Orbital Angular Momentum at Small x

Yuri Kovchegov
The Ohio State University

arXiv:1901.07453 [hep-ph]

Outline

- Introduction and goal: evaluating quark and gluon OAM at small x.
- Tools: brief review of quark and gluon helicity evolution at small x. The 3-step “simplify-evolve-solve” prescription.
- OAM at small x:
  - Quark OAM
  - Gluon OAM

Introduction and goals

Proton Spin Puzzle

- Helicity sum rule (Jaffe-Manohar form): $$\frac{1}{2} = S_q + L_q + S_g + L_g$$

  with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

- The helicity parton distributions are

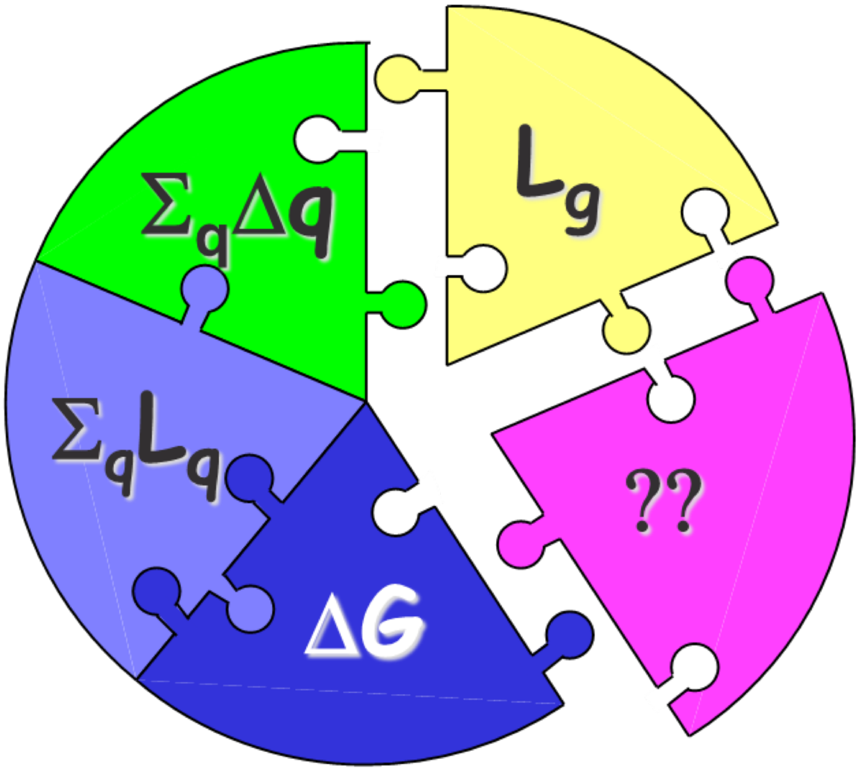
$$\Delta f(x, Q^2) \equiv f^+(x, Q^2) - f^-(x, Q^2)$$

  with the net quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

- L_q and L_g are the quark and gluon orbital angular momenta

Piecing Together the Proton Spin



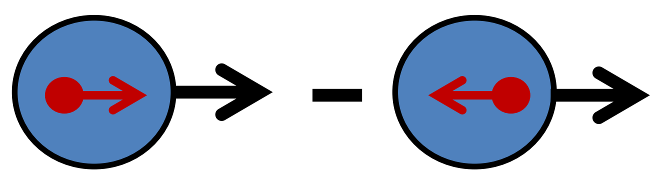
$$S = \frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L$$

- Simple picture of proton composed of three valence quarks is superseded by complex interaction of quarks, antiquarks, and gluons
- The proton's spin must arise from combination of spin and orbital angular momenta of quarks and gluons. Does it? – the proton spin puzzle

$$\Delta\Sigma = \int (\Delta u + \Delta d + \Delta s + \Delta\bar{u} + \Delta\bar{d} + \Delta\bar{s} + \cdots) dx$$

