GPD modeling with Continuum QCD

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Meson GPD modeling with Continuum QCD

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This conference series began in 1970 at Duke University. The Seville meeting follows the successful events in Glasgow (2013) and Tallahassee (2016); and canvass similar themes. It will highlight the physics of baryons and related subjects in astro-, nuclear- and particle-physics, developing our understanding of the spectrum, structure and reactions of baryons using all available tools. Particular emphasis will be placed on elucidating the role of confinement and emergent mass, key non-perturbative phenomena within the Standard Model. Recent developments at existing facilities and those anticipated from the next-generation will also be showcased.

Main Topics Include:

- Spectroscopy of Light / Heavy Flavor Hadrons.
- Structure of Hadrons.
- Recent Approaches to Non-Perturbative QCD.
- Exotic Baryons.
- Hadrons at Finite Density and Temperature.
- Electromagnetic and Weak Interactions.
- Hadron-Hadron Interactions.
- New Facilities and Instrumentation.
Hadron Physics. General Motivation.

The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!

What happens down here?

\[ Q^2 \]

\[ \pi, \rho, \omega \ldots \]

\[ N, N^*, \Lambda, \Lambda^* \ldots \]

\[ 3q \text{-core} + MB \text{-cloud} \]
The QCD Holy Grail: the understanding of hadrons in terms of its elementary excitations; namely, quarks and gluons!

**Confinement**

Colored bound states have never been seen to exist as particles in nature

**Q^2**

What happens down here?

**DCSB**

Chiral symmetry appears dynamically violated in the Hadron spectrum

Emergent phenomena playing a dominant role in the real world dominated by the IR dynamics of QCD.
Antecedents:

GPD definition:

\[
H^q_{\pi}(x, \xi, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{i x P^+ z^-} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q} \left( -\frac{z}{2} \right) \gamma^+ q \left( \frac{z}{2} \right) \right| \pi, P - \frac{\Delta}{2} \right\rangle_{z^+ = 0, z_\perp = 0}
\]

with \( t = \Delta^2 \) and \( \xi = -\Delta^+/(2P^+) \).

References


- From isospin symmetry, all the information about pion GPD is encoded in \( H^u_{\pi^+} \) and \( H^d_{\pi^+} \).
- Further constraint from charge conjugation:
  \( H^u_{\pi^+}(x, \xi, t) = -H^d_{\pi^+}(-x, \xi, t) \).
Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

\[ \langle x^m \rangle_q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \middle| \bar{q}(0)\gamma^+(iD^+)^mq(0) \middle| \pi, P - \frac{\Delta}{2} \right\rangle \]

- Compute **Mellin moments** of the pion GPD \( H \).
Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

\[ \langle x^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0) \gamma^+ (iD^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle \]

- Compute **Mellin moments** of the pion GPD \( H \).
- Triangle diagram approx.
Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

\[ \langle \chi^m \rangle^q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0) \gamma^+ (i D^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle \]

- Compute *Mellin moments* of the pion GPD $H$.
- Triangle diagram approx.
- Resum *infinitely many* contributions.

Dyson - Schwinger equation

\[ -1 = -1 + \text{Diagram} \]
Antecedents:

GPDs in the Schwinger-Dyson and Bethe-Salpeter approach

\[
\langle x^m \rangle_q = \frac{1}{2(P^+)^{n+1}} \left\langle \pi, P + \frac{\Delta}{2} \left| \bar{q}(0) \gamma^+ (i D^+)^m q(0) \right| \pi, P - \frac{\Delta}{2} \right\rangle
\]

- Compute **Mellin moments** of the pion GPD $H$.
- Triangle diagram approx.
- Resum **infinitely many** contributions.

**Bethe-Salpeter equation**
Antecedents:

**GPD asymptotic algebraic model:**

- Expressions for vertices and propagators:

  \[ S(p) = \left[ -i \gamma \cdot p + M \right] \Delta_M(p^2) \]
  \[ \Delta_M(s) = \frac{1}{s + M^2} \]
  \[ \Gamma_\pi(k, p) = i \gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} dz \rho_\nu(z) \left[ \Delta_M(k_{+z}^2) \right]^{\nu} \]
  \[ \rho_\nu(z) = R_\nu (1 - z^2)^\nu \]

  with \( R_\nu \) a normalization factor and \( k_{+z} = k - p(1 - z)/2 \).


- Only two parameters:
  - Dimensionful parameter \( M \).
  - Dimensionless parameter \( \nu \). **Fixed to 1** to recover asymptotic pion DA.
Antecedents:

GPD asymptotic algebraic model:

\[ H'_{x \geq \xi}(x, \xi, 0) = \frac{48}{5} \left\{ \frac{3 \left( -2(x - 1)^4 \left( 2x^2 - 5\xi^2 + 3 \right) \log(1 - x) \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( +4\xi \left( 15x^2(x + 3) + (19x + 29)\xi^4 + 5(x(x(x + 11) + 21) + 3)\xi^2 \right) \tanh^{-1} \left( \frac{(x - 1)}{x - \xi^2} \right) \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( x^3(x(2(x - 4)x + 15) - 30) - 15(2x(x + 5) + 5)\xi^4 \right) \log \left( x^2 - \xi^2 \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( -5x(x(x + 2) + 36) + 18)\xi^2 - 15\xi^6 \right) \log \left( x^2 - \xi^2 \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( 2(x - 1) \left( (23x + 58)\xi^4 + (x(x + 67) + 112) + 6\xi^2 + x((5 - 2x)x + 15) + \xi^4 \right) \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( (15(2x(x + 5) + 5)\xi^4 + 10x(3x(x + 5) + 11)\xi^2 \right) \log \left( 1 - \xi^2 \right)}{20 \left( \xi^2 - 1 \right)^3} \right. \\
\left. + \frac{3 \left( 2x(5x(x + 2) - 6) + 15\xi^6 - 5\xi^2 + 3 \right) \log \left( 1 - \xi^2 \right)}{20 \left( \xi^2 - 1 \right)^3} \right\} \\
\]

Antecedents:

GPD asymptotic algebraic model (completion):

The full model:

\[
2(P \cdot n)^{m+1} \left< x^m \right>_u = \text{tr}_{CFD} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) i\gamma \cdot n S(k + \frac{\Delta}{2}) \\
\tau_- i\Gamma_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P),
\]

\[
2(P \cdot n)^{m+1} \left< x^m \right>_u = \text{tr}_{CFD} \int \frac{d^4 k}{(2\pi)^4} (k \cdot n)^m \tau_+ i\Gamma_\pi \left( \eta(k - P) + (1 - \eta) \left( k - \frac{\Delta}{2} \right), P - \frac{\Delta}{2} \right) \\
S(k - \frac{\Delta}{2}) \tau_- \frac{\partial}{\partial k} \Gamma_\pi \left( (1 - \eta) \left( k + \frac{\Delta}{2} \right) + \eta(k - P), P + \frac{\Delta}{2} \right) S(k - P)
\]

Antecedents:

GPD asymptotic algebraic model (completion):

Antecedents:

GPD asymptotic algebraic model (completion):

\[ q(x) = H^q(x, 0, 0) \]

PDF forward limit
Antecedents:

**GPD overlap approach:** The pion light front wave function

\[ |H; P, \lambda\rangle = \sum_{N, \beta} \int [dx]_N [d^2k_{\perp}]_N |\Psi_{N, \beta}^\lambda(\Omega)\rangle |N, \beta, k_1 \cdots k_N\rangle \]

\[ \Omega = (x_1, k_{\perp 1}, \cdots, x_N, k_{\perp N}) \]

\[ [dx]_N = \prod_{i=1}^N dx_i \delta \left( 1 - \sum_{i=1}^N x_i \right) , \]

\[ [d^2k_{\perp}]_N = \frac{1}{(16\pi^3)^{N-1}} \prod_{i=1}^N d^2k_{\perp i} \delta^2 \left( \sum_{i=1}^N k_{\perp i} - P_{\perp} \right) \]

\[ \sum_{N, \beta} \int [dx]_N [d^2k_{\perp}]_N |\Psi_{N, \beta}^\lambda(\Omega)|^2 = 1. \]

