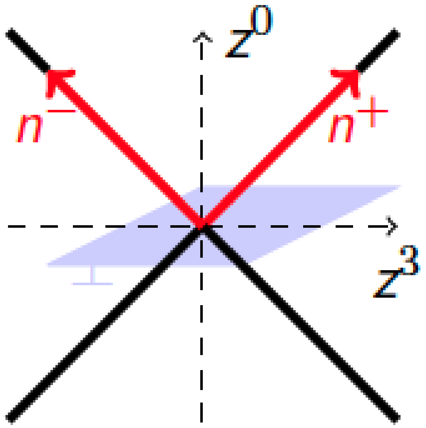
Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.

Antecedents:

GPD definition:

$$H_\pi^q(x,\xi,t) \;=\; \frac{1}{2}\int \frac{dz^-}{2\pi}\, e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2} \right| \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \left| \pi, P-\frac{\Delta}{2} \right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi=-\Delta^+/(2P^+)$.



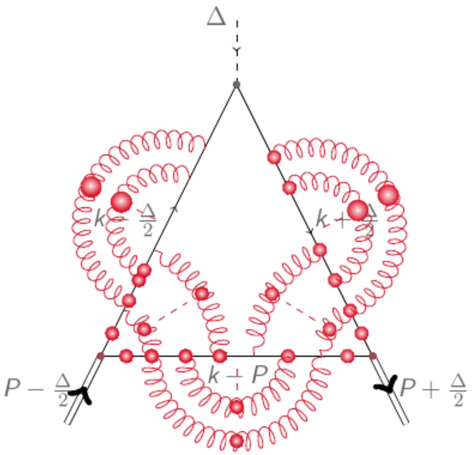
References

Muller et al., Fortchr. Phys. **42**, 101 (1994)
Radyushkin, Phys. Lett. **B380**, 417 (1996)
Ji, Phys. Rev. Lett. **78**, 610 (1997)

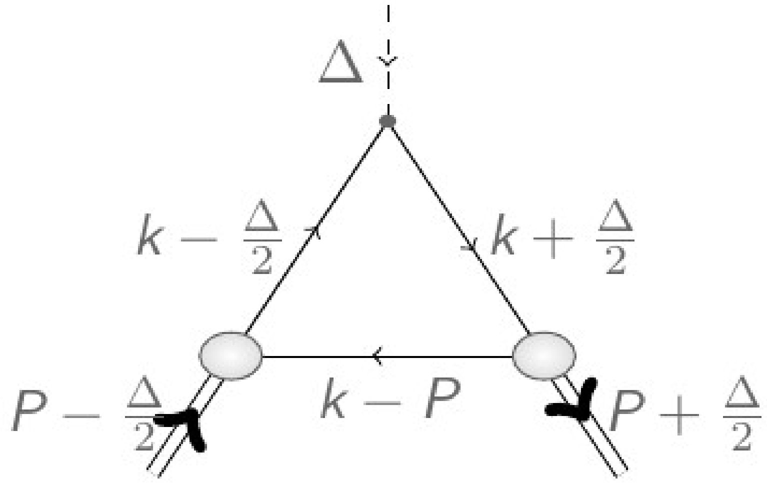
- From **isospin symmetry**, all the information about pion GPD is encoded in $H^u_{\pi^+}$ and $H^d_{\pi^+}$.
- Further constraint from **charge conjugation**: $H^u_{\pi^+}(x,\xi,t) = -H^d_{\pi^+}(-x,\xi,t)$.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



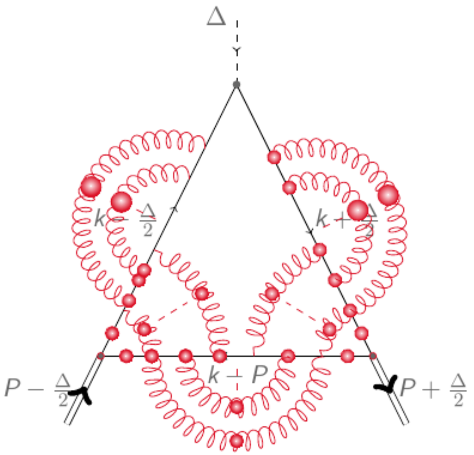
$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



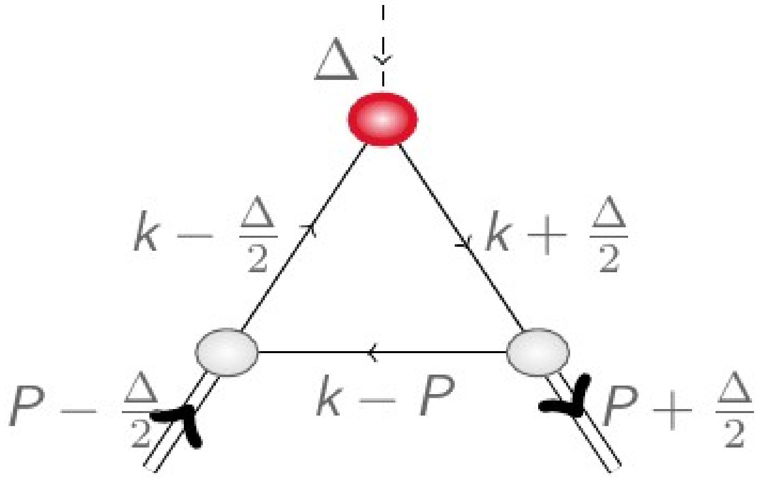
- Compute **Mellin moments** of the pion GPD H.

Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0)\gamma^+ (i\overleftrightarrow{D}^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle$$



- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.

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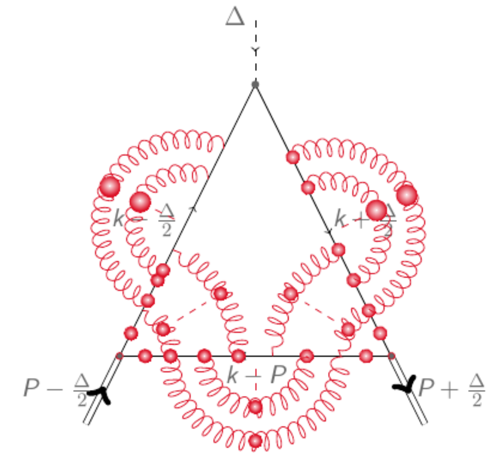
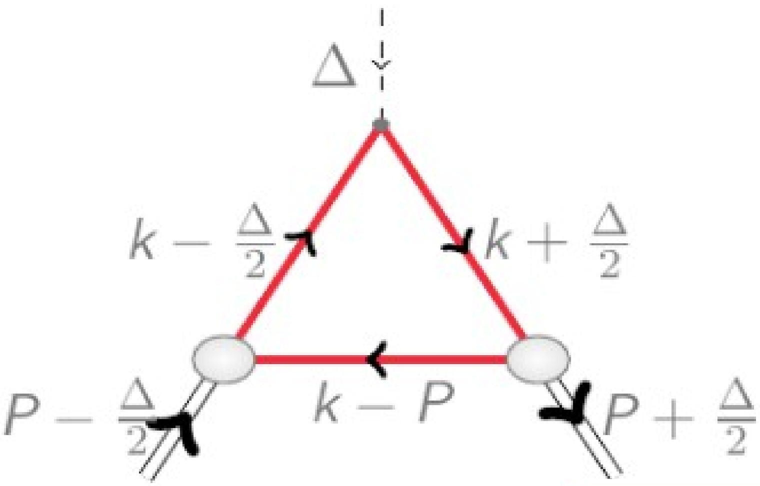
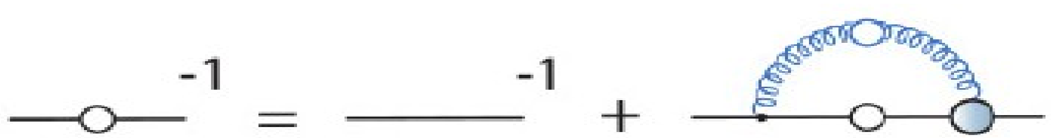


$$\langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+ (i \overleftrightarrow{D}^+)^m q(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle$$



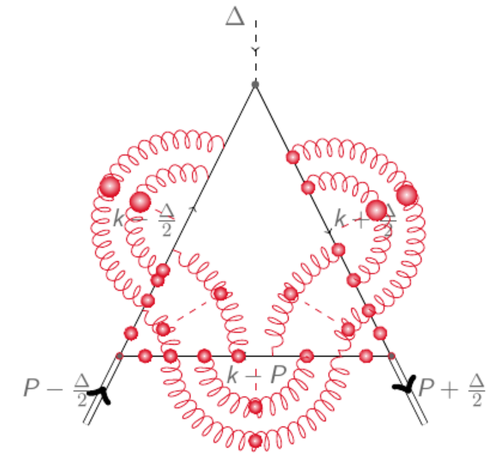
- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Dyson - Schwinger equation



Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach



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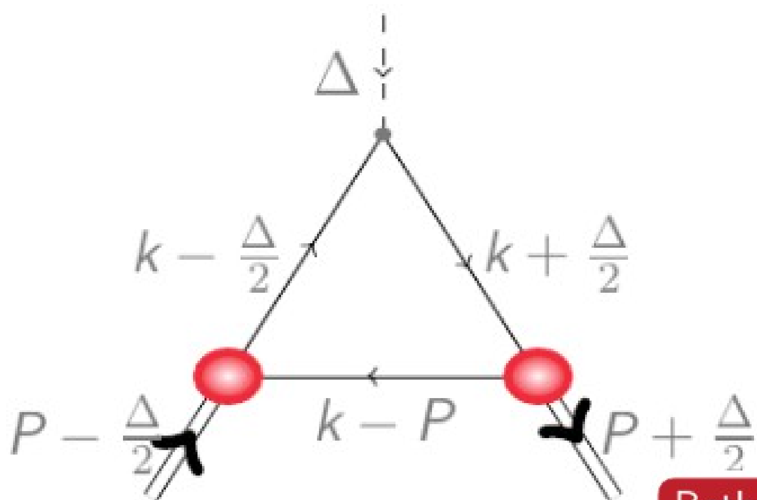
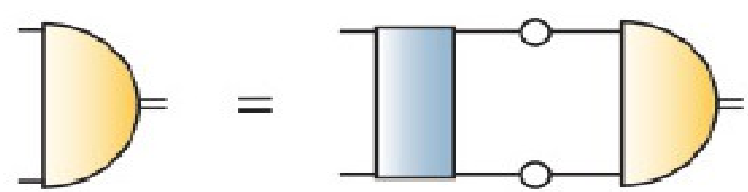


- Compute **Mellin moments** of the pion GPD H.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

Bethe - Salpeter equation



Antecedents:

GPD asymptotic algebraic model:

- Expressions for vertices and propagators:

$$
\begin{aligned}
S(p) &= \left[-i\gamma \cdot p + M\right]\Delta_M(p^2) \\
\Delta_M(s) &= \frac{1}{s+M^2} \\
\Gamma_\pi(k,p) &= i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} \mathrm{d}z\, \rho_\nu(z) \,\left[\Delta_M(k_{+z}^2)\right]^\nu \\
\rho_\nu(z) &= R_\nu (1-z^2)^\nu
\end{aligned}
$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

- Only two parameters:
  - Dimensionful parameter M.
  - Dimensionless parameter ν. **Fixed to 1** to recover asymptotic pion DA.

Antecedents:

GPD asymptotic algebraic model:

- **Analytic expression** in the DGLAP region.

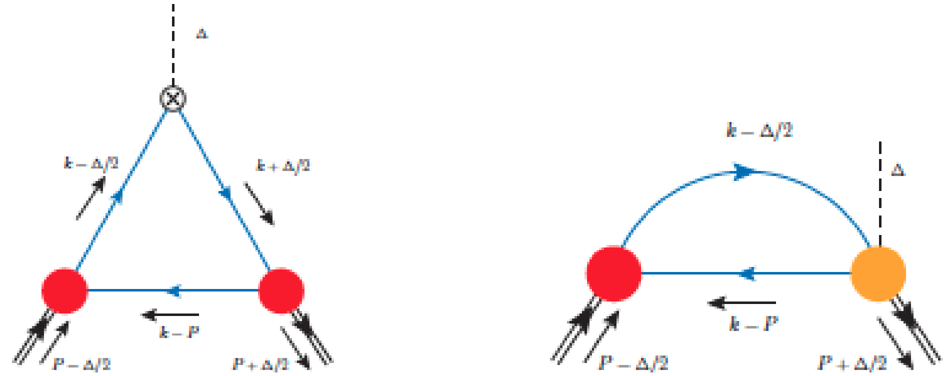
$$
\begin{aligned}
H^{u}_{x\geq\xi}(x,\xi,0) \quad = \quad & \frac{48}{5}\Bigg\{\frac{3\left(-2(x-1)^4\left(2x^2-5\xi^2+3\right)\log(1-x)\right)}{20\left(\xi^2-1\right)^3} \\
& \frac{3\left(+4\xi\left(15x^2(x+3)+(19x+29)\xi^4+5(x(x(x+11)+21)+3)\xi^2\right)\tanh^{-1}\left(\frac{(x-1)}{x-\xi^2}\right.\right.}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(x^3(x(2(x-4)x+15)-30)-15(2x(x+5)+5)\xi^4\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(-5x(x(x(x+2)+36)+18)\xi^2-15\xi^6\right)\log\left(x^2-\xi^2\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(2(x-1)\left((23x+58)\xi^4+(x(x(x+67)+112)+6)\xi^2+x(x((5-2x)x+15)+\right.\right.}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(\left(15(2x(x+5)+5)\xi^4+10x(3x(x+5)+11)\xi^2\right)\log\left(1-\xi^2\right)\right)}{20\left(\xi^2-1\right)^3} \\
& +\frac{3\left(2x(5x(x+2)-6)+15\xi^6-5\xi^2+3\right)\log\left(1-\xi^2\right)}{20\left(\xi^2-1\right)^3}\Bigg\}
\end{aligned}
$$

C. Mezrag et al., PLB741(2015)190; ArXiv:1406.7425[hep-ph]

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:



$$2(P\cdot n)^{m+1}\left\langle x^m\right\rangle^u = \mathrm{tr}_{CFD}\int \frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\; i\gamma\cdot n\; S(k+\frac{\Delta}{2})$$
$$\tau_- i\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P),$$

$$2(P\cdot n)^{m+1}\left\langle x^m\right\rangle^u = \mathrm{tr}_{CFD}\int \frac{d^4k}{(2\pi)^4}(k\cdot n)^m\,\tau_+ i\Gamma_\pi\left(\eta(k-P)+(1-\eta)\left(k-\frac{\Delta}{2}\right),P-\frac{\Delta}{2}\right)$$
$$S(k-\frac{\Delta}{2})\tau_-\;\frac{\partial}{\partial k}\bar{\Gamma}_\pi\left((1-\eta)\left(k+\frac{\Delta}{2}\right)+\eta(k-P),P+\frac{\Delta}{2}\right)S(k-P)$$

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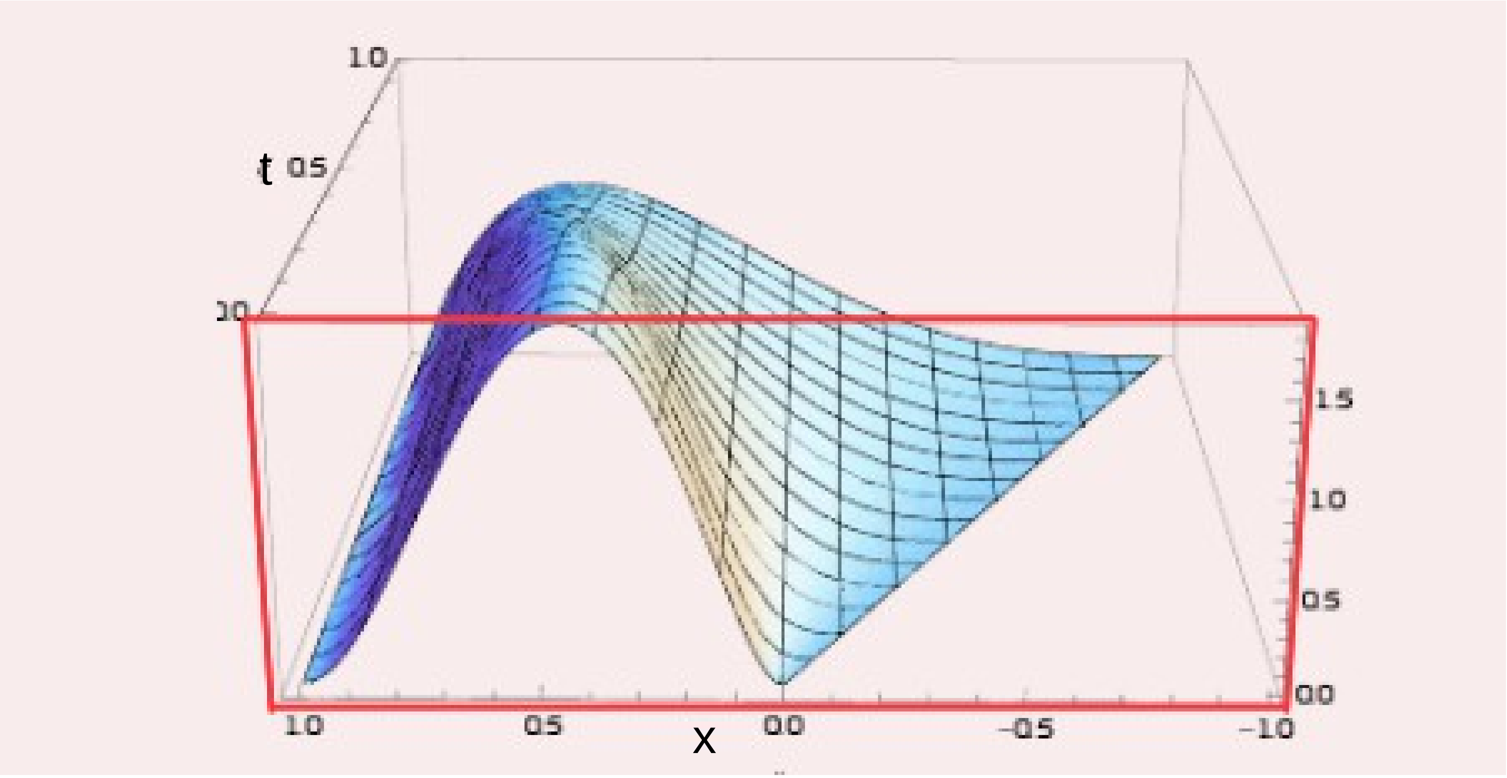

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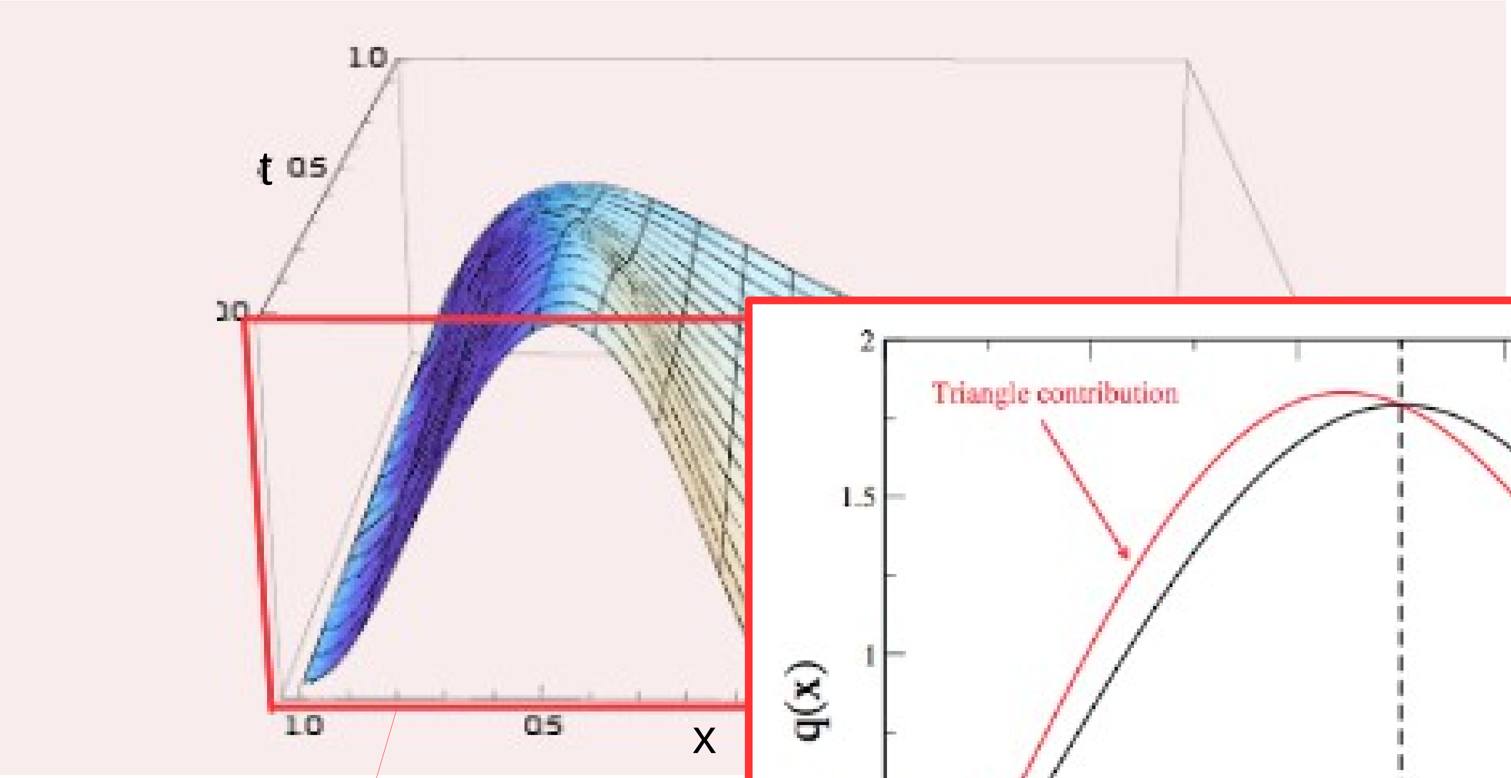

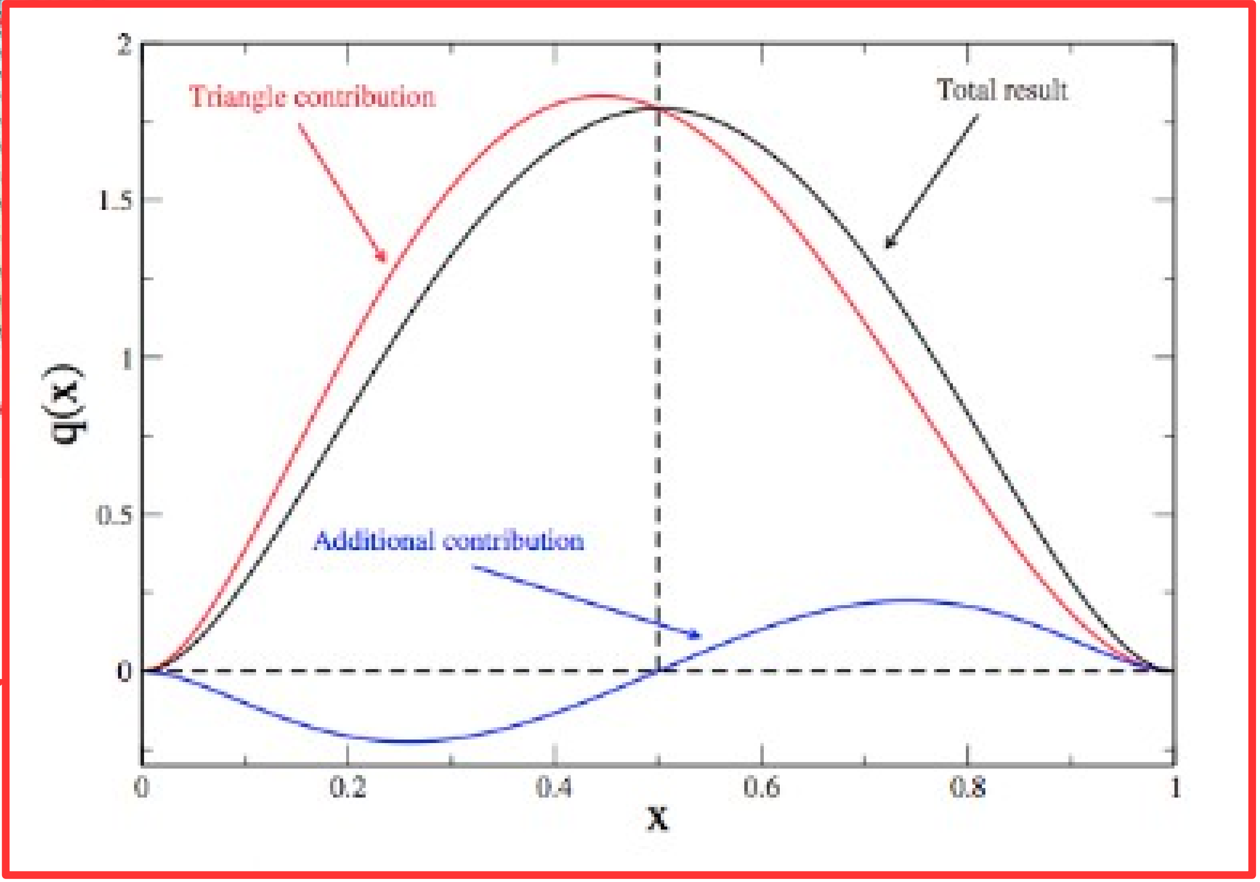

$q(x) = H^q(x, 0, 0)$

PDF forward limit

Antecedents:

GPD overlap approach: The pion light front wave function

$$|H;P,\lambda\rangle = \sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N \Psi^{\lambda}_{N,\beta}(\Omega)\,|N,\beta,k_1\cdots k_N\rangle$$

$$\Omega = (x_1, \mathbf{k}_{\perp 1}, \cdots, x_N, \mathbf{k}_{\perp N})$$

$$[\mathrm{d}x]_N = \prod_{i=1}^{N} \mathrm{d}x_i\, \delta\left(1-\sum_{i=1}^{N} x_i\right),$$

$$[\mathrm{d}^2\mathbf{k}_\perp]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^{N} \mathrm{d}^2\mathbf{k}_{\perp i}\, \delta^2\left(\sum_{i=1}^{N} \mathbf{k}_{\perp i} - \mathbf{P}_\perp\right)$$

N-partons LCWF for the hadron H

$$\sum_{N,\beta} \int [\mathrm{d}x]_N [\mathrm{d}^2\mathbf{k}_\perp]_N |\Psi^{\lambda}_{N,\beta}(\Omega)|^2 = 1.$$

Let's consider the two-body pion LCWF:

$$|\pi^+,P\rangle|^{\text{2-body}}_{\uparrow\downarrow} = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^3} \frac{\mathrm{d}x}{\sqrt{x(1-x)}} \Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) \Big[b^\dagger_{u\uparrow}(x,\mathbf{k}_\perp) d^\dagger_{d\downarrow}(1-x,-\mathbf{k}_\perp) + b^\dagger_{u\downarrow}(x,\mathbf{k}_\perp) d^\dagger_{d\uparrow}(1-x,-\mathbf{k}_\perp) \Big] |0\rangle\,,$$

$$\Gamma_\pi(k,P) = S^{-1}(-k_2)\,\chi(k,P)\,S^{-1}(k_1).$$

BS wave function

$$2P^+\Psi_{\uparrow\downarrow}(k^+,\mathbf{k}_\perp) = \int \frac{\mathrm{d}k^-}{2\pi} \mathrm{Tr}\left[\gamma^+\gamma_5\chi(k,P)\right]$$

Antecedents:

GPD overlap approach: The pion light front wave function

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- Expressions for vertices and propagators:

$$S(p) = \left[-i\gamma\cdot p + M\right]\Delta_M(p^2)$$

$$\Delta_M(s) = \frac{1}{s+M^2}$$

$$\Gamma_\pi(k,p) = i\gamma_5\frac{M}{f_\pi}M^{2\nu}\int_{-1}^{+1}\mathrm{d}z\,\rho_\nu(z)\,\left[\Delta_M(k_{+z}^2)\right]^\nu$$

$$\rho_\nu(z) = R_\nu(1-z^2)^\nu$$

with R_ν a normalization factor and $k_{+z} = k - p(1-z)/2$.

