Quasi TMDPDFs and Collins-Soper Kernel from Lattice QCD

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[ME, Stewart, Zhao; 1811.00026]
[ME, Stewart, Zhao; 1901.03685]

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Outline

1. Review of TMD factorization
2. Towards Quasi TMDPDFs
3. Collins-Soper kernel from Lattice QCD
4. Conclusions

Review of TMD factorization

Review of TMD factorization

- TMD factorization (e.g. for Drell-Yan, $pp \to Z/\gamma^* \to l^+l^-$): [Collins, Soper, Sterman '85; Becher, Neubert '10; Collins '11; Echevarria, Idilbi, Scimemi '11; Chiu, Jain, Neill, Rothstein '12]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\vec{q}_T} = H(Q) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, f_n^{\mathrm{TMD}}(x_1, \vec{b}_T)\, f_{\bar{n}}^{\mathrm{TMD}}(x_2, \vec{b}_T)\left[1 + \mathcal{O}(q_T^2/Q^2)\right]$$

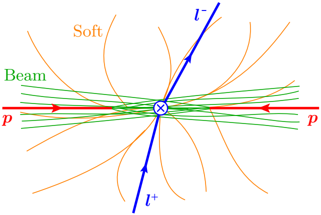
- On closer look: $f_n^{\mathrm{TMD}} = B_n\sqrt{S}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\vec{q}_T} = H(Q) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{b}_T\cdot\vec{q}_T} B_n(x_1, \vec{b}_T) B_{\bar{n}}(x_2, \vec{b}_T) S(b_T)\left[1 + \mathcal{O}(q_T^2/Q^2)\right]$$

- Hard function H: underlying hard process

$$q\bar{q} \to Z/\gamma^* \to l^+l^-$$

- Beam functions $B_{n,\bar{n}}$: collinear radiation
  - Factorize into n and $\bar{n}$ functions
- Soft function S: soft radiation
  - Depends on *both* n and $\bar{n}$
  - Universal: same soft function for DIS



Definitions of TMDPDFs

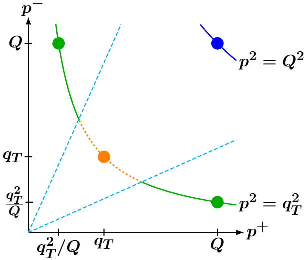
- TMDPDF defined with soft subtraction:

$$f_n^{\mathrm{TMD}}(x,\vec b_T,\mu,\zeta) = \lim_{\substack{\epsilon\to 0\\ \tau\to 0}} Z_{\mathrm{uv}}(\epsilon,\mu,\zeta) B_n(x,\vec b_T,\epsilon,\tau) S(b_T,\epsilon,\tau)$$

  - ϵ: UV regulator $\to$ scale μ
  - τ: rapidity regulator $\to$ Collins-Soper scale ζ

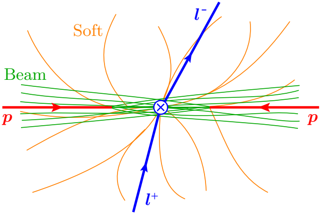
$$\int_{q_T}^{Q}\frac{\mathrm{d}k^+}{k^+} = \int_0^Q \frac{\mathrm{d}k^+}{k^+}R(k^+,\tau,\nu) + \int_{q_T}^{\infty}\frac{\mathrm{d}k^+}{k^+}R(k^+,\tau,\nu)$$

$$= \left(-\frac{1}{\tau}+\ln\frac{Q}{\nu}\right) + \left(\frac{1}{\tau}+\ln\frac{\nu}{q_T}\right) = \ln\frac{Q}{q_T}$$



- Many definitions / schemes in the literature:
  - Wilson lines off light cone [Collins '11]
  - Δ regulator [Echevarria, Idilbi, Scimemi '11]
  - Analytic regulator [Becher, Bell '12]
  - η regulator [Chiu, Jain, Neill, Rothstein '12]
  - Exponential regulator [Li, Neill, Zhu '16]

- but TMDPDF is scheme independent
  - B_n and S not meaningful alone
  - Must determine f^{TMD} on lattice



Definitions of TMDPDFs

$$n = (1, 0, 0, 1)$$

$$b^{\pm} = b^0 \mp b^z$$

- TMDPDF defined with soft subtraction:

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \lim_{\substack{\epsilon \to 0 \\ \tau \to 0}} Z_{\mathrm{uv}}(\epsilon, \mu, \zeta) B_n(x, \vec{b}_T, \epsilon, \tau) S(b_T, \epsilon, \tau)$$

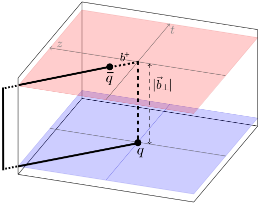
- Beam function definition: $B_n(x, \vec{b}_T, \epsilon, \tau) = \int \frac{\mathrm{d}b^+}{4\pi} e^{-\frac{\mathrm{i}}{2} b^+ (x P_n^-)} B_n(b^+, \vec{b}_T, \epsilon, \tau)$

$$B_n(b^+, \vec{b}_T, \epsilon, \tau) = \langle P(P_n) \big| \bar{q}(b^+, \vec{b}_T) W^{(0, \vec{0}_T)}_{(b^+, \vec{b}_T)} \frac{\gamma^-}{2} q(0) \big| P(P_n) \rangle_\tau$$

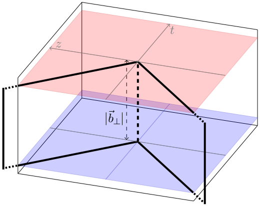
- Soft function definition:

$$S(b_T, \epsilon, \tau) = \langle 0 \big| [S^\dagger_{\bar{n}} S_n](\vec{b}_T) S^{(0, \vec{b}_T)}_{\perp, -\infty n} [S^\dagger_n S_{\bar{n}}](\vec{0}_T) S^{(0, \vec{b}_T)}_{\perp, -\infty \bar{n}} \big| 0 \rangle_\tau$$

- Wilson line paths:



Beam function



Soft function

TMD evolution equations

- Renormalization scale dependence:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}f_q^{\mathrm{TMD}}(x,\vec{b}_T,\mu,\zeta)=\gamma_\mu^q(\mu,\zeta)f_q^{\mathrm{TMD}}(x,\vec{b}_T,\mu,\zeta)$$

  - Anomalous dimension: $\gamma_\mu^q(\mu,\zeta)=\Gamma_C^q[\alpha_s(\mu)]\ln\frac{\mu^2}{\zeta}+\gamma_\mu^q[\alpha_s(\mu)]$
  - Evolution perturbative for $\mu\gtrsim\Lambda_{\mathrm{QCD}}$

- Collins-Soper evolution:

$$\boxed{\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}f_q^{\mathrm{TMD}}(x,\vec{b}_T,\mu,\zeta)=\frac{1}{2}\gamma_\zeta^q(\mu,b_T)f_q^{\mathrm{TMD}}(x,\vec{b}_T,\mu,\zeta)}$$

  - Anomalous dimension: (independent of hadron state!)

$$\gamma_\zeta^q(\mu,b_T)=-2\int_{1/b_T}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma_C^q[\alpha_s(\mu')]+\gamma_\zeta^q[\alpha_s(1/b_T)]$$

  - Nonperturbative for $b_T\sim\Lambda_{\mathrm{QCD}}^{-1}$, *independently* of μ,ζ

- Collins-Soper scale fixed to $\zeta_n\zeta_{\bar{n}}=Q^4$ → think $\zeta_n=(x_nP_n^-)^2$
- Nonperturbative f^{TMD}, determined at low scales $\mu\sim\sqrt{\zeta}\sim\mathrm{GeV}$, must be evolved to collider scales $\mu\sim\sqrt{\zeta}\sim Q$
  - Requires nonperturbative knowledge of γ_ζ^q

Towards Quasi TMDPDFs

1. Calculate nonperturbative Collins-Soper kernel $\gamma_\zeta^q(\mu, b_T)$
   [ME, Stewart, Zhao; 1811.00026]
2. Calculate nonperturbative TMDPDF $f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)$
   [ME, Stewart, Zhao; 1901.03865]
   (see also [Ji, Jin, Yuan, Zhang, Zhao; 1801.05930])

Reminder: Collinear quasi PDF

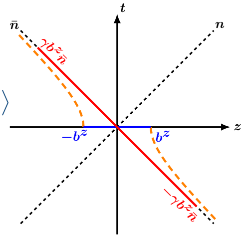
$$n = (1, 0, 0, 1)$$
$$b^\pm = b^0 \mp b^z$$