Let's consider the two-body pion LCWF:

\[ |\pi^+, P\rangle_{2\text{-body}}^{\uparrow\downarrow} = \int \frac{d^2k_{\perp}}{(2\pi)^3} \frac{dx}{\sqrt{x(1-x)}} \psi_{\uparrow\downarrow}(k^+, k_{\perp}) \left[ b_{u\uparrow}^\dagger(x, k_{\perp}) d_{d\downarrow}^\dagger(1-x, -k_{\perp}) + b_{u\downarrow}^\dagger(x, k_{\perp}) d_{d\uparrow}^\dagger(1-x, -k_{\perp}) \right] |0\rangle, \]

\[ \Gamma_\pi(k, P) = S^{-1}(-k_2) \chi(k, P) S^{-1}(k_1). \]

\[ 2P^+ \psi_{\uparrow\downarrow}(k^+, k_{\perp}) = \int \frac{dk^-}{2\pi} \text{Tr} \left[ \gamma^+ \gamma_5 \chi(k, P) \right] \]
Antecedents:

**GPD overlap approach:** The pion light front wave function

\[ 2P^+ \Psi_{\uparrow\downarrow}^+(k^+, k_\perp) = \int \frac{dk^-}{2\pi} \text{Tr} \left[ \gamma^+ \gamma_5 \chi(k, P) \right] \]

- Expressions for vertices and propagators:
  \[ S(p) = \left[ -i\gamma \cdot P + M \right] \Delta_M(p^2) \]
  \[ \Delta_M(s) = \frac{1}{s + M^2} \]
  \[ \Gamma_\pi(k, p) = i\gamma_5 \frac{M}{f_\pi} M^{2\nu} \int_{-1}^{+1} dz \rho_\nu(z) \left[ \Delta_M(k_z^2) \right]^\nu \]
  \[ \rho_\nu(z) = R_\nu (1 - z^2)^\nu \]

with \( R_\nu \) a normalization factor and \( k_{+z} = k - p(1 - z)/2 \).


Keeping so contact with the previous "covariant" approach" based on DSE and BSE.

\[ \Gamma_\pi(k, P) = S^{-1}(-k_2) \chi(k, P) S^{-1}(k_1). \]
Antecedents:

**GPD overlap approach:**

Helicity-0 two-body pion LCWF:

\[
\Psi_{\uparrow \downarrow}(x, \mathbf{k}\perp) = -\frac{\Gamma(v + 1)}{\Gamma(v + 2)} \frac{M^{2v+1}R_v}{[k\perp^2 + M^2]^{v+1}} x^v (1 - x)^v.
\]

In DGLAP kinematics: \( \zeta \leq x \leq 1 \)

GPD in the overlap approach:

\[
H(x, \xi, t) = \sqrt{2} \sum_{N,N'} \sum_{\beta, \beta'} \int [d\tilde{x}'_{N'}]_{N'} [d^2 \tilde{k}'\perp]_{N'} [d\tilde{x}]_N [d^2 \tilde{k}\perp]_N \psi_{N', \beta'}^{\ast} (\hat{\Omega}') \psi_{N, \beta} (\hat{\tilde{\Omega})} \\
\times \int \frac{d^2 z^-}{2\pi} e^{iP^+ z^-} \langle N', \beta, k_1' \cdots k_N' \mid \phi^{q\ast} \left( \frac{z}{2} \right) \phi^{q} \left( \frac{z}{2} \right) \mid N, \beta, k_1 \cdots k_N \rangle \\
= \sum_N \sqrt{1 - \xi}^{2-N} \sqrt{1 + \xi}^{2-N} \sum_{\beta=\beta'} \sum_j \delta_{s,j,q} \\
\times \int [d\tilde{x}]_N [d^2 \tilde{k}\perp]_N \delta(x - \tilde{x}_j) \psi_{N, \beta}^{\ast} (\hat{\Omega}) \psi_{N, \beta} (\hat{\tilde{\Omega}}) \\
= \int [d\tilde{x}]_2 [d^2 \tilde{k}\perp]_2 \delta(x - \tilde{x}_j) \psi_{\uparrow \downarrow}^{\ast} (\hat{\Omega}') \psi_{\uparrow \downarrow} (\hat{\tilde{\Omega}}) \\
+ \text{Helicity-1 component} \\
= \frac{\Gamma(2v + 2)}{\Gamma(v + 2)^2} \int du dv u^v v^v \delta(1 - u - v) \frac{(2M^{2v+1}R_v)^2}{(t uv (1-x)^2 + M^2)^{2v+1}} \\
\times \hat{x}^v (1 - \hat{x})^v \tilde{x}^v (1 - \tilde{x})^v.
\]

In the pion 2-body case:

\[ \frac{x - \zeta}{1 - \zeta} \quad \frac{x + \zeta}{1 + \zeta} \]
Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

\[
\Psi_{\uparrow \downarrow}(x, k_\perp) = -\frac{\Gamma(v + 1)}{\Gamma(v + 2)} \frac{M^{2v+1} R_v}{[k_\perp^2 + M^2]^{v+1}} x^v (1-x)^v.
\]

GPD in the overlap approach:

\[
H(x, \xi, t) = \frac{\Gamma(2v + 2)}{\Gamma(v + 2)^2} \int du dv u^v v^v \delta(1-u-v) \frac{(2M^{2v} R_v)^2 \hat{x}^v (1-\hat{x})^v \check{x}^v (1-\check{x})^v}{\left(t uv \frac{(1-x)^2}{1-\xi^2} + M^2\right)^{2v+1}}, \quad \xi \leq x \leq 1
\]

\[
= \frac{30}{(1-x)^2(1-\xi^2)^2} \frac{1}{(1+z)^2} \left(\frac{3}{4} + \frac{1}{4} \frac{1-2z}{1+z} \arctanh \sqrt{\frac{z}{1+z}}\right)
\]

\[
z = \frac{t}{4M^2} \frac{(1-x)^2}{1-\xi^2}
\]

Encoding the correlations of kinematical variables

N. Chouika et al., PLB780(2018)287
Antecedents:

GPD overlap approach:

Helicity-0 two-body pion LCWF:

\[ \Psi_{\uparrow \downarrow}(x, k_\perp) = -\frac{\Gamma(v + 1)}{\Gamma(v + 2)} \frac{M^{2v+1} R^v}{[k_\perp^2 + M^2]^v} x^v (1-x)^v. \]

GPD in the overlap approach:

\[
H(x, \xi, t) = 30 \frac{(1-x)^2(1-\xi^2)}{(1-\xi^2)^2} \frac{1}{(1+z)^2} \left[ \frac{3}{4} + \frac{1}{4} \frac{1-2z}{1+z} \frac{\text{arctan} \left( \frac{z}{\sqrt{1+z}} \right)}{\sqrt{1+z}} \right].
\]

Forward limit

\[ 0 \leq x \leq 1 \]

PDF:

\[ H(x, 0, 0) = q(x) = 30 x^2 (1-x)^2 \]

Consists numerically very well with the results obtained from the Triangle diagram!!!

N. Chouika et al., PLB780(2018)287
Pion (kaon maybe) realistic picture:

- The pseudoscalar LFWF can be written:
  \[ f_K \psi_{K}^{\uparrow\downarrow}(x, k_{\perp}^2) = \text{tr}_{CD} \int_{dk_{\parallel}} \delta(n \cdot k - xn \cdot P_K) \gamma_5 \gamma_\cdot n \chi_{K}^{(2)}(k_{\perp}; P_K). \]

- The moments of the distribution are given by:
  \[
  < x^m >_{\psi_{K}^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_{K}^{\uparrow\downarrow}(x, k_{\perp}^2) = \frac{1}{f_K n \cdot P} \int_{dk_{\parallel}} \left[ \frac{n \cdot k}{n \cdot P} \right]^m \gamma_5 \gamma \cdot n \chi_{K}^{(2)}(k_{\perp}; P_K)
  \]

  \[
  \int_0^1 d\alpha \alpha^m \left[ \frac{12}{f_K} \mathcal{V}_K(\alpha; \sigma^2) \right], \quad \mathcal{V}_K(\alpha; \sigma^2) = [M_u(1 - \alpha) + M_s \alpha] \mathcal{X}(\alpha; \sigma^2^2). \]

Uniqueness of Mellin moments

\[
\psi_{K}^{\uparrow\downarrow}(x, k_{\perp}^2) = \frac{12}{f_K} \mathcal{V}_K(x; \sigma^2^2)
\]

\[
\chi_{K}(\alpha; \sigma^3) = \left[ \int_{-1}^{1-2\alpha} d\omega \int_{1+2\alpha}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{e^{-1+2\alpha}}^{1} dv \right] \frac{\rho_K(\omega) \Lambda_K^2}{n_K \sigma^3^2}.
\]

The spectral density \(\rho_K(z)\) can be modelled...