Chang *et al.*, Phys. Rev. Lett. **110**, 132001 (2013)

Keeping so contact with the previous "covariant" approach" based on DSE and BSE.

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{[\mathbf{k}_\perp^2+M^2]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \sqrt{2}\sum_{N,N'}\sum_{\beta,\beta'}\int[\mathrm{d}\hat{x}']_{N'}[\mathrm{d}^2\hat{\mathbf{k}}'_\perp]_{N'}[\mathrm{d}\tilde{x}]_N[\mathrm{d}^2\tilde{\mathbf{k}}_\perp]_N\Psi^*_{N',\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$
$$\times\int\frac{\mathrm{d}z^-}{2\pi}e^{iP^+z^-}\langle N',\beta,k'_1\cdots k'_N|\,\phi^{q\dagger}\left(-\frac{z}{2}\right)\phi^q\left(\frac{z}{2}\right)|N,\beta,k_1\cdots k_N\rangle$$

$$=\sum_N\sqrt{1-\xi}^{\,2-N}\sqrt{1+\xi}^{\,2-N}\sum_{\beta=\beta'}\sum_j\delta_{s_jq}$$
$$\times\int[\mathrm{d}\bar{x}]_N[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_N\delta(x-\bar{x}_j)\Psi^*_{N,\beta'}(\hat{\Omega}')\Psi_{N,\beta}(\tilde{\Omega})$$

In DGLAP kinematics: $\zeta\leqslant x\leqslant 1$

$$=\int[\mathrm{d}\bar{x}]_2[\mathrm{d}^2\bar{\mathbf{k}}_\perp]_2\delta(x-\bar{x}_j)\Psi^*_{\uparrow\downarrow}(\hat{\Omega}')\Psi_{\uparrow\downarrow}(\tilde{\Omega})$$

In the pion 2-body case

$+$ Helicity-1 component

$\frac{x-\zeta}{1-\zeta}$ $\qquad$ $\frac{x+\zeta}{1+\zeta}$

$$=\frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int\mathrm{d}u\mathrm{d}v\;u^\nu v^\nu\delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2\hat{x}^\nu(1-\hat{x})^\nu\tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}.$$

Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)}\frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2+M^2\right]^{\nu+1}}x^\nu(1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2}\int \mathrm{d}u\mathrm{d}v\, u^\nu v^\nu \delta(1-u-v)\frac{\left(2M^{2\nu}4^\nu R_\nu\right)^2 \hat{x}^\nu(1-\hat{x})^\nu \tilde{x}^\nu(1-\tilde{x})^\nu}{\left(t\,uv\frac{(1-x)^2}{1-\xi^2}+M^2\right)^{2\nu+1}}, \qquad \xi \leqslant x \leqslant 1$$

$$\hat{x} = \frac{x-\xi}{1-\xi}, \qquad \tilde{x} = \frac{x+\xi}{1+\xi}$$

$$= 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\,\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\,\frac{1-2z}{1+z}\,\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right)$$

$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

N. Chouika et al., PLB780(2018)287

Antecedents:

GPD overlap approach:

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$$\Psi_{\uparrow\downarrow}(x,\mathbf{k}_\perp) = -\frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} \frac{M^{2\nu+1}4^\nu R_\nu}{\left[\mathbf{k}_\perp^2 + M^2\right]^{\nu+1}} x^\nu (1-x)^\nu.$$

GPD in the overlap approach:

$$H(x,\xi,t) = 30\frac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}\frac{1}{(1+z)^2}\left(\frac{3}{4}+\frac{1}{4}\frac{1-2z}{1+z}\frac{\operatorname{arctanh}\sqrt{\frac{z}{1+z}}}{\sqrt{\frac{z}{1+z}}}\right) \qquad 0 \leqslant x \leqslant 1$$

0

1

Forward limit

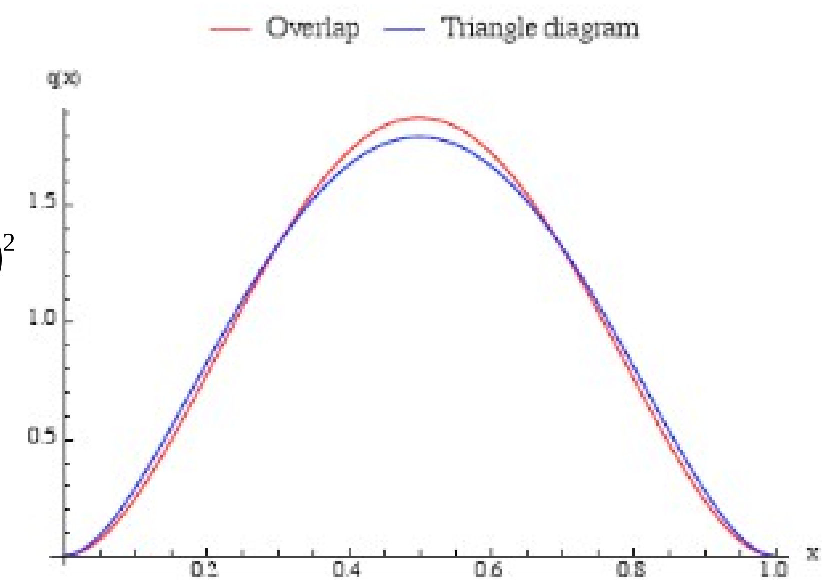
$$z = \frac{t}{4M^2}\frac{(1-x)^2}{1-\xi^2}$$

Encoding the correlations of kinematical variables

PDF:

$$H(x,0,0) = q(x) = 30\,x^2(1-x)^2$$

Compares numerically very well with the results obtained from the Triangle diagram!!!



Consistent descriptions from both approaches!!! (tested with a simple model)

N. Chouika et al., PLB780(2018)287

Pion (kaon maybe) realistic picture:

- The pseudoscalar LFWF can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \mathrm{tr}_{CD} \int_{dk_{||}} \delta(n\cdot k - x n \cdot P_K)\gamma_5 \gamma \cdot n \chi_K^{(2)}(k_-^K; P_K) \ .$$

- The moments of the distribution are given by:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \frac{1}{f_K n\cdot P} \int_{dk_{||}} \left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n \chi_K^{(2)}(k_-^K; P_K)$$

$$\int_0^1 d\alpha \alpha^m \left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right] \ , \quad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha) + M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2) \ .$$

Uniqueness of Mellin moments ⟶

$$\psi_K^{\uparrow\downarrow}(x, k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)$$

$$\chi_K(\alpha;\sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right] \frac{\rho_K(\omega)}{n_K}\frac{\Lambda_K^2}{\sigma^3} \ .$$

The spectral density $\rho_K(z)$ can be modelled...
...Or taken with BSE solutions as an input!

$$\Rightarrow \psi_K^{\uparrow\downarrow}(x, k_\perp^2) \sim \int d\omega \ \cdots \rho_K(\omega) \cdots$$

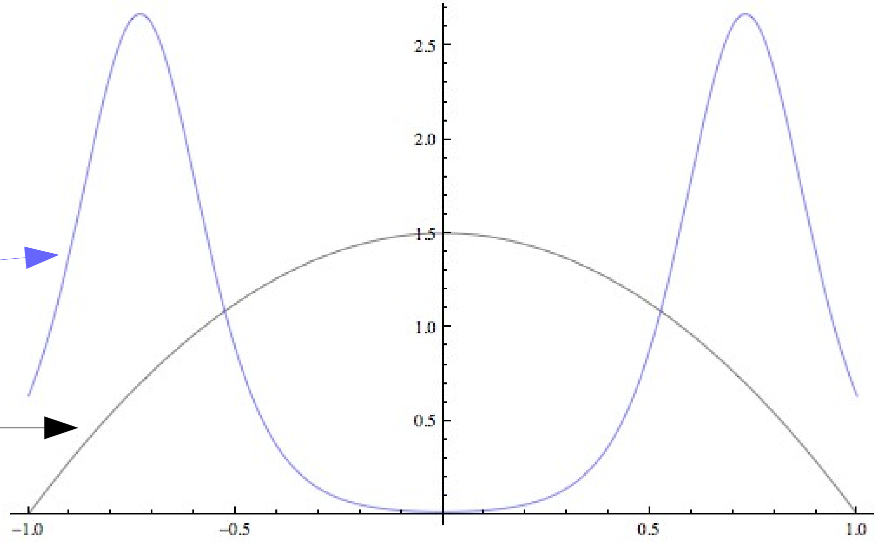
Pion realistic picture:

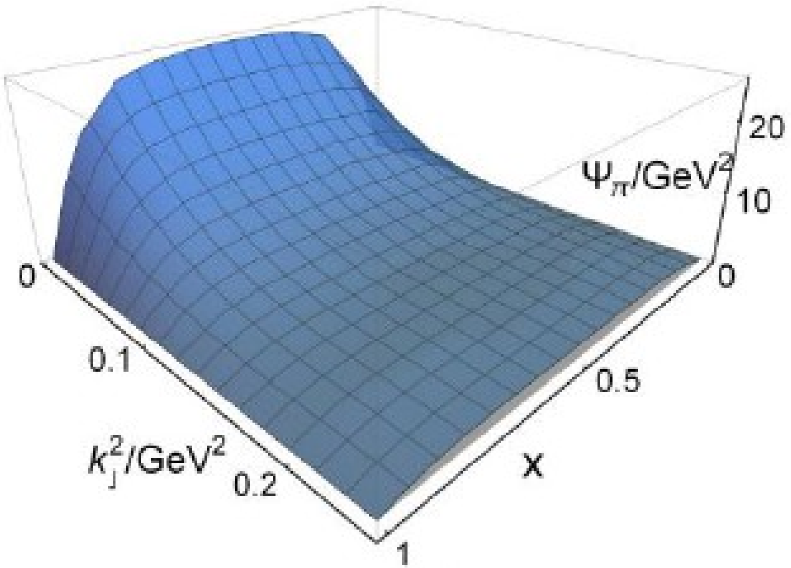
- Spectral density is chosen as:

$$u_G\rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega-\omega_0^G}{2b_0^G}\right)+\mathrm{sech}^2\left(\frac{\omega+\omega_0^G}{2b_0^G}\right)\right]$$

Phenomelogical model: $b_0^\pi=0.1, b_0^\pi=0.73;$

Asymptotic case: $\rho(\omega;\nu)\sim(1-\omega^2)^\nu$



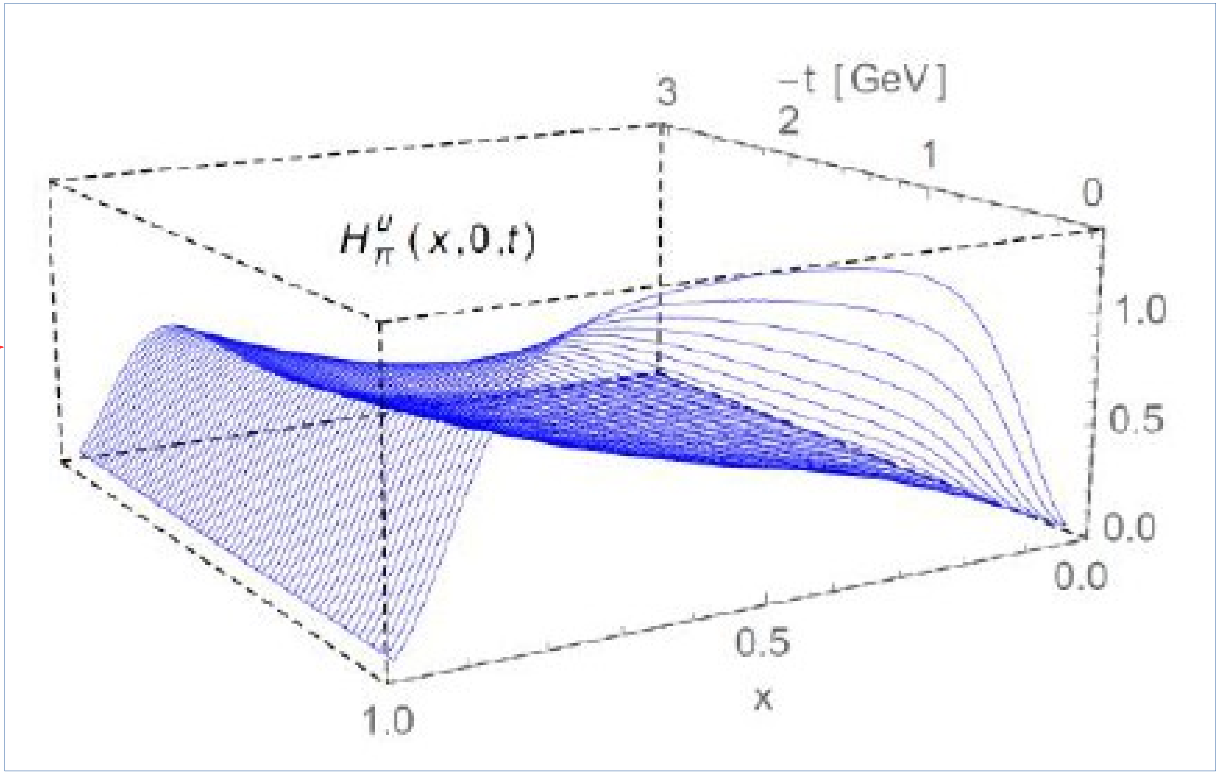


Pion realistic picture:

GPD overlap representation:

$$H^q_M(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

Phenomenological model



Pion realistic picture: PDF as benchmark

GPD overlap representation: forward limit

$$H_M^q(x,\xi,t) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

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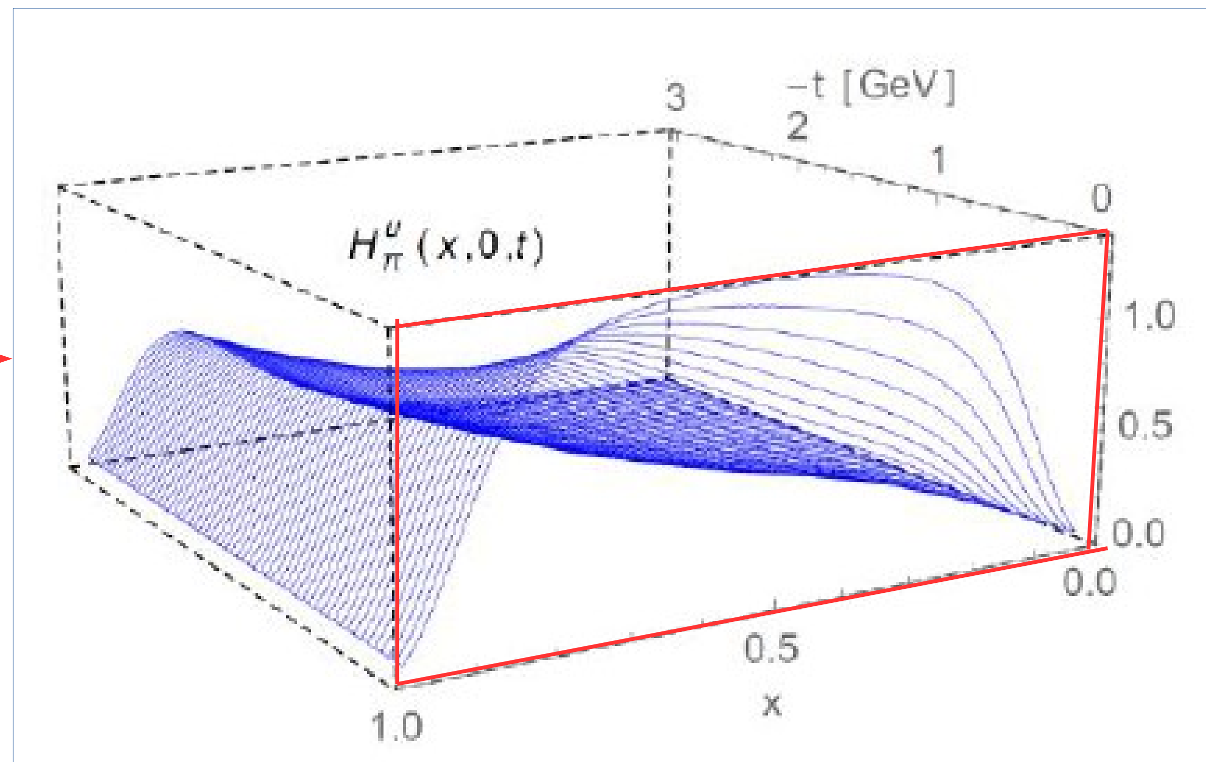

Pion realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator that, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\overline{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\overline{f}}(x,\mathbf{k}_\perp)$$

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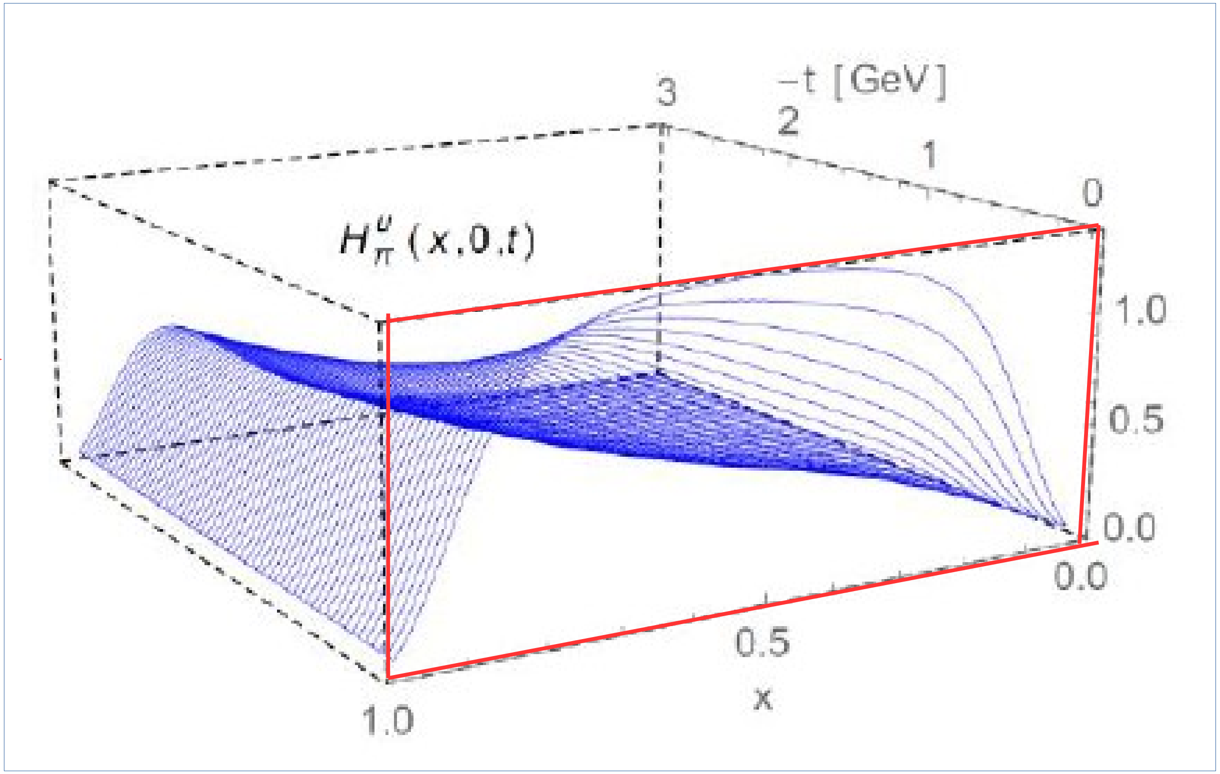

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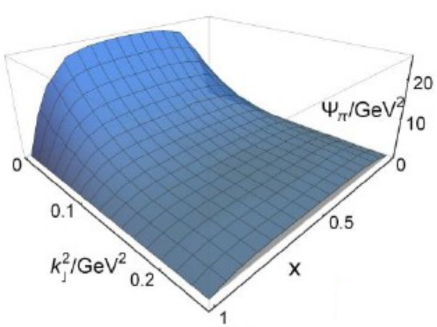
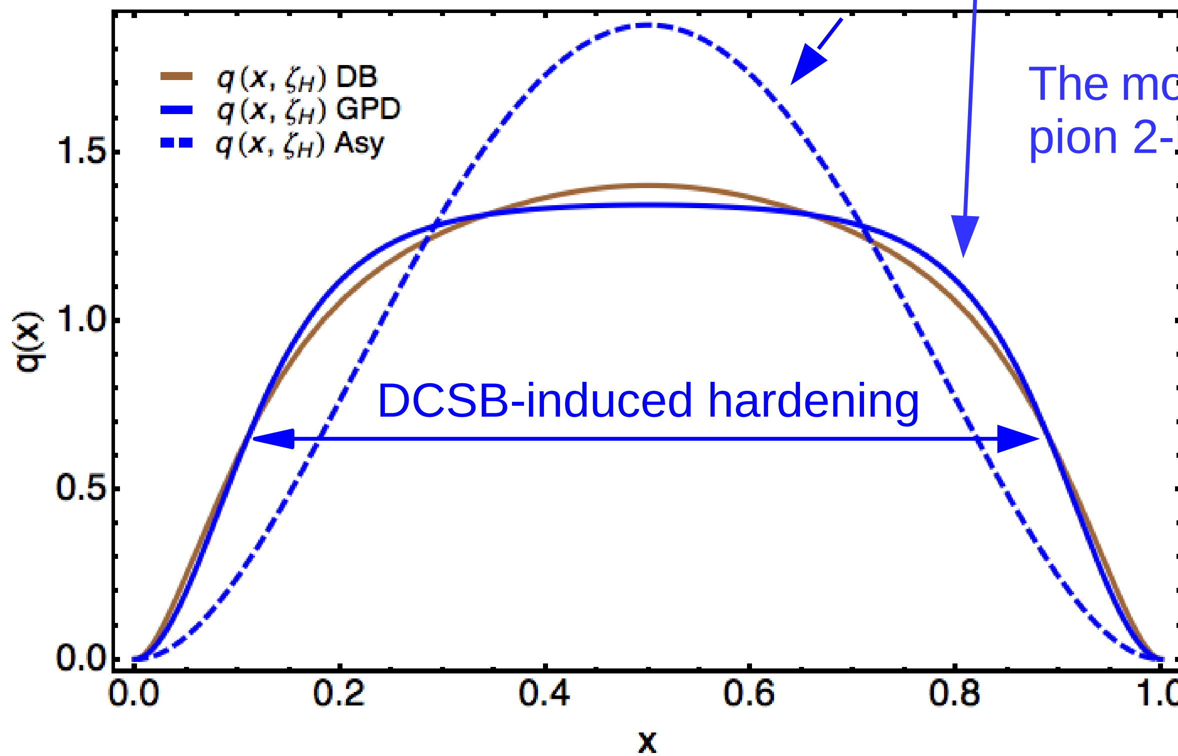
LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$