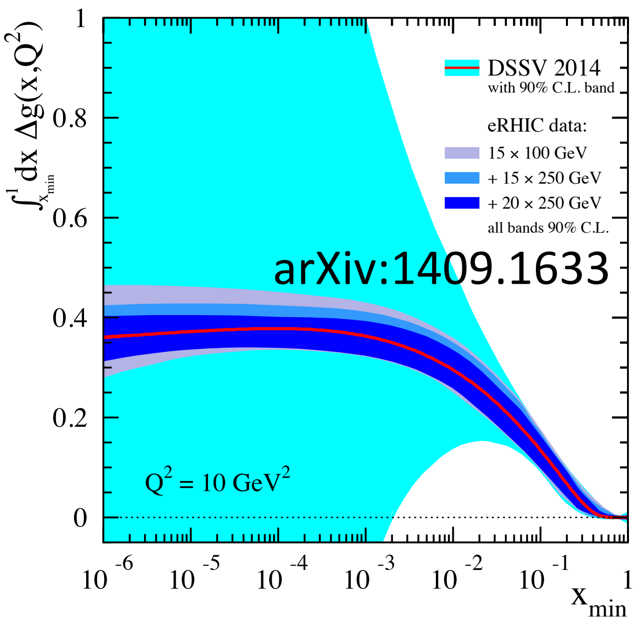
$$\Delta G = \int \Delta g(x)\, dx$$

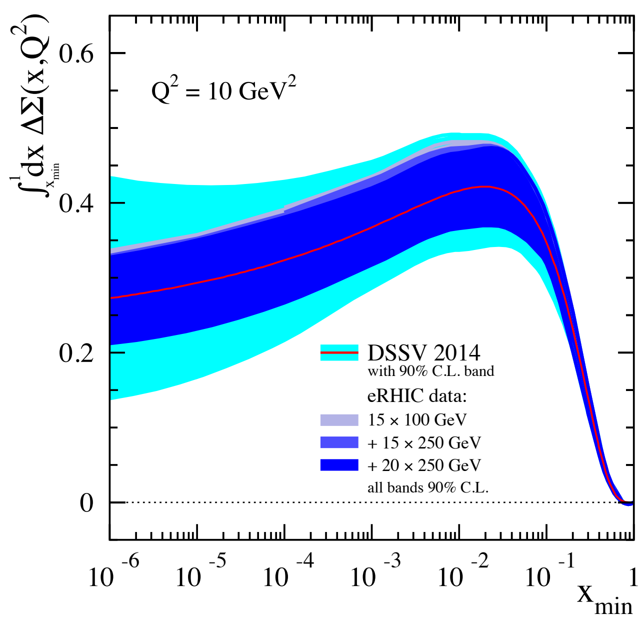
Helicity Distributions: Δq, Δg

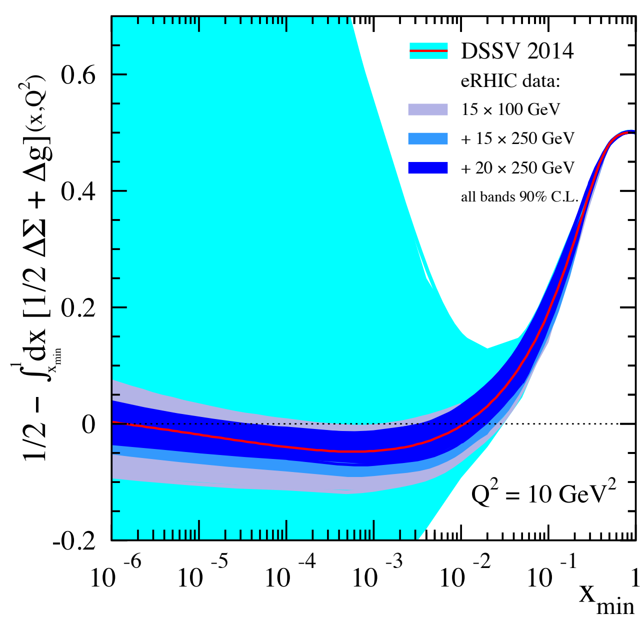


``Borrowed" from Brian Page

EIC: Solving the Spin Puzzle







1/2 - Gluon - Quarks = orbital angular momentum

- Above plot shows the running integral of $\Delta g(x,Q^2)$ from x_{min} to 1 as a function of x_{min}
- Large reduction in uncertainty on ΔG from EIC can be seen

- EIC will also reduce the uncertainty on the quark contribution to the proton spin

Constraints on gluon and quark contributions will provide information on the orbital angular momentum component of proton spin

``Borrowed” from Brian Page

OAM at small x

- Quark and gluon OAM are an integral part of spin puzzle

- Theoretical input is needed to constrain the amount of OAM at small x

- Theoretical predictions for OAM at small x may be compared to the data to be generated by the EIC.

Tools

Quark Helicity at Small x
(flavor-singlet case)

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph], arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph], arXiv:1703.05809 [hep-ph], arXiv:1808.09010 [hep-ph]

Quark Helicity TMD

- We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

$$g_{1L}^q(x, k_T^2) = \frac{1}{(2\pi)^3} \, \frac{1}{2} \sum_{S_L} S_L \, \int d^2 r \, dr^- \, e^{ik\cdot r} \langle p, S_L | \bar{\psi}(0) \, \mathcal{U}[0, r] \, \frac{\gamma^+ \gamma^5}{2} \, \psi(r) | p, S_L \rangle_{r^+ = 0}$$

- At small-x, in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in A$^-$=0 gauge for the + moving proton)

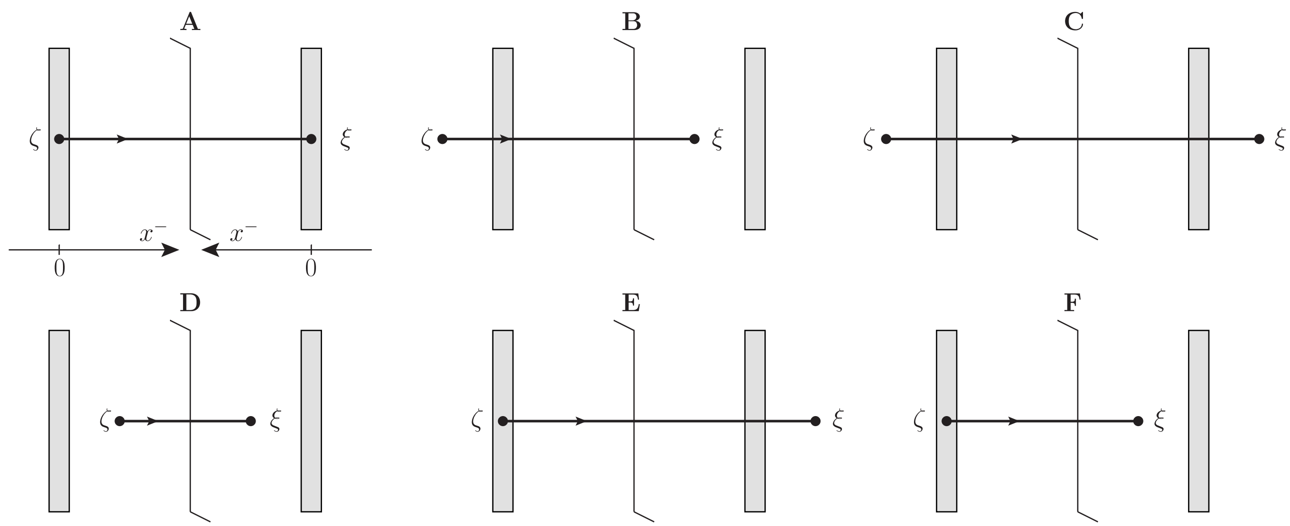
$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \, \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta - \xi)} \left(\frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \;\; V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

where the fundamental light-cone Wilson line is

$$V_{\underline{x}}[b^-, a^-] = \mathrm{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$

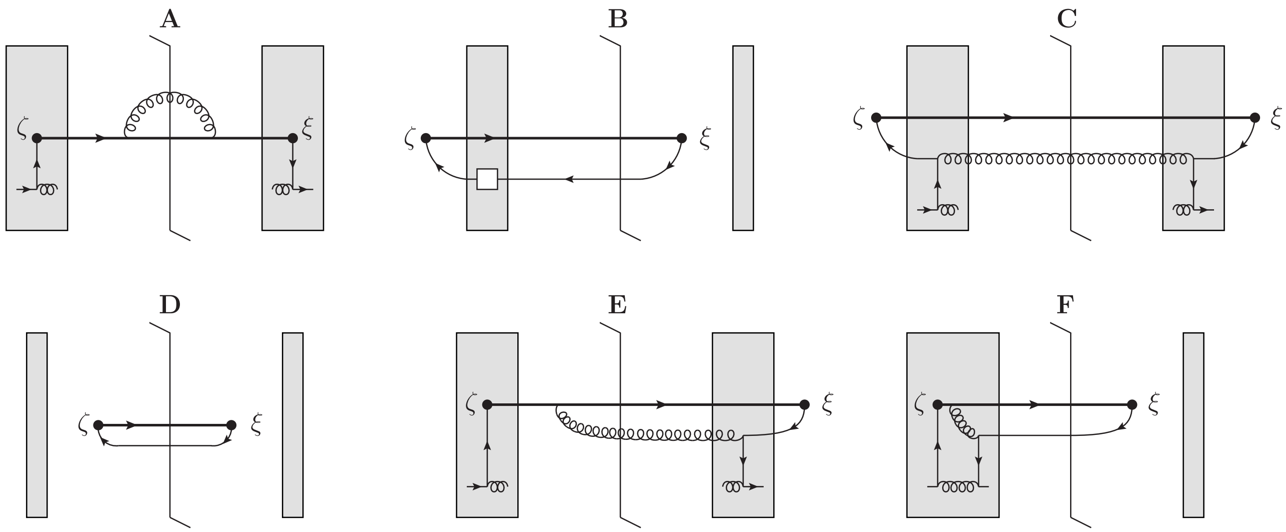
Quark Helicity TMD at Small x

- At high energy/small-x the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:



$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \sum_X \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\tfrac{1}{2}\gamma^+\gamma^5\right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \, |X\rangle \; \langle X| \, V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \right\rangle$$

Quark Helicity TMD at Small x



- Diagram D does not transfer spin information from the target. Diagram C is canceled as we move t-channel quarks across the cut.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other at the leading (DLA) order (Ward identity).
- We are left with the diagram B.