- PDF:

$$f_q(x, \mu) = \int \frac{\mathrm{d}b^+}{4\pi} e^{\mathrm{i} b^+ (x P_n^-)} \langle P(P_n) | \bar{q}(b^+) W_n(b^+, 0)\, (\gamma_0 + \gamma_3)\, q(0) | P(P_n) \rangle$$

- Quasi PDF: Equal-time correlator [Ji '13, '14]

$$\tilde{f}_q(x, P_z, \mu) = \int \frac{\mathrm{d}b^z}{4\pi} e^{\mathrm{i} b^z (x P_z)} \langle P(P_{b^z}) | \bar{q}(b^z) W_{\hat{z}}(b^z, 0) \gamma_3 q(0) | P(P_z) \rangle$$



- Factorization theorem:
  [Xiong, Ji, Zhang, Zhao '13; Ma, Qiu '14 '17;
  Izubuchu, Ji, Jin, Stewart, Zhao '18]

$$\underbrace{\boxed{\tilde{f}_i(x, P_z, \tilde{\mu})}}_{\text{Simulation/renormalization on lattice}} = \int_{-1}^{1} \frac{\mathrm{d}y}{y} \underbrace{\boxed{C_{ij}\left(\frac{x}{y}, \frac{\tilde{\mu}}{P_z}, \frac{\mu}{y P_z}\right)}}_{\text{Perturbative matching coefficient}} \underbrace{\boxed{f_j(y, \mu)}}_{\text{PDF}} + \underbrace{\boxed{\mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{x^2 P_z^2}\right)}}_{\text{Higher-twist correction}}$$

Towards Quasi TMDPDFs

Roadmap:

1. Construct quasi beam function $\tilde{B}_q(x, \vec{b}_T, \dots)$
   and quasi soft function $\tilde{S}^q(b_T, \dots)$
   - Must be computable on lattice
   - Must regulate UV and rapidity divergences on lattice
2. Combine into quasi TMDPDF

$$\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \dots) = \tilde{Z}_{\mathrm{uv}}(\dots)\tilde{B}_q(x, \vec{b}_T, \dots)\tilde{S}^q(b_T, \dots)$$

   - Rapidity divergences must cancel
3. Derive *perturbative* matching

$$\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \dots) = (C^{\mathrm{TMD}} \otimes f_q^{\mathrm{TMD}})(x, \vec{b}_T, \mu, \zeta)$$

Previous work:

- Importance of soft subtraction to construct quasi TMDPDFs [Ji, Sun, Xiong, Yuan '15]
- Regularization of rapidity divergences by finite lattice size [Ji, Jin, Yuan, Zhang, Zhao '18]

Constructing the quasi beam function

$$n = (1, 0, 0, 1)$$
$$b^{\pm} = b^0 \mp b^z$$

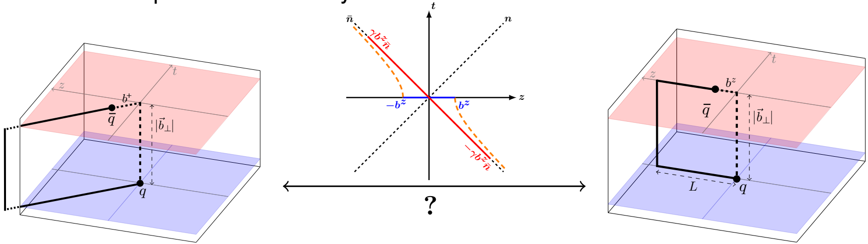
- Beam function: (light-cone correlator)

$$B_q(x, \vec{b}_T, \dots) = \int \frac{\mathrm{d}b^+}{4\pi} e^{-\frac{\mathrm{i}}{2} b^+ (xP^-)} \Big\langle p(P) \Big| \bar{q}(b^\mu) W^{(0,\vec{0}_T)}_{(b^+,\vec{b}_T)} \frac{\gamma^-}{2} q(0) \Big| p(P) \Big\rangle$$

