# Optimized Ogata quadrature and applications to the Sivers Asymmetry

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Fit to Sivers asymmetry done in collaboration with

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Optimized Ogata method is done in collaboration with

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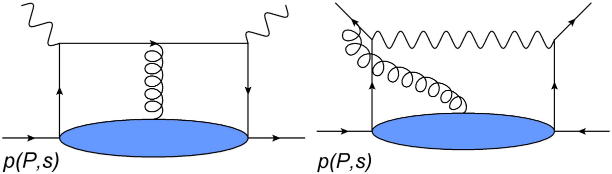
# Overview

# Why Sivers Asymmetry

By measuring the Sivers asymmetry, one probes quark Sivers functions




Modified universality (sign change between SIDIS and DY)

$$f_{1T}^{\perp q}(x, k_\perp)|_{\rm SIDIS} = -f_{1T}^{\perp q}(x, k_\perp)|_{\rm DY}$$

$$f_1^q(x, \vec{k}_\perp, \vec{S}) = f_1^q(x, k_\perp) - \frac{1}{M} f_{1T}^{\perp q}(x, k_\perp) \vec{S} \cdot (\hat{p} \times \vec{k}_\perp)$$

# SIDIS: $e(\ell) + p(P, \vec{s}_\perp) \to e(\ell') + h(P_h) + X$

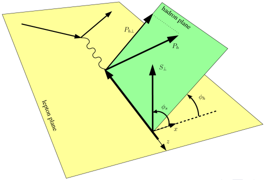
The differential cross section with TMD factorization

$$\frac{d\sigma}{dx_B dy dz_h d^2 q_\perp} = \sigma_0^{\rm DIS} \left[ F_{UU} + \sin(\phi_h - \phi_s) F_{UT}^{\sin(\phi_h - \phi_s)} \right]$$

$$A_{UT}^{\sin(\phi_h - \phi_s)} = \frac{F_{UT}^{\sin(\phi_h - \phi_s)}}{F_{UU}}$$

$$F_{UU}(q_\perp, Q) = \; H(Q, \mu) \sum_q e_q^2 \int d^2\mathbf{k}_\perp d^2\mathbf{p}_\perp f_{q/p}(x_B, k_\perp^2)\, D_{h/q}(z_h, p_\perp^2) \delta^{(2)} \left( \mathbf{k}_\perp + \frac{\mathbf{p}_\perp}{z_h} - \mathbf{q}_\perp \right)$$

$$F_{UT}^{\sin(\phi_h - \phi_s)}(q_\perp, Q) = - \; H(Q, \mu) \sum_q e_q^2 \int d^2\mathbf{k}_\perp d^2\mathbf{p}_\perp \frac{k_\perp}{M} f_{1T}^{\perp q}(x_B, k_\perp^2)\, D_{h/q}(z_h, p_\perp^2) \; \delta^{(2)} \left( \mathbf{k}_\perp + \frac{\mathbf{p}_\perp}{z_h} - \mathbf{q}_\perp \right)$$



# $F_{UU}$ in $b_\perp$ space

Unpolarized structure function in the TMD formalism

$$F_{UU}(Q, q_\perp) = \int_0^\infty \frac{b_\perp db_\perp}{2\pi} \; J_0(b_\perp q_\perp) \; \widetilde{F}_{UU}(Q, b_\perp)$$

$$\widetilde{F}_{UU}(b_\perp; x, z, Q) \equiv H^{DIS}(\mu, Q) C_{q\leftarrow i} \otimes f_A^i\,(x_B, \mu_b)\, \hat{C}_{j\leftarrow q} \otimes D^i_{h/j}\,(z_h, \mu_b)$$
$$\times \exp\left(\int_{\mu_{b_*}}^{Q} \frac{d\mu'}{\mu'} \left[2\gamma(\mu') - \ln\left(\frac{Q^2}{\mu'^2}\right) \gamma_K(\mu')\right] + \widetilde{K}(b_\perp, \mu_{b_*}) \ln\left(\frac{Q^2}{\mu_{b_*}^2}\right)\right)$$
$$\times \exp\left(-g_f(x, b_\perp) - g_D(z, b_\perp) - 2g_K(b_\perp, b_{max}) \ln\left(\frac{Q}{Q_0}\right)\right)$$

