Conformal properties of TMD rapidity evolution

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Outline

1. Reminder: rapidity evolution of TMDs from low to moderate x
   - Rapidity factorization for particle production.
   - One-loop evolution of gluon TMD
   - DGLAP, Sudakov and BK limits of TMD evolution equation
2. In works: conformal properties of TMD factorization:
   - Conformal group and "TMD" subgroup
   - Conformal rapidity evolution of TMDs in the Sudakov region
3. Conclusions and outlook

TMD factorization

Factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_f \int d^2b_\perp e^{i(q,b)_\perp} \mathcal{D}_{f/A}(x_A, b_\perp, \eta)\mathcal{D}_{f/B}(x_B, b_\perp, \eta)\sigma(ff \to H)$$

$$+ \text{power corrections} + \text{“Y} - \text{terms}''$$

where η is the rapidity, $\mathcal{D}_{f/A}(x, z_\perp, \eta)$ is the TMD density of a parton f in hadron A, and $\sigma(ff \to H)$ is the cross section of production of particle H of invariant mass $m_H^2 = Q^2$ in the scattering of two partons.

Suppose we produce a scalar particle (Higgs) in a gluon-gluon fusion.

For simplicity, assume the vertex is local:

$$\mathcal{L}_\Phi = g_\Phi \int dz\, \Phi(z) F^2(z), \qquad F^2 \equiv F^a_{\mu\nu} F^{\mu\nu}_a$$

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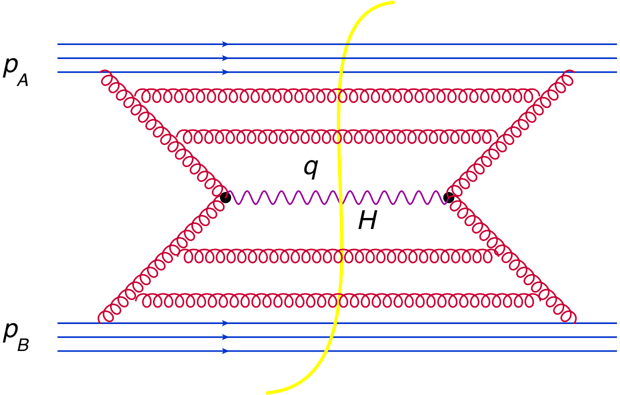


$s >> Q^2 >> q_\perp^2$

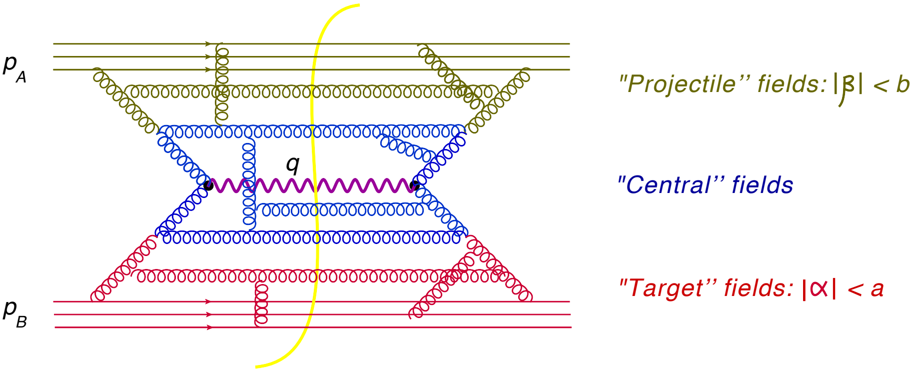
$q^2 = Q^2 = M_H^2$

Rapidity factorization for particle production

Sudakov variables:

$$p \;=\; \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A,\;\; p_2 \simeq p_B,\;\;\; p_1^2 = p_2^2 = 0$$

$$p_2 \cdot x \;=\; \sqrt{\frac{s}{2}}x^+, \qquad p_1 \cdot x \;=\; \sqrt{\frac{s}{2}}x^-$$



We integrate over “central” fields in the background of projectile and target fields.