...Or taken with BSE solutions as an input!

\[
\Rightarrow \psi_{K}^{\uparrow\downarrow}(x, k_{\perp}^2) \sim \int d\omega \cdots \rho_K(\omega) \cdots
\]
Pion realistic picture:

- Spectral density is chosen as:
  \[ u_G \rho_G(\omega) = \frac{1}{2b_0^G} \left[ \sech^2 \left( \frac{\omega - \omega_0^G}{2b_0^G} \right) + \sech^2 \left( \frac{\omega + \omega_0^G}{2b_0^G} \right) \right] \]

Phenomelogical model: \( b_0^\pi = 0.1, b_0^\pi = 0.73 \);

Asymptotic case: \( \rho(\omega; \nu) \sim (1 - \omega^2)^\nu \)
Pion realistic picture:

GPD overlap representation:

\[ H^q_{M}(x, \xi, t) = \int \frac{d^2 k_{\perp}}{16 \pi^3} \Psi^*_{uf} \left( \frac{x - \xi}{1 - \xi} k_{\perp} + \frac{1 - x \Delta_{\perp}}{2} \right) \Psi_{uf} \left( \frac{x + \xi}{1 + \xi} k_{\perp} - \frac{1 - x \Delta_{\perp}}{2} \right) \]
Pion realistic picture: PDF as benchmark

GPD overlap representation: forward limit

\[ H^q_{M} (x, \xi, t) = \int \frac{d^2k_\perp}{16 \pi^3} \Psi^*_{uf} \left( \frac{x - \xi}{1 - \xi}, k_\perp + \frac{1 - x}{1 - \xi} \frac{\Delta_\perp}{2} \right) \Psi_{uf} \left( \frac{x + \xi}{1 + \xi}, k_\perp - \frac{1 - x}{1 + \xi} \frac{\Delta_\perp}{2} \right) \]
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator that, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale.

\[
q^{\pi}(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \left| \bar{\psi}^q(-z)\gamma^+ \psi^q(z) \right| P \right\rangle_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{uj}(x, k_\perp) \Psi_{uj}(x, k_\perp)
\]
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale.

\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \left| \bar{\psi}^q(-z)\gamma^+\psi^q(z) \right| P \right\rangle \mid_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{uJ}(x, k_\perp) \Psi_{uJ}(x, k_\perp) \]

LFWF leading to asymptotic PDAs

\[ q_{sf}(x) \approx 30 x^2 (1 - x)^2 \]

The more realistic pion 2-body LFWF

DCSB-induced hardening
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale.

\[
q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^{-}}{2\pi} e^{ixPz^{-}} \left\langle P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+ = 0, z_\perp = 0} = \int \frac{d^2k_\perp}{16\pi^3} \frac{\Psi_{uJ}^*(x, k_\perp)}{\Psi_{uJ}^*(x, k_\perp)}
\]

LFWF leading to asymptotic PDAs

\[q_{sf}(x) \approx 30x^2(1-x)^2\]

A more realistic pion 2-body LFWF

DCSB-induced hardening

Direct computation of Mellin moments:

\[
(x^m)_H^\pi = \int_0^1 dx x^m q^\pi(x; \zeta_H) = \frac{N_c}{n \cdot P} \text{tr} \int \frac{dk}{n \cdot P} \Gamma_{<}(k_{\eta}P) S(k_{\eta}) n \cdot \partial_{k_{\eta}} [\Gamma_{<}(k_{\eta}, -P) S(k_{\eta})]
\]

\[
x \cdot q^\pi(x; \zeta_H) = 213.32x^2(1-x)^2 \\
\times [1 - 2.9342\sqrt{x(1-x)} + 2.2911x(1-x)]
\]
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale:

\[
q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP+z^-} \left\langle P \left| \bar{\psi}^u(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+=0, z^\perp = 0} = \int \frac{d^2k_\perp}{16\pi^3} \Psi^*_{uJ}(x, k_\perp) \Psi_{uJ}(x, k_\perp)
\]

LFWF leading to asymptotic PDAs

\[q_{sf}(x) \approx 30 x^2 (1 - x)^2\]

A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

\[
(x^m)^\pi_{\zeta_H} = \int_0^1 dx x^n q^\pi(x; \zeta_H)
\]

\[
= \frac{N_c}{n \cdot P} \text{tr} \left( \frac{n \cdot k_\eta}{n \cdot P} \right)^m \Gamma_{\pi}(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma_{\pi}(k_\eta, -P) S(k_\eta)]
\]

\[
xq^\pi(x; \zeta_H) = 213.32 x^2 (1 - x)^2 \times [1 - 2.9342 \sqrt{x(1 - x)} + 2.2911 x(1 - x)]
\]
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale:

\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^+}{2\pi} e^{ix^+z^+} \left( P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right) \bigg|_{z^+ = 0, z^z = 0} = \int \frac{d^2k}{16\pi^3} \bar{\Psi}^*_{u\bar{f}}(x, k) \Psi_{u\bar{f}}^J(x, k) \]

LFWF leading to asymptotic PDAs:

\[ q_{sf}(x) \approx 30 x^2 (1 - x)^2 \]

A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

\[ \langle x^m \rangle^\pi_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H) \]

\[ = \frac{N_c}{n \cdot P} \text{tr} \int dk \left[ \frac{n \cdot k}{n \cdot P} \right]^m \Gamma_\pi(k_{\eta}, P) S(k_{\eta}) n \cdot \partial_{k_{\eta}} [\Gamma_\pi(k_{\eta}, -P) S(k_{\eta})] \]

\[ \times 213.32 x^2 (1 - x)^2 \]

\[ \times [1 - 2.9342 \sqrt{x(1 - x)} + 2.2911 x(1 - x)] \]
Pion realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale

\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixPz^-} \left\langle P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right\rangle \bigg|_{z^+=0, z_-=0} = \int \frac{d^2 k_\perp}{16\pi^3} \Psi^*_{uj}(x, k_\perp) \Psi_{uj}(x, k_\perp) \]

LFWF leading to asymptotic PDAs

\[ \zeta_H \rightarrow \zeta_2 = 5.2 \text{ GeV} \]

\[ q_{sf}(x) \approx 30 x^2 (1 - x)^2 \]

A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

\[ (x^m)^{\pi}_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H) = \frac{N_c}{n \cdot P} \text{tr} \int_{dk} \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot k_\eta [\Gamma_\pi(k_\eta, -P) S(k_\eta)] \]

\[ q^\pi(x; \zeta_H) = 213.32 x^2 (1 - x)^2 \times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)] \]
Pion realistic picture: DGLAP evolution

\[ M_n(t) = \int_0^1 dx \, x^n q(x,t) \]

\[ t = \ln \left( \frac{\zeta_2}{\zeta_0} \right) \]

Moments' evolution (1-loop):

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots \]
A master equation for the (1-loop) moments' evolution:

\[
\frac{d}{dt} q(x, t) = \frac{\alpha(t)}{4\pi} \int_{x}^{1} dy \frac{y}{y} q(y, t) P\left(\frac{x}{y}\right) + \ldots
\]