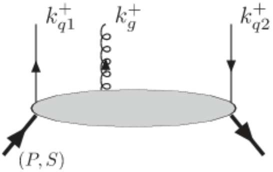
## $F_{UT}^{\sin\phi_h-\phi_s}$ in $b_\perp$ space

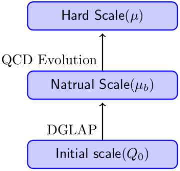
Polarized structure function in the TMD formalism

$$F_{UT}^{\sin(\phi_h-\phi_s)}(Q,q_\perp) = \int_0^\infty \frac{b_\perp^2\, db_\perp}{4\pi}\, J_1(b_\perp q_\perp)\, \widetilde{F}_{UT}^{\sin(\phi_h-\phi_s)}(Q,b_\perp)$$

$$\begin{aligned}\widetilde{F}_{UT}^{\sin(\phi_h-\phi_s)}(b_\perp;x,z,Q) \equiv{}& H^{DIS}(\mu,Q)\Delta C_{q\leftarrow i}\otimes f_{1T}^{\perp(1)}(x_B,\mu_{b_*})\hat{C}_{j\leftarrow q}\otimes D_{h/j}^{i}(z_h,\mu_{b_*})\\ &\times \exp\left(\int_{\mu_{b_*}}^{Q}\frac{d\mu'}{\mu'}\left[2\gamma(\mu')-\ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\mu')\right]+\widetilde{K}(b_\perp,\mu_{b_*})\ln\left(\frac{Q^2}{\mu_{b_*}^2}\right)\right)\\ &\times \exp\left(-g_T(x,b_\perp)-g_D(z,b_\perp)-2g_K(b_\perp,b_{max})\ln\left(\frac{Q}{Q_0}\right)\right)\end{aligned}$$

# Qiu-Sterman function




$$f_{1T}^{\perp(1)}(x_B, \mu_{b_*}) = -\frac{1}{2M} T_{q,F}(x_B, x_B, \mu_{b_*}).$$

Initial condition

$$T_{q/F}(x, x, Q_0) = N_q \frac{(\alpha_q + \beta_q)^{(\alpha_q + \beta_q)}}{\alpha_q^{\alpha_q} \beta_q^{\beta_q}} x^{\alpha_q} (1-x)^{\beta_q} f_{q/A}(x, Q_0)$$

fit parameters: $\alpha_u, \alpha_d, N_u, N_d$ are the valence fit parameters and $\alpha_{sea}, N_{\bar{u}}, N_{\bar{d}}, N_s$, and $N_{\bar{s}}$ are the sea fit paramters and $\beta_q = \beta$ is the same for all flavors.
Evolve the Qiu-Sterman function according to the following approximate form

$$\frac{\partial}{\partial \ln \mu^2} T_{q,F}\left(x, x; \mu^2\right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dx'}{x'} P_{q \leftarrow q}^{QS}\left(x', \alpha_s\left(\mu^2\right)\right) T_{q,F}\left(\frac{x}{x'}, \frac{x}{x'}; \mu^2\right)$$

The splitting kernel is

$$P_{q \leftarrow q}^{QS}(z) = P_{q \leftarrow q}(z) - N_C \delta(1-z).$$

# NLO and NNLL

**Coefficient functions at NLO**, e.g., NLO TMD PDF coefficient functions

$$C_{q\leftarrow q'}(x,\mu_b) = \delta_{qq'}\left[\delta(1-x) + \frac{\alpha_s}{\pi}\frac{C_F}{2}(1-x)\right] \qquad C_{q\leftarrow g}(x,\mu_b) = \frac{\alpha_s}{\pi}T_R x(1-x)\,.$$

The quark-Sivers coefficient function

$$\Delta C^T_{q\leftarrow q'}(x,\mu_b) = \delta_{qq'}\left[\delta(1-x) - \frac{\alpha_s}{2\pi}\frac{1}{4N_C}(1-x)\right]$$