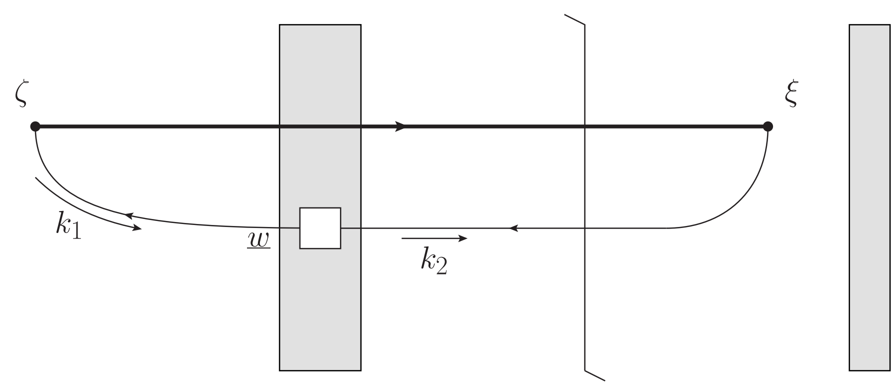
Quark Helicity TMD at Small x

- Evaluating diagram B we arrive at

$$g_{1L}^q(x, k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta \, d^2w \, d^2y \, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2} \, G_{\underline{w},\underline{\zeta}}(zs)$$

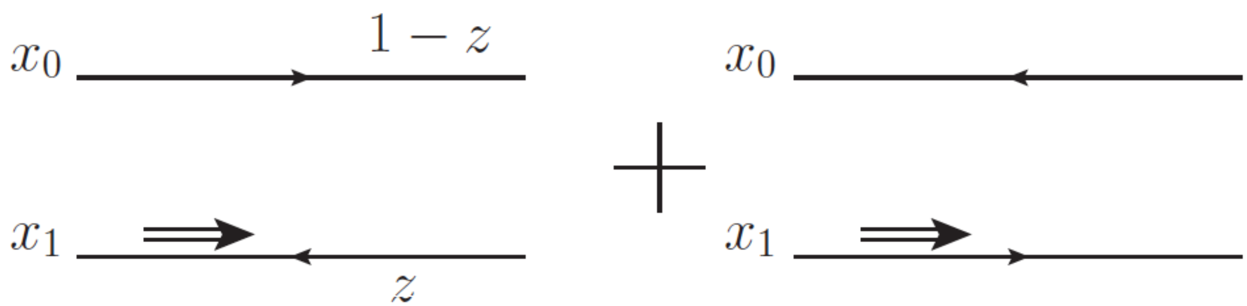
  where $G_{w\zeta}$ is the polarized dipole amplitude (defined on the next slide).

- Here s is the cms energy squared, Λ is some IR cutoff, underlining denotes transverse vectors, z = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark



Polarized Dipole

- All flavor-singlet small-x helicity observables depend on one object, "polarized dipole amplitude":



$$G_{10}(z) \equiv \frac{1}{2N_c} \text{Re} \left\langle\!\!\left\langle \text{T tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] + \text{T tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

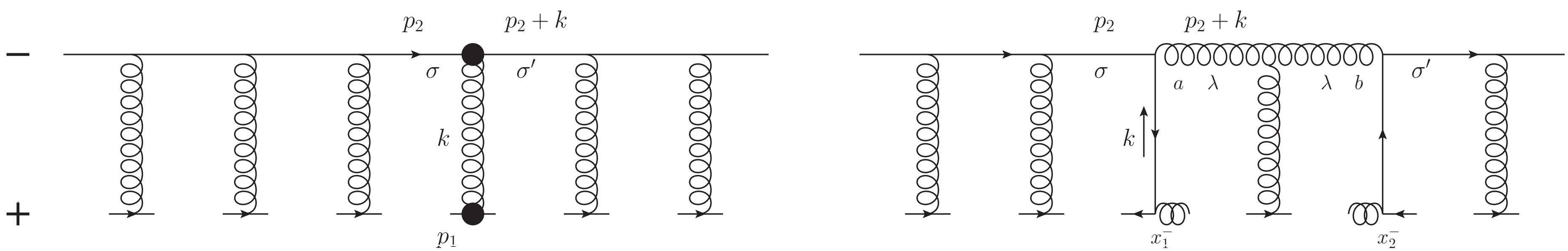
$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle (z) \equiv zs \left\langle \mathcal{O} \right\rangle (z)$$

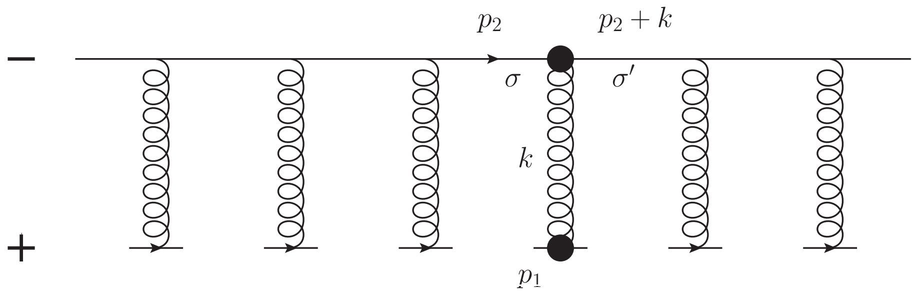
Polarized fundamental "Wilson line"

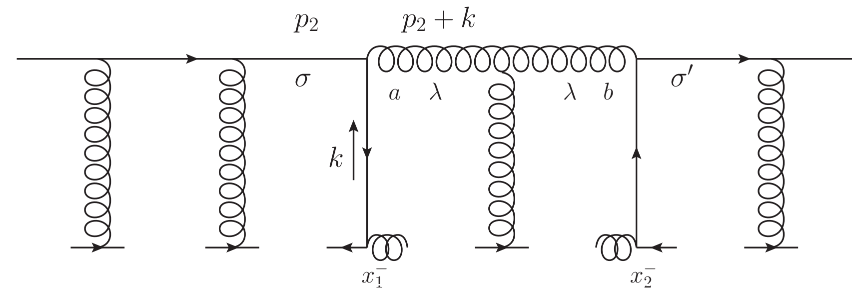
- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized "Wilson line" V^{pol}, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.