The more realistic pion 2-body LFWF





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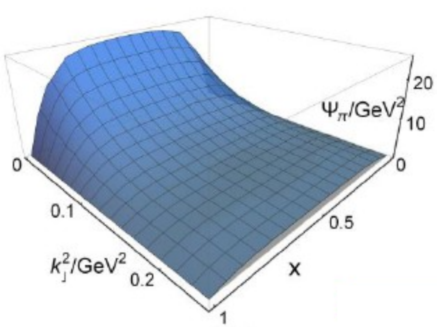

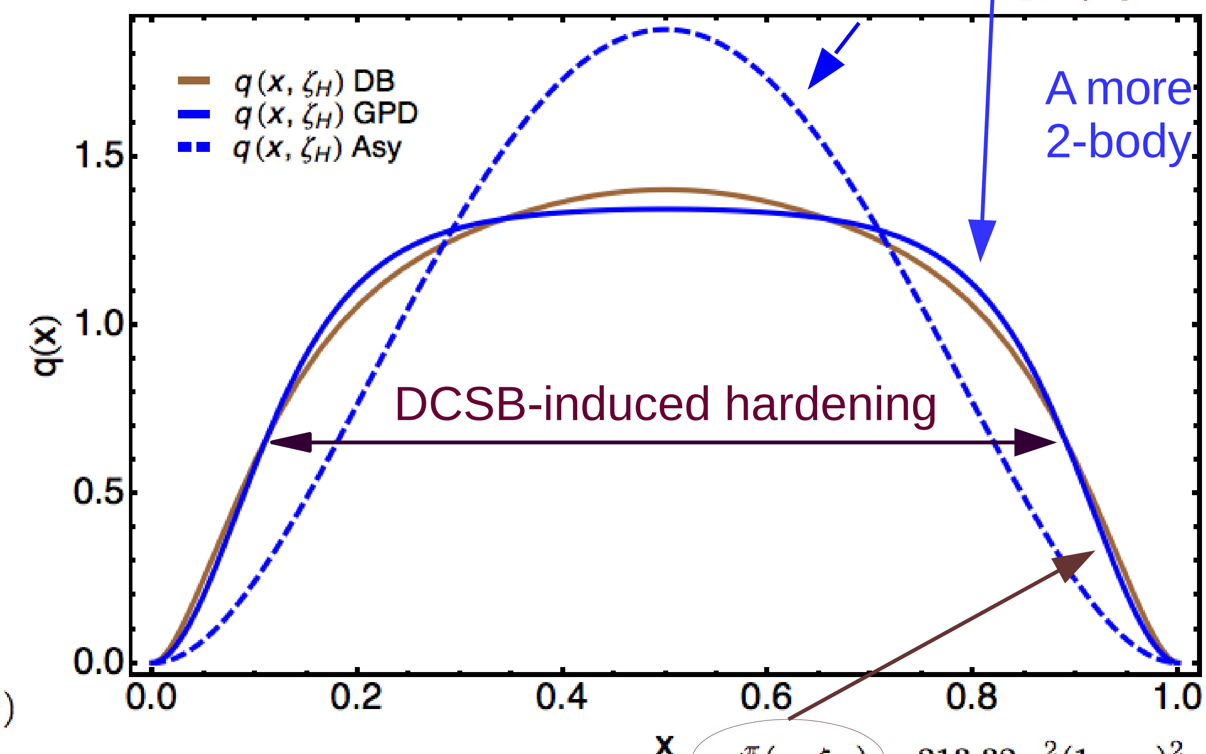

Direct computation of Mellin moments:

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$$q^\pi(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times \left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\,x(1-x)\right]$$

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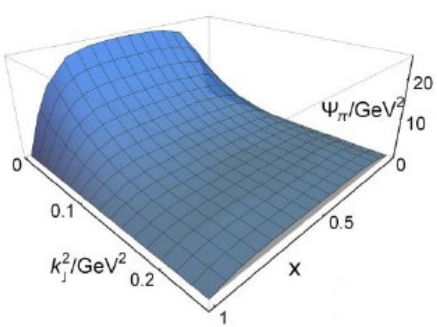

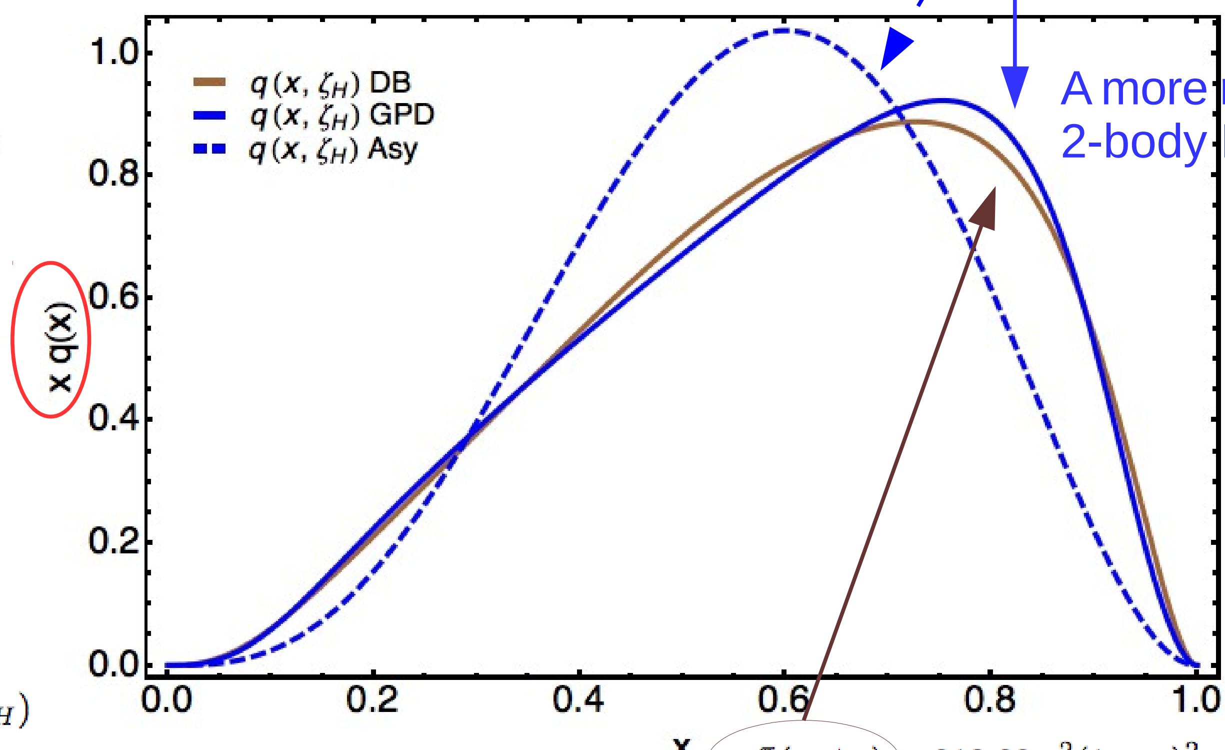

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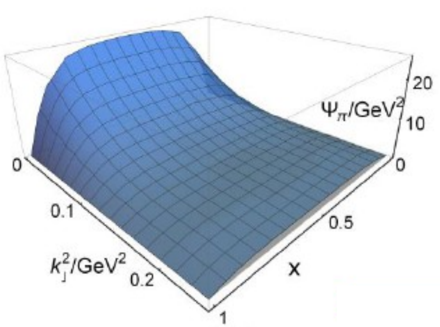

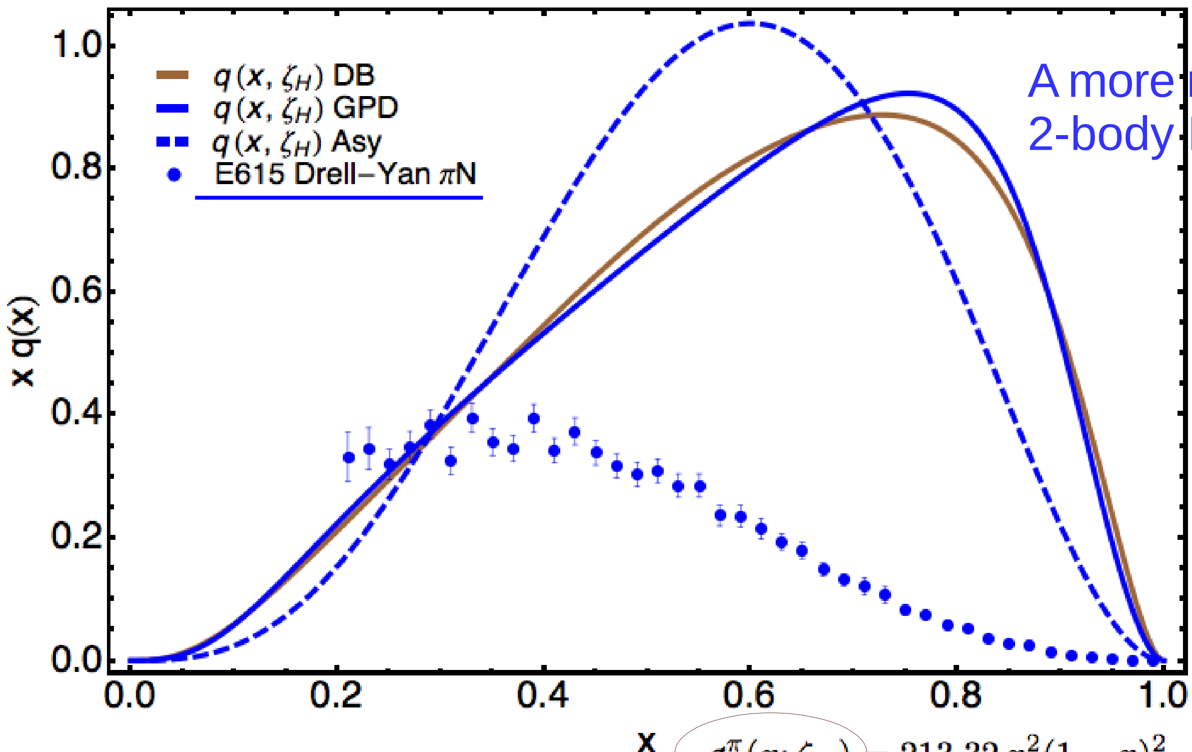

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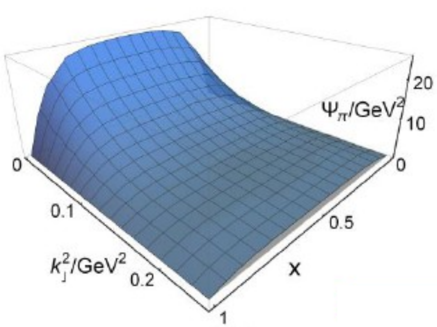

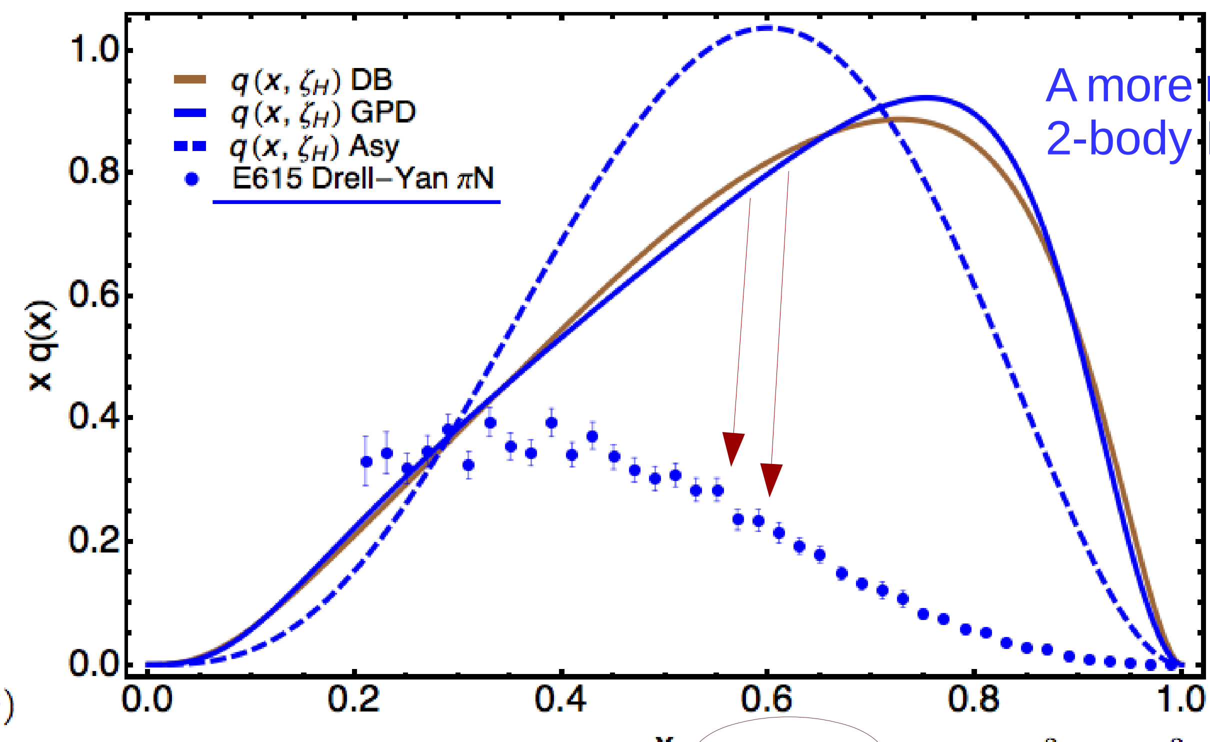

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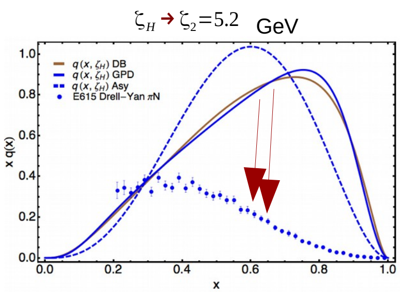
Pion realistic picture: DGLAP evolution

$$M_n(t)=\int_0^1 dx\, x^n q(x,t)$$

$$t=\ln\left(\frac{\zeta^2}{\zeta_0^2}\right)$$

Moments' evolution (1-loop):

$$\frac{d}{dt}M_n(t)=-\frac{\alpha(t)}{4\pi}\gamma_0^n M_n(t)+\ldots$$



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A master equation for the (1-loop) moments' evolution:

$$\frac{d}{dt} q(x,t) = \frac{\alpha(t)}{4\pi} \int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

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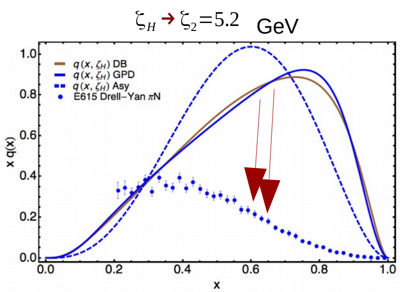

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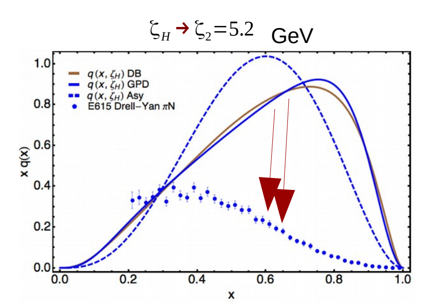
$$\gamma_n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - 4\sum_{i=1}^{n+1}\frac{1}{i}\right)$$



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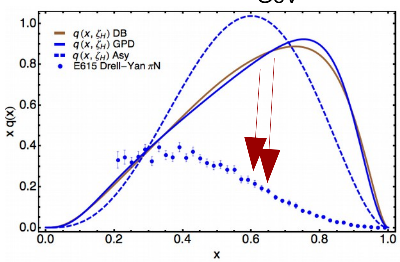
$$\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)}+\ldots$$

$$t_\Lambda=\ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)$$

$\zeta_H \rightarrow \zeta_2=5.2$ GeV



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$$M_n(t) = M_n(t_0)\left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}$$

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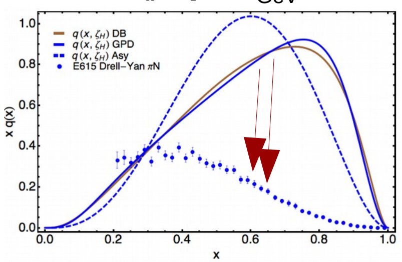

Pion realistic picture: DGLAP evolution

Which value of Lambda?

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

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$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

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The evolution will thus depend on the scheme *via* the perturbative truncation

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The evolution will thus depend on the scheme *via* the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

PDG2018:
[PRD98(2018)030001]

$$\Lambda^{(5)}_{\overline{MS}} = (210 \pm 14)\ \text{MeV}, \qquad (9.24b)$$

$$\Lambda^{(4)}_{\overline{MS}} = (292 \pm 16)\ \text{MeV}, \qquad (9.24c)$$

$$\Lambda^{(3)}_{\overline{MS}} = (332 \pm 17)\ \text{MeV}, \qquad (9.24d)$$

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$$\langle x^m \rangle^{\pi}_{\zeta_H} = \int_0^1 dx\, x^m q^{\pi}(x;\zeta_H)$$
$$= \frac{N_c}{n\cdot P} \mathrm{tr} \int_{dk} \left[\frac{n\cdot k_\eta}{n\cdot P} \right]^m \Gamma_\pi(k_{\bar\eta}, P)\, S(k_{\bar\eta})\, n\cdot \partial_{k_\eta} \left[\Gamma_\pi(k_\eta, -P) S(k_\eta) \right]$$

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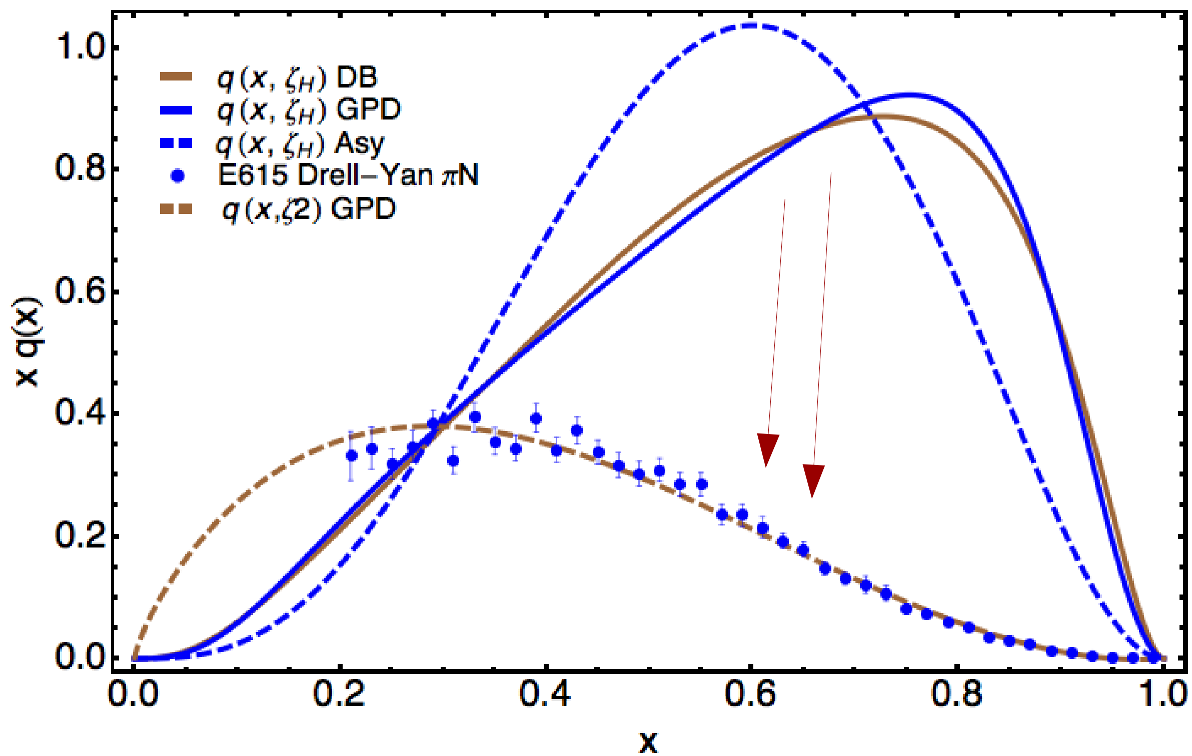
$$\zeta_H \rightarrow \zeta_2 = 5.2\ \mathrm{GeV}$$

Optimal best-fitting parameters:

$$\Lambda_{QCD} = 0.234\ \mathrm{GeV}\ ;$$
$$\zeta_H = 0.349\ \mathrm{GeV}\ .$$



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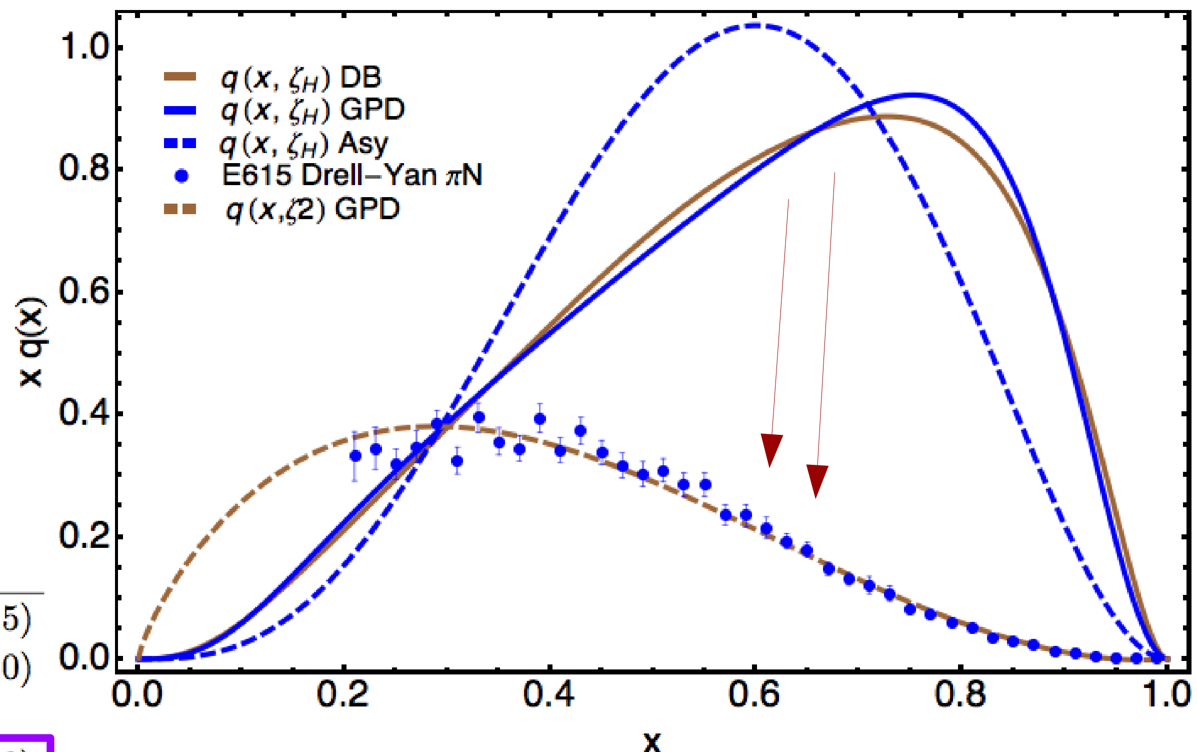

ζ_2	$\langle x\rangle^\pi_u$	$\langle x^2\rangle^\pi_u$	$\langle x^3\rangle^\pi_u$
Ref. [33]	0.24(2)	0.09(3)	0.053(15)
Ref. [34]	0.27(1)	0.13(1)	0.074(10)
Ref. [35]	0.21(1)	0.16(3)	
average	0.24(2)	0.13(4)	0.064(18)
Herein	0.24(2)	0.098(10)	0.049(07)