**Perturbative Sudakov to NNLL** ($\gamma_K$ to $\mathcal{O}(\alpha_s^3)$ and $\gamma$ to $\mathcal{O}(\alpha_s^2)$)

$$\exp\left(\int_{\mu_{b_*}}^{Q}\frac{d\mu'}{\mu'}\left[2\gamma(\mu') - \ln\left(\frac{Q^2}{\mu'^2}\right)\gamma_K(\mu')\right]\right)$$

# Non-perturbative Parameterization

Unpolarized SIDIS parameterization

$$\exp\left(-g_f(x,b_\perp) - g_D(z,b_\perp) - 2g_K(b_\perp,b_{max})\ln\left(\frac{Q}{Q_0}\right)\right)$$

$$g_f(x,b_\perp) = 0.106\ b_\perp^2 \qquad g_D(z,b_\perp) = 0.042\ b_\perp^2/z_h^2 \qquad g_K = 0.42\ \ln\left(\frac{b_\perp}{b_*}\right)$$

Sivers function

$$\exp\left(-g_T(x_1,b_\perp) - g_K(b_\perp,b_{max})\ln\left(\frac{Q}{Q_0}\right)\right) \qquad g_T(x_1,b_\perp) = g_S b_\perp^2$$

$g_S$ is fit parameter.
P. Sun, J. Isaacson, C. P. Yuan, and F. Yuan, Int. J. Mod. Phys. A33 , 1841006 (2018), arXiv:1406.3073.
Z.-B. Kang, A. Prokudin, P. Sun, and F. Yuan, Phys. Rev. D93 , 014009 (2016), arXiv:1505.05589.

# A need for fast Fourier-Bessel Transforms

For global analysis, need to perform the following types of integrals many times

$$F_{UU}(Q, q_\perp) = \frac{1}{2\pi} \int_0^\infty db_\perp \; b_\perp J_0(b_\perp q_\perp) \widetilde{F}_{UU}(Q, b_\perp)$$

$$F_{UT}(Q, q_\perp) = \frac{1}{4\pi} \int_0^\infty db_\perp \; b_\perp^2 J_1(b_\perp q_\perp) \widetilde{F}_{UT}(Q, b_\perp)$$

$b_\perp$ space functions are expensive to call.

# Ogata Formalism

Original Ogata's quadrature formula

$$\int_{-\infty}^{\infty} dx|x|^{2n+1} f(x) = h \sum_{j=-\infty, j\neq 0}^{\infty} w_{nj}|x_{nj}|^{2n+1} f(x_{nj}) + \mathcal{O}\left(e^{-c/h}\right)$$

For an even $f(x)$

$$\int_{0}^{\infty} dx|x|^{2n+1} f(x) = h \sum_{j=0}^{\infty} w_{nj}|x_{nj}|^{2n+1} f(x_{nj}) + \mathcal{O}\left(e^{-c/h}\right)$$

The nodes and weights

$$x_{nj} = h\xi_{nj} \ , \qquad w_{nj} = \frac{2}{\pi^2 \xi_{n|j|} J_{n+1}(\pi \xi_{n|j|})}$$

$\pi\xi_{nj}$ are zeros of the Bessel function $J_n(x) : J_n\left(\pi\xi_{nj}\right) = 0$.

# Double exponential transformation

Ogata makes the following change of variables

$$x = \frac{\pi}{h}\psi(t) \qquad \text{with } \psi(t) = t\tanh\left(\tfrac{\pi}{2}\sinh t\right)$$

Quadrature becomes

$$\int_0^\infty dx\, f(x)\, J_n(x) \approx \pi \sum_{j=1}^{N} w_{nj}\, f\left(\frac{\pi}{h}\psi(x_{nj})\right)\, J_n\left(\frac{\pi}{h}\psi(x_{nj})\right)\, \psi'(x_{nj}),$$

Asymptotic behavior

$$\frac{\pi}{h}\psi\left(h\xi_{nj}\right) \approx \pi\xi_{nj}\left[1 - 2\exp\left(-\frac{\pi}{2}e^{h\xi_{nj}}\right)\right]$$

$$J_n\left(\frac{\pi}{h}\psi(x_{nj})\right) \approx 2\pi\xi_{nj} J_{n+1}(\pi\xi_{nj}) \exp\left(-\frac{\pi}{2}e^{h\xi_{nj}}\right).$$