Polarized fundamental "Wilson line"





- In the end one arrives at (cf. Chirilli '18)

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

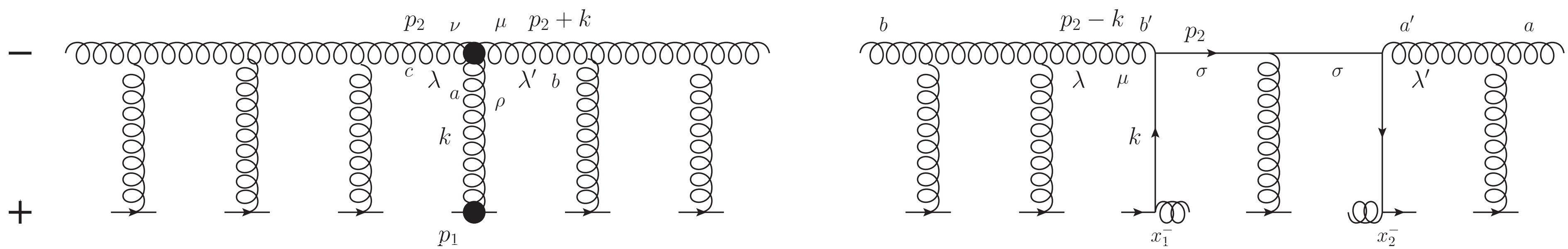
$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5\right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

- The first term on the right (the gluon exchange contribution) was known before (KPS '17), the second term (quark exchange) is new.
- We have employed an adjoint light-cone Wilson line

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.



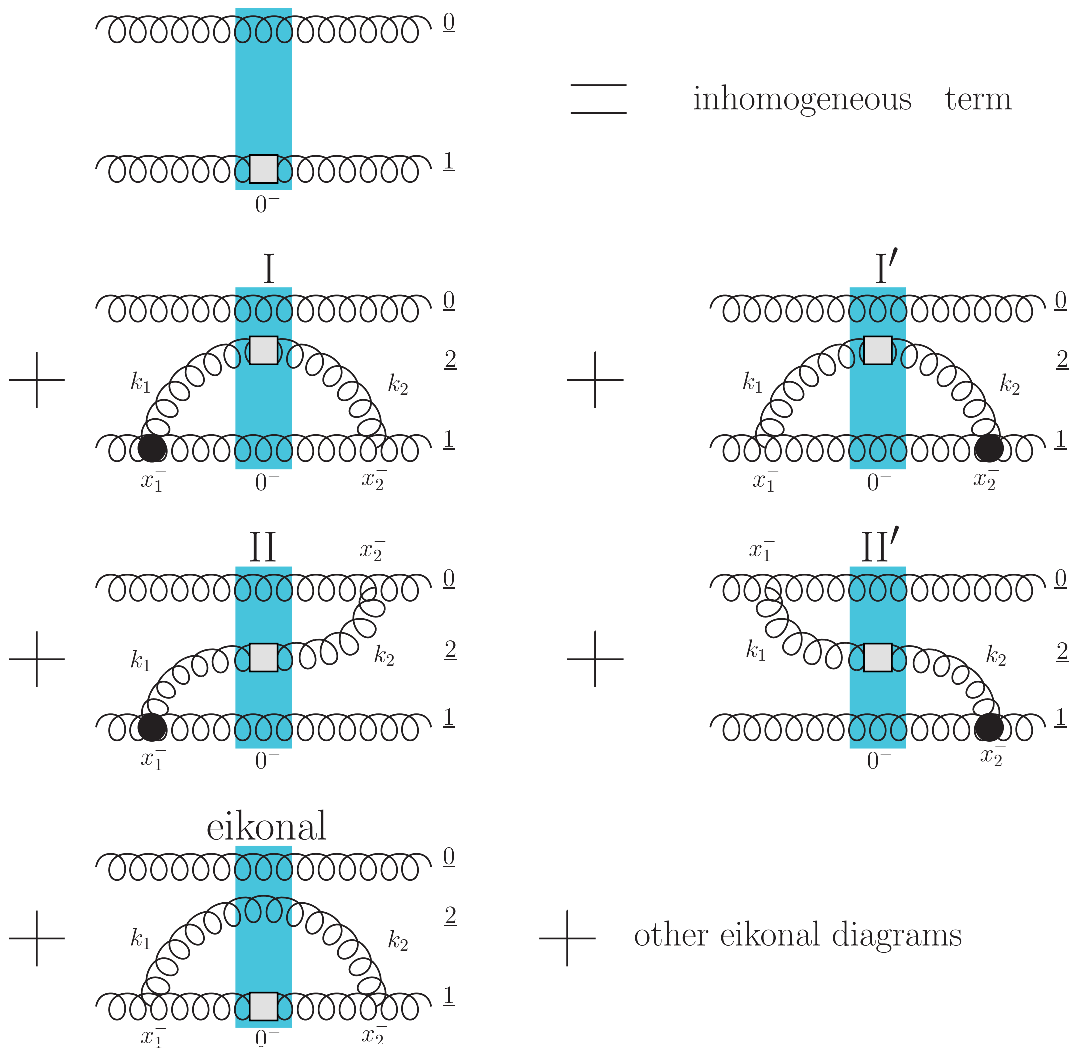
- The calculation is similar to the quark scattering case. It yields (cf. Chirilli ‘18)

$$(U_{\underline{x}}^{pol})^{ab} = \frac{2i\, g\, p_1^+}{s} \int\limits_{-\infty}^{+\infty} dx^- \ \left(U_{\underline{x}}[+\infty, x^-]\, \mathcal{F}^{12}(x^+ = 0, x^-, \underline{x})\, U_{\underline{x}}[x^-, -\infty] \right)^{ab}$$

$$- \frac{g^2\, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^-\, U_{\underline{x}}^{aa'}[+\infty, x_2^-]\, \bar{\psi}(x_2^-, \underline{x})\, t^{a'}\, V_{\underline{x}}[x_2^-, x_1^-]\, \frac{1}{2}\, \gamma^+ \gamma_5\, t^{b'}\, \psi(x_1^-, \underline{x})\, U_{\underline{x}}^{b'b}[x_1^-, -\infty] - \text{c.c.}$$

Small-x Evolution at large N_c

- We need to sum the following diagrams (box denotes the polarized "Wilson lines"):



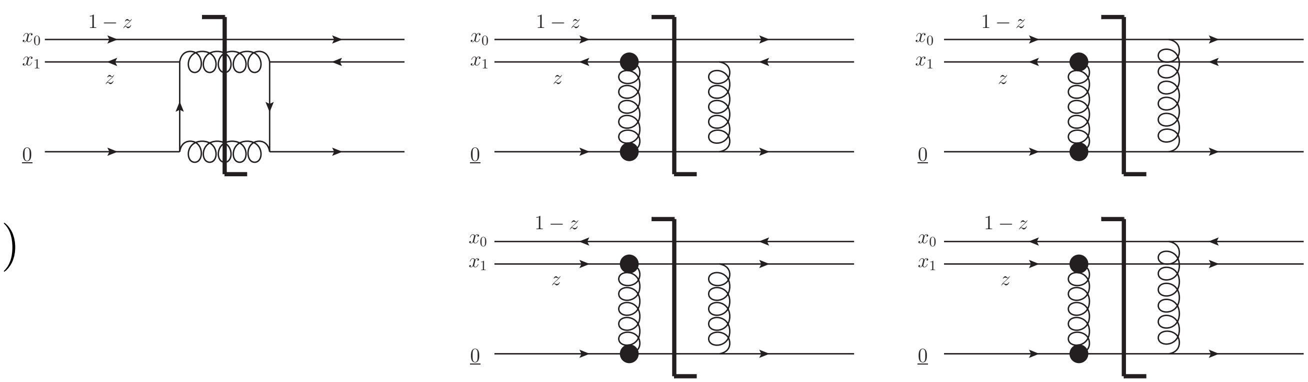
Large-N_c Evolution

- In the strict DLA limit (S=1) and at large N_c we get (here Γ is an auxiliary function we call the 'neighbor dipole amplitude') (KPS '15)