TMD factorization

At the tree level, the "hadronic tensor"

$$W(p_A,p_B,q) \stackrel{\rm def}{=} \langle p_A,p_B|F^2(x)F^2(0)|p_A,p_B\rangle$$

in the region $s \gg Q^2 \gg Q_\perp^2$ has the form

$$W(p_A,p_B,q) = \frac{16}{N_c^2-1}\int d^2x_\perp\, e^{i(q,x)_\perp}\int dx_+dx_-\, e^{-i\sqrt{\frac{s}{2}}(\alpha_q x_- + \beta_q x_+)}$$
$$\times\Big\{\langle p_A|F_+^{mi}(x_-,x_\perp)F_+^{mj}(0)|p_A\rangle\langle p_B|F^n_{-i}(x_+,x_\perp)F^n_{-j}(0)|p_B\rangle + x\leftrightarrow 0\Big\}$$

Rapidity evolution: one loop

We study evolution of $\tilde{\mathcal{F}}_i^{a\eta}(x_\perp, x_B)\mathcal{F}_j^{a\eta}(y_\perp, x_B)$ with respect to rapidity cutoff η

$$\mathcal{F}_i^{a(\eta)}(z_\perp, x_B) \equiv \sqrt{\frac{2}{s}}\int dz_+ \, e^{i\sqrt{\frac{s}{2}}x_B z_+}[-\infty, z_+]_z^{am} F^m_{\bullet i}(z_+, z_\perp)$$

$$A^\eta_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$

Matrix element of $\tilde{\mathcal{F}}_i^a(k'_\perp, x'_B)\mathcal{F}^{ai}(k_\perp, x_B)$ at one-loop accuracy:
diagrams in the “external field” of gluons with rapidity $< \eta$.

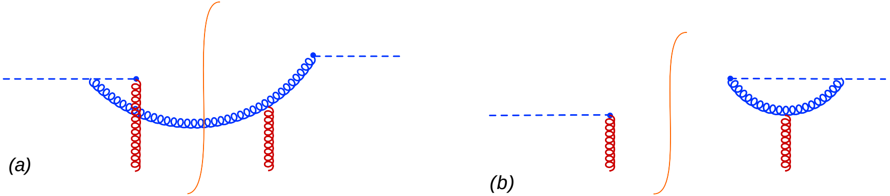
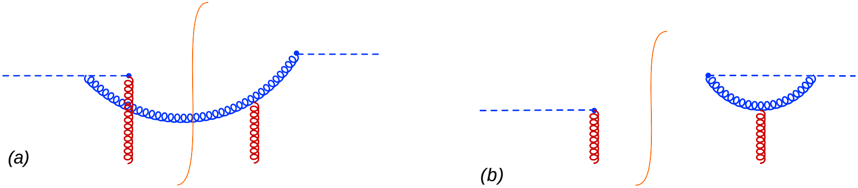


Figure : Typical diagrams for one-loop contributions to the evolution of gluon TMD.

Method of calculation

We calculate one-loop diagrams in the fast-field background



in the following way:

if $k_\perp \sim k_\perp \Rightarrow$ propagators in the shock-wave background

if $k_\perp \gg k_\perp \Rightarrow$ light-cone expansion of propagators

We compute one-loop diagrams in these two cases and write down "interpolating" formulas correct both at $k_\perp \sim k_\perp$ and $k_\perp \gg k_\perp$

$$\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp) \tag{4.1}$$
$$= -\alpha_s \mathrm{Tr}\Big\{\int d^2k_\perp (x_\perp|\Big\{U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i}-2k_\mu^\perp k_i}{\sigma\beta_B s+k_\perp^2}$$
$$-2k_\mu^\perp g_{ik}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U+\frac{2k_\mu^\perp}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)$$
$$\times (k_\perp|\mathcal{F}^l(\beta_B+\frac{k_\perp^2}{\sigma s})\Big\{\frac{\sigma\beta_B s\delta_j^\mu-2k_\perp^\mu k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U$$
$$-2k_\perp^\mu g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2\delta_l^\mu U^\dagger\frac{p_j}{\sigma\beta_B s+p_\perp^2}U+2g_{jl}\frac{k_\perp^\mu}{k_\perp^2}\Big\}|y_\perp)$$
$$+2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta_m^k\delta_j^l-g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U$$
$$+\mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp)$$
$$+2(x_\perp|-U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta_i^k\delta_m^l-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}$$
$$+\tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\}+O(\alpha_s^2)$$