# Optimization Scheme: how to choose $h$

Need to optimize parameters $h$ such that error terms are small for small $N$.

$$\int_0^\infty dx f(x)\, J_n(x) = h \sum_{j=0}^{N} w_{nj} f(h\xi_{nj})\, J_n(h\xi_{nj}) + \ldots$$

$$\int_0^\infty dx f(x)\, J_n(x) \approx \pi \sum_{j=1}^{N} w_{nj}\, f\left(\frac{\pi}{h}\psi(h\xi_{nj})\right)\, J_n\left(\frac{\pi}{h}\psi(h\xi_{nj})\right)\, \psi'(h\xi_{nj}) + \ldots$$

Make the first node contribute the most

$$\frac{\partial}{\partial h}\left(h f(h\xi_{n1})\right) = 0\,,$$

Can make sure that errors in both formulas are similar

$$h_u \xi_{nN} = \frac{\pi}{h}\psi(h\xi_{nN})\,.$$

# Efficiency for Toy TMDs

Want to test the convergence. Need a function with analytic

$$W(q_\perp) = \int_0^\infty db_\perp b_\perp\, \widetilde{W}(b_\perp)\, J_0(b_\perp q_\perp)$$

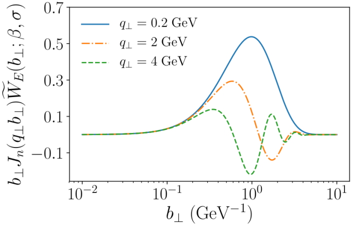
Let's define the toy TMDs

$$\widetilde{W}_E(b_\perp;\beta,\sigma) = \frac{1}{b_\perp}\left(\frac{\beta b_\perp}{\sigma^2}\right)^{\beta^2/\sigma^2} \frac{1}{\Gamma(\frac{\beta^2}{\sigma^2})} e^{-\frac{\beta b_\perp}{\sigma^2}}\ .$$

The function peaks at

$$b_\perp = \frac{1}{Q} = \frac{\beta^2-\sigma^2}{\beta}$$

Take $Q = 2$ and $\sigma = 1$.

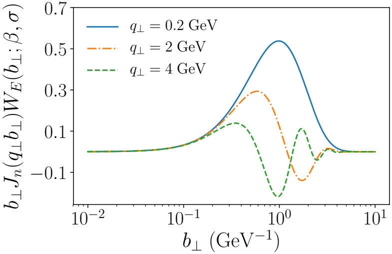


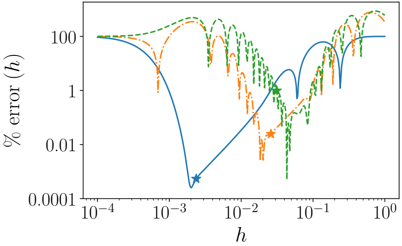
# How well was $h$ determined?

$$\%error = \frac{\text{exact} - \text{Ogata result}(\text{N} = 15)}{\text{exact}} * 100$$

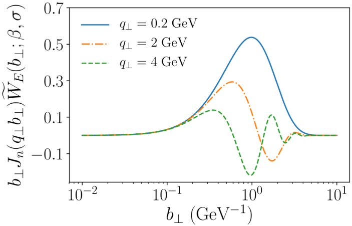
Optimization conditions

$$\frac{\partial}{\partial h}\left(hf(h\xi_{n1})\right) = 0\,, \qquad h_u \xi_{nN} = \frac{\pi}{h}\psi(h\xi_{nN})\,.$$

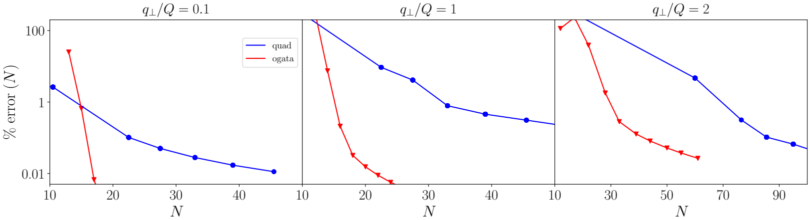


# How does this method compare with other numerical methods?




We benchmark against adaptive quadrature (quad)