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z') \right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'') \right]$$

- The initial conditions are given by the Born-level graphs



$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2\, \ln(zs\, x_{10}^2) \right]$$

Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

- Helicity evolution resummation parameter is double-logarithmic (DLA):

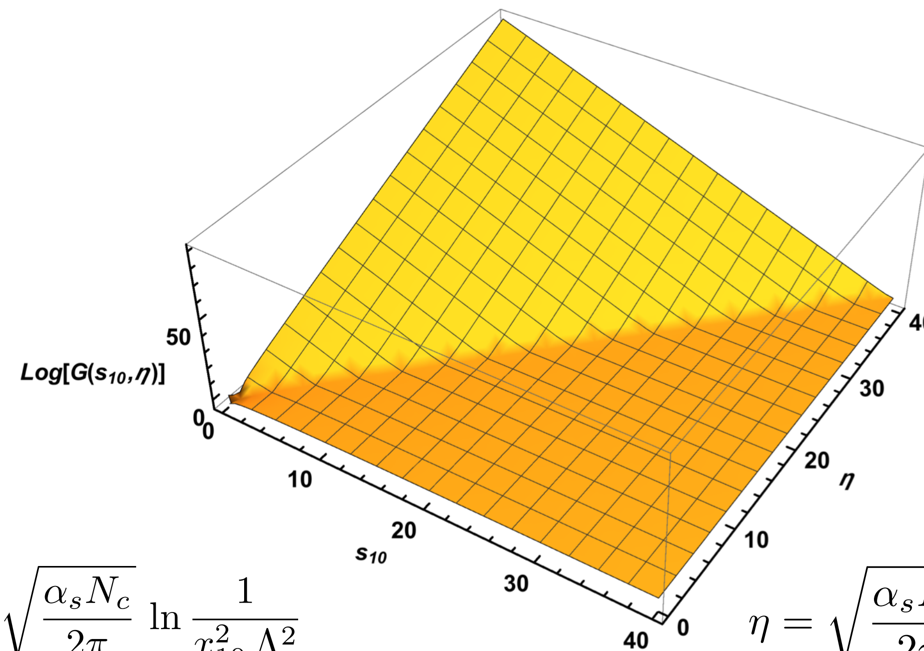
$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Gluon Helicity at Small x

Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

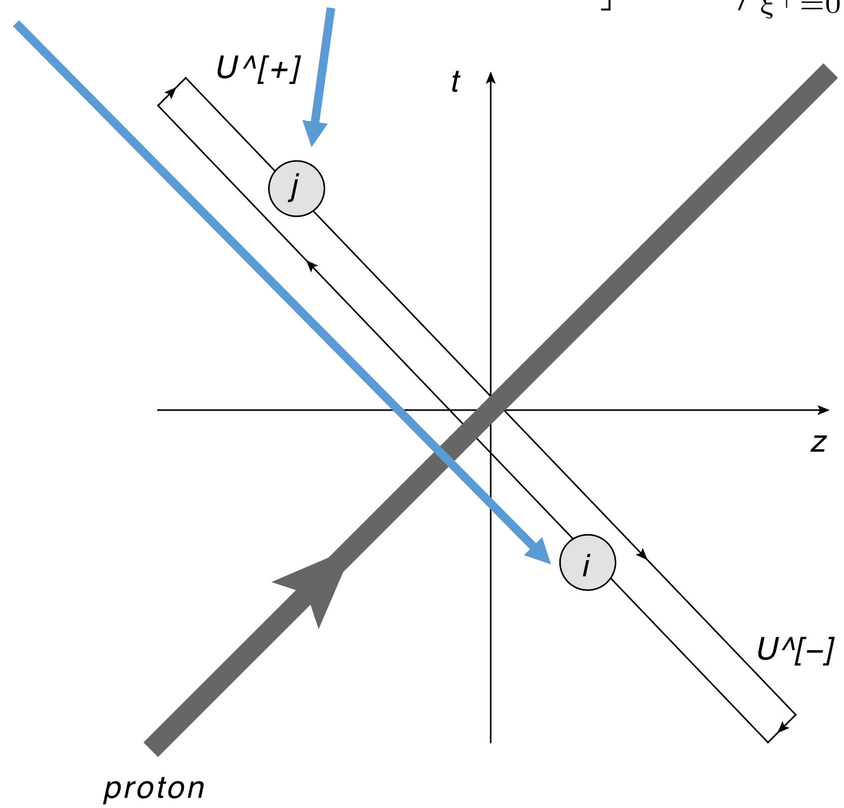
Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x, k_T^2) = \frac{-2i\, S_L}{x\, P^+} \int \frac{d\xi^- \, d^2\xi}{(2\pi)^3} \, e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P, S_L | \epsilon_T^{ij} \, \mathrm{tr} \left[F^{+i}(0) \, \mathcal{U}^{[+]\dagger}[0, \xi] \, F^{+j}(\xi) \, \mathcal{U}^{[-]}[\xi, 0] \right] | P, S_L \right\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past Wilson line staples (hence the name `dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

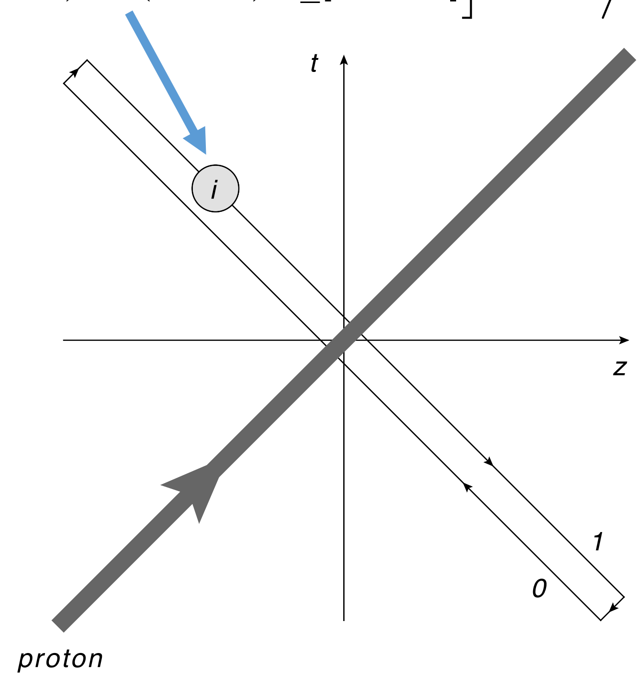
- At small x, the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G\,dip}(x,k_T^2) = \frac{8i\,N_c\,S_L}{g^2(2\pi)^3} \int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\, k_\perp^i \epsilon_T^{ij} \left[\int d^2b_{10}\, G_{10}^j(zs = \frac{Q^2}{x})\right]$$

- Here we obtain a new operator, which is a transverse vector (written here in A^-=0 gauge):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr}\left[V_{\underline{0}}[\infty,-\infty] V_{\underline{1}}[-\infty,x^-]\,(-ig)\,\tilde{A}^i(x^-,\underline{x})\,V_{\underline{1}}[x^-,\infty]\right] + \mathrm{c.c.}\right\rangle(z)$$

- Note that $k_\perp^i\,\epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-,\underline{x})$ -- different from the polarized dipole amplitude!