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\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots
\]

\[
M_n(t) = \int_{0}^{1} dx x^n q(x, t)
\]

\[
t = \ln \left(\frac{\zeta_2^2}{\zeta_0^2}\right)
\]

\[
-\int_{0}^{1} dx P(x) = \gamma_0^n
\]
A master equation for the (1-loop) moments' evolution:

\[ \frac{d}{dt} q(x, t) = \frac{\alpha(t)}{4\pi} \int_0^1 dy \frac{q(y, t) P\left( \frac{x}{y} \right)}{y} + \ldots \]

Moments' evolution (1-loop):

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots \]

\[ M_n(t) = \int_0^1 dx x^n q(x, t) \]

\[ t = \ln \left( \frac{\zeta_2}{\zeta_2^0} \right) \]

\[ -\int_0^1 dx P(x) = \gamma_0^n \]

\[ P(x) = \frac{8}{3} \left( \frac{1+z^2}{(1-x)_+} + \frac{3}{2} \delta(x-1) \right) \]

\[ \gamma_n = -\frac{4}{3} \left( 3 + \frac{2}{(n+2)(n+3)} - 4 \sum_{i=1}^{n+1} \frac{1}{i} \right) \]
A master equation for the (1-loop) moments' evolution:

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\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) + \ldots
\]

\[
\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots
\]

\[
\alpha(t) = \frac{4\pi}{\beta_0(t - t_{\Lambda})} + \ldots
\]

\[
t_{\Lambda} = \ln\left(\frac{\Lambda^2}{\zeta_0^2}\right)
\]

\[
M_n(t) = \int_{0}^{1} dx x^n q(x, t)
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t = \ln\left(\frac{\zeta^2}{\zeta_0^2}\right)
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\[
\zeta_0 \rightarrow \zeta_0 = 5.2 \text{ GeV}
\]
A master equation for the (1-loop) moments' evolution:

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\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots
\]

\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots
\]

\[
t_\Lambda = \ln\left(\frac{\Lambda^2}{\xi_0^2}\right)
\]

\[
M_n(t) = M_n(t_0) \left(\frac{\alpha(t)}{\alpha(t_0)}\right)^{\gamma_0^n/\beta_0}
\]

\[
M_n(t) = \int_0^1 dx x^n q(x, t)
\]

\[
t = \ln\left(\frac{\xi^2}{\xi_0^2}\right)
\]

\[
\gamma_0^n = -\frac{4}{3} \left(3 + \frac{2}{(n+2)(n+3)} - 4 \sum_{i=1}^{n+1} \frac{1}{i}\right)
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\[
\xi_0 \rightarrow \xi_0 = 5.2 \text{ GeV}
\]
Pion realistic picture: DGLAP evolution

Which value of Lambda?

\[ \alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + ... = \frac{4\pi}{\beta_0 \ln \left( \frac{\xi^2}{\Lambda^2} \right)} + ... \]
Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

\[ \alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\xi^2}{\Lambda^2}\right)} + \ldots \]

\[ \ln\left(\frac{\Lambda^2}{\Lambda'}^2\right) = \frac{4\pi}{\beta_0} \left( \frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)} \right) + \ldots = \frac{4\pi c}{\beta_0} \]

Pion realistic picture: DGLAP evolution
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\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0} \ln\left(\frac{\zeta^2}{\Lambda^2}\right) + \ldots
\]

\[
\ln\left(\frac{\Lambda^2}{\Lambda^2}\right) = \frac{4\pi}{\beta_0} \left(\frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}
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The evolution will thus depend on the scheme via the perturbative truncation

\[
\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots
\]
Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots
\]

\[
\ln\left(\frac{\Lambda^2}{\overline{\Lambda}^2}\right) = \frac{4\pi}{\beta_0} \left(\frac{1}{\alpha(t)} - \frac{1}{\overline{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}
\]

\[
\frac{d}{dt} M_n(t) = -\frac{\overline{\alpha(t)}}{4\pi} \gamma_0^n M_n(t) + \ldots
\]

\[
\frac{d}{dt} \overline{\alpha}(t) = -\frac{\overline{\alpha}^2(t)}{4\pi} \beta_0 + \ldots
\]

The evolution will thus depend on the scheme via the perturbative truncation and the usual prejudice is that truncation errors are optimally small in MS scheme.

PDG2018: [PRD98(2018)030001]

\[
\Lambda^{(5)}_{MS} = (210 \pm 14) \text{ MeV,} \quad (9.24b)
\]

\[
\Lambda^{(4)}_{MS} = (292 \pm 16) \text{ MeV,} \quad (9.24c)
\]

\[
\Lambda^{(3)}_{MS} = (332 \pm 17) \text{ MeV,} \quad (9.24d)
\]
Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[
(q^m_{\zeta_H})_{\zeta_H} = \int_0^1 dx x^m \bar{q}^\zeta(x; \zeta_H) \\
= \frac{N_c}{n \cdot P} \text{tr} \int \frac{dk}{n \cdot P} [n \cdot k_\eta \Gamma(\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma(\eta, P) S(k_\eta)]]
\]

\[q^0(x; \zeta_H) = 213.32 x^2 (1-x)^2 \times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]\]

\[\zeta_H \rightarrow \zeta_2 = 5.2 \text{ GeV}\]

Optimal best-fitting parameters:
\[\Lambda_{QCD} = 0.234 \text{ GeV} ; \quad \zeta_H = 0.349 \text{ GeV} .\]
Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[
\langle x^m \rangle_{\zeta_H} \equiv \int_0^1 dx \, x^m q^\pi(x; \zeta_H) = \frac{N_c}{n \cdot P} \text{tr} \int dk \left[ \frac{n \cdot k_{\eta}}{n \cdot P} \right]^m \Gamma_{\pi}(k_{\eta}, P) S(k_{\eta}) n \cdot \delta_{k_{\eta}} \left[ \Gamma_{\pi}(k_{\eta}, -P) S(k_{\eta}) \right]
\]

\[q^\pi(x; \zeta_H) = 213.32 x^2 (1-x)^2 \times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]\]

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**Comparison with the three first moments obtained from lQCD**

<table>
<thead>
<tr>
<th>(\zeta_2)</th>
<th>\langle x \rangle_{u}^{\pi}</th>
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<th>\langle x^3 \rangle_{u}^{\pi}</th>
</tr>
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<tbody>
<tr>
<td>Ref. [33]</td>
<td>0.24(2)</td>
<td>0.09(3)</td>
<td>0.053(15)</td>
</tr>
<tr>
<td>Ref. [34]</td>
<td>0.27(1)</td>
<td>0.13(1)</td>
<td>0.074(10)</td>
</tr>
<tr>
<td>Ref. [35]</td>
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<td>0.16(3)</td>
<td></td>
</tr>
<tr>
<td><strong>average</strong></td>
<td><strong>0.24(2)</strong></td>
<td><strong>0.13(4)</strong></td>
<td><strong>0.064(18)</strong></td>
</tr>
<tr>
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\[
\langle x^m \rangle_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H)
\]

\[
= \frac{N_c}{n \cdot P} \text{tr} \int_0^1 \left[ \frac{n \cdot k_{\eta}}{n \cdot P} \right]^m \Gamma_\pi(k_{\eta}, P) S(k_{\eta}) n \cdot \delta_{k_{\eta}} [\Gamma_\pi(k_{\eta}, -P) S(k_{\eta})]
\]

\[
q^\pi(x; \zeta_H) = 213.32 x^2 (1 - x)^2 
\times [1 - 2.9342 \sqrt{x(1 - x)} + 2.2911 x(1 - x)]
\]

\[\zeta_H \rightarrow \zeta_L = 2 \text{ GeV} \rightarrow \zeta_2 = 5.2 \text{ GeV}\]

Optimal best-fitting parameters:

\[\Lambda_{QCD} = 0.234 \text{ GeV} ; \quad \zeta_H = 0.349 \text{ GeV} .\]

\[\Lambda_{QCD} = 0.234 \text{ GeV} ; \quad \zeta_H = 0.374 \text{ GeV} .\]

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Matching the three first moments obtained from IQCD
Pion realistic picture: DGLAP evolution

Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\zeta^2}{\Lambda^2}\right)} + \ldots \quad \alpha(t)=\bar{\alpha}(t)(1 + c \bar{\alpha}(t)+\ldots)
\]

\[
\ln\left(\frac{\Lambda^2}{\bar{\Lambda}^2}\right) = \frac{4\pi}{\beta_0} \left(\frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}
\]

\[
\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma^0 M_n(t) + \ldots
\]

\[
\frac{d}{dt} \alpha(t) = -\frac{\alpha^2(t)}{4\pi} \beta_0 + \ldots
\]

The evolution will thus depend on the scheme via the perturbative truncation

The use of \( \Lambda=0.234 \) GeV can be thus interpreted as the choice of particular scheme, differing from MS.
Which value of Lambda? It depends on the scheme... Indeed, at the one-loop level, its value defines by itself the scheme!!!

\[
\alpha(t) = \frac{4\pi}{\beta_0(t-t_\Lambda)} + \ldots = \frac{4\pi}{\beta_0 \ln\left(\frac{\xi^2}{\Lambda^2}\right)} + \ldots
\]

\[
\ln\left(\frac{\Lambda^2}{\tilde{\Lambda}^2}\right) = \frac{4\pi}{\beta_0} \left(\frac{1}{\alpha(t)} - \frac{1}{\bar{\alpha}(t)}\right) + \ldots = \frac{4\pi c}{\beta_0}
\]

\[
\frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t)
\]

The evolution will thus depend on the scheme via the perturbative truncation

The use of \(\Lambda=0.234\) GeV can be thus interpreted as the choice of particular scheme, differing from MS. Beyond this, the scheme can be defined in such a way that one-loop DGLAP is exact at all orders (Grunberg's effective charge).
Pion realistic picture: DGLAP evolution

\[ \alpha(t) = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2} \right)} = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + k^2}{\Lambda^2} \right)} \]

\[ t = \ln \left( \frac{k^2}{\zeta_0^2} \right) \]
\[ \alpha(t) = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2}\right)} = \frac{4\pi}{\beta_0 \ln\left(\frac{m_\alpha^2 + k^2}{\Lambda^2}\right)} \]

\[ \alpha(0) = \alpha_{PI}(0) \rightarrow m_\alpha = 0.300 \text{ GeV} \]

D. Binosi et al., PRD96(2017)054026
J. R-Q et al., FBS59(2018)121
M. Ding et al., ArXiv:1905.05208 [nucl-th]
Pion realistic picture: DGLAP evolution

\[ \alpha(t) = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2} \right)} = \frac{4\pi}{\beta_0 \ln \left( \frac{m_\alpha^2 + k^2}{\Lambda^2} \right)} \]

\[ \alpha(0) = \alpha_{PI}(0) \rightarrow m_\alpha = 0.300 \text{ GeV} \]

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) \]

Numerical integration with the effective charge

\[ M_n(t) = M_n(t_0) \exp \left( -\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} dz \alpha(z) \right) \]

\[ \gamma_0^n = -\frac{4}{3} \left( 3 + \frac{2}{(n+2)(n+3)} \right) - \sum_{i=1}^{n+1} \frac{1}{i} \]
Pion realistic picture: DGLAP evolution

\[ \alpha(t) = \beta_0 \ln \left( \frac{m_\alpha^2 + \zeta_0^2 \exp(t)}{\Lambda^2} \right) = \beta_0 \ln \left( \frac{m_\alpha^2 + k^2}{\Lambda^2} \right) \]

\[ \alpha(0) = \alpha_{PI}(0) \Rightarrow m_\alpha = 0.300 \text{ GeV} \]

\[ \frac{d}{dt} M_n(t) = -\frac{\alpha(t)}{4\pi} \gamma_0^n M_n(t) \]

Numerical integration with the effective charge

\[ M_n(t) = M_n(t_0) \exp \left( -\frac{\gamma_0^n}{4\pi} \int_{t_0}^{t} \alpha(z) dz \right) \]

If one identifies: \( m_\alpha \equiv \zeta_H \), all the scales (and the evolution between them) appear thus fixed, apart from \( \Lambda_{QCD} \) (fixed by the scheme).
Pion realistic picture: DGLAP evolution

Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

\[ \Lambda_{QCD} = 0.234 \text{ GeV}; \ \zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2 \text{ GeV} \]

If one identifies: \( m_\alpha \equiv \zeta_H \), all the scales (and the evolution between them) appear thus fixed, apart from \( \Lambda_{QCD} \) (fixed by the scheme). And the agreement with E615 data is perfect!!!
Then, one can evolve the pion PDF, e.g. the one obtained by direct computation of Mellin moments, by using DGLAP evolution from one unknown hadronic scale up to the relevant one for the E615 experiment:

$$\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2 \text{ GeV}$$

$$\Lambda_{QCD} = 0.234 \text{ GeV}$$

The same is obtained from the overlap of realistic pion 2-body LFWFs

and after integration of the DGLAP master equation

$$\frac{d}{dt} q(x, t) = -\frac{\alpha(t)}{4\pi} \int_\frac{1}{x} dy q(y, t) P\left(\frac{x}{y}\right) + \ldots$$

If one identifies: $$m_\alpha \equiv \zeta_H$$, all the scales (and the evolution between them) appear thus fixed, apart from $$\Lambda_{QCD}$$ (fixed by the scheme). And the agreement with E615 data is perfect!!!
Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

\[
\zeta^2 \frac{d}{d\zeta^2} \left( \frac{q^S(x;\zeta)}{G^S(x;\zeta)} \right) = \frac{\alpha(\zeta^2)}{4\pi} \int_x^1 \frac{dy}{y} \begin{pmatrix}
P_{0,qq}^S \left( \frac{x}{y} \right) & 2n_f P_{0,qG}^S \left( \frac{x}{y} \right) \\
P_{0,Gq}^S \left( \frac{x}{y} \right) & P_{0,GG}^S \left( \frac{x}{y} \right)
\end{pmatrix} \begin{pmatrix}
q^S(y;\zeta) \\
G^S(y;\zeta)
\end{pmatrix}
\]

[Altarelli, Parisi; NPB126(1977)298]
Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

\[
\alpha(\zeta^2) \frac{d}{d\zeta^2} \begin{pmatrix}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta)
\end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \begin{pmatrix}
\gamma_{0,qq}^{S,(m)} & 2nf\,\gamma_{0,qG}^{S,(m)} \\
\gamma_{0,Gq}^{S,(m)} & \gamma_{0,GG}^{S,(m)}
\end{pmatrix} \begin{pmatrix}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta)
\end{pmatrix}
\]

[Altarelli,Parisi; NPB126(1977)298]
Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

\[
\frac{\zeta^2}{d\zeta^2} \left( \begin{array}{c}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta)
\end{array} \right) = -\frac{\alpha_s(\zeta^2)}{4\pi} \left( \begin{array}{cc}
\gamma_{0,qq}^{S,(m)} & 2n_f \gamma_{0,qG}^{S,(m)} \\
\gamma_{0,Gq}^{S,(m)} & \gamma_{0,GG}^{S,(m)}
\end{array} \right) \left( \begin{array}{c}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta)
\end{array} \right)
\]

\[
P^{-1} \Gamma_0^{S,(m)} P = \left( \begin{array}{cc}
\lambda_+^{(m)} & 0 \\
0 & \lambda_-^{(m)}
\end{array} \right)
\]

\[
\Gamma_0^{S,(m)} = \left( \begin{array}{c}
\Gamma_0^{S,(m)} \\
\Gamma_D^{(m)}
\end{array} \right)
\]

\[
\gamma_{0,AB}^{S,(m)} = -\int_0^1 dx \, x^m P_0^{S,AB}(x)
\]
Let us also consider the singlet components:

(an almost textbook exercise)

\[ \gamma_{0,AB}^{S,(m)} = - \int_0^1 dx \, x^m \, P_{0,AB}^S(x) \]

\[ \zeta^2 \frac{d}{d \zeta^2} \left( \begin{array}{c} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{array} \right) = \frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1} \left( \begin{array}{c} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{array} \right) \]

\[ P^{-1} \Gamma_0^{S,(m)} P = \left( \begin{array}{cc} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{array} \right) \]
Pion realistic picture: **DGLAP evolution**