- Quasi beam function: (equal-time correlator)

$$\tilde{B}_q(x, \vec{b}_T, \dots) = \int \frac{\mathrm{d}b^z}{2\pi} e^{\mathrm{i} b^z (xP^z)} \Big\langle p(P) \Big| \bar{q}(b^\mu) W^{(0,\vec{0}_T)}_{(b^z,\vec{b}_T)} \frac{\gamma^3}{2} q(0) \Big| p(P) \Big\rangle$$

- Wilson line path:
  - Finite lattice size requires to truncate at length L
  - Bare operators related by Lorentz boost



Constructing the quasi soft function

$$n = (1, 0, 0, 1)$$
$$\bar{n} = (1, 0, 0, -1)$$

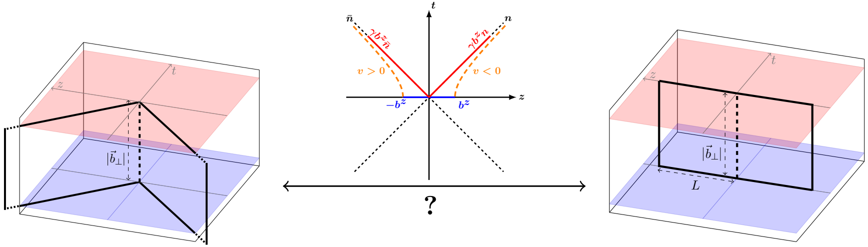
- Soft function: (light-cone correlator)

$$S^q(b_T) = \langle 0 \big| [S_n^\dagger S_T S_{\bar{n}}](\vec{b}_T)[S_{\bar{n}}^\dagger S_T^\dagger S_n](\vec{0}_T) \big| 0 \rangle$$

- Quasi soft function: (equal-time correlator)

$$\tilde{S}^q(b_T) = \langle 0 \big| [S_{\hat{z}}^\dagger S_T S_{-\hat{z}}](\vec{b}_T)[S_{-\hat{z}}^\dagger S_T S_{\hat{z}}](\vec{0}_T) \big| 0 \rangle$$

- Wilson line path:
  - Finite lattice size requires to truncate at length L
  - Bare operators *not* related by Lorentz boost (more on this later)



Constructing the quasi TMDPDF

- Recall physical TMDPDF:

$$f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \lim_{\substack{\epsilon\to 0\\ \tau\to 0}} Z_{\mathrm{uv}}(\epsilon, \mu, \zeta) B_q(x, \vec{b}_T, \epsilon, \tau, xP) S^q(b_T, \epsilon, \tau)$$

- On the lattice:

$$\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z) = \int \frac{\mathrm{d}b^z}{2\pi} e^{\mathrm{i}b^z(xP^z)} \lim_{\substack{a\to 0\\ L\to\infty}} \tilde{Z}'_q(b^z, \mu, \tilde{\mu}) \tilde{Z}^q_{\mathrm{uv}}(b^z, \tilde{\mu}, a) \times \tilde{B}_q(b^z, \vec{b}_T, a, L, P^z) \tilde{S}^q(b_T, a, L)$$

- Lattice regulators:
  - Lattice spacing a acts as UV regulator
  - Wilson line length L acts as rapidity regulator
  - Must extrapolate $L \to \infty, a \to 0$
- Renormalization on lattice:
  - Need to subtract Wilson-line self energies (vanish on light-cone)
  - Subtraction $\tilde{Z}^q_{\mathrm{uv}}$ multiplicative in b^z-space
  - $\tilde{Z}'_q$ converts from lattice renormalization scheme ($\tilde{\mu}$) to $\overline{\mathrm{MS}}$ scheme (μ)

Relating quasi TMDPDF and TMDPDF

- Goal: perturbative matching between TMDPDF and quasi TMDPDF
- Due to soft mismatch: expect *nonperturbative* relation

$$\overset{\text{Quasi-TMD from lattice}}{\boxed{\tilde{f}_i^{\rm TMD}(x, \vec{b}_T, \mu, P^z)}} = \overset{\text{Perturbative kernel}}{\boxed{C_{ij}^{\rm TMD}(\mu, xP^z)}} \overset{\text{TMDPDF}}{\boxed{f_j^{\rm TMD}(x, \vec{b}_T, \mu, \zeta)}}$$

$$\times \underset{\text{Soft mismatch}}{\boxed{g_{ij}^S(b_T, \mu)}} \underset{\text{Collins-Soper kernel (ensures } \zeta \text{ independence)}}{\boxed{\exp\left[\frac{1}{2} \ln \frac{(2xP^z)^2}{\zeta} \gamma_\zeta^j(\mu, b_T)\right]}}$$