# Does it work for real TMDs?

SIDIS COMPASS multiplicity given by

$$M(q_\perp; x, z, Q) = \frac{\pi}{z^2} \frac{d\sigma}{dx dy dz d^2 q_\perp} \Big/ \frac{d\sigma}{dx dy} \, ,$$

We normalize the data so that at the first point in each bin the theory is equal to the data

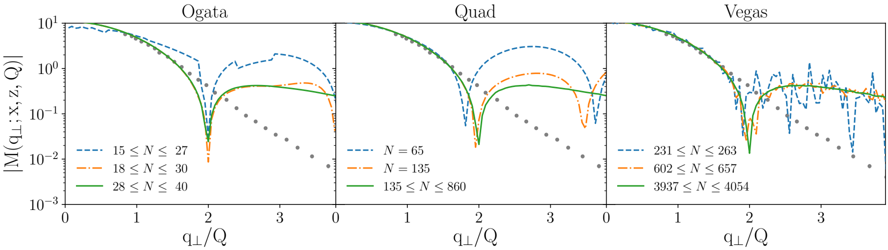
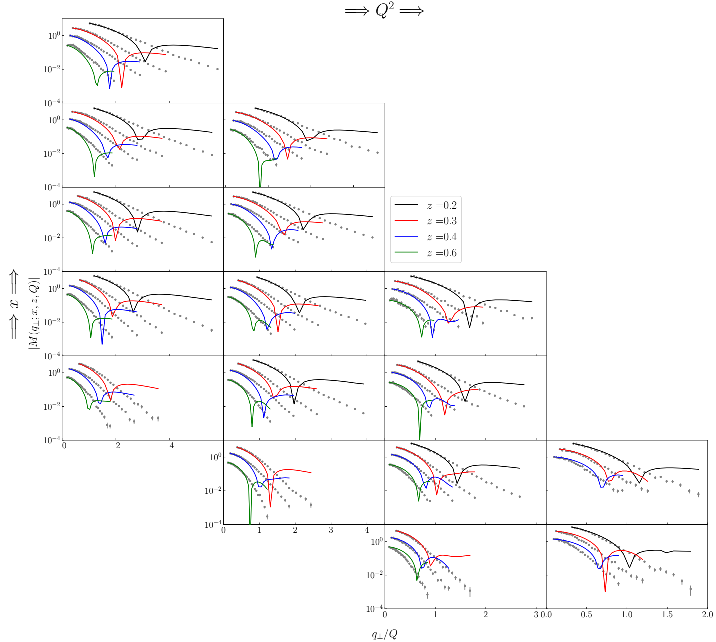



Figure: Multiplicity for Ogata, quad, and Vegas Monte Carlo.

# Application to Unpolarized SIDIS Continued



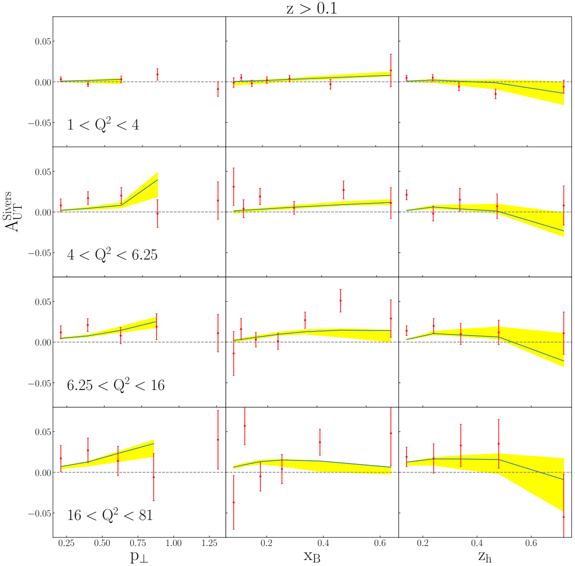
# Back to Sivers!