Let us also consider the singlet components:
( an almost textbook exercise )

\[
\gamma_{0,AB}^{S,(m)} = - \int_0^1 dx \, x^m \, P_0^{S,AB}(x)
\]

\[
\zeta^2 \frac{d}{d\zeta^2} \, P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = \frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} \, P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix}
\]

\[
P^{-1} \Gamma_0^{S,(m)} \, P = \begin{pmatrix} \lambda_+^{(m)} & 0 \\ 0 & \lambda_-^{(m)} \end{pmatrix}
\]

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom
Let us also consider the singlet components: (an almost textbook exercise)

\[ \zeta^2 \frac{d}{d\zeta^2} P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix} \]

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

\[ P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{pmatrix} = \exp \left( -\Gamma_D^{(m)} \int_{\ln \zeta^2_H}^{\ln \zeta^2} d\ln z^2 \frac{\alpha(z^2)}{4\pi} \right) P^{-1} \begin{pmatrix} M_q^{(m)}(\zeta_H) \\ 0 \end{pmatrix} \]
Let us also consider the singlet components:
(an almost textbook exercise)

\[
\begin{pmatrix}
  w_{11} & w_{12} \\
  w_{21} & w_{22}
\end{pmatrix}
\begin{pmatrix}
  \xi^2 \frac{d}{d\xi^2} P^{-1} 
\end{pmatrix}
\begin{pmatrix}
  M_q^{(m)}(\xi) \\
  M_G^{(m)}(\xi)
\end{pmatrix}
= -\frac{\alpha(\xi^2)}{4\pi} \Gamma_D^{(m)} P^{-1}
\begin{pmatrix}
  M_q^{(m)}(\zeta_H) \\
  0
\end{pmatrix}
\]

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

\[
M_q^{(m)}(\zeta) = M_q^{(m)}(\zeta_H)
\times \frac{w_{11}w_{22}}{\text{Det}(P)} \exp \left( -\frac{\lambda^+_1}{4\pi} \int_{\ln \zeta_H^2}^{\ln \xi^2} dt \alpha(t) \right) - \frac{w_{12}w_{21}}{\text{Det}(P)} \exp \left( -\frac{\lambda^-_1}{4\pi} \int_{\ln \zeta_H^2}^{\ln \xi^2} dt \alpha(t) \right)
\]

\[
M_G^{(m)}(\zeta) = M_q^{(m)}(\zeta_H) \frac{w_{22}w_{21}}{\text{Det}(P)}
\times \exp \left( -\frac{\lambda^+_1}{4\pi} \int_{\ln \zeta_H^2}^{\ln \xi^2} dt \alpha(t) \right) - \exp \left( -\frac{\lambda^-_1}{4\pi} \int_{\ln \zeta_H^2}^{\ln \xi^2} dt \alpha(t) \right)
\]
Pion realistic picture: DGLAP evolution

Let us also consider the singlet components:
(an almost textbook exercise)

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

Case $m=1$
($n_f=4$)

\[
\begin{pmatrix}
1 & 3/4 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
\zeta^2 \frac{d}{d\zeta^2} P^{-1} \begin{pmatrix}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta)
\end{pmatrix} = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma^{(m)}_D P^{-1} \begin{pmatrix}
M_q^{(m)}(\zeta_H) \\
0
\end{pmatrix}
\end{pmatrix}
\]

\[
P^{-1} \Gamma^{S,(1)}_0 \frac{1}{P} = \begin{pmatrix}
56/9 & 0 \\
0 & 0
\end{pmatrix}
\]
Let us also consider the singlet components: (an almost textbook exercise)

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

Case m=1
(nf=4)
Let us also consider the singlet components:
(an almost textbook exercise)

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\begin{pmatrix}
1 & 3/4 \\
-1 & 1 \\
\end{pmatrix}
\]

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

\[
\frac{\zeta^2}{d\zeta^2} P^{-1} \begin{pmatrix}
M_q^{(m)}(\zeta) \\
M_G^{(m)}(\zeta) \\
\end{pmatrix}
= \frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} P^{-1}
\begin{pmatrix}
M_q^{(m)}(\zeta_H) \\
0 \\
\end{pmatrix}
\]

Case \( m=1 \)
\((nf=4)\)

\[
M_q^{(1)}(\zeta) = M_q^{(1)}(\zeta_H) \left[ \frac{3}{7} + \frac{4}{7} \exp \left( -\frac{56}{36\pi} \int_{\ln\zeta_H^2}^{\ln\zeta^2} dt \alpha(t) \right) \right]
\]

\[
M_G^{(1)}(\zeta) = \frac{4}{7} M_q^{(1)}(\zeta_H) \left[ 1 - \exp \left( -\frac{56}{36\pi} \int_{\ln\zeta_H^2}^{\ln\zeta^2} dt \alpha(t) \right) \right]
\]

\[
M_q^{(1)}(\zeta) + M_G^{(1)}(\zeta) = M_q^{(1)}(\zeta_H)
\]
Let us also consider the singlet components: (an almost textbook exercise)

\[ \zeta^2 \frac{d}{d\zeta^2} P^{-1} \left( \begin{array}{c} M_q^{(m)}(\zeta) \\ M_G^{(m)}(\zeta) \end{array} \right) = -\frac{\alpha(\zeta^2)}{4\pi} \Gamma_D^{(m)} \left( \begin{array}{c} M_q^{(m)}(\zeta_H) \\ 0 \end{array} \right) \]

Initial conditions at the hadronic scale, where only valence-quarks are assumed to be the correct degrees-of-freedom, can be evolved and shown to produce non-zero gluon and sea-quark components.

<table>
<thead>
<tr>
<th>(\zeta_5)</th>
<th>(\langle x \rangle^\pi_u)</th>
<th>(\langle x^2 \rangle^\pi_u)</th>
<th>(\langle x^3 \rangle^\pi_u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. [31]</td>
<td>0.17(1)</td>
<td>0.060(9)</td>
<td>0.028(7)</td>
</tr>
<tr>
<td>Herein</td>
<td>0.21(2)</td>
<td>0.076(9)</td>
<td>0.036(5)</td>
</tr>
</tbody>
</table>

\(\langle x \rangle^\pi_g = 0.45(1)\), \(\langle x \rangle^\pi_{\text{sea}} = 0.14(2)\).

M. Ding et al., ArXiv:1905.05208 [nucl-th]
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale.

\[ q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle P \left| \bar{\psi}^q(-z)\gamma^+\psi^q(z) \right| P \right\rangle \bigg|_{z^+, z^-=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \frac{1}{4} \Psi^{*uJ}_{uJ}(x, k_\perp) \Psi_{uJ}(x, k_\perp) \]

LFWF leading to asymptotic PDAs

\[ q_{sf}(x) \approx 30 x^2 (1 - x)^2 \]

A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

\[ (x^m)^\pi_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H) \]

\[ = \frac{N_c}{n \cdot P} \text{tr} \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_{\pi}(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma_{\pi}(k_\eta, -P) S(k_\eta)] \]

\[ \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911 x(1-x)] \]
Pion (more) realistic picture: PDF as benchmark

The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale.