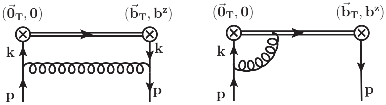
- Perturbative matching requires
  - $\zeta = (2xP^z)^2$
  - Quasi-soft functions such that $g^S = 1$
- or to take ratios such that g^S, γ_ζ cancel
- Relation not proven, but verified at NLO

Verification of matching at NLO

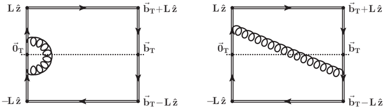
- Test matching relation perturbatively at NLO
  - Work in $\overline{\rm MS}$ scheme, not lattice renormalization
  - On-shell external quark state
  - Ignore mixing with gluons (→ isovector quark)
- Quasi-TMDPDF becomes simple product:

$$\tilde{f}_q^{\rm TMD}(x, \vec{b}_T, \mu, P^z) = \lim_{\substack{\epsilon\to0\\ L\to0}} \tilde{Z}^q_{\rm uv}(\mu, P^z, \epsilon) \frac{\tilde{B}_q(x, \vec{b}_T, \epsilon, L, P^z)}{\sqrt{\tilde{S}^q(b_T, \epsilon, L)}}$$

  - Precise form fixed by cancellation of rapidity divergences L/b_T
- Example diagrams:



Quasi beam function



Quasi soft function

Verification of matching at NLO

- Result at one loop: (nonsinglet $q = u - d$)

$$\frac{\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z)}{f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)} = 1 + \frac{\alpha_s C_F}{2\pi}\Big[-\frac{1}{2}\ln^2\frac{(2xP^z)^2}{\mu^2} + \ln\frac{(2xP^z)^2}{\mu^2} + \ln(b_T^2\mu^2) - \ln(b_T^2\mu^2)\ln\frac{(2xP^z)^2}{\zeta} + \cdots\Big]$$

- Compare to matching formula:

$$\frac{\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z)}{f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)} = C_q^{\mathrm{TMD}}(\mu, xP^z)\, g_q^S(b_T, \mu)\, \exp\Big[\frac{1}{2}\ln\frac{(2xP^z)^2}{\zeta}\gamma_\zeta^j(\mu, b_T)\Big]$$

- Comparison:
  - Perturbative kernel C_q^{TMD}
  - Nonperturbative Collins-Soper kernel γ_ζ^q
    $\Rightarrow$ NLO results confirms that $\zeta = (2xP^z)^2$
  - Leftover nonperturbative logarithm $\ln(b_T^2\mu^2)$
- Interpretation: remnant of failure of relating soft factors S^q and $\tilde{S}^q$

Bent soft function

- Recall soft and quasi soft function:

$$S^q(b_T) = \langle 0 \big| [S_n^\dagger S_T S_{\bar n}](\vec b_T)[S_{\bar n}^\dagger S_T^\dagger S_n](\vec 0_T) \big| 0 \rangle$$

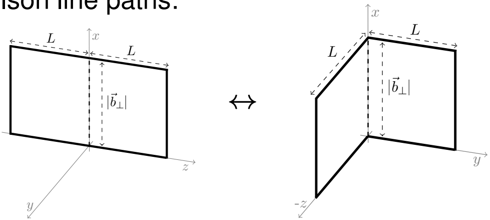
$$\tilde S^q(b_T) = \langle 0 \big| [S_{\hat z}^\dagger S_T S_{-\hat z}](\vec b_T)[S_{-\hat z}^\dagger S_T S_{\hat z}](\vec 0_T) \big| 0 \rangle$$

- $S^q(b_T)$ and $\tilde S^q(b_T)$ *not* related through Lorentz boost
- Define "bent" soft function to remove boost-violating diagrams at NLO:

$$\tilde S^q_{\mathrm{bent}}(b_T) = \langle 0 \big| [S_{\hat z}^\dagger S_T S_{-\bar n_\perp}](\vec b_T)[S_{-\bar n_\perp}^\dagger S_T S_{\hat z}](\vec 0_T) \big| 0 \rangle$$

- Compare Wilson line paths:

$\tilde S^q(b_T) =$



$= \tilde S^q_{\mathrm{bent}}(b_T)$

- Yields a perturbative matching relation at NLO
  - Proof beyond NLO required

Collins-Soper kernel from Lattice QCD

[ME, Stewart, Zhao; 1811.00026]

Collins-Soper kernel

- Recall TMD factorization for $pp \to Z/\gamma^* \to l^+ l^-$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^2\vec{q}_T} = H(Q,\mu) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, f_n^{\mathrm{TMD}}(x_1, \vec{b}_T, \mu, \zeta_1)\, f_{\bar{n}}^{\mathrm{TMD}}(x_2, \vec{b}_T, \mu, \zeta_2)$$

- Recall TMDPDF evolution equations:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \gamma_\mu^q(\mu, \zeta) f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)$$

$$\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta} f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \frac{1}{2}\gamma_\zeta^q(\mu, b_T) f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)$$

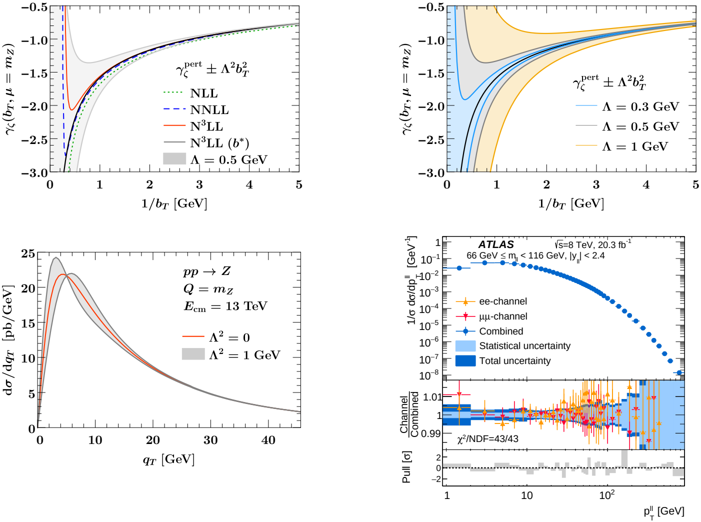
- Combined solution:

$$f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta) = \exp\left[\int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \gamma_\mu^q(\mu', \zeta_0)\right] \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{\zeta}{\zeta_0}\right] f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu_0, \zeta_0)$$

- Solution resums large logarithms $\ln(Q^2 b_T^2) \sim \ln\dfrac{Q^2}{q_T^2}$

- γ_ζ^q required to connect lattice calculations / nonperturbative extractions at $\mu_0 \sim P^z \sim \mathcal{O}(\mathrm{GeV})$ to scales in factorization theorem $\mu \sim Q, P^z \sim Q/x$

Collins-Soper kernel



Nonperturbative knowledge of $\gamma_\zeta^q(\mu, b_T)$ important for (LHC) phenomenology.

Collins-Soper kernel from Lattice QCD

- Reminder: Isovector quark (quasi) TMDPDFs related by (at NLO)

$$\frac{\tilde f_q^{\rm TMD}(x, \vec b_T, \mu, P^z)}{f_q^{\rm TMD}(x, \vec b_T, \mu, \zeta)} = C_q^{\rm TMD}(\mu, xP^z)\, g_q^S(b_T, \mu) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{(2xP^z)^2}{\zeta}\right]$$

- Nonperturbative factor $g_q^S(b_T, \mu)$ cancels in ratios with same b_T and μ
- Factor out Collins-Soper kernel by varying proton momenta $P_1^z \neq P_2^z$:

$$\begin{aligned} \gamma_\zeta^q(\mu, b_T) &= \frac{1}{\ln(P_1^z/P_2^z)} \ln\frac{\tilde f_q^{\rm TMD}(x, \vec b_T, \mu, P_1^z)\, C_q^{\rm TMD}(\mu, xP_2^z)}{\tilde f_q^{\rm TMD}(x, \vec b_T, \mu, P_2^z)\, C_q^{\rm TMD}(\mu, xP_1^z)} \\ &= \frac{1}{\ln(P_1^z/P_2^z)} \ln\frac{C_q^{\rm TMD}(\mu, xP_2^z) \int {\rm d}b^z e^{{\rm i}b^z xP_1^z}\, \tilde Z' \tilde Z_{\rm uv} \tilde B_q(b^z, \vec b_T, a, L, P_1^z)}{C_q^{\rm TMD}(\mu, xP_1^z) \int {\rm d}b^z e^{{\rm i}b^z xP_2^z}\, \tilde Z' \tilde Z_{\rm uv} \tilde B_q(b^z, \vec b_T, a, L, P_2^z)} \end{aligned}$$

- Independent of hadron state: Can use pion state for simplicity
- Soft factor cancels in ratio → sufficient to calculate beam function
- First exploratory studies already in progress

Important universal QCD quantity within reach of lattice QCD!