Dipole TMD vs dipole amplitude

- Note that the operator for the dipole gluon helicity TMD

$$G^i_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

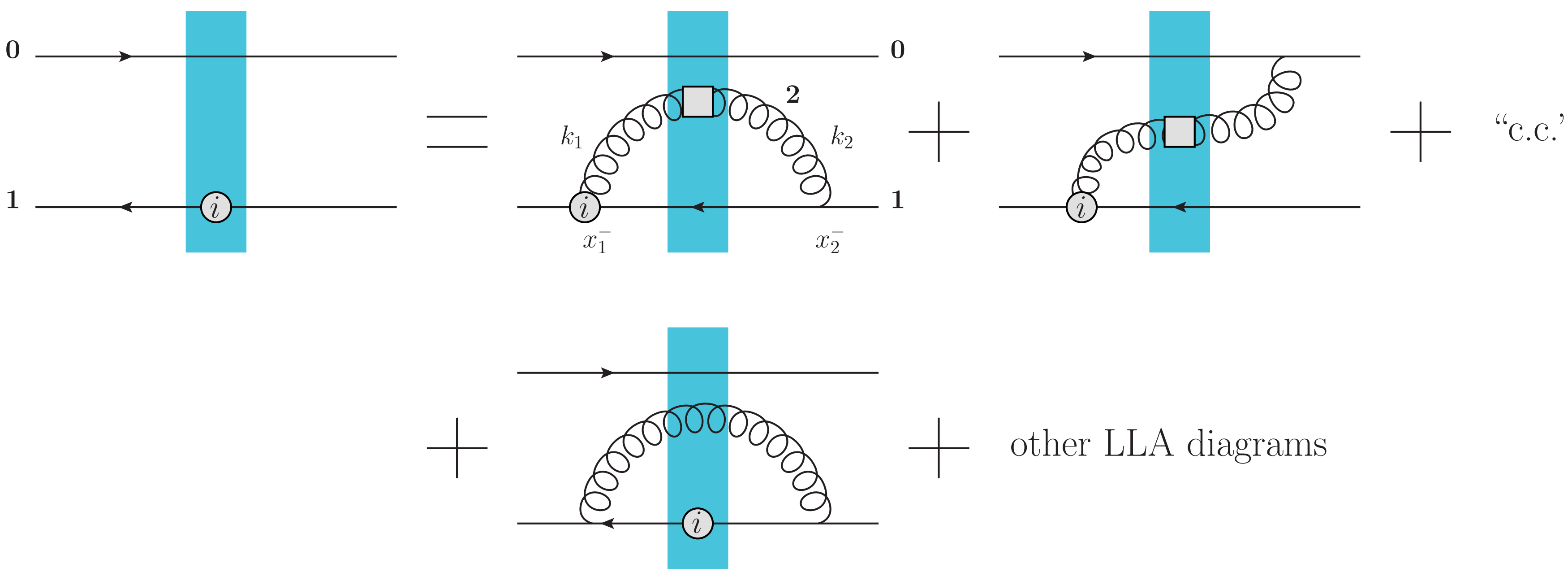
is different from the polarized dipole amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMDs.)
- This is different from the unpolarized gluon TMD case.

Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{31})^j_\perp}{x_{31}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{30})^j_\perp}{x_{30}^2} \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2\,,\,x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s)\right].$$

Large-N_c Evolution: Equations

- Here

$$\Gamma^{gen}_{20,21}(z's) = \theta(x_{20} - x_{21})\, \Gamma_{20,21}(z's) + \theta(x_{21} - x_{20})\, G_{20}(z's)$$

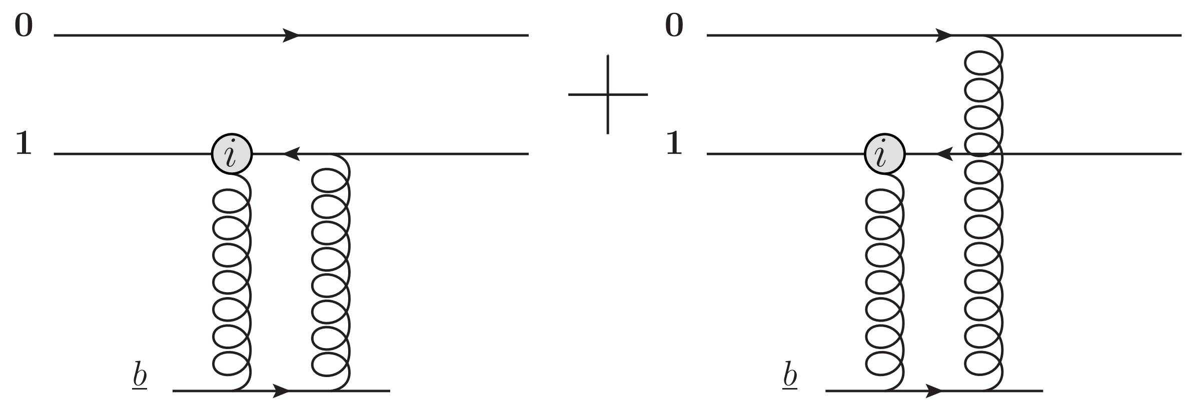
is an object which we know from the quark helicity evolution, as the latter gives us G and Γ.

- Note that our evolution equations mix the gluon (G^i) and quark (G) small-x helicity evolution operators:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\, (x_{21})^j_\perp}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's) \right]$$
$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\, (x_{20})^j_\perp}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's) \right]$$
$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's) \right]$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2 b_{10}\, G^{i\,(0)}_{10}(zs) = \int d^2 b_{10}\, \Gamma^{i\,(0)}_{10,21}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\, \pi\, \epsilon^{ij}\, x^j_{10}\, \ln \frac{1}{x_{10}\, \Lambda}$$

- Note that these initial conditions have no ln s, unlike the initial conditions for the quark evolution:

$$\int d^2 b_{10}\, G^{(0)}_{10}(zs) = \int d^2 b_{10}\, \Gamma^{(0)}_{10,21}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\pi\, \ln(zs\, x_{10}^2)$$

Large-N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small-x gluon helicity intercept

$$\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We obtain the small-x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G \, dip}(x, k_T^2) \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