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A more realistic pion 2-body LFWF

Direct computation of Mellin moments:

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\[ = \frac{N_c}{n \cdot P} \text{tr} \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} [\Gamma_\pi(k_\eta, -P) S(k_\eta)] \]

\[ q^\pi(x; \zeta_H) = 213.32 x^2(1 - x)^2 \times [1 - 2.9342\sqrt{x(1 - x)} + 2.2911 x(1 - x)] \]
Pion (more) realistic picture: PDF as benchmark

- Spectral density is chosen as:
  
  \[ u_G \rho_G(\omega) = \frac{1}{2b_0^G} \left[ \text{sech}^2 \left( \frac{\omega - \omega_0^G}{2b_0^G} \right) + \text{sech}^2 \left( \frac{\omega + \omega_0^G}{2b_0^G} \right) \right] \]

  **Phenomelogical model:** \( b_0^\pi = 0.1, \omega_0^\pi = 0.73 \);  
  **Realistic case:** \( b_0^\pi = 0.275, b_0^\pi = 1.23 \); 

  **Asymptotic case:** \( \rho(\omega; \nu) \sim (1 - \omega^2)^\nu \)
The pion PDF can be computed as the lightfront projection of the hadronic matrix element of a bilocal operator and, in the overlap representation at low Fock states, can be expressed in terms of 2-body LFWFs at a given hadronic scale:

$$q^\pi(x; \zeta_H) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP+z^-} \left< P \left| \bar{\psi}^q(-z) \gamma^+ \psi^q(z) \right| P \right> \bigg|_{z^+=0, z_\perp=0} = \int \frac{d^2k_\perp}{16\pi^3} \overline{\Psi}^*_{u\overline{J}}(x, k_\perp) \Psi_{u\overline{J}}(x, k_\perp)$$

$$\zeta_H \equiv m_\alpha \rightarrow \zeta_2 = 5.2 \text{ GeV}$$

Direct computation of Mellin moments:

$$\langle x^m \rangle^{\pi}_{\zeta_H} = \int_0^1 dx x^m q^\pi(x; \zeta_H)$$

$$= \frac{N_c}{n \cdot P} \text{tr} \int dk \left[ \frac{n \cdot k_\eta}{n \cdot P} \right]^m \Gamma_\pi(k_\eta, P) S(k_\eta) n \cdot \partial_{k_\eta} \left[ \Gamma_\pi(k_\eta, -P) S(k_\eta) \right]$$

$$\times [1 - 2.9342 \sqrt{x(1-x)} + 2.2911 x(1-x)]$$
Pion (more) realistic picture: GPD

\[ H^q_{M}(x, \xi, t) = \int \frac{d^2k_\perp}{16 \pi^3} \Psi_{\alpha f}^* \left( \frac{x - \xi}{1 - \xi}, \frac{1 - x \Delta_\perp}{1 - \xi} \right) \Psi_{\alpha \bar{f}} \left( \frac{x + \xi}{1 + \xi}, \frac{1 - x \Delta_\perp}{1 + \xi} \right) \]

Phenomenological model

Realistic case
Pion (more) realistic picture: DGLAP evolution

\[ H_M^q(x, \xi, t) = \int \frac{d^2k_\perp}{16 \pi^3} \Psi^*_{u \bar{f}} \left( \frac{x - \xi}{1 - \xi}, k_\perp + \frac{1 - x \Delta_\perp}{1 - \xi} \frac{1}{2} \right) \Psi_{u \bar{f}} \left( \frac{x + \xi}{1 + \xi}, k_\perp - \frac{1 - x \Delta_\perp}{1 + \xi} \frac{1}{2} \right) \]

\[ \xi_0 = \xi_H = 0.3 \text{ GeV} \rightarrow \xi_2 = 1.0 \text{ GeV} \]
Pion (more) realistic picture: Elect. Form Factor

\[ F_M(\Delta^2) = e_u F_M^u(\Delta^2) + e_f F_M^f(\Delta^2) \text{, } F_M^u(-t = \Delta^2) = \int_{-1}^{1} dx \, H_M^u(x, \xi, t) \]

Electric charges

Blue: Computed from GPD
Green: Computed from HS formula
Red: ‘Evolved’ form factor

\[ r_{\pi} \sim 0.68 \text{ fm} \]

\[ Q^2 F_{\pi}(Q^2) \]

\[ Q^2 [\text{GeV}^2] \]
One word (or two) about ERBL covariant extension

The GPD can be cast (because of fundamental symmetries as Lorentz invariance) as a Radon transform of a double distribution:

\[ H(x, \xi, t) = \int_\Omega d\beta d\alpha \left[ F(\beta, \alpha, t) + \xi G(\beta, \alpha, t) \right] \delta(x - \beta - \alpha \xi) \]

(Polyakov-Weiss)

So-called “gauge” transformations

1 component DD!!!

(Pobylitsa “gauge”)
One word (or two) about ERBL covariant extension

The GPD can be cast (because of fundamental symmetries as Lorentz invariance) as a Radon transform of a double distribution:

\[ H(x, \xi, t) - \text{sgn}(\xi) D(x/\xi, t) = \int_{\Omega} d\beta d\alpha F_D(\beta, \alpha, t) \delta(x - \beta - \alpha \xi) \]

(Polyakov-Weiss)

So-called “gauge” transformations

\[ H(x, \xi, t) = \int_{\Omega} d\beta d\alpha [F(\beta, \alpha, t) + \xi G(\beta, \alpha, t)] \delta(x - \beta - \alpha \xi) \]

1 component DD!!!
(Pobylitsa “gauge”)

\[ F(\beta, \alpha, t) = (1 - \beta) h(\beta, \alpha, t) + \delta(\beta) D^+(\alpha, t), \]
\[ G(\beta, \alpha, t) = -\alpha h(\beta, \alpha, t) + \delta(\beta) D^-(\alpha, t), \]

Radon transform inversion

Is DGLAP GPD information enough to get the DD? (and thus extend the GPD to ERBL region)
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D-term ambiguities  
(not living in DGLAP region)

Radon transform inversion

Is DGLAP GPD information enough to get the DD? (and thus extend the GPD to ERBL region) 

The answer is yes... up to D-term ambiguities ... and issues related to the fact that, mathematically, the inversion problem is ill-posed (in the Hadamard sens)

PDA and LFWF evolution

Standard PDA evolution:

- We project PDA onto a 3/2-Gegenbauer polynomial basis. Such that it evolves, from an initial scale $\zeta_0$ to a final scale $\zeta$, according to the corresponding ERBL equations:

$$
\phi(x; \zeta) = 6x(1-x) \left[ 1 + \sum_{n=1} a_n(\zeta) C_n^{3/2}(2x - 1) \right],
$$

$$
a_n(\zeta) = a_n(\zeta_0) \left[ \frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)} \right] \gamma_0^n / \beta_0^n, \quad \gamma_0^n = -\frac{4}{3} \left[ 3 + \frac{2}{(n + 1)(n + 2)} - 4 \sum_{k=1}^{n+1} \frac{1}{k} \right].
$$

- Thus, any PDA at hadronic scale evolves logarithmically towards its conformal distribution, $\phi(x) = 6x(1-x)$.

- Quark mass and flavor become irrelevant. Broad PDA becomes narrower, skewed PDA becomes symmetric.
PDA and LFWF evolution

LFWF evolution:

\[ \phi(x) = \frac{1}{16\pi^3} \int d^2 k_\perp \psi^{\uparrow\downarrow}(x, k_\perp^2) \]

- We look for a way to evolve the LFWF.

- First, let’s assume that the LFWF admits a similar Gegenbauer expansion. That is:

\[ \psi(x, k_\perp^2; \zeta) = 6x(1-x) \left[ \sum_{n=0} b_n(k_\perp^2; \zeta) C_n^{3/2}(2x - 1) \right], \]

\[ a_n(\zeta) = \frac{1}{16\pi^3} \int d^2 k_\perp b_n(k_\perp^2; \zeta) \text{ (for } n \geq 1) , \quad \frac{1}{16\pi^3} \int d^2 k_\perp b_0(k_\perp^2; \zeta) = 1. \]

- 1-loop ERBL evolution of \( a_n(\zeta) \) implies:

\[ \frac{1}{a_n(\zeta)} \frac{d}{d \ln \zeta^2} a_n(\zeta) = \frac{\int d^2 k_\perp \frac{d}{d \ln \zeta^2} b_n(k_\perp^2; \zeta)}{\int d^2 k_\perp b_n(k_\perp^2; \zeta)} , \]
PDA and LFWF evolution

LFWF evolution:

\[ \phi(x) = \frac{1}{16\pi^3} \int d^2k_\perp \psi^{\uparrow\downarrow}(x, k^2_\perp) \]

- Now, if we take a factorization assumption, we arrive at:

\[ \frac{b_n(k^2_\perp; \zeta)}{b_n(k^2_\perp; \zeta_0)} = \frac{\hat{b}_n(\zeta)}{\hat{b}_n(\zeta_0)} = \left[ \frac{\alpha(\zeta^2)}{\alpha(\zeta_0^2)} \right]^{\gamma_0^\pi/\beta_0}, \quad b_n(k^2_\perp; \zeta) \equiv \hat{b}_n(\zeta) \chi_n(k^2_\perp). \]

- Supplemented by the condition \( \chi_n(k^2_\perp) \equiv \chi(k^2_\perp) \), one gets \( \hat{b}_n(\zeta) \equiv a_n(\zeta) \).