Ratios of TMDPDFs

- Reminder: Isovector quark (quasi) TMDPDFs related by (at NLO)

$$\frac{\tilde{f}_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z)}{f_q^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)} = C_q^{\mathrm{TMD}}(\mu, xP^z)\, g_q^S(b_T, \mu) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{(2xP^z)^2}{\zeta}\right]$$

- Nonperturbative factor $g_q^S(b_T, \mu)$ cancels in ratios with same b_T and μ
- In general: Collins-Soper kernel γ_ζ^q must cancel as well
  - Must choose same values for x, P^z, ζ
- Ex.: ratios of different hadron states $h_1 \neq h_2$ (e.g. proton, neutron, pion):

$$\frac{\tilde{f}_{q/h_1}^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z)}{\tilde{f}_{q/h_2}^{\mathrm{TMD}}(x, \vec{b}_T, \mu, P^z)} = \frac{C_{q/h_1}^{\mathrm{TMD}}(\mu, xP^z)}{C_{q/h_2}^{\mathrm{TMD}}(\mu, xP^z)} \frac{f_{q/h_1}^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)}{f_{q/h_2}^{\mathrm{TMD}}(x, \vec{b}_T, \mu, \zeta)}$$

- Similarly: ratios with different spin structures
  - Similar ratios have already been considered using moments in x
    [Musch et al ’10 ’12; Engelhardt et al ’15; Yoon et al ’17]
  - See talk my M. Engelhardt later

Ratios of TMDPDFs

- Reminder: Isovector quark (quasi) TMDPDFs related by (at NLO)

$$\frac{\tilde{f}_q^{\rm TMD}(x, \vec{b}_T, \mu, P^z)}{f_q^{\rm TMD}(x, \vec{b}_T, \mu, \zeta)} = C_q^{\rm TMD}(\mu, xP^z)\, g_q^S(b_T, \mu) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{(2xP^z)^2}{\zeta}\right]$$

- Nonperturbative factor $g_q^S(b_T, \mu)$ cancels in ratios with same b_T and μ
- In general: Collins-Soper kernel γ_ζ^q must cancel as well
  - Must choose same values for x, P^z, ζ
- Ex.: ratios of different hadron states $h_1 \neq h_2$ (e.g. proton, neutron, pion):

$$\frac{\tilde{f}_{q/h_1}^{\rm TMD}(x, \vec{b}_T, \mu, P^z)}{\tilde{f}_{q/h_2}^{\rm TMD}(x, \vec{b}_T, \mu, P^z)} = \frac{C_{q/h_1}^{\rm TMD}(\mu, xP^z)}{C_{q/h_2}^{\rm TMD}(\mu, xP^z)}\, \frac{f_{q/h_1}^{\rm TMD}(x, \vec{b}_T, \mu, \zeta)}{f_{q/h_2}^{\rm TMD}(x, \vec{b}_T, \mu, \zeta)}$$

- Similarly: ratios with different spin structures
  - Similar ratios have already been considered using moments in x

    [Musch et al '10 '12; Engelhardt et al '15; Yoon et al '17]

Lattice can give important constraints on physical TMDPDFs after all!