Comparison with the three first moments obtained from lQCD

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$$\langle x^m \rangle^{\pi}_{\zeta_H} = \int_0^1 dx\, x^m q^{\pi}(x;\zeta_H)$$
$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

$$q^{\pi}(x;\zeta_H) = 213.32\,x^2(1-x)^2 \times\left[1-2.9342\sqrt{x(1-x)}+2.2911\,x(1-x)\right]$$

$$\zeta_H \rightarrow \zeta_L = 2\ \mathrm{GeV} \rightarrow \zeta_2 = 5.2\ \mathrm{GeV}$$

Optimal best-fitting parameters:

$$\Lambda_{QCD} = 0.234\ \mathrm{GeV}\ ;\quad \zeta_H = 0.349\ \mathrm{GeV}\ .$$

$$\Lambda_{QCD} = 0.234\ \mathrm{GeV};\quad \zeta_H = 0.374\ \mathrm{GeV}\ .$$

ζ_2	$\langle x\rangle^\pi_u$	$\langle x^2\rangle^\pi_u$	$\langle x^3\rangle^\pi_u$
Ref. [33]	0.24(2)	0.09(3)	0.053(15)
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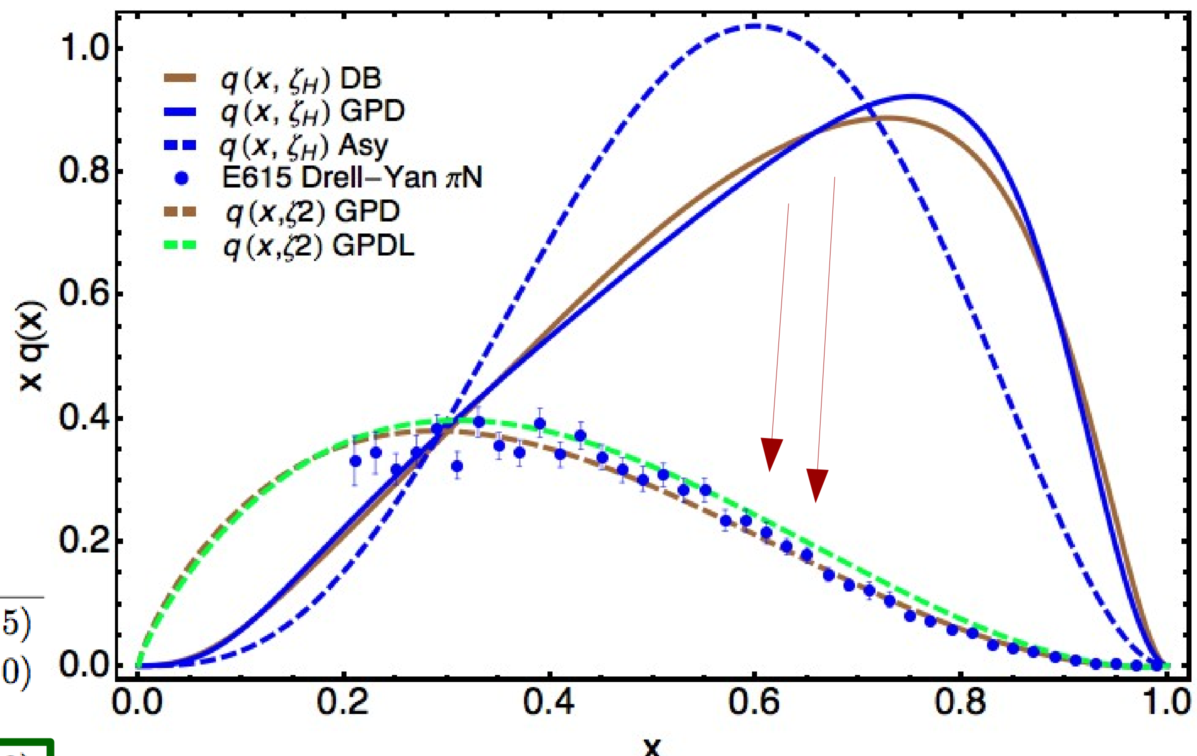

Matching the three first moments obtained from lQCD

Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

$$\alpha(t) = \frac{4\pi}{\beta_0 (t - t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots$$

$$\alpha(t) = \overline{\alpha}(t)\left(1 + c\,\overline{\alpha}(t) + \ldots\right)$$

$$\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0}\left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}$$

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots$$

$$\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots$$

The evolution will thus depend on the scheme *via* the perturbative truncation

The use of $\Lambda = 0.234$ GeV can be thus interpreted as the choice of particular scheme, differing from MS.

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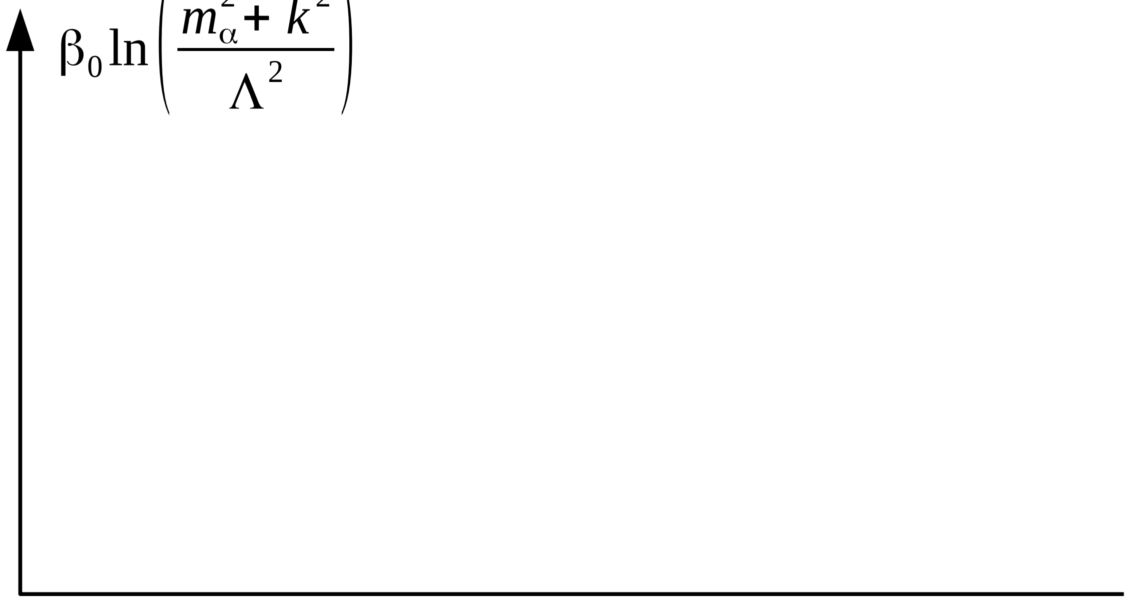
The evolution will thus depend on the scheme *via* the perturbative truncation

The use of $\Lambda = 0.234$ GeV can be thus interpreted as the choice of particular scheme, differing from MS. Beyond this, the scheme can be defined in such a way that one-loop DGLAP is exact at all orders (Grunberg's effective charge).

Pion realistic picture: DGLAP evolution

$$\alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\dfrac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\dfrac{m_\alpha^2 + k^2}{\Lambda^2}\right)}$$



$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

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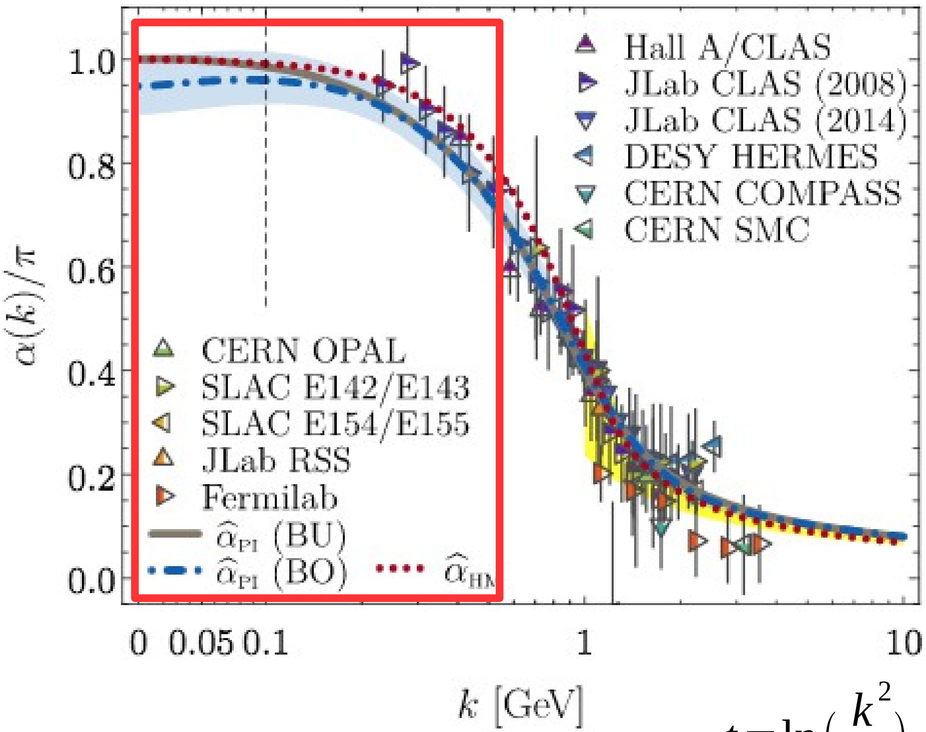
$$\alpha(0) = \alpha_{PI}(0) \rightarrow m_\alpha = 0.300 \text{ GeV}$$

D. Binosi et al., PRD96(2017)054026
J. R-Q et al., FBS59(2018)121
M. Ding et al., ArXiv:1905.05208 [nucl-th]



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Pion realistic picture: DGLAP evolution

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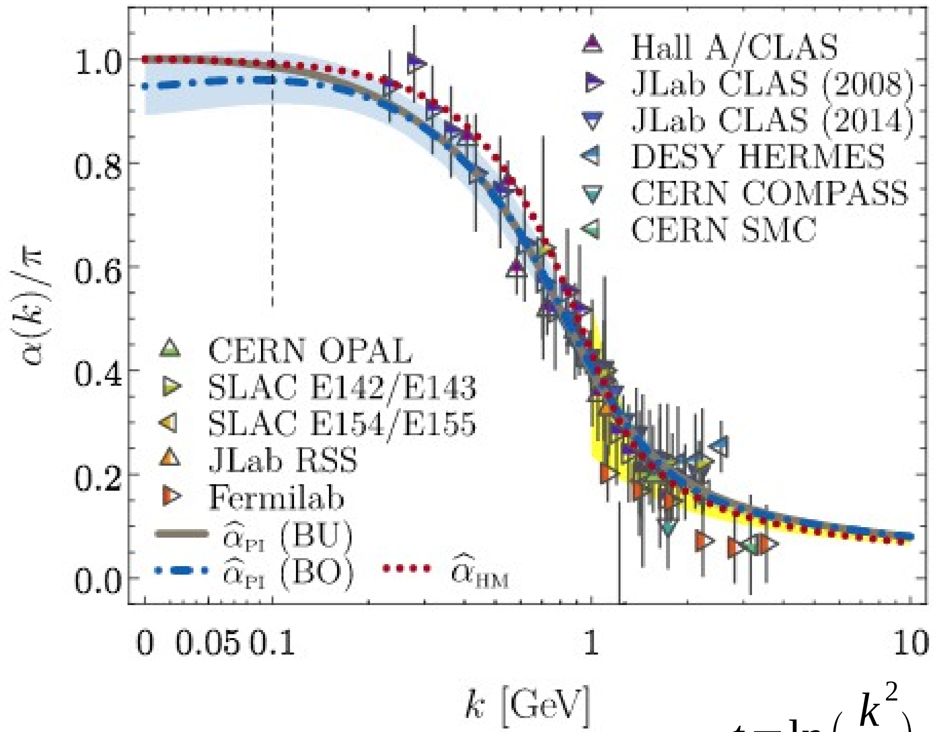

$$\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)$$

$$t = \ln\left(\frac{k^2}{\zeta_0^2}\right)$$

Numerical integration with the effective charge

$$M_n(t) = \int_0^1 dx\, x^n q(x,t)$$

$$M_n(t) = M_n(t_0) \exp\left(-\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz\, \alpha(z)\right)$$

$$\gamma_0^n = -\frac{4}{3}\left(3 + \frac{2}{(n+2)(n+3)} - \sum_{i=1}^{n+1} \frac{1}{i}\right)$$

Pion realistic picture: DGLAP evolution

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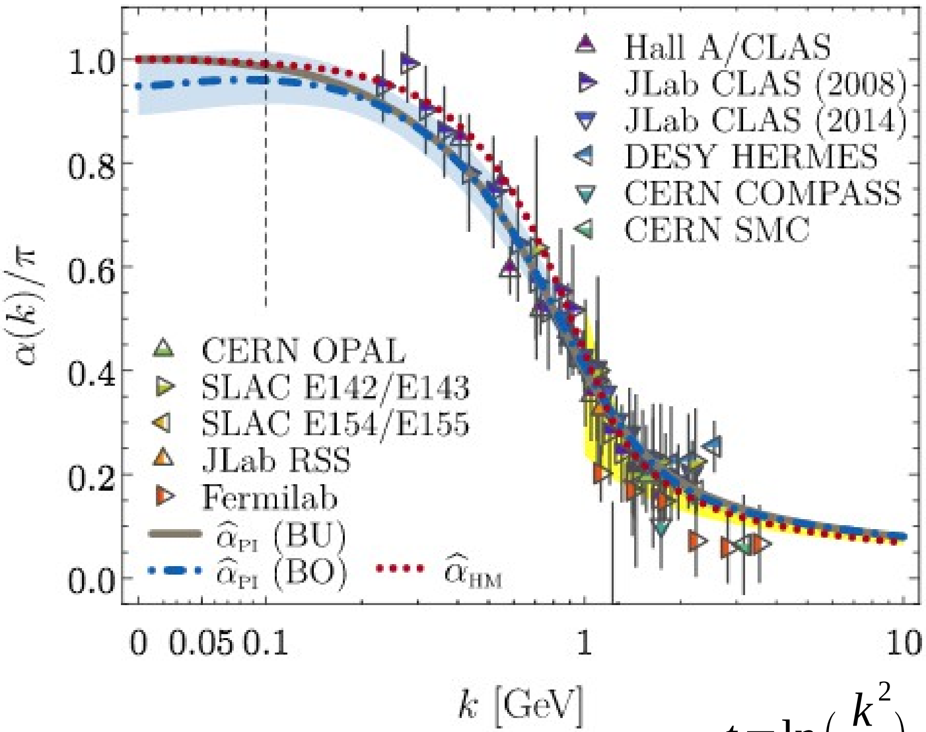

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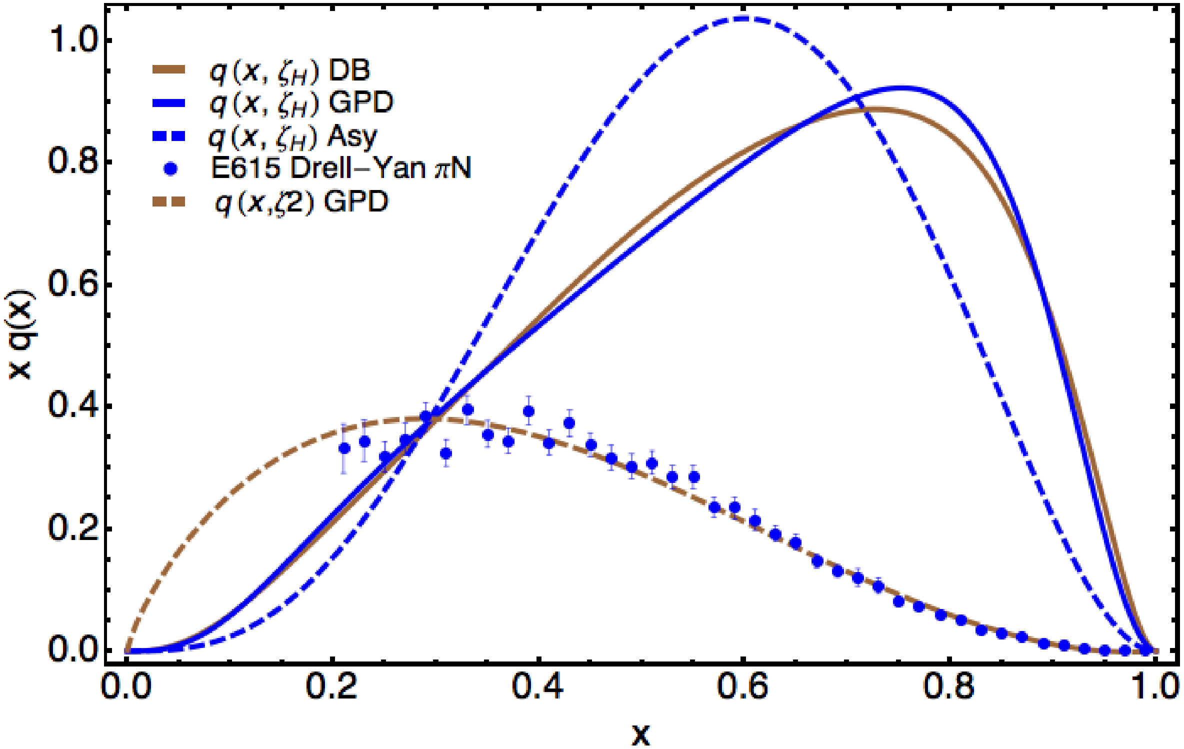
If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme).

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$\Lambda_{QCD} = 0.234$ GeV; $\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV

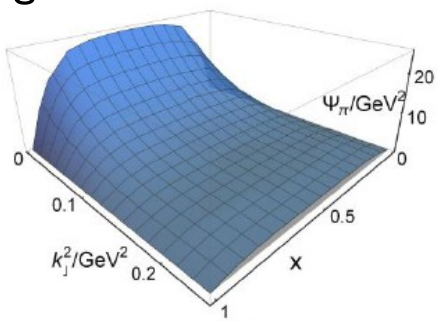


If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme). And the agreement with E615 data is perfect!!!

Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

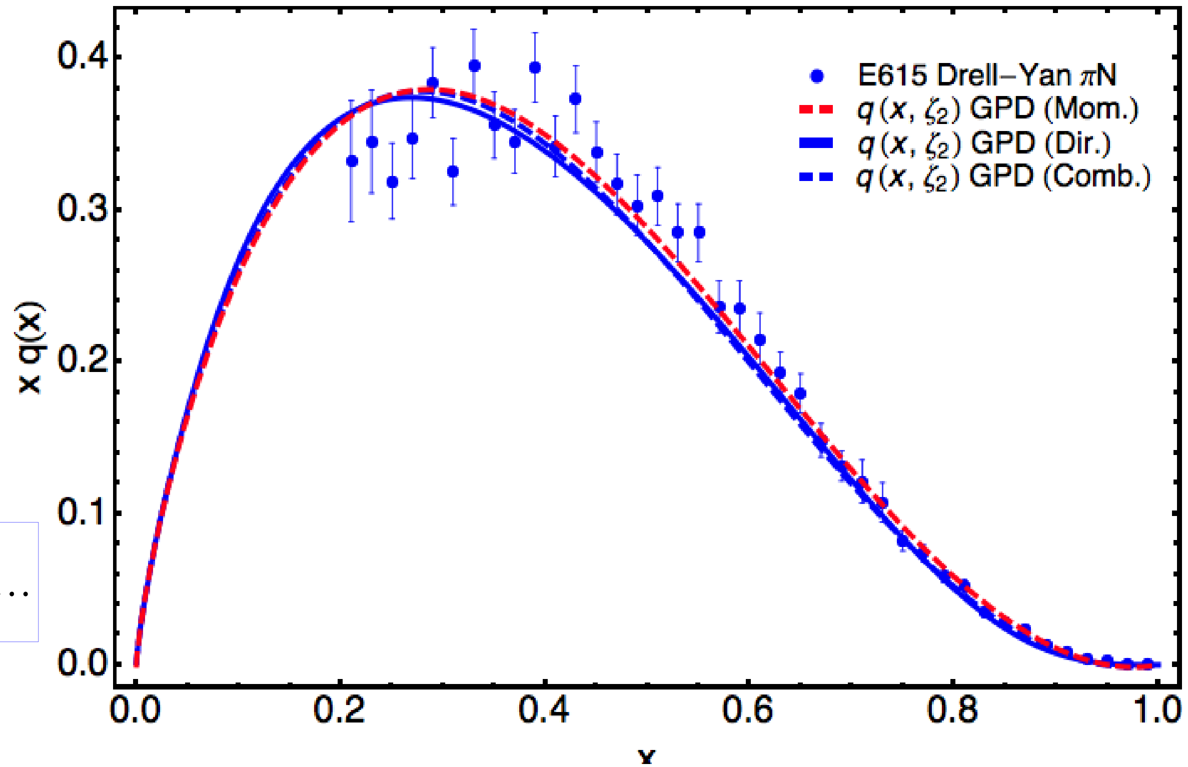
The same is obtained from the overlap of realistic pion 2-body LFWFs



and after integration of the DGLAP master equation

$$\frac{d}{dt}q(x,t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y,t) P\left(\frac{x}{y}\right) + \ldots$$

$\Lambda_{QCD} = 0.234$ GeV; $\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV



If one identifies: $m_\alpha \equiv \zeta_H$, all the scales (and the evolution between them) appear thus fixed, apart from Λ_{QCD} (fixed by the scheme). And the agreement with E615 data is perfect!!!