Quark and Gluon OAM at Small x

Quark OAM: Definition

- We begin by writing the (Jaffe-Manohar) quark OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp \, db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k, b)$$

with the quark SIDIS Wigner distribution

$$W^{q,SIDIS}(k,b) = 2 \sum_X \int d^2 r \, dr^- \; e^{ik \cdot r} \left\langle \bar{\psi}_\alpha \left(b - \tfrac{1}{2} r\right) \, V_{\underline{b} - \frac{1}{2} \underline{r}} \left[b^- - \tfrac{1}{2} r^-, \infty\right] |X\rangle \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \right.$$
$$\left. \times \; \langle X| \, V_{\underline{b} + \frac{1}{2} \underline{r}} \left[\infty, b^- + \tfrac{1}{2} r^-\right] \; \psi_\beta \left(b + \tfrac{1}{2} r\right) \right\rangle$$

- Here, and above, the angle brackets denote ”CGC averaging” in the (polarized) proton target:

$$\left\langle \hat{\mathcal{O}}(b,r) \right\rangle = \frac{1}{2P^+} \int \frac{d^2 \Delta \, d\Delta^+}{(2\pi)^3} \, e^{ib \cdot \Delta} \left\langle P + \frac{\Delta}{2} \right| \hat{\mathcal{O}}(0,r) \left| P - \frac{\Delta}{2} \right\rangle$$

Quark OAM: small-x simplifications

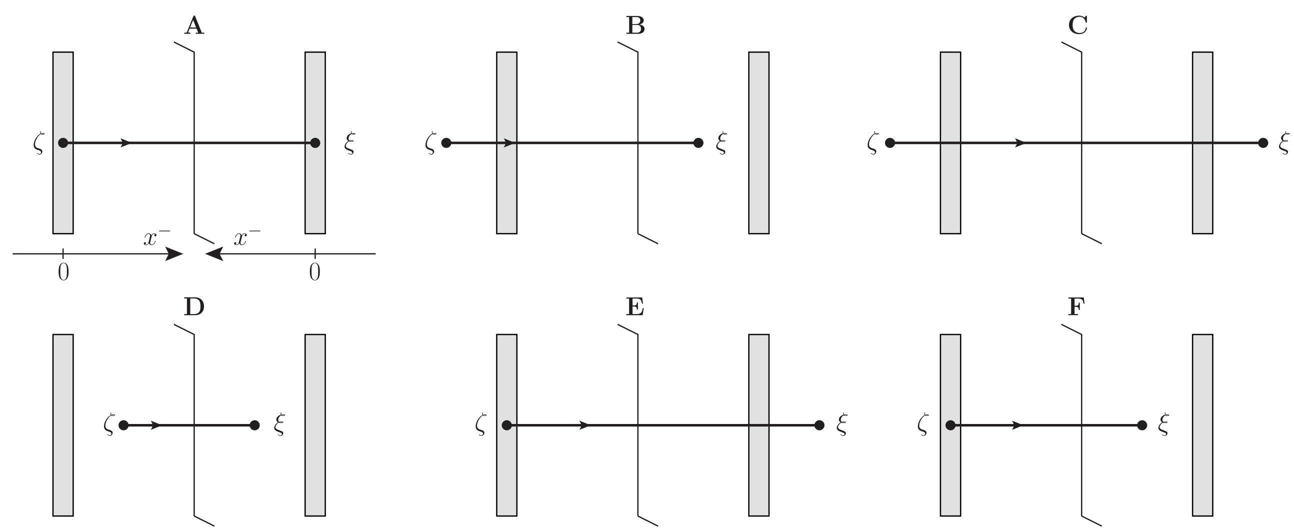
- The resulting quark OAM "PDF" is

$$L_q(x,Q^2) = \frac{2P^+}{(2\pi)^3} \sum_X \int d^2k_\perp \, d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\frac{\underline{\zeta} + \underline{\xi}}{2} \times \underline{k} \right) \Big\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \, |X\rangle \, \left(\tfrac{1}{2}\gamma^+\right)_{\alpha\beta} \times \langle X| \, V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \Big\rangle$$

- This can be compared to quark helicity,

$$g_{1L}^q(x,k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta \, d\zeta^- \, d^2\xi \, d\xi^- \, e^{ik\cdot(\zeta-\xi)} \left(\frac{1}{2}\gamma^+\gamma^5 \right)_{\alpha\beta} \Big\langle \bar{\psi}_\alpha(\xi) \, V_{\underline{\xi}}[\xi^-, \infty] \;\; V_{\underline{\zeta}}[\infty, \zeta^-] \, \psi_\beta(\zeta) \Big\rangle$$

- The operators are different, but the structure is similar. The quark OAM can be evaluated in the same way as the quark helicity operator: only diagram B survives.



Quark OAM: small-x expression

- After some algebra we arrive at the following small-x expression for quark OAM:

$$L_{q+\bar{q}}(x, Q^2) = \frac{8N_c}{(2\pi)^5} \int d^2 k_\perp \, d^2 x_{10} \, d^2 x_1 \, e^{i\underline{k}\cdot\underline{x}_{10}} \, \frac{\underline{x}_{10}}{x_{10}^2} \times \frac{\underline{k}}{\underline{k}^2} \, \underline{x}_1 \times \underline{k} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \, G_{10}(zs) - \sum_f [\Delta q^f(x, Q^2) + \Delta\bar{q}^f(x, Q^2)]$$

- The result is written in terms of the polarized dipole amplitude G_{10} (z). It seems we are done, right?
- This is almost correct. The remaining minor technicality is that the above quark OAM depends on the "first moment" of the polarized dipole amplitude

$$I^k(\underline{x}_{10}, zs) = \int d^2 x_1 \, x_1^k \, G_{10}(zs)$$

while all our earlier results for the quark helicity were derived for the "zeroth moment", the impact-parameter integrated polarized dipole amplitude

$$G(x_{10}^2, zs) = \int d^2 x_1 \, G_{10}(zs)$$

Quark OAM: small-x asymptotics

- It turns out that the "first moment" of the polarized amplitude is subleading. It grows with energy as a smaller power of energy

$$I^k(\underline{x}_{10}, zs) \sim \left(zs x_{10}^2\right)^{2\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

than the flavor-singlet quark helicity distribution

$$\Delta\Sigma(x, Q^2) = \sum_f [\Delta q^f(x, Q^2) + \Delta\bar{q}^f(x, Q^2)] \sim \left(\frac{1}{x}\right)^{\alpha_h^q} = \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \approx \left(\frac{1}{x}\right)^{2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Since 2.31 > 2, we get (cf. Y. Hatta & D.-J. Yang, 2018)

$$L_{q+\bar{q}}(x, Q^2) = -\Delta\Sigma(x, Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Note that this is not a complete cancellation, the contribution to the proton spin is

$$\frac{1}{2}\,\Delta\Sigma(x, Q^2) + L_{q+\bar{q}}(x, Q^2) = -\frac{1}{2}\,\Delta\Sigma(x, Q^2)$$

Gluon OAM: definition

- The gluon OAM story is similar. We start with the Wigner distribution definition

$$L_z = \int \frac{d^2 b_\perp db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k, b)$$

with the dipole Wigner distribution for gluons

$$W^{G\,dip}(k,b) = \frac{4}{xP^+} \int d\xi^- \, d^2\xi_\perp \;\; e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \mathrm{tr}\left[F^{+i}(b - \tfrac{1}{2}\xi)\, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi)\, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