- Such that, the following factorised form is obtained:

\[ \psi(x, k^2_\perp; \zeta) \equiv \phi(x; \zeta) \chi(k^2_\perp) \]

- Which is far from being a general result, but an useful approximation instead.
PDA and LFWF evolution

Testing the factorization ansatz:

\[ \psi(x, k_1^2; \zeta) \equiv \phi(x; \zeta) \chi(k_1^2) \]

- A first validation of the factorized ansatz is addressed in *Phys. Rev. D97* (2018) no.9, 094014:

\[ k^2 = 0, k^2 = 0.2 \text{ GeV}, k^2 = 0.8 \text{ GeV}, k^2 = 3.2 \text{ GeV} \]

\[ \int dx \psi(x, k) = 1 \]

- If the factorized ansatz is a good approximation, then the plotted ratio must be 1. For the pion, it slightly deviates from 1; for the kaon, the deviation is much larger.
Testing the factorization ansatz:

1) Compute LFWF and ERBL running of PDA
2) ERBL running of LFWF and compute PDA

Notably, 1) and 2) are equivalent. Factorization assumption and evolution seem reasonable.
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:

1) Obtained from ERBL evolution of LFWF
2) Obtained from DGLAP evolution of GPD

Clearly, 1) and 2) are not equivalent.
PDA and LFWF evolution

How ERBL and DGLAP evolutions make contact:

1) Obtained from ERBL evolution of LFWF
2) Obtained from DGLAP evolution of GPD

Clearly, 1) and 2) are not equivalent.

Sea-quark and gluon content incorporated to the parton distribution by DGLAP are obviously not present in the valence-quark PDF from LFWFs!!!
Conclusions

Owing to a sensible parametrisation of the BSA grounded on the so-called Nakanishi representation, one is left with a flexible algebraic model for the LFWF in terms of a spectral density.
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A direct calculation of the PDF from realistic quark gap and Bethe-Salpeter equations' solutions (in the forward kinematical limit) delivers a benchmark result to identify the spectral density which corresponds to the realistic LFWF.
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A recently proposed PI effective charge can be used to make the DGLAP GPD evolve from the hadronic scale (where quasi-particle DSE's solutions are the correct degrees-of-freedom) up to any other relevant scale.
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Thank you!!
Backslides
PI-effective charge from lattice data with Nf=3 flavors at the physical point

Preliminary results:

\[ \hat{\alpha}_{\text{PI}}(q^2) = \frac{\hat{d}(q^2)}{D(q^2)} \approx \frac{\alpha_T(q^2)}{q^2 [1 - L(q^2, \zeta^2)F(q^2, \zeta^2)]^2} \frac{m_0^2 \Delta_F(0, \zeta^2)}{\Delta_F(q^2, \zeta^2)} \]

\[ = \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2)F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2)m_0^2 \]

The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function

\[ F(q^2) \]

\[ q \ [\text{GeV}] \]
PI-effective charge from lattice data with Nf=3 flavors at the physical point

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- The ghost dressing function
- The PT-BFM function L
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- The ghost dressing function
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Its strength depends also on the saturation point at zero-momentum of the gluon propagator.
PI-effective charge from lattice data with Nf=3 flavors at the physical point

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\]

\[
= \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2) F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2) m_0^2
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The IR running of the PI effective charge with momenta only depends on:
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Its strength depends also on the saturation point at zero-momentum of the gluon propagator and on the Taylor coupling.
PI-effective charge from lattice data with Nf=3 flavors at the physical point

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\]

\[
= \alpha_T(\zeta^2) \frac{F(q^2, \zeta^2)}{[1 - L(q^2, \zeta^2) F(q^2, \zeta^2)]^2} \Delta_F(0, \zeta^2)m_0^2
\]

The IR running of the PI effective charge with momenta only depends on:

- The ghost dressing function
- The PT-BFM function L

Its strength depends also on the saturation point at zero-momentum of the gluon propagator and on the Taylor coupling.

All put together:

Less uncertainties (that of the gluon mass is only left here) and still a better agreement with the world data for the experimental determination of the Bjorken sum-rule effective charge.
A word about GPD polynomiality first:

- Express Mellin moments of GPDs as matrix elements:
  \[ \int_{-1}^{+1} dx x^m H^q(x, \xi, t) = \frac{1}{2 (P^+)^{m+1}} \left\langle P + \frac{\Delta}{2} \left| q(0) \bar{q}(0) (i \not{D}^+)^m q(0) \right| P - \frac{\Delta}{2} \right\rangle \]

- Identify the Lorentz structure of the matrix element:
  linear combination of \((P^+)^{m+1-k}(\Delta^+)^k\) for \(0 \leq k \leq m+1\)

- Remember definition of skewness \(\Delta^+ = -2\xi P^+\).

- Select even powers to implement time reversal.

- Obtain polynomiality condition:
  \[ \int_{-1}^{1} dx x^m H^q(x, \xi, t) = \sum_{i=0}^{m} (2\xi)^i C^q_{mi}(t) + (2\xi)^{m+1} C^q_{mm+1}(t) . \]
A word about bout gravitational Form Factors

Definition and evaluation:

- Pion gravitational form factors are defined through*: 
  \[ J_{\pi^+}(-t, \xi) \equiv \int_{-1}^{1} dx \; x H_{\pi^+}(x, \xi, t) = \Theta_2(t) - \Theta_1(t) \xi^2. \]

- Taking \( \xi = 0 \) + isospin symmetric limit, one can readily compute:
  \[ \Theta_2(t) = \int_{0}^{1} dx \; x [H_{\pi^+}^u(x, 0, t) + H_{\pi^+}^d(x, 0, t)] = \int_{0}^{1} dx \; 2x H_{\pi^+}^u(x, 0, t). \]

- To obtain \( \Theta_1(t) \), we need to take a non zero value of \( \xi \); hence requiring the knowledge of the GPD in the ERBL region.

- Nevertheless, one can approximate \( \Theta_1(t) \), by estimating the derivative of \( J_{\pi^+}(-t, \xi) \) with respect to \( \xi^2 \) as:
  \[ D(\xi + \Delta/2) \equiv \frac{J(\xi + \Delta) - J(\xi)}{2(\xi + \Delta/2)\Delta}, \; \Delta \to 0. \]

A word about gravitational Form Factors

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A word about bout gravitational Form Factors

Definition and evaluation:

- To get a clearer picture, let’s split \( J(-t, \xi) \) as follows:

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J(-t, \xi) = \int_{-\xi}^{1} dx \ 2x H(x, \xi, t) = \left[ \int_{-\xi}^{\xi} dx + \int_{\xi}^{1} dx \right] 2x H(x, \xi, t)
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\[
\Rightarrow J(-t, \xi) = J_{ERBL}(-t, \xi) + J_{DGLAP}(-t, \xi)
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- Notice that, because of the polinomiality of the complete GPD:

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J_{DGLAP}(-t, \xi) = \Theta_2(t) - \xi^2 \Theta_1(t)_{DGLAP} + \sum_{i=1}^{\infty} c_i(t) \xi^{2+i}
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Definition and evaluation:

- The extensión to ERBL region is then needed. Taking advantage of the soft-pion theorem, one can connect PDA with $J(-t, \xi)^{ERBL}$ and thus with $\Theta_1(t)^{ERBL}$.

- Nonetheless, polinomiality of GPD is not fulfilled without the ERBL región. Such extensión is necessary to provide a more reliable computation of $\Theta_1$.

\[ m_\pi \approx 0.45 \text{ GeV} \]

\[ \Theta_2(t) \]


GPD + Ding et al.


$\Theta_2(0)/2 = <x> = 0.261(5)$

$\Theta_2(0)/2 = <x> = 0.242(20)$
A word about gravitational Form Factors

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