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

$$\zeta^2 \frac{d}{d\zeta^2} \left(\begin{array}{c} q^S(x;\zeta) \\ G^S(x;\zeta) \end{array} \right) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \left(\begin{array}{cc} P^S_{0,qq}\left(\frac{x}{y}\right) & 2n_f P^S_{0,qG}\left(\frac{x}{y}\right) \\ P^S_{0,Gq}\left(\frac{x}{y}\right) & P^S_{0,GG}\left(\frac{x}{y}\right) \end{array} \right) \left(\begin{array}{c} q^S(y;\zeta) \\ G^S(y;\zeta) \end{array} \right)$$

[Altarelli,Parisi; NPB126(1977)298]

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
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$$\gamma_{0,AB}^{S,(m)} = - \int_0^1 dx \, x^m P_{0,AB}^S(x)$$

$$\zeta^2 \frac{d}{d\zeta^2} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix} \gamma_{0,qq}^{S,(m)} & 2n_f \gamma_{0,qG}^{S,(m)} \\ \gamma_{0,Gq}^{S,(m)} & \gamma_{0,GG}^{S,(m)} \end{pmatrix} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix}$$

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$$P^{-1} \Gamma_0^{S,(m)} P = \underbrace{\begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}}_{\Gamma_D^{(m)}}$$

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Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom

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Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

$$P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = \exp\left(-\Gamma_D^{(m)} \int_{\ln \zeta_H^2}^{\ln \zeta^2} d\ln z^2\, \frac{\alpha(z^2)}{4\pi} \right) P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$$

$$P^{-1} \Gamma_0^{S,(m)} P = \begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}$$

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

$$\begin{aligned} M_q^{(m)}(\zeta) &= M_q^{(m)}(\zeta_H) \\ &\times \left[\frac{w_{11}w_{22}}{\mathrm{Det}\,(P)} \exp\left(-\frac{\lambda_+^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) - \frac{w_{12}w_{21}}{\mathrm{Det}\,(P)} \exp\left(-\frac{\lambda_-^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right] \\ M_G^{(m)}(\zeta) &= M_q^{(m)}(\zeta_H) \frac{w_{22}w_{21}}{\mathrm{Det}\,(P)} \\ &\times \left[\exp\left(-\frac{\lambda_+^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) - \exp\left(-\frac{\lambda_-^{(m)}}{4\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right] \end{aligned}$$

Pion realistic picture: DGLAP evolution

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$$\begin{pmatrix} 1 & 3/4 \\ -1 & 1 \end{pmatrix}$$

$$P^{-1}\, \Gamma_0^{S,(1)}\, P = \begin{pmatrix} 56/9 & 0 \\ 0 & 0 \end{pmatrix}$$

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

Case m=1
(nf=4)

$$M_q^{(1)}(\zeta) = M_q^{(1)}(\zeta_H) \left[\frac{3}{7} + \frac{4}{7} \exp\left(-\frac{56}{36\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right]$$

$$M_G^{(1)}(\zeta) = \frac{4}{7} M_q^{(1)}(\zeta_H) \left[1 - \exp\left(-\frac{56}{36\pi} \int_{\ln \zeta_H^2}^{\ln \zeta^2} dt\, \alpha(t) \right) \right]$$

Pion realistic picture: DGLAP evolution

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$$M_q^{(1)}(\zeta) + M_G^{(1)}(\zeta) = M_q^{(1)}(\zeta_H)$$

Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
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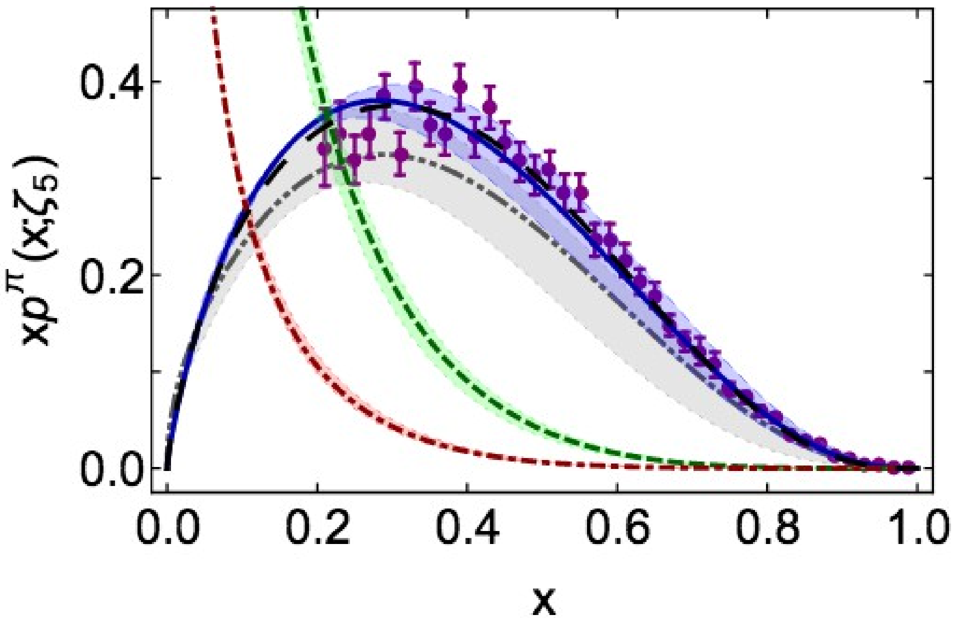
Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

ζ_5	$\langle x\rangle_u^\pi$	$\langle x^2\rangle_u^\pi$	$\langle x^3\rangle_u^\pi$
Ref. [31]	0.17(1)	0.060(9)	0.028(7)
Herein	0.21(2)	0.076(9)	0.036(5)

$$\langle x\rangle_g^\pi = 0.45(1)\,, \quad \langle x\rangle_{\rm sea}^\pi = 0.14(2)\,.$$



M. Ding et al., ArXiv:1905.05208 [nucl-th]

Pion realistic picture: PDF as benchmark

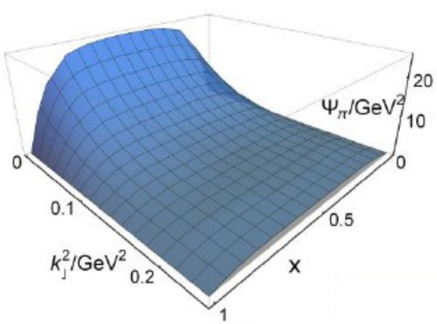
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

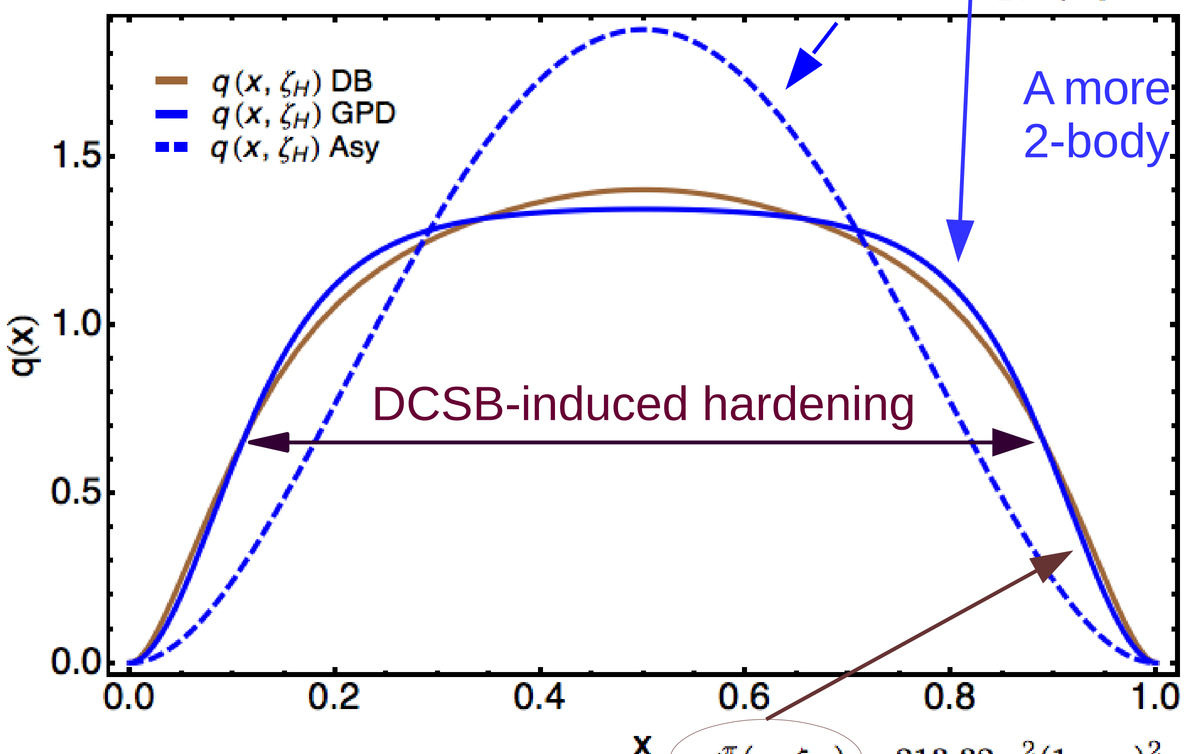
$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P\right\rangle\Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3}\Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\,\Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$

A more realistic pion 2-body LFWF





Direct computation of Mellin moments:

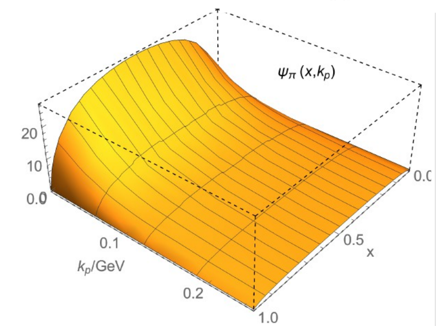
$$\langle x^m\rangle^\pi_{\zeta_H} = \int_0^1 dx\, x^m q^\pi(x;\zeta_H)$$
$$= \frac{N_c}{n\cdot P}\mathrm{tr}\int_{dk}\left[\frac{n\cdot k_\eta}{n\cdot P}\right]^m \Gamma_\pi(k_{\bar\eta},P)\,S(k_{\bar\eta})\,n\cdot\partial_{k_\eta}\left[\Gamma_\pi(k_\eta,-P)S(k_\eta)\right]$$

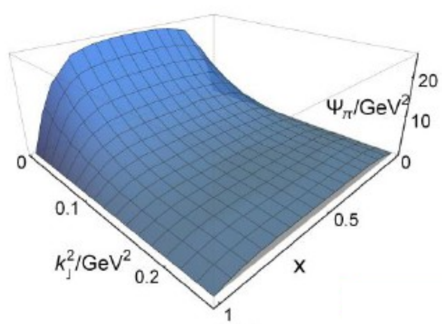
$$q^\pi(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times\left[1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)\right]$$

Pion (more) realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

$$q^\pi(x;\zeta_H) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \middle| \overline{\psi}^q(-z)\gamma^+\psi^q(z) \middle| P \right\rangle \Big|_{z^+=0,z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{u\bar{f}}(x,\mathbf{k}_\perp)\, \Psi_{u\bar{f}}(x,\mathbf{k}_\perp)$$

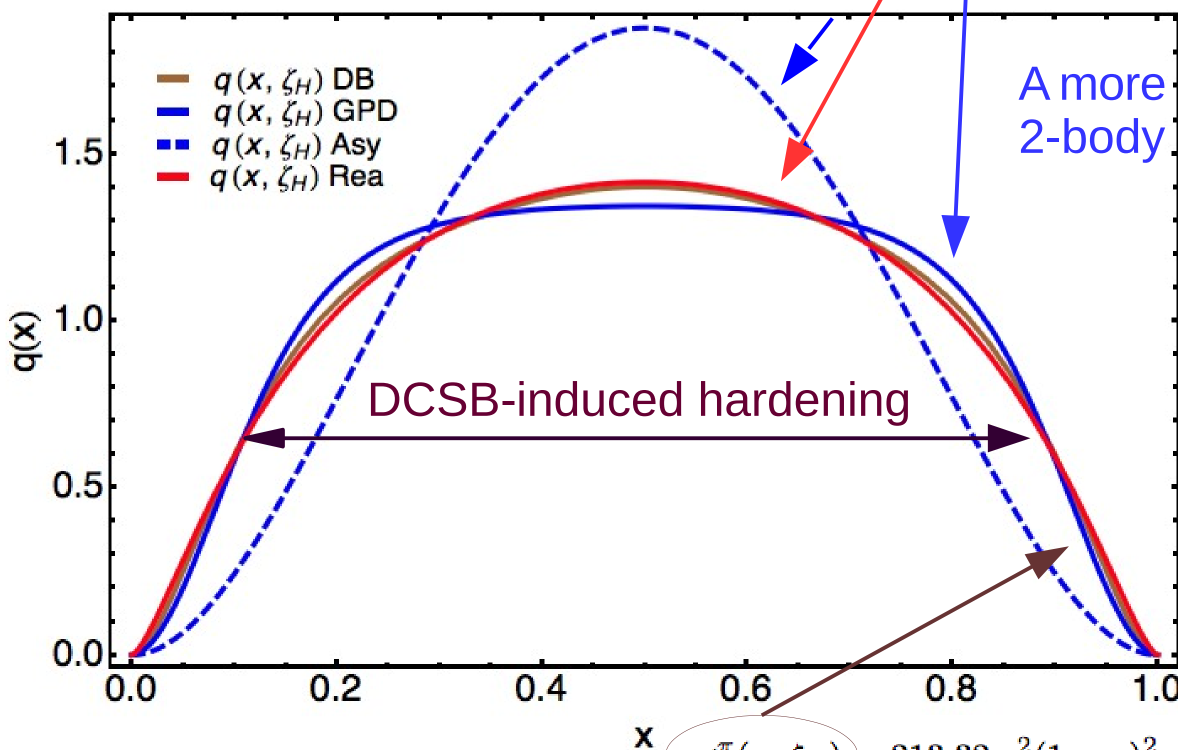




LFWF leading to asymptotic PDAs

$$q_{\rm sf}(x) \approx 30\, x^2(1-x)^2$$

A more realistic pion 2-body LFWF



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Pion (more) realistic picture: PDF as benchmark

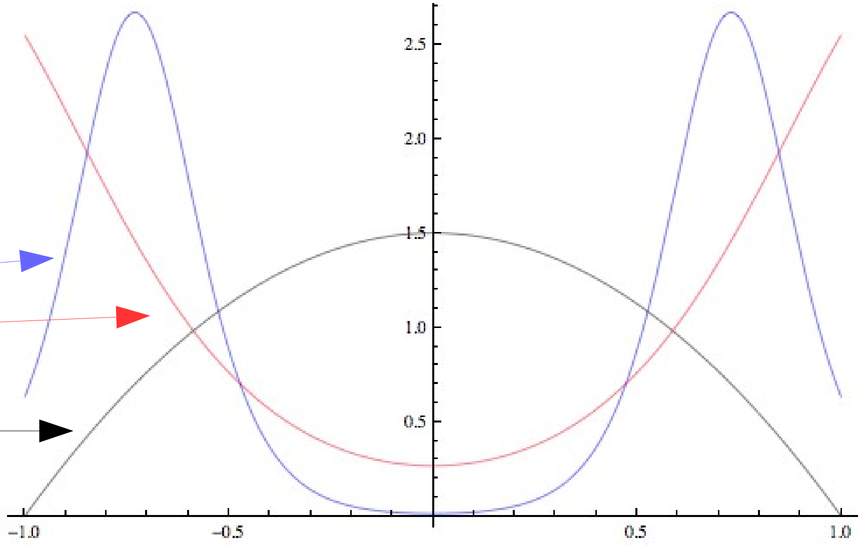
- Spectral density is chosen as:

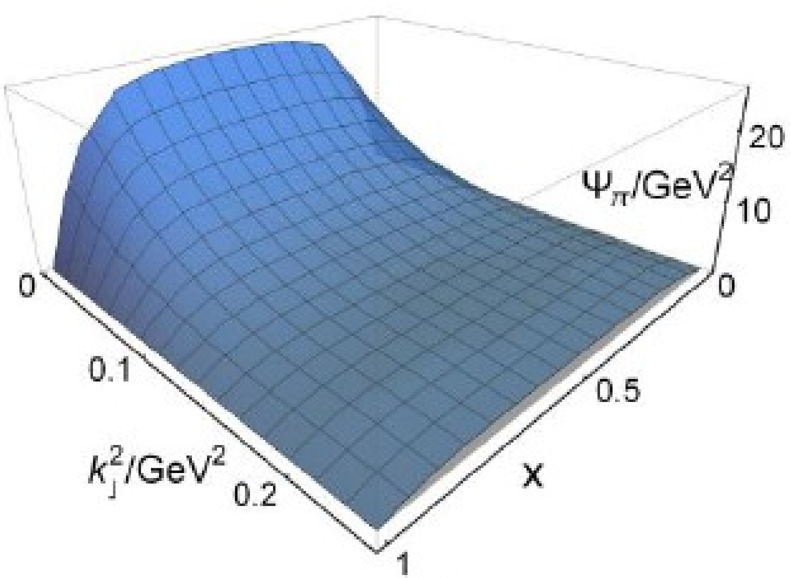
$$u_G \rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega-\omega_0^G}{2b_0^G}\right)+\mathrm{sech}^2\left(\frac{\omega+\omega_0^G}{2b_0^G}\right)\right]$$

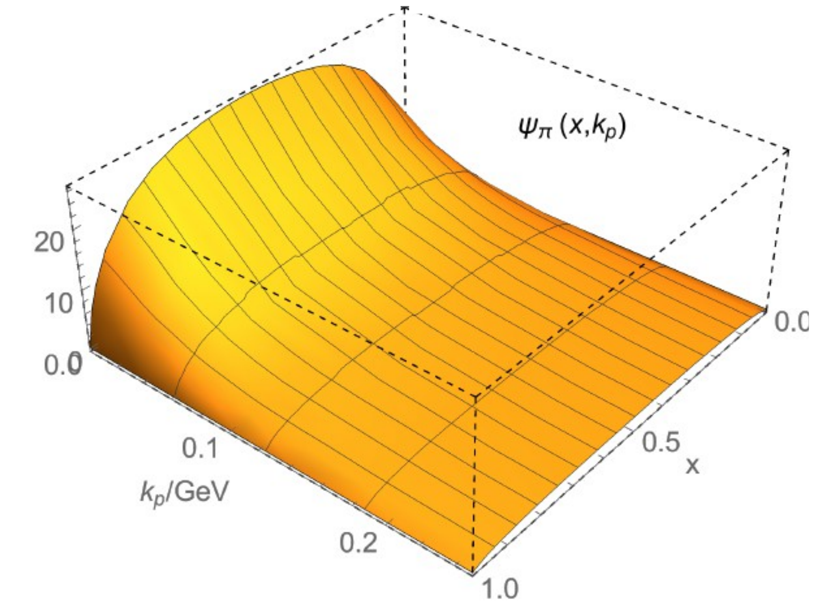
Phenomelogical model: $b_0^\pi = 0.1, w_0^\pi = 0.73$;

Realistic case: $b_0^\pi = 0.275, b_0^\pi = 1.23$;

Asymptotic case: $\rho(\omega;\nu) \sim (1-\omega^2)^\nu$







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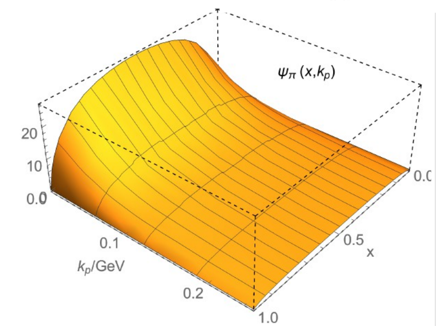
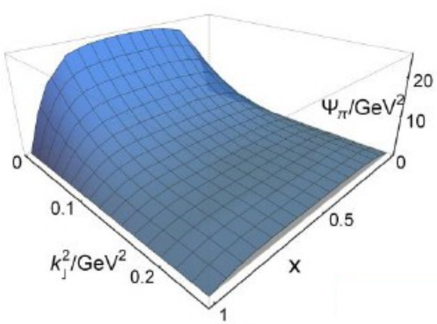
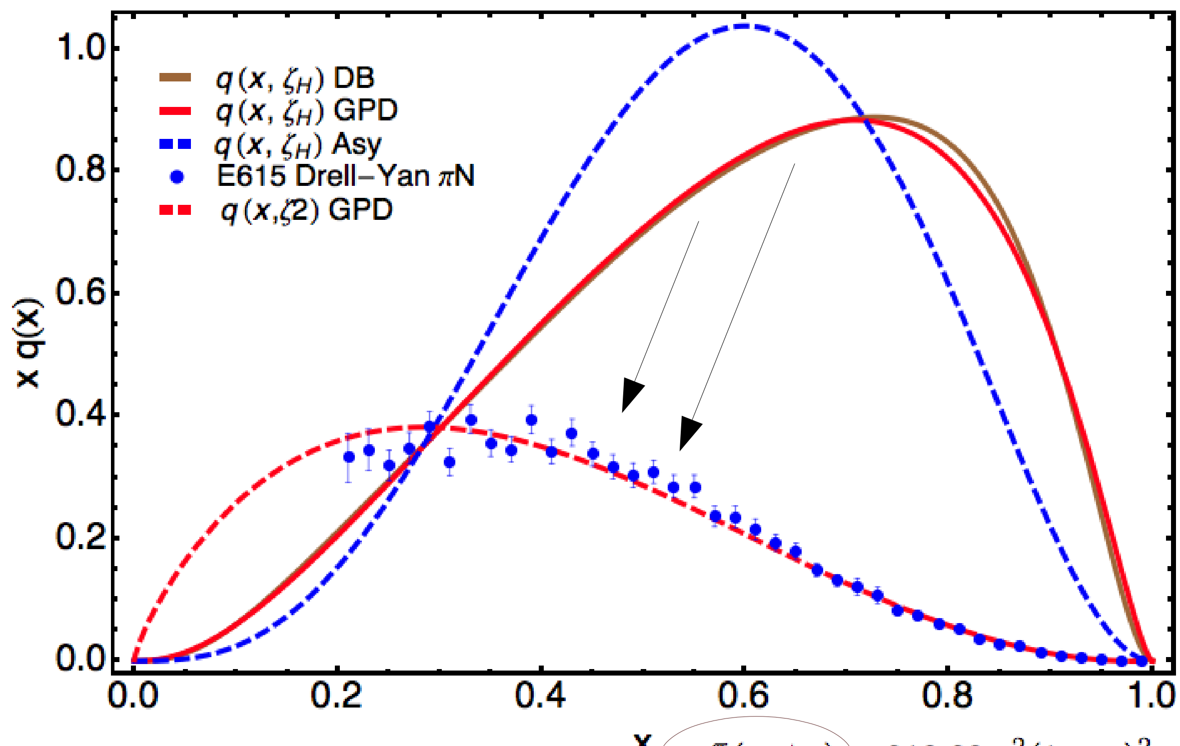




$\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2$ GeV



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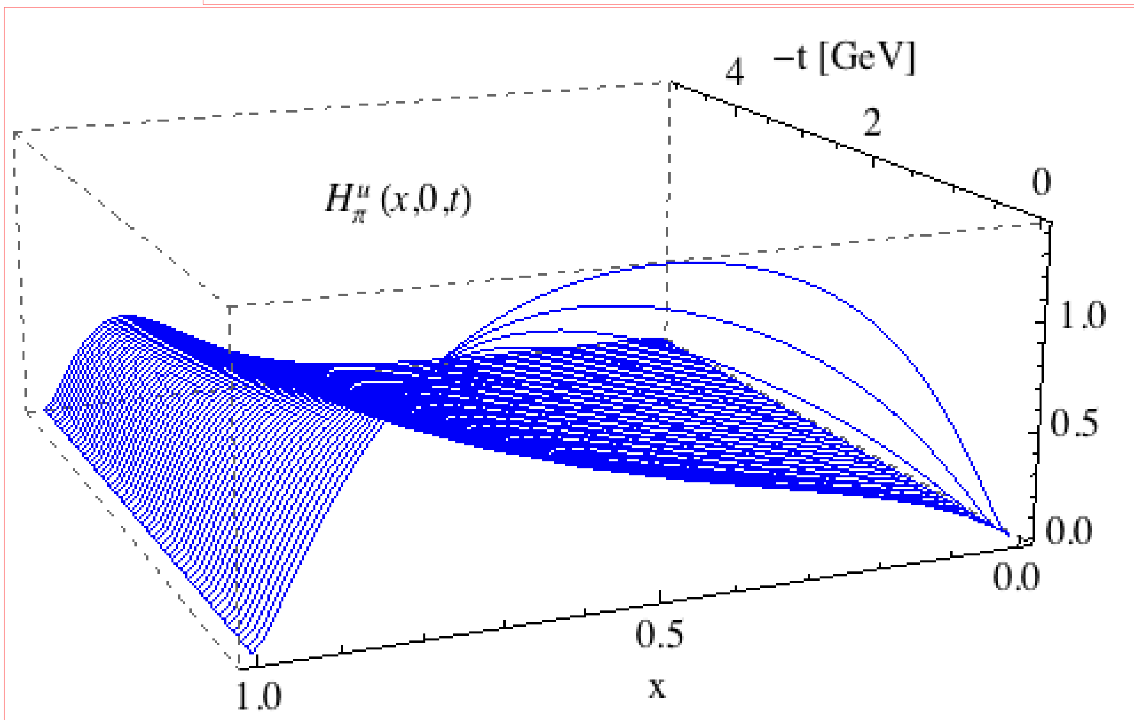
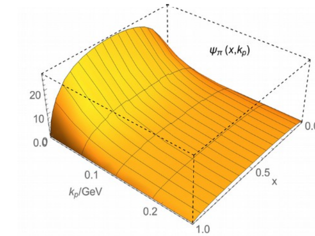
Pion (more) realistic picture: GPD

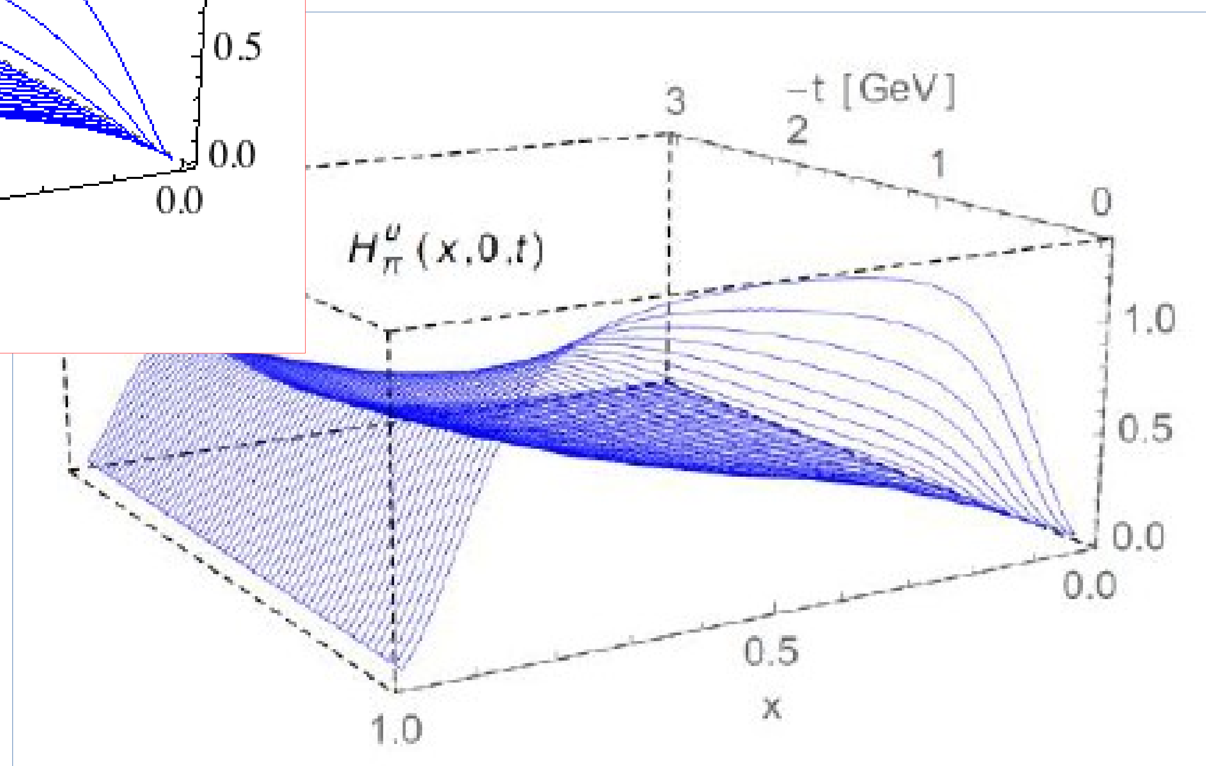
$$H^q_M(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3}\Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi},\mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right)\Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi},\mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$