- We obtain the following expression for the gluon OAM “PDF” (cf. Hatta et al, 2016)

$$L_G(x, Q^2) = \frac{4}{(2\pi)^3 \, x} \int d^2 b_\perp db^- \, d^2 k_\perp \, d\xi^- \, d^2\xi_\perp \;\; (\underline{b} \times \underline{k}) \;\; e^{ixP^+\,\xi^- - i\underline{k}\cdot\underline{\xi}} \times \left\langle \mathrm{tr}\left[F^{+i}(b - \tfrac{1}{2}\xi)\, \mathcal{U}^{[+]}[b - \tfrac{1}{2}\xi, b + \tfrac{1}{2}\xi] \; F^{+i}(b + \tfrac{1}{2}\xi)\, \mathcal{U}^{[-]}[b + \tfrac{1}{2}\xi, b - \tfrac{1}{2}\xi] \right] \right\rangle$$

Gluon OAM: small-x expression

- Gluon OAM at small x can (similarly to the quark OAM) be rewritten in terms of the "moment" of the polarized dipole amplitude G_{10}^i for the gluon helicity TMD. This object is different from the polarized amplitude for the quark.
- We get

$$L_G(x, Q^2) = -\frac{8iN_c}{g^2\,(2\pi)^3} \int d^2x_{10}\, d^2k_\perp\, e^{i\underline{k}\cdot\underline{x}_{10}}\ (\underline{k}\cdot\underline{x}_{10})\ G_5\left(x_{10}^2, zs = \frac{Q^2}{x}\right)$$

where

$$\int d^2x_1\ x_1^j\, \nabla_{10}^i\, G_{10}^i(zs) = x_{10}^j\, G_4(x_{10}^2, zs) + \epsilon^{jk} x_{10}^k\, G_5(x_{10}^2, zs)$$

- We write down and solve the equations for G_5.

Gluon OAM: small-x asymptotics

- We arrive at the following relation

$$L_G(x, Q^2) = \left(\frac{\alpha_h^q}{4} \, \ln \frac{Q^2}{\Lambda^2} \right) \Delta G(x, Q^2)$$

where

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

- We conclude that

$$L_G(x, Q^2) \sim \Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \sim \left(\frac{1}{x} \right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}}} \sim \left(\frac{1}{x} \right)^{1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}}$$

- Note that with the DLA accuracy we could also simply conclude that

$$|L_G| \ll |\Delta G|$$

Conclusions

- We have constructed the small-x asymptotics of the quark and gluon OAM (in the Jaffe-Manohar decomposition).

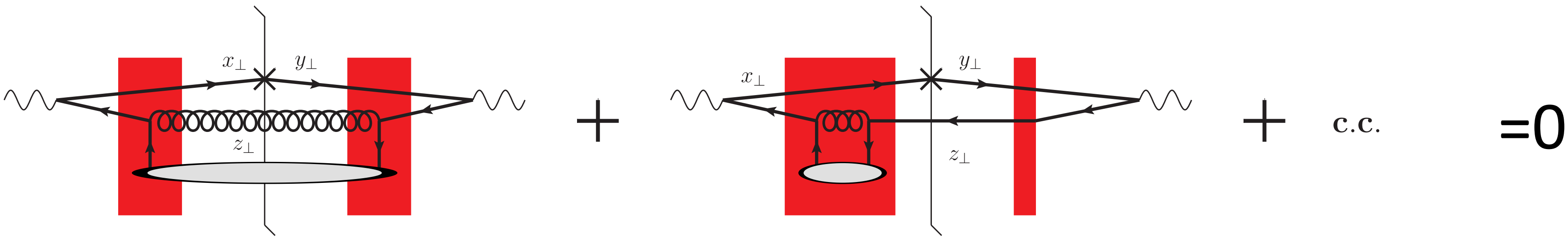
- In the large-N_C limit we obtain

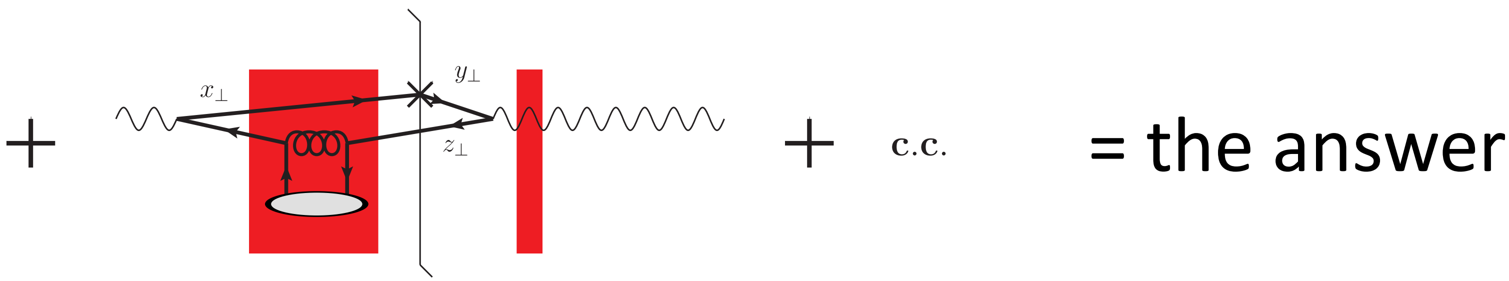
$$L_{q+\bar{q}}(x,Q^2) = -\Delta\Sigma(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}},$$
$$L_G(x,Q^2) \sim \Delta G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Backup Slides

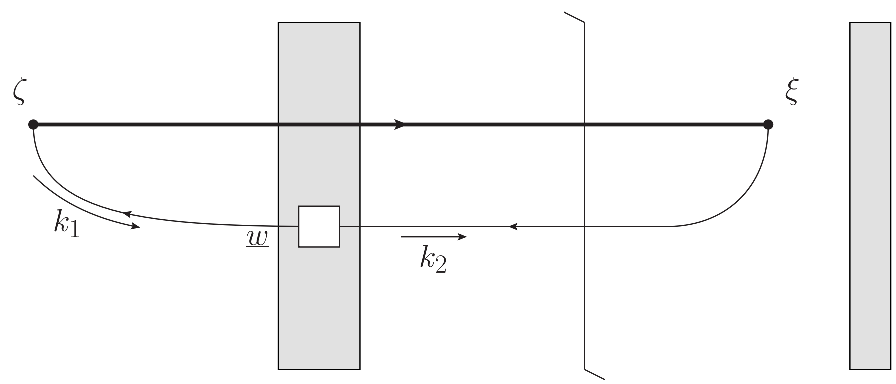
Quark Helicity TMD at Small x

- Dominance of diagram B can also be obtained by applying crossing symmetry to the SIDIS process (KS ‘15):





- Compare the last line to the diagram B: reflecting the cc amplitude into the amplitude reduces the above diagram to the one on the right.



Small-x Evolution at large N_c

- At large N_c the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large N_c the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4\, G_{10}(z)$$

(Note that the factor is 4, not 2 like in the unpolarized dipole case.)