| Collaboration | Scattering event | Number of points | Year |
|---|---|---|---|
| COMPASS | $e + P \to e + h^+$ | 128 | 2017 |
| | $e + P \to e + h^-$ | 128 | 2017 |
| | $e + P \to e + h^+$ | 23 | 2012 |
| | $e + P \to e + h^-$ | 23 | 2012 |
| | $e + D \to e + \pi^+$ | 23 | 2008 |
| | $e + D \to e + \pi^-$ | 23 | 2008 |
| | $e + D \to e + K^+$ | 23 | 2008 |
| | $e + D \to e + K^-$ | 23 | 2008 |
| | $\pi^- + NH^3 \to \gamma^* + X$ | 15 | 2017 |
| hermes | $e + P \to e + K^-$ | 19 | 2009 |
| | $e + P \to e + K^+$ | 19 | 2009 |
| | $e + P \to e + \pi^0$ | 19 | 2009 |
| | $e + P \to e + \pi^+$ | 19 | 2009 |
| | $e + P \to e + \pi^-$ | 19 | 2009 |
| Jefferson Lab | $e + P \to e + \pi^+$ | 4 | 2011 |
| | $e + P \to e + \pi^-$ | 4 | 2011 |
| RHIC | $P + P \to W^+$ | 8 | 2015 |
| | $P + P \to W^-$ | 8 | 2015 |
| | $P + P \to Z$ | 1 | 2015 |

529 points in total, $\chi^2/dof \approx 1.4$

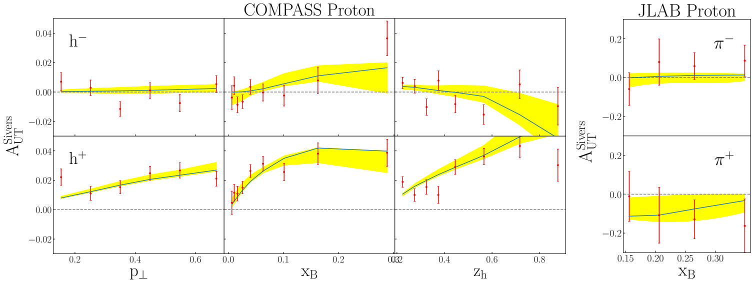
# COMPASS 2017: $e^- + P_{s_\perp} \to e^- + h^-$

128 points in total



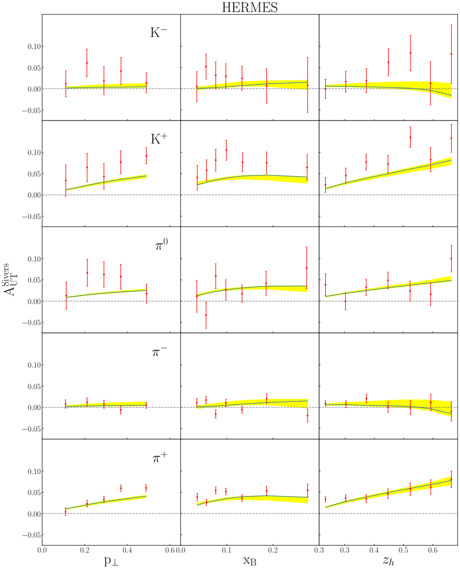
# COMPASS $P$ 2012 and JLab 2011

COMPASS 56 points
JLab 8 points

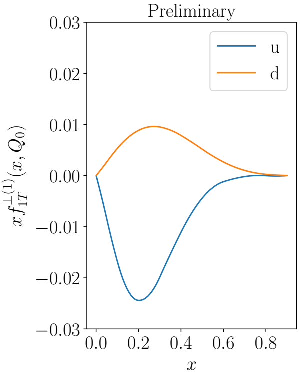


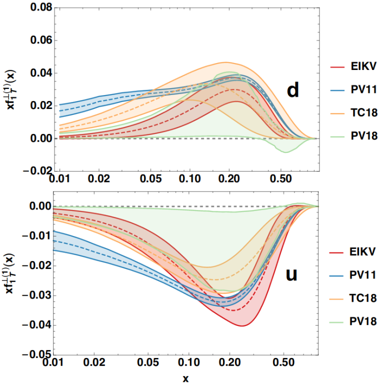
# HERMES 2009

95 points



# Preliminary extraction for Sivers function



# Conclusion

- We generated an optimized Ogata method, highly numerically efficient
- The method can be used for global analysis
- Ogata algorithm will be available open source in the future
- We can use the NLO/NNLL parameterization to describe SIDIS Sivers data
- We hope to soon describe the DY and vector boson data

Thanks!!