Realistic case



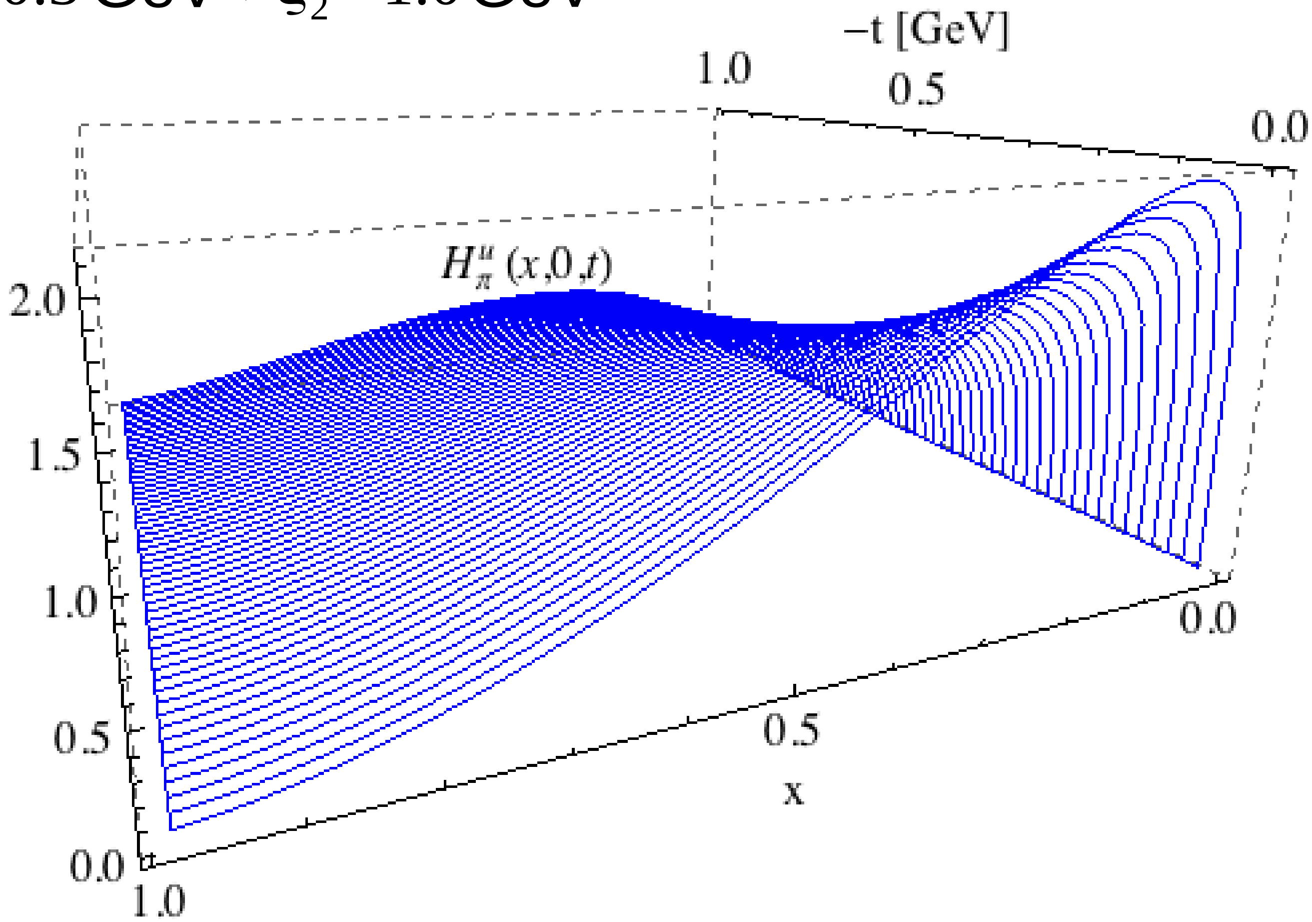
Phenomenological model



Pion (more) realistic picture: DGLAP evolution

$$H_M^q(x,\xi,t) = \int \frac{d^2\mathbf{k}_\perp}{16\,\pi^3} \Psi^*_{u\bar{f}}\left(\frac{x-\xi}{1-\xi}, \mathbf{k}_\perp + \frac{1-x}{1-\xi}\frac{\Delta_\perp}{2}\right) \Psi_{u\bar{f}}\left(\frac{x+\xi}{1+\xi}, \mathbf{k}_\perp - \frac{1-x}{1+\xi}\frac{\Delta_\perp}{2}\right)$$

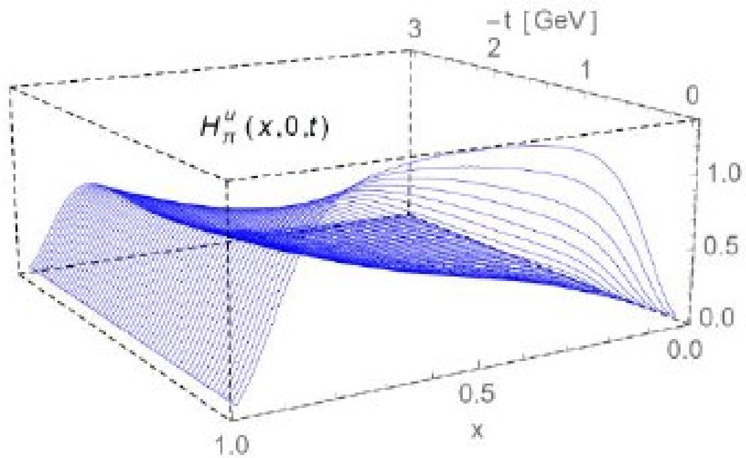
$\zeta_0 = \zeta_H = 0.3\,\text{GeV} \rightarrow \zeta_2 = 1.0\,\text{GeV}$

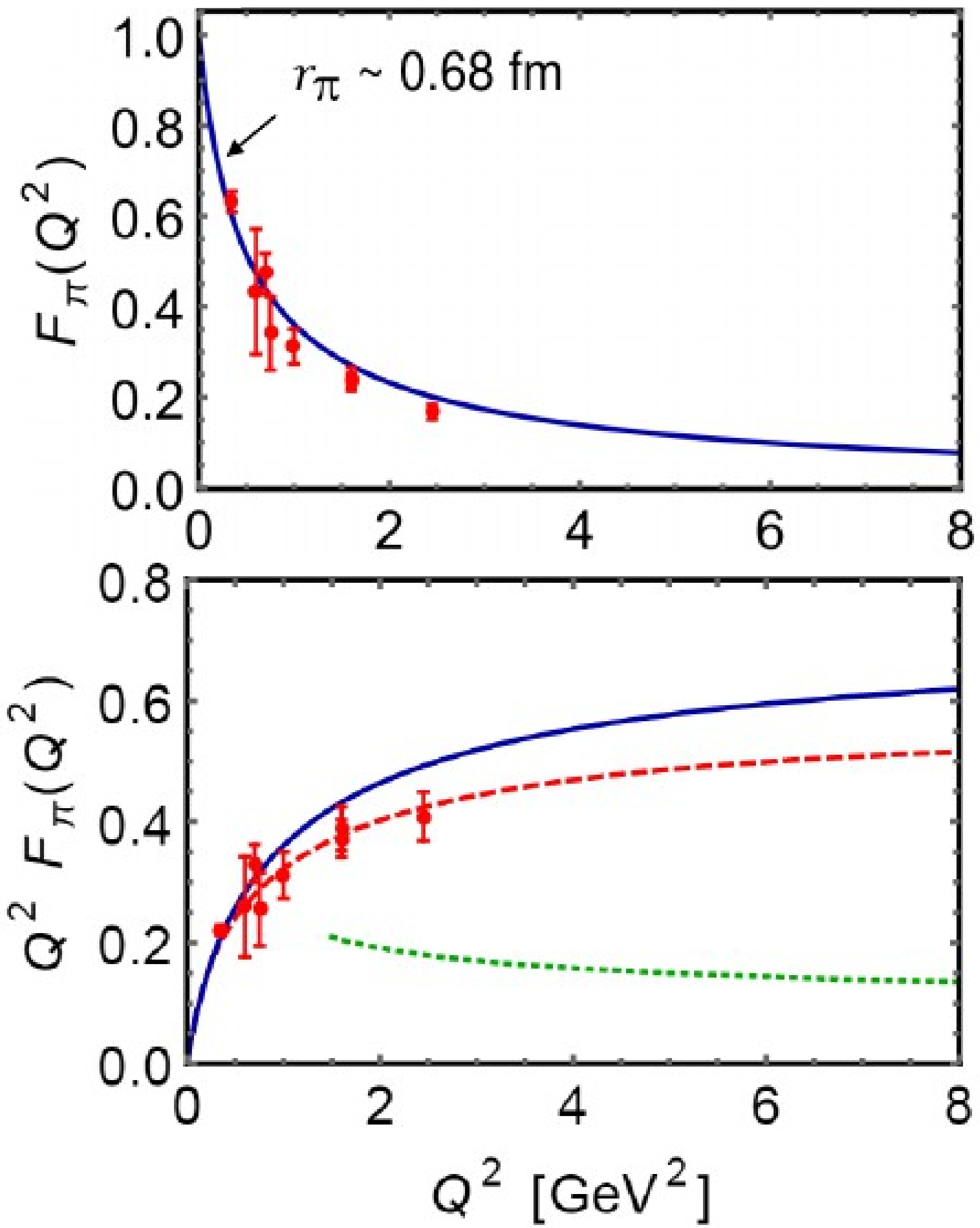


Pion (more) realistic picture: Elect. Form Factor

$$F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2)\ ,\ F_M^q(-t=\Delta^2) = \int_{-1}^{1} dx\ H_M^q(x,\xi,t)$$

Electric charges





Blue: Computed from GPD

Green: Computed from HS formula

Red: '*Evolved*' form factor

One word (or two) about ERBL covariant extension

The GPD can be cast (because of fundamental symmetries as Lorentz invariance) as a Radon transform of a double distribution:

(Polyakov-Weiss)

$$H(x,\xi,t) - \mathrm{sgn}(\xi) D(x/\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, F_D(\beta,\alpha,t)\delta(x-\beta-\alpha\xi)$$

So-called "gauge" transformations

$$H(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \left[F(\beta,\alpha,t) + \xi \, G(\beta,\alpha,t) \right] \delta(x-\beta-\alpha\xi)$$

1 component DD!!!
(Pobylitsa "gauge")

$$\begin{aligned} F(\beta,\alpha,t) &= (1-\beta) h(\beta,\alpha,t) + \delta(\beta) \, D^+(\alpha,t) \, , \\ G(\beta,\alpha,t) &= -\alpha h(\beta,\alpha,t) + \delta(\beta) \, D^-(\alpha,t) \, , \end{aligned}$$

$$H(x,\xi,t) = (1-x) \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, h(\beta,\alpha,t) \delta(x-\beta-\alpha\xi) + \frac{1}{|\xi|} D^+\left(\frac{x}{\xi},t\right) + \mathrm{sgn}(\xi) \, D^-\left(\frac{x}{\xi},t\right)$$

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Radon transform inversion

Is DGLAP GPD information enough to get the DD? (and thus extend the GPD to ERBL region)

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Radon transform inversion

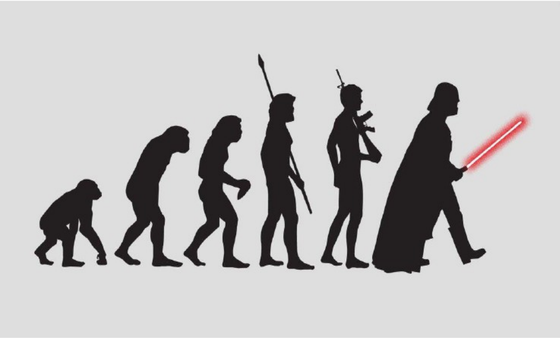
D-term ambiguities
(not living in DGLAP region)

Is DGLAP GPD information enough to get the DD? (and thus extend the GPD to ERBL region)

The answer is yes... up to D-term ambiguities … and issues related to the fact that, mathematically, the inversion problem is ill-posed (in the Hadamard sens)

N. Chouika, C. Mezrag, H. Moutarde, J. R-Q, Eur.Phys.J. C77 (2017) no.12, 906,
Phys.Lett. B780 (2018) 287-293

PDA and LFWF evolution



Standard PDA evolution:

- We project **PDA** onto a 3/2-Gegenbauer polynomial basis. Such that it **evolves**, from an initial scale ζ_0 to a final scale ζ, **according to** the corresponding **ERBL equations**:

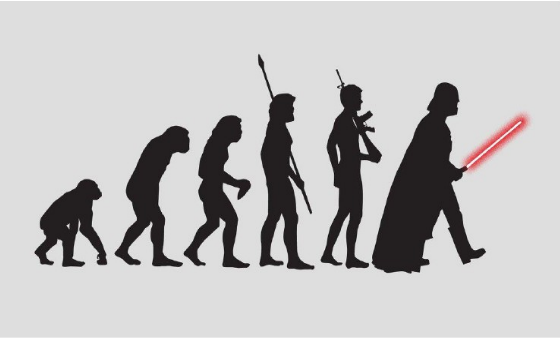
$$\phi(x;\zeta) = 6x(1-x)\left[1 + \sum_{n=1} a_n(\zeta) C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) = a_n(\zeta_0)\left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0} , \quad \gamma_0^n = -\frac{4}{3}\left[3 + \frac{2}{(n+1)(n+2)} - 4\sum_{k=1}^{n+1}\frac{1}{k}\right] .$$

- Thus, any PDA at hadronic scale evolves logarithmically towards its conformal distribution, $\phi(x)=6x(1-x)$.

  - ➢ Quark mass and flavor become irrelevant. Broad PDA becomes narrower, skewed PDA becomes symmetric.

PDA and LFWF evolution



LFWF evolution: $$\phi(x) = \frac{1}{16\pi^3}\int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x,k_\perp^2)$$

- We look for a way to evolve the LFWF.
- First, let's assume that the LFWF admits a similar Gegenbauer expansion. That is:

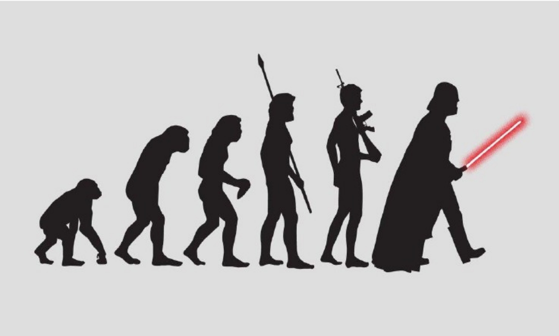
$$\psi(x,k_\perp^2;\zeta) = 6x(1-x)\left[\sum_{n=0} b_n(k_\perp^2;\zeta)\, C_n^{3/2}(2x-1)\right] ,$$

$$a_n(\zeta) \;=\; \frac{1}{16\pi^3}\int d^2\vec{k}_\perp\, b_n(k_\perp^2;\zeta)\ (\text{for } n\geq 1) ,\ \frac{1}{16\pi^3}\int d^2\vec{k}_\perp\, b_0(k_\perp^2;\zeta) = 1 .$$

- 1-loop ERBL evolution of $a_n(\zeta)$ implies:

$$\frac{1}{a_n(\zeta)}\frac{d}{d\ln\zeta^2}a_n(\zeta) \;=\; \frac{\displaystyle\int d^2\vec{k}_\perp\, \frac{d}{d\ln\zeta^2}b_n(k_\perp^2;\zeta)}{\displaystyle\int d^2\vec{k}_\perp\, b_n(k_\perp^2;\zeta)} ,$$

PDA and LFWF evolution



LFWF evolution:

$$\phi(x) = \frac{1}{16\pi^3}\int d^2\vec{k}_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2)$$

- Now, if we take a factorization assumtion, we arrive at:

$$\frac{b_n(k_\perp^2;\zeta)}{b_n(k_\perp^2;\zeta_0)} = \frac{\widehat{b}_n(\zeta)}{\widehat{b}_n(\zeta_0)} = \left[\frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)}\right]^{\gamma_0^n/\beta_0}, \quad b_n(k_\perp^2;\zeta) \equiv \widehat{b}_n(\zeta)\chi_n(k_\perp^2) \; .$$

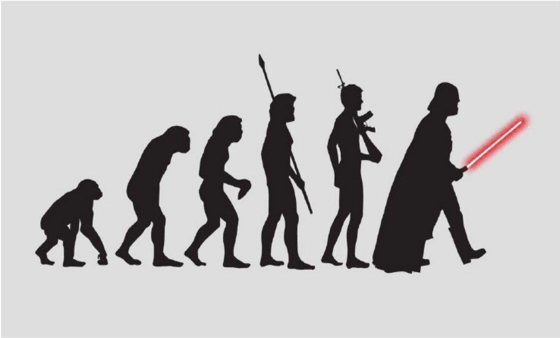
- Suplemented by the condition $\chi_n(k_\perp^2) \equiv \chi(k_\perp^2)$, one gets $\widehat{b}_n(\zeta) \equiv a_n(\zeta)$.
- Such that, the followiong factorised form is obtained:

$$\psi(x, k_\perp^2;\zeta) \equiv \phi(x;\zeta)\,\chi(k_\perp^2)$$

→ **LFWF Evolves like PDA**

- Which is far from being a general result, but an useful approximation instead.

PDA and LFWF evolution

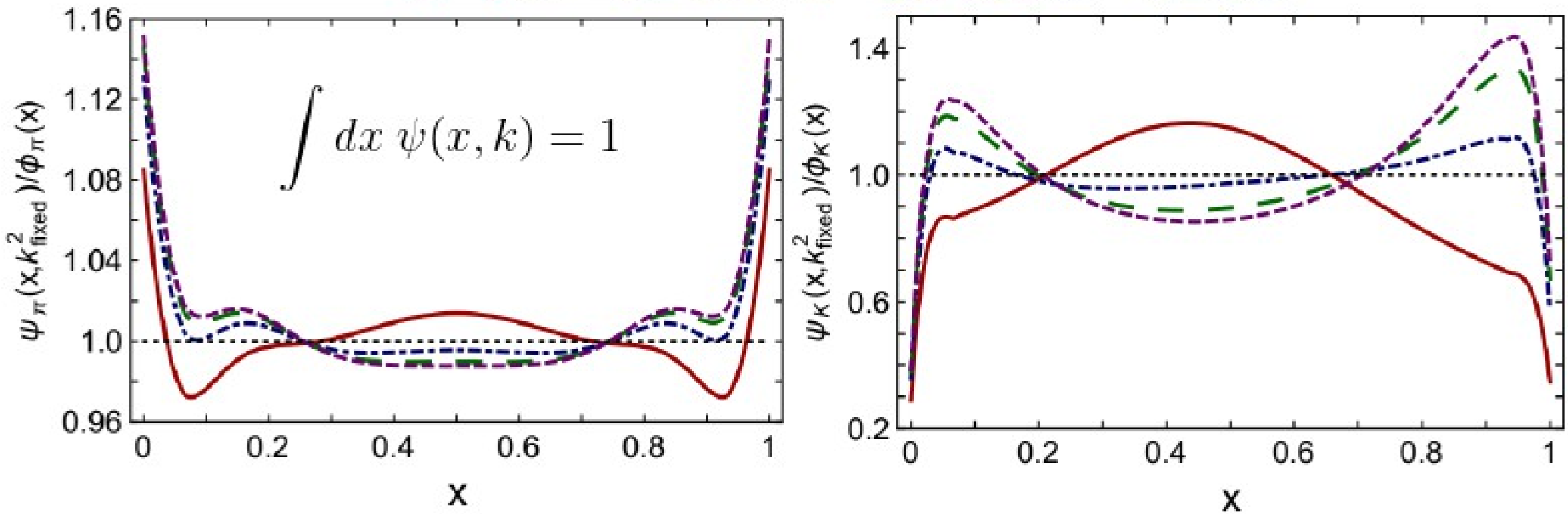


Testing the factorization ansatz:

$$\psi(x, k_\perp^2; \zeta) \equiv \phi(x; \zeta)\, \chi(k_\perp^2)$$

- A first validation of the factorized ansätz is addressed in **Phys.Rev. D97 (2018) no.9, 094014**:

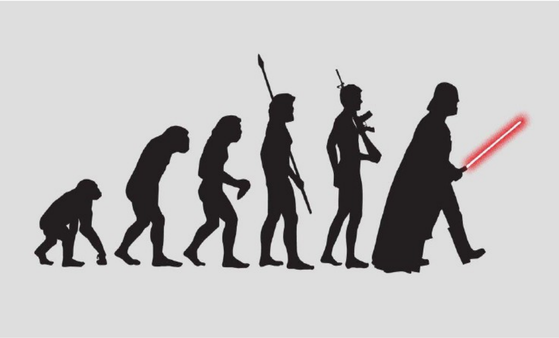
k^2=0, k^2=0.2 GeV, k^2=0.8 GeV, k^2=3.2 GeV

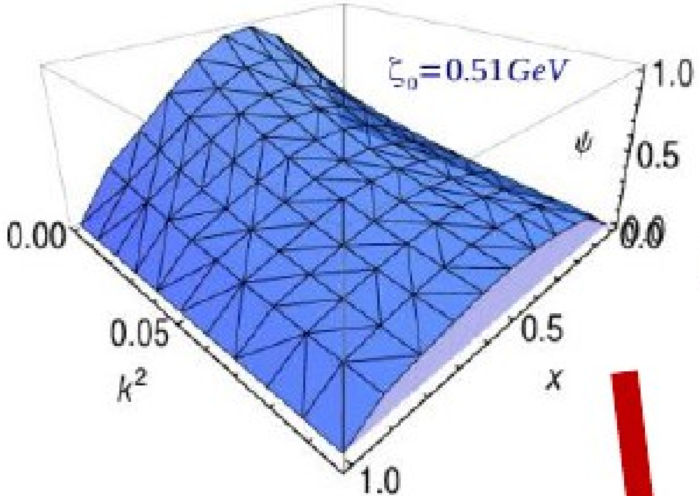


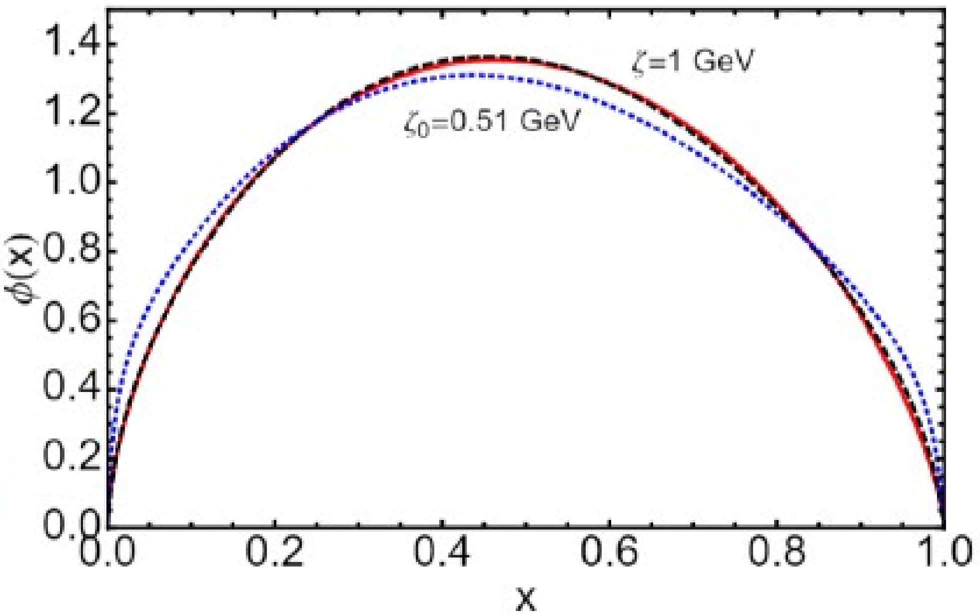
- If the factorized ansatz is a good approximation, then the plotted ratio must be 1. For the pion, it slightly deviates from 1; for the kaon, the deviation is much larger.