Conclusions

Conclusions

Summary:

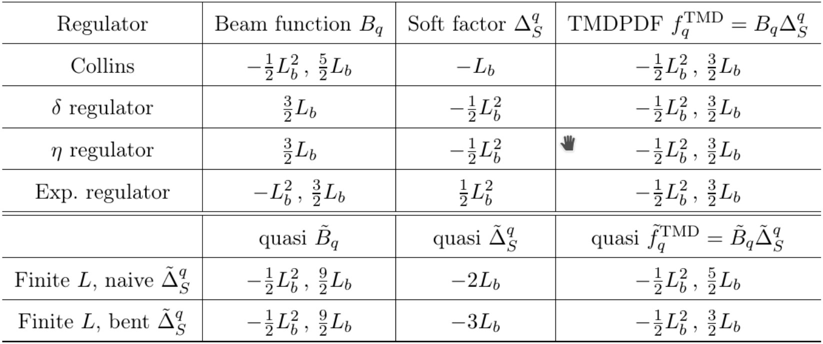
- TMDs are in general much less constrained from experiment than PDFs
  - Determination from lattice desired
- TMD factorization more complicated:
  - Wilson line paths
  - Rapidity divergences
  - Combining hadronic & soft vacuum matrix matrix element
- In general: nonperturbative relation between quasi-TMDPDF and TMDPDF
  - Proposed perturbative relation using bent soft function at NLO)
- Proposed method to determine Collins-Soper kernel from Lattice
- More general: can study ratios of TMDPDFs on lattice (→ see also M. Engelhardt's talk)

Outlook:

- Need proof for matching relation (beyond one loop)
- Calculate γ_ζ^q from lattice (→ see Y. Zhao's talk in the afternoon)

Backup slides

Overview of rapidity regulators



Regulator	Beam function B_q	Soft factor Δ_S^q	TMDPDF $f_q^{\mathrm{TMD}} = B_q \Delta_S^q$
Collins	$-\frac{1}{2}L_b^2\,,\ \frac{5}{2}L_b$	$-L_b$	$-\frac{1}{2}L_b^2\,,\ \frac{3}{2}L_b$
δ regulator	$\frac{3}{2}L_b$	$-\frac{1}{2}L_b^2$	$-\frac{1}{2}L_b^2\,,\ \frac{3}{2}L_b$
η regulator	$\frac{3}{2}L_b$	$-\frac{1}{2}L_b^2$	$-\frac{1}{2}L_b^2\,,\ \frac{3}{2}L_b$
Exp. regulator	$-L_b^2\,,\ \frac{3}{2}L_b$	$\frac{1}{2}L_b^2$	$-\frac{1}{2}L_b^2\,,\ \frac{3}{2}L_b$
	quasi $\tilde{B}_q$	quasi $\tilde{\Delta}_S^q$	quasi $\tilde{f}_q^{\mathrm{TMD}} = \tilde{B}_q \tilde{\Delta}_S^q$
Finite L, naive $\tilde{\Delta}_S^q$	$-\frac{1}{2}L_b^2\,,\ \frac{9}{2}L_b$	$-2L_b$	$-\frac{1}{2}L_b^2\,,\ \frac{5}{2}L_b$
Finite L, bent $\tilde{\Delta}_S^q$	$-\frac{1}{2}L_b^2\,,\ \frac{9}{2}L_b$	$-3L_b$	$-\frac{1}{2}L_b^2\,,\ \frac{3}{2}L_b$

- Dependence of beam and soft function on $L_b = \ln \dfrac{b_T^2 \mu^2}{4e^{-2\gamma_E}}$ depends on regulator
- No agreement between beam function and quasi beam function in any regulator
- Dependence of TMDPDF on L_b is independent of regulator

Rapidity divergences at finite L

- Recall: Rapidity divergences arise from integrals of type

$$I_{\rm div} = \int \frac{{\rm d}k^+ {\rm d}k^-}{(k^+k^-)^\epsilon} \frac{f(k^+k^-)}{k^+k^-} = \int \frac{{\rm d}(k^+/k^-)}{2\,k^+/k^-} \int \frac{{\rm d}(k^+k^-)}{(k^+k^-)^\epsilon} \frac{f(k^+k^-)}{k^+k^-}$$

  - Integrand depends only on product k^+k^-

- Eikonal propagator for $L < \infty$:

$$\frac{1}{k^\pm + {\rm i}0} \to \frac{1 - e^{{\rm i}k^\pm L}}{k^\pm}$$

- With finite L:

$$I_{\rm div} \to \int {\rm d}k^+ {\rm d}k^- \frac{f(k^+k^-)}{(k^+k^-)^\epsilon} \frac{1 - e^{{\rm i}k^+ L}}{k^+} \frac{1 - e^{-{\rm i}k^- L}}{k^-}$$

  - Finite L fully regulates $k^\pm \to 0$
  - No rapidity divergences for finite L