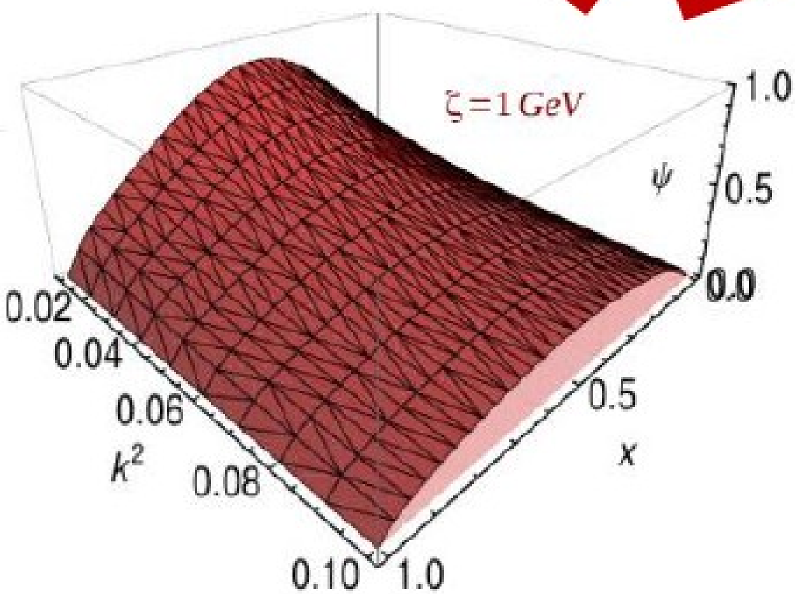
PDA and LFWF evolution

Testing the factorization ansatz:









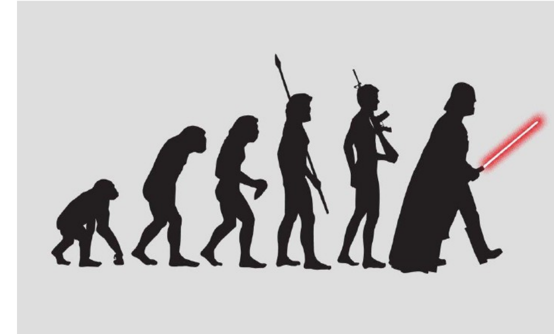
1) Compute LFWF and ERBL running of PDA

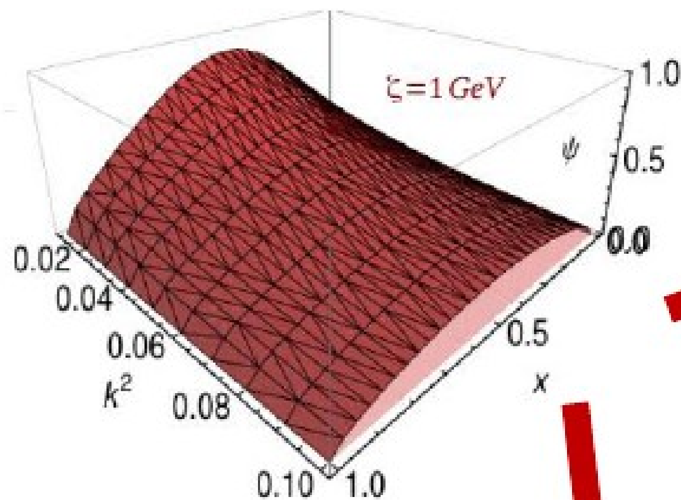
2) ERBL running of LFWF and compute PDA

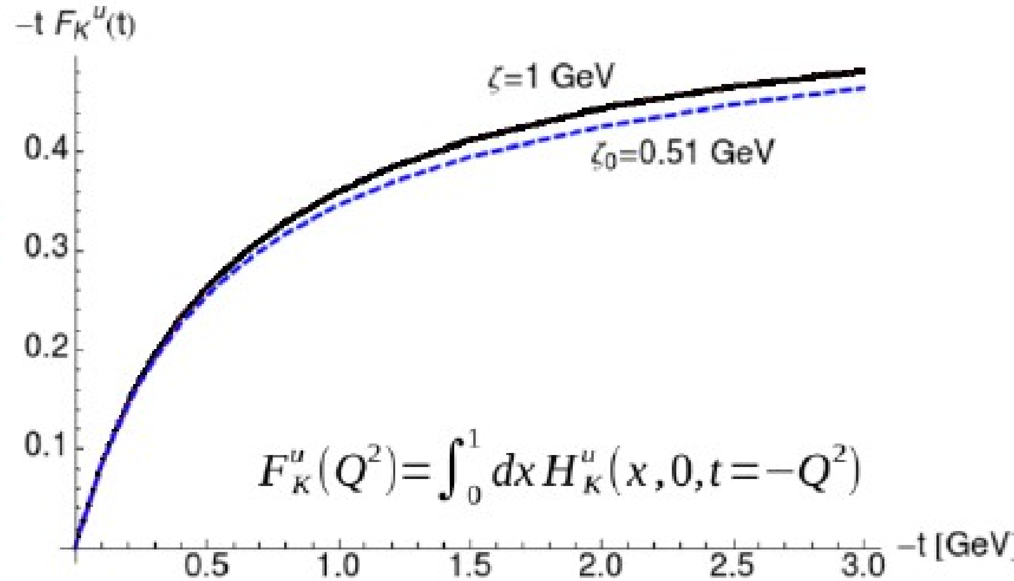
Notably, **1)** and **2)** are **equivalent**. Factorization assumption and evolution seem reasonable.

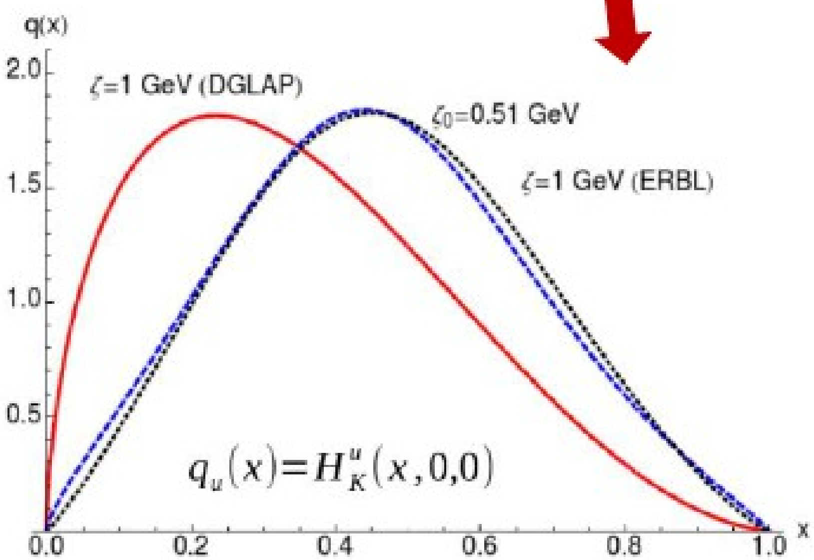
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:









1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

PDA and LFWF evolution

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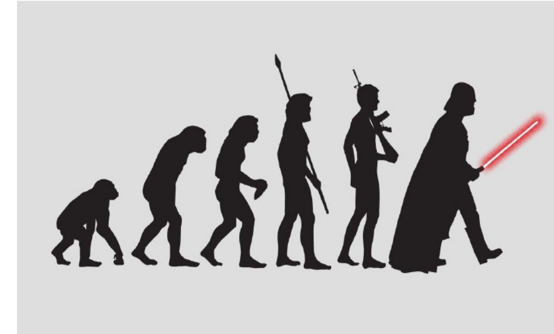
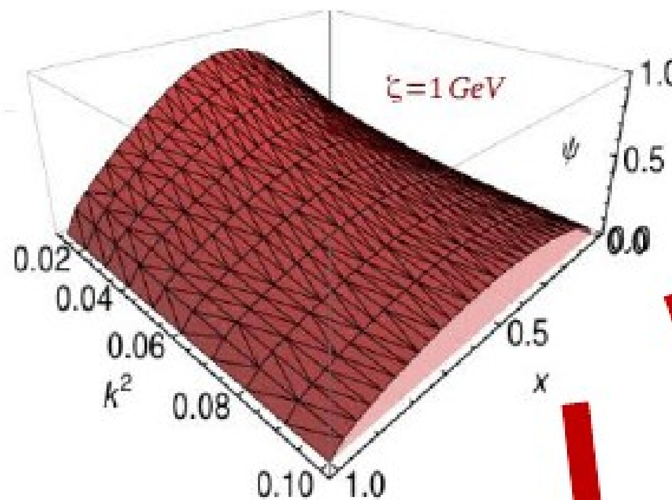
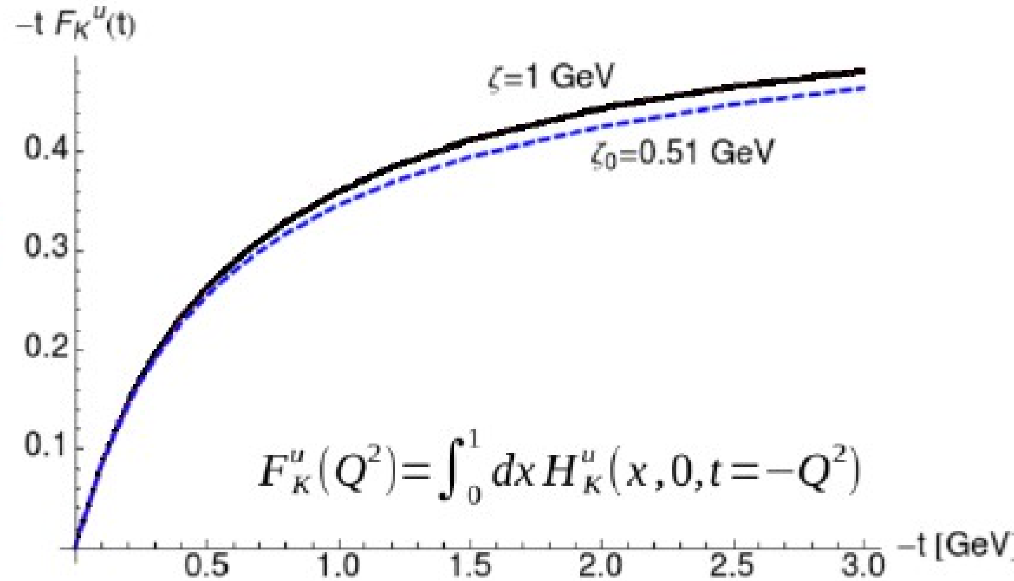
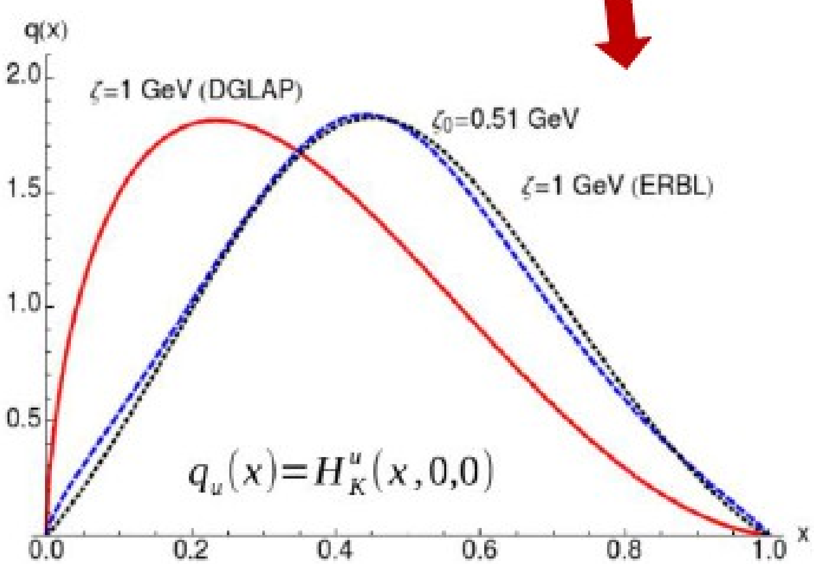








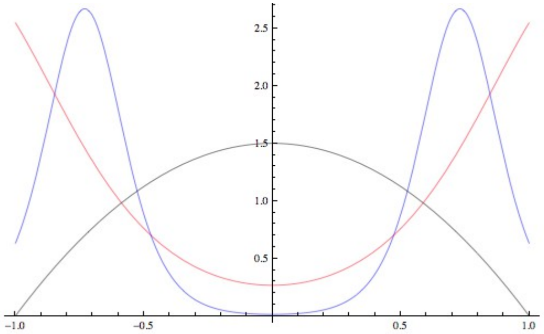
1) Obtained from ERBL evolution of LFWF

2) Obtained from DGLAP evolution of GPD

Clearly, **1)** and **2)** are **not equivalent**.

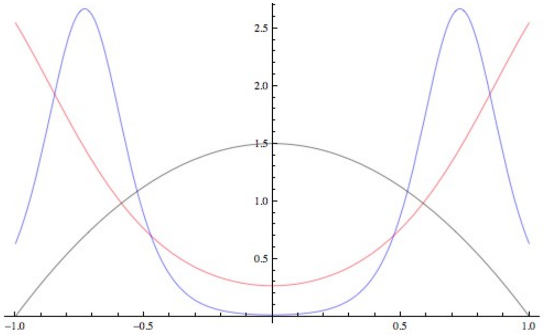
Sea-quark and gluon content incorporated to the parton distribution by DGLAP are obviously not present in the valence-quark PDF from LFWFs!!!

Conclusions



Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.

Conclusions



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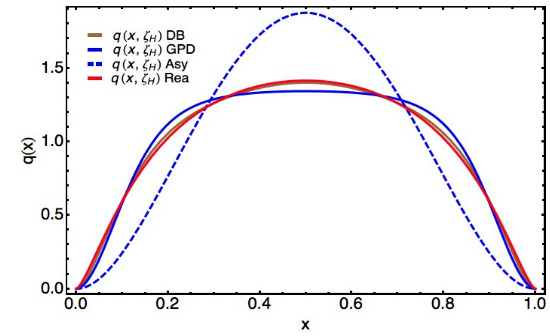

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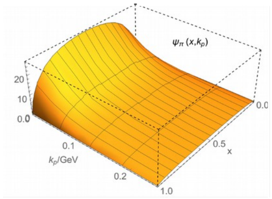

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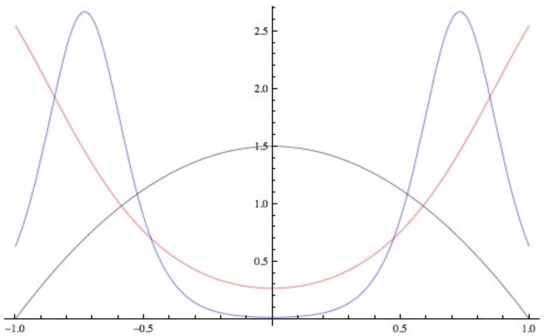

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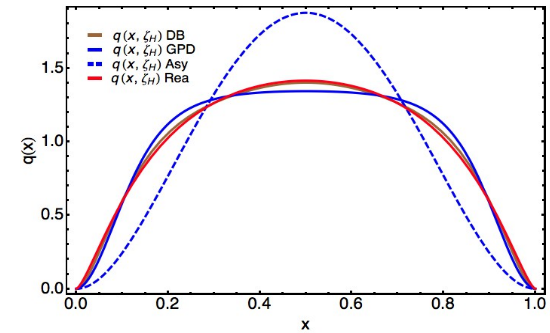

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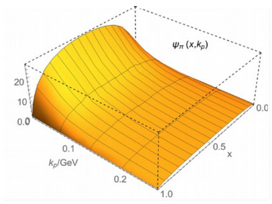

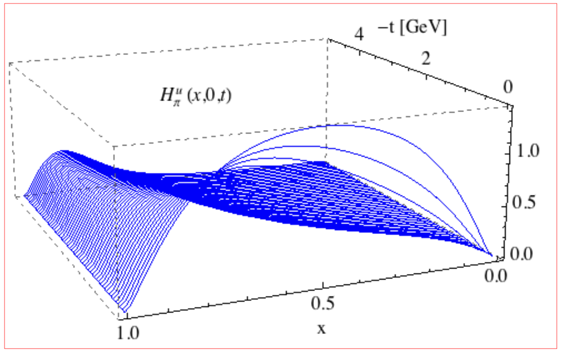

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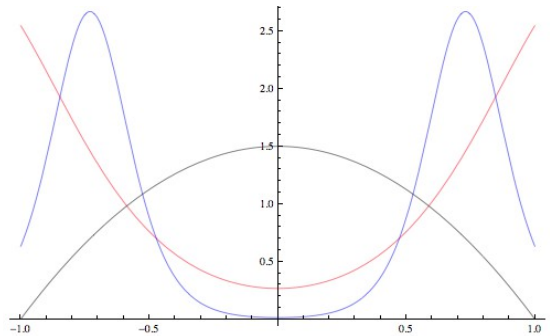

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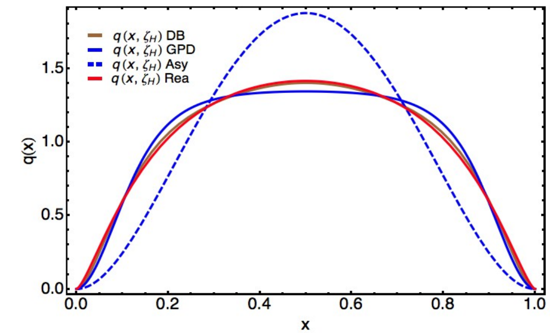

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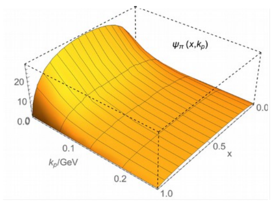

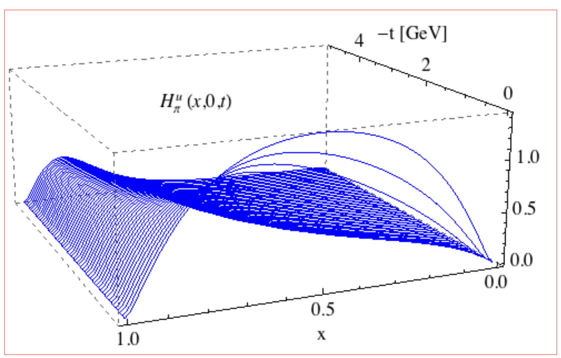

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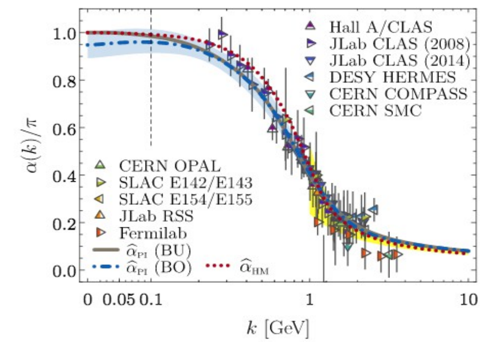

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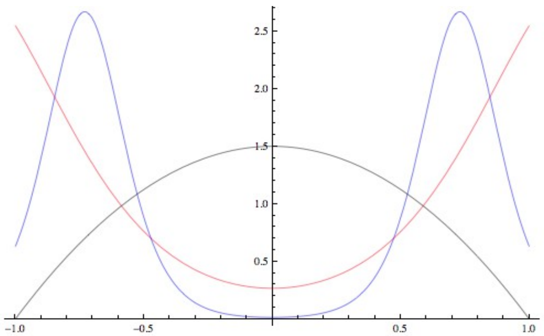

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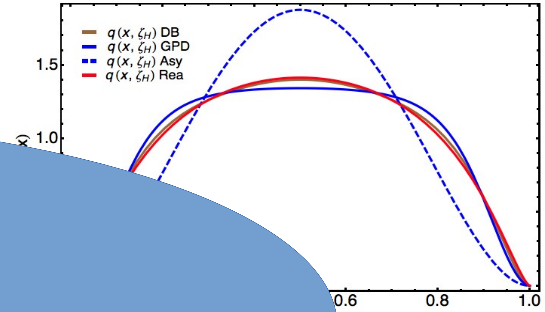

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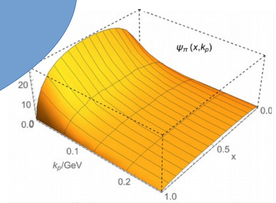
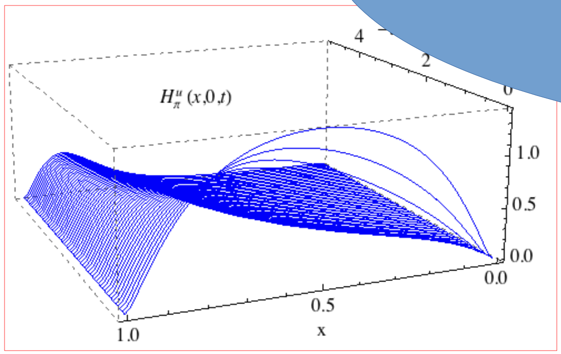
Thank you!!





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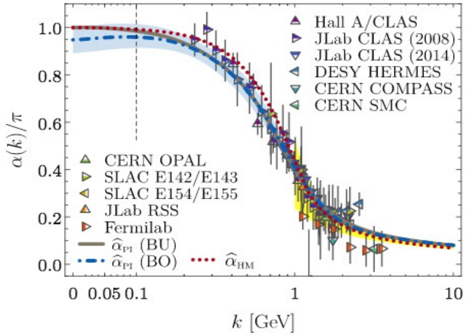

Backslides

PI-effective charge from lattice data with Nf=3 flavors at the physical point

Preliminary results:

$$\widehat{\alpha}_{\rm PI}(q^2) = \frac{\widehat{d}(q^2)}{\mathcal{D}(q^2)} \simeq \frac{\alpha_T(q^2)}{q^2\,[1 - L(q^2,\zeta^2)F(q^2,\zeta^2)]^2}\,\frac{m_0^2\Delta_F(0,\zeta^2)}{\Delta_F(q^2,\zeta^2)}$$

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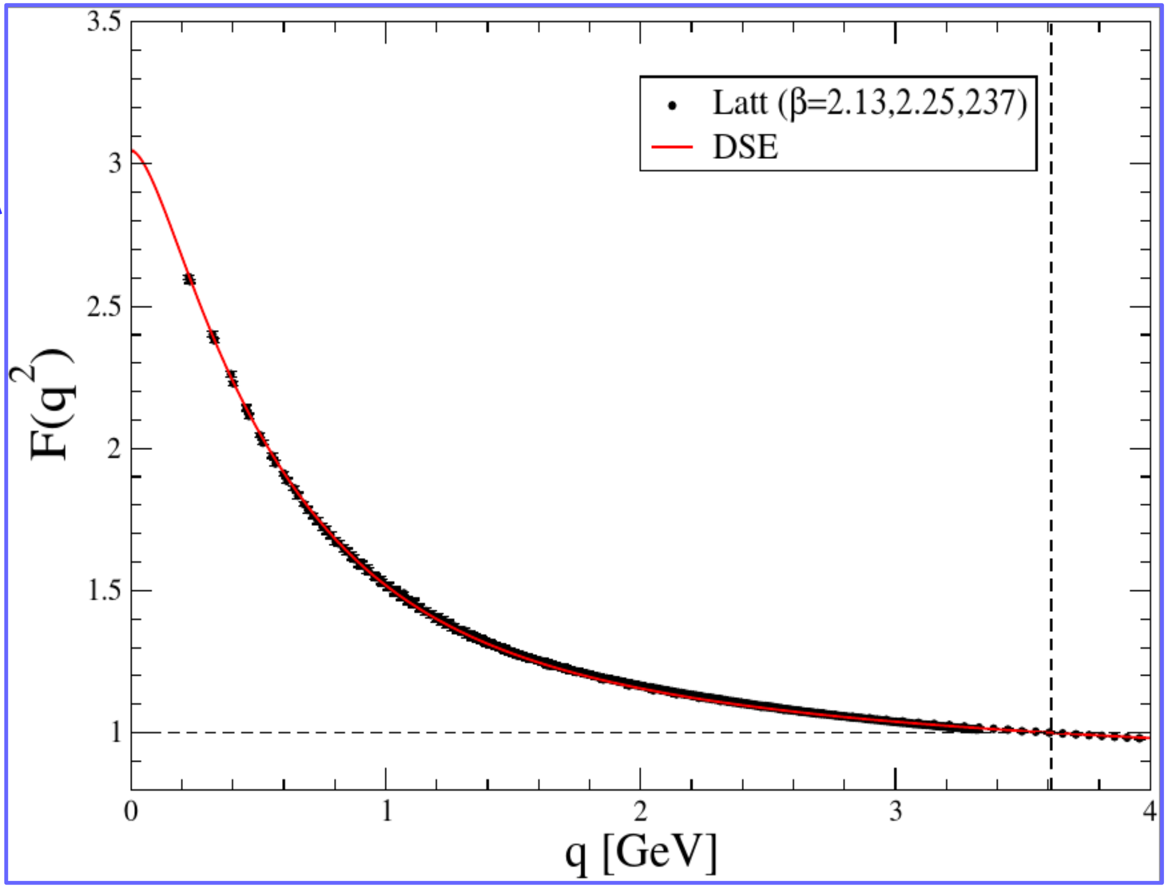
The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function



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The IR running of the PI effective charge with momenta only depends on:

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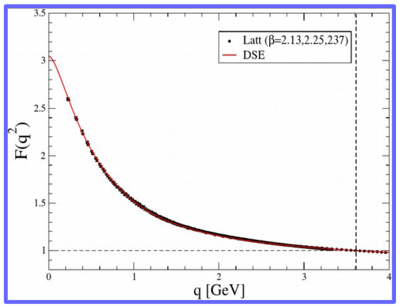

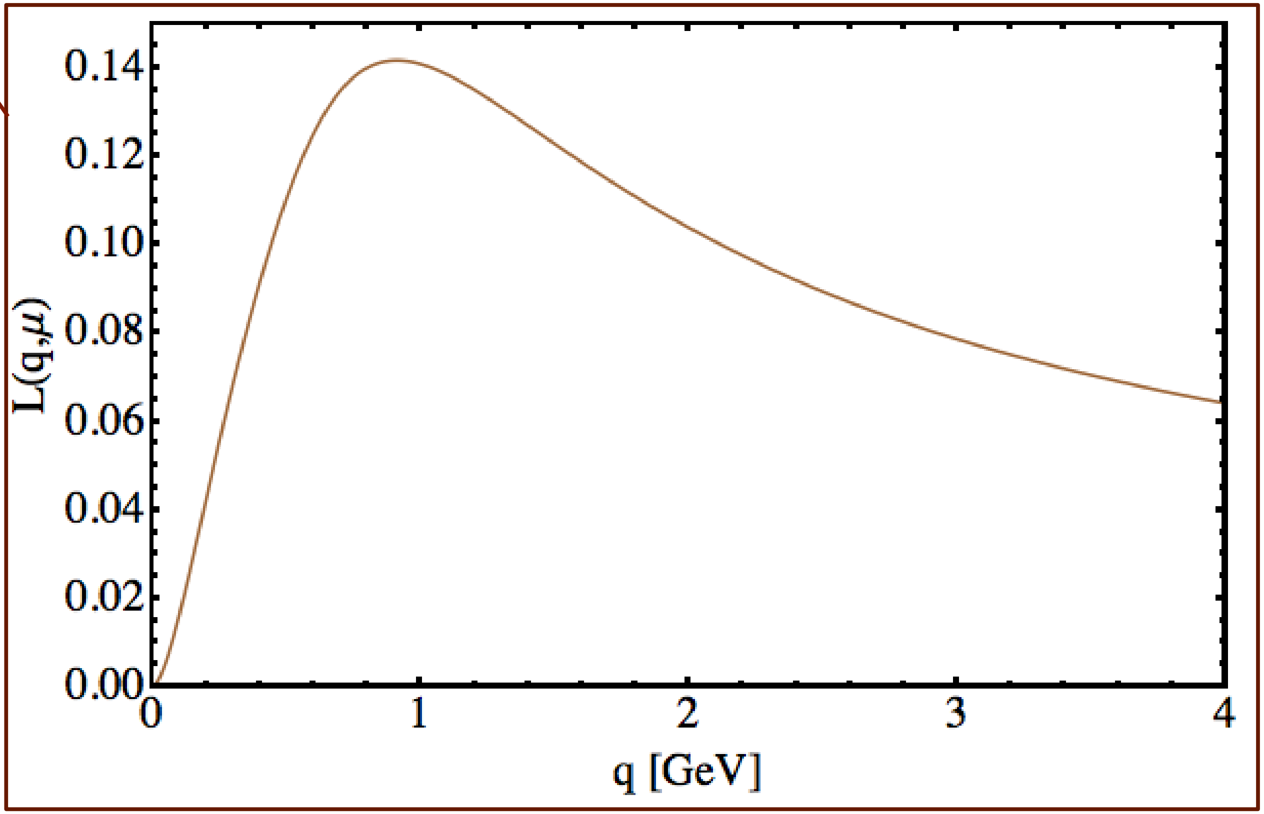

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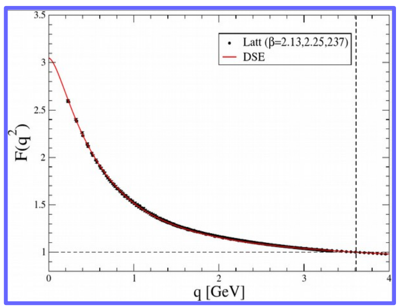

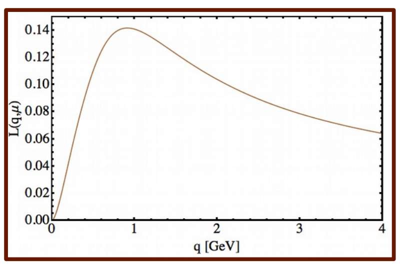

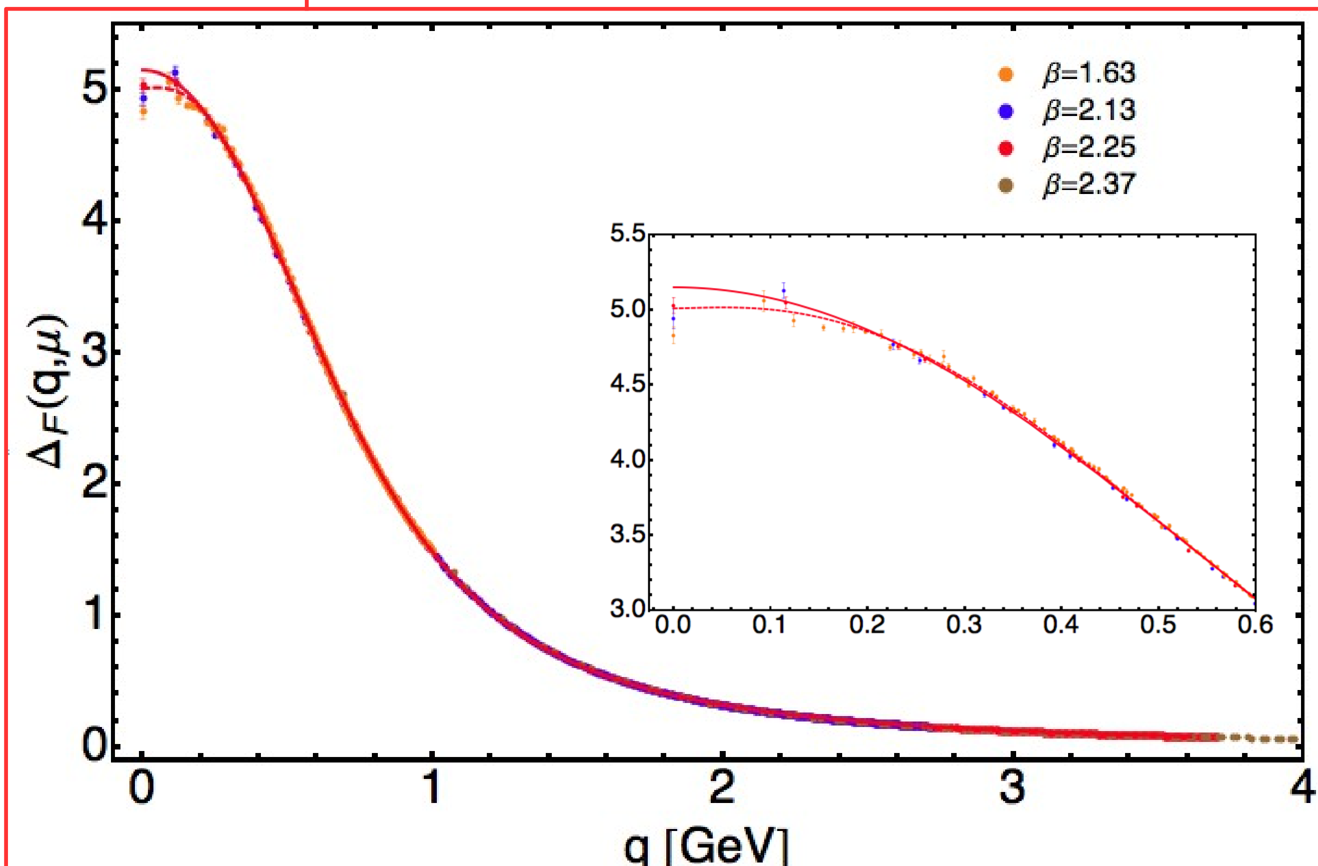

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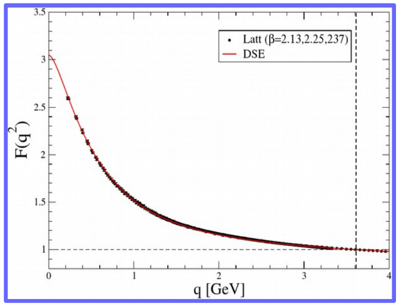

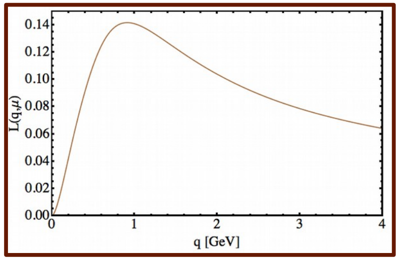

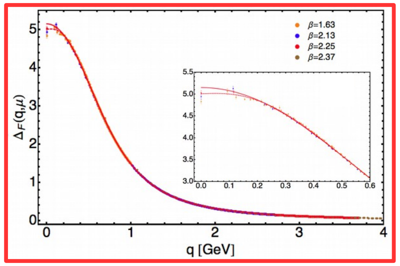

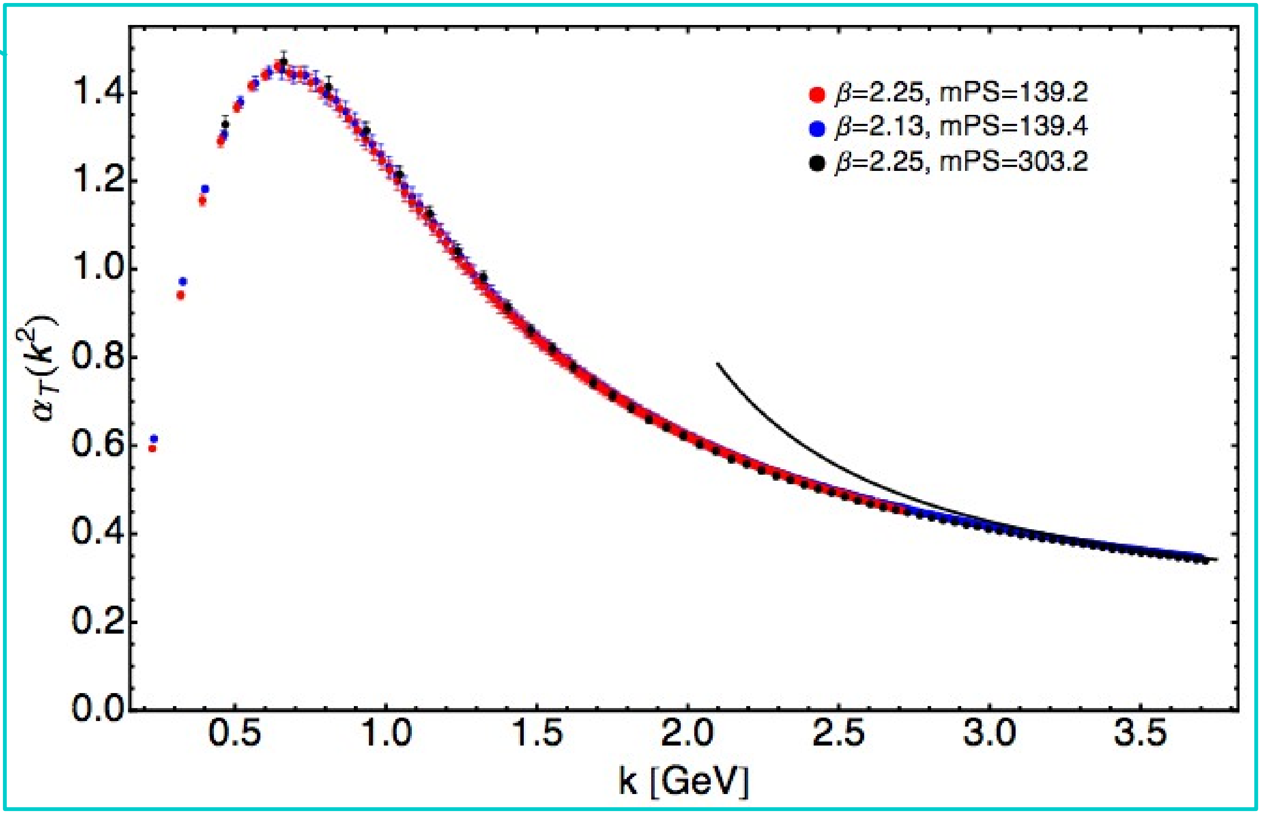

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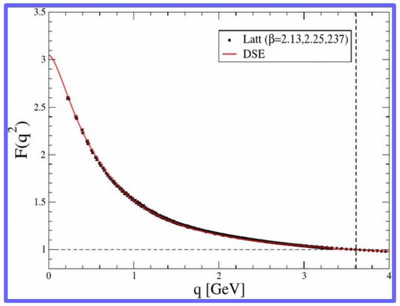
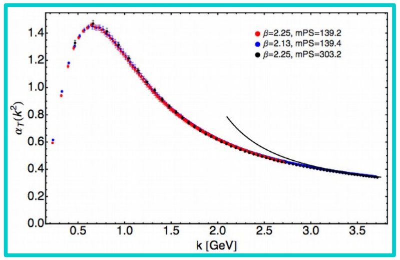
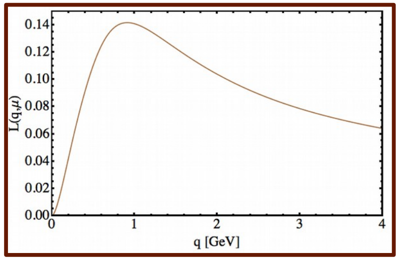
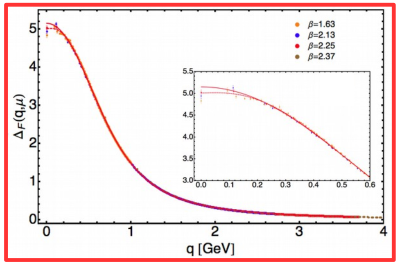


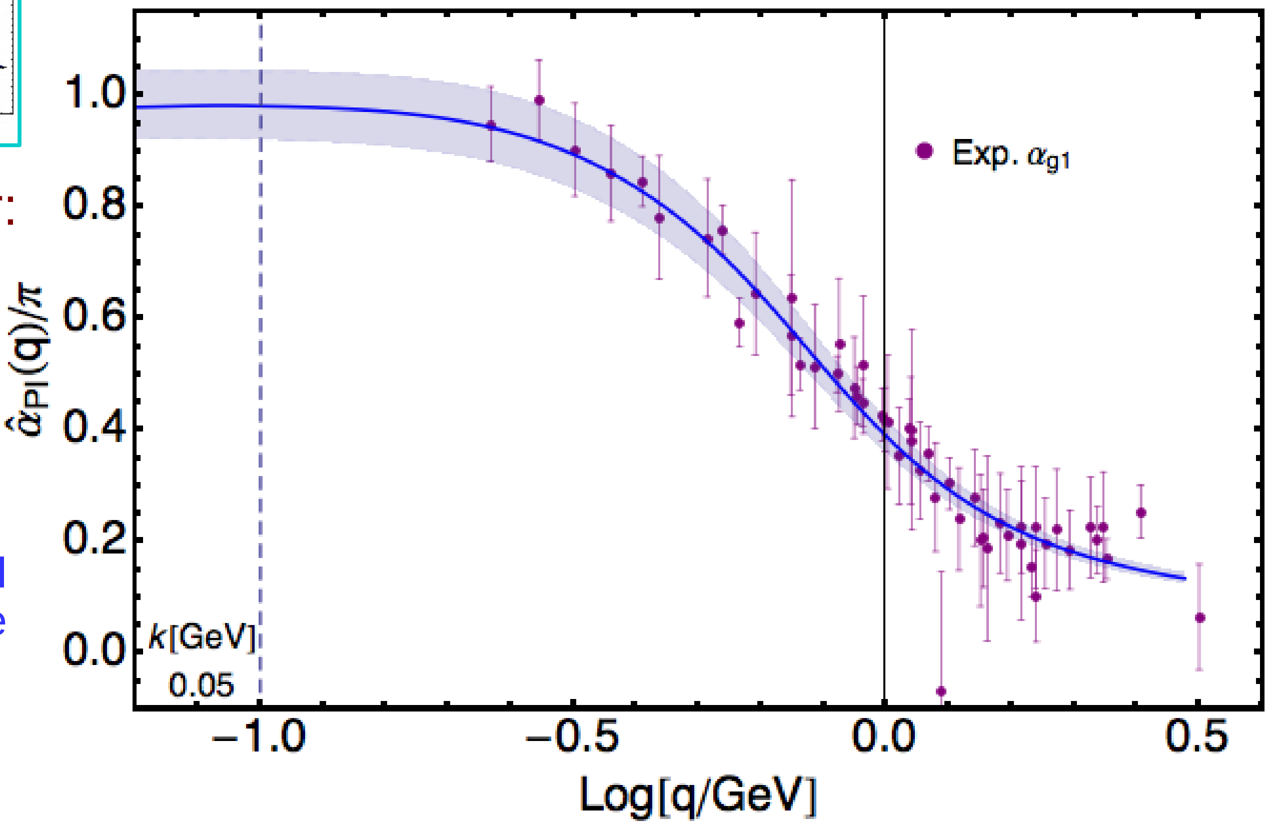






All put together:

Less uncertainties (that of the gluon mass is only left here) and still a better agreement with the world data for the experimental determination of the Bjorken sum-rule effective charge.



A word about bout gravitational Form Factors

A word about GPD polinomiality first:

- Express Mellin moments of GPDs as **matrix elements**:

$$\int_{-1}^{+1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \frac{1}{2(P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(i\overleftrightarrow{D}^+)^m q(0) \middle| P - \frac{\Delta}{2} \right\rangle$$

- Identify the **Lorentz structure** of the matrix element:

$$\text{linear combination of } (P^+)^{m+1-k}(\Delta^+)^k \text{ for } 0 \leq k \leq m+1$$

- Remember definition of **skewness** $\Delta^+ = -2\xi P^+$.
- Select **even powers** to implement time reversal.
- Obtain **polynomiality condition**:

$$\int_{-1}^{1} \mathrm{d}x\, x^m H^q(x,\xi,t) = \sum_{\substack{i=0 \\ \text{even}}}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t)\ .$$

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Definition and evaluation:

- Pion gravitational form factors are defined through*: Polinomiality!

$$J_{\pi^+}(-t,\xi) \equiv \int_{-1}^{1} dx\; x H_{\pi^+}(x,\xi,t) = \Theta_2(t) - \Theta_1(t)\xi^2 \;.$$

- Taking ξ=0 + isospin symmetric limit, one can readily compute:

$$\Theta_2(t) = \int_0^1 dx\; x[H^u_{\pi^+}(x,0,t) + H^d_{\pi^+}(x,0,t)] = \int_0^1 dx\; 2x H^u_{\pi^+}(x,0,t) \;.$$

- To obtain $\Theta_1(t)$, we need to take a non zero value of ξ; hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate $\Theta_1(t)$, by estimating the derivative of $J_{\pi^+}(-t,\xi)$ with respect to ξ^2 as:

$$D(\xi+\Delta/2) \equiv \frac{J(\xi+\Delta) - J(\xi)}{2(\xi+\Delta/2)\Delta} \;,\; \Delta \to 0 \,.$$

***Phys.Rev. D78 (2008) 094011.**

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$$\Rightarrow J(-t,\xi) = J^{\mathrm{ERBL}}(-t,\xi) + J^{\mathrm{DGLAP}}(-t,\xi)\ ,$$

- Notice that, because of the polinomiality of the *complete* GPD:

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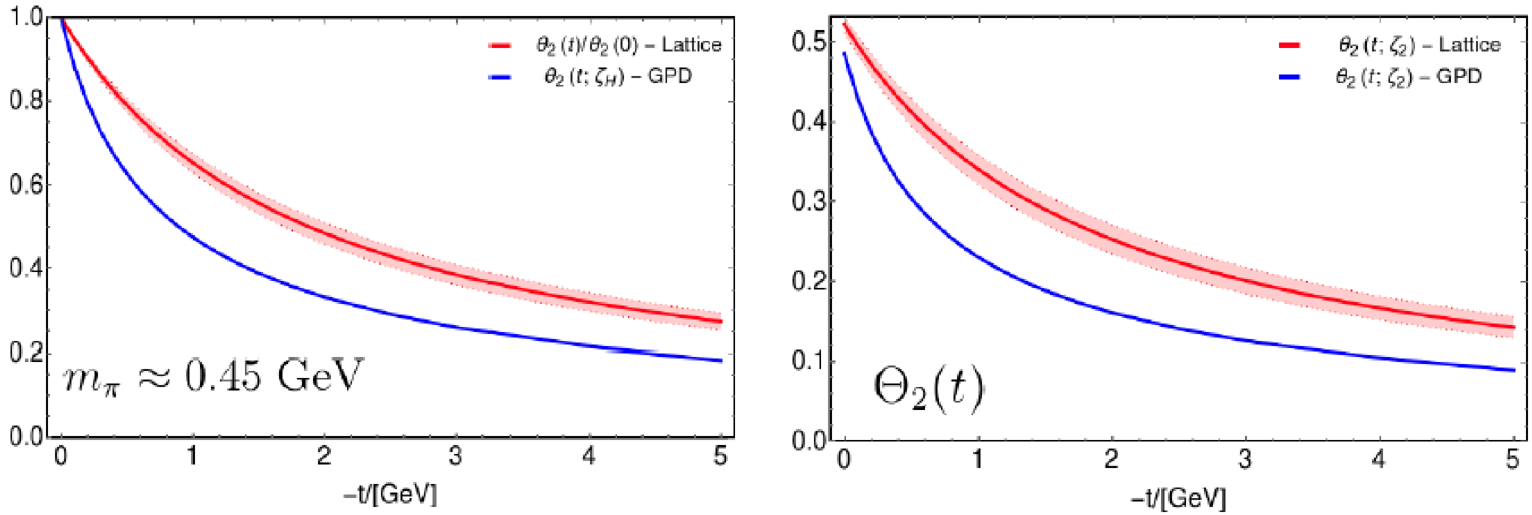

Lattice: (2007) Brömmel's dissertation. $\Theta_2(0)/2 = \langle x \rangle = 0.261(5)$

GPD + Ding et al. $\Theta_2(0)/2 = \langle x \rangle = 0.242(20)$

Latt.: D. Brommel, Ph.D. thesis, University of Regensburg, Regensburg, Germany (2007), DESY-THESIS-2007-023

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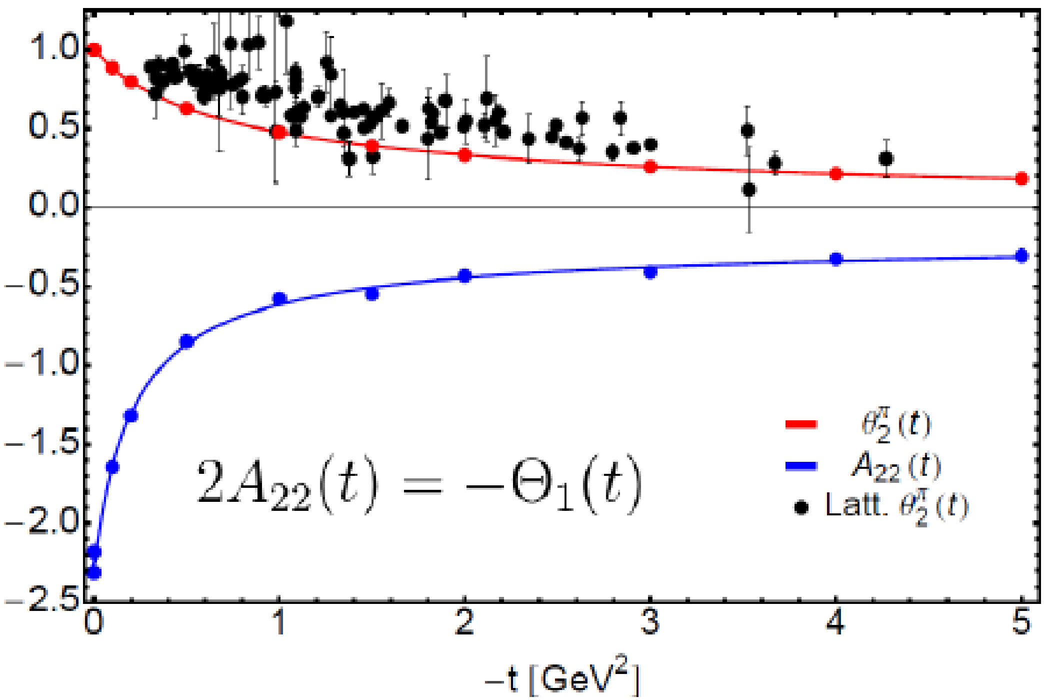

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