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \qquad (4.1)\\
&= -\alpha_s \mathrm{Tr}\Big\{\int d\!\!\!{}^-{}^2k_\perp (x_\perp|\Big\{U^\dagger \frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2}\\
&\quad - 2k^\perp_\mu g_{ik}U^\dagger \frac{1}{\sigma\beta_B s+p_\perp^2}U - 2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U + \frac{2k^\perp_\mu}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times (k_\perp|\mathcal{F}^l\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)\Big\{\frac{\sigma\beta_B s\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U\\
&\quad - 2k^\mu_\perp g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U - 2\delta^\mu_l U^\dagger \frac{p_j}{\sigma\beta_B s+p_\perp^2}U + 2g_{jl}\frac{k^\mu_\perp}{k_\perp^2}\Big\}|y_\perp)\\
&\quad + 2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U\\
&\quad + \mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp)\\
&\quad + 2(x_\perp| - U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}\\
&\quad + \tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\}+O(\alpha_s^2)
\end{aligned}
$$

Valid at all Bjorken $x_B \equiv \beta_B$ and all $k_\perp$ but complicated and not unique

Light-cone limit

$$\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma} = \frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-x_B-\beta)$$
$$\times\Big[\frac{1}{\beta}-\frac{2x_B}{(x_B+\beta)^2}+\frac{x_B^2}{(x_B+\beta)^3}-\frac{x_B^3}{(x_B+\beta)^4}\Big]\langle p|\tilde{\mathcal{F}}^n_i(x_B+\beta,x_\perp)$$
$$\times\,\mathcal{F}^{ni}(x_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{x_B}{\beta(x_B+\beta)}\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}$$

In the LLA the cutoff in $\sigma \Leftrightarrow$ cutoff in transverse momenta:

$$\langle p|\tilde{\mathcal{F}}^n_i(x_B,x_\perp)\mathcal{F}^{in}(x_B,x_\perp)|p\rangle^{k_\perp^2<\mu^2} = \frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\ \{\text{same}\}$$

$\Rightarrow$ DGLAP equation $\Rightarrow$ $(z'\equiv\frac{x_B}{x_B+\beta})$

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(x_B,0_\perp,\eta) \qquad\qquad \text{DGLAP kernel}$$
$$=\frac{\alpha_s}{\pi}N_c\int_{x_B}^1\frac{dz'}{z'}\ \Big[\Big(\frac{1}{1-z'}\Big)_+ + \frac{1}{z'} - 2 + z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{x_B}{z'},0_\perp,\eta\Big)$$

Low-x case: BK evolution of the WW distribution

Low-x regime: $x_B = 0$ + characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$

$\Rightarrow$ in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1$

$\Rightarrow$ the kinematical constraint $\theta\big(1 - \frac{k_\perp^2}{\alpha s}\big)$ can be omitted

$\Rightarrow$ non-linear evolution equation

$$\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)$$
$$= -\frac{g^2}{8\pi^3}\mathrm{Tr}\Big\{(-i\partial_i^{z_1} + \tilde{U}_i^{z_1})\Big[\int d^2 z_3 (\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger - 1)\frac{z_{12}^2}{z_{13}^2 z_{23}^2}(U_{z_3}U_{z_2}^\dagger - 1)\Big](i\overleftarrow{\partial}_j^{z_2} + U_j^{z_2})\Big\}$$

where $\eta \equiv \ln\sigma$ and $\frac{z_{12}^2}{z_{13}^2 z_{23}^2}$ is the dipole kernel

Sudakov double logs

Sudakov limit: $x_B \equiv x_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV.

$$\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle$$

$$= \; 4\alpha_s N_c \int \frac{d\!\!{}^-{}^2 p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(x_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(x_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle$$

$$- \; \frac{\sigma x_B s}{\sigma x_B s + p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(x_B,x_\perp)\mathcal{F}_j^a(x_B,y_\perp)|p\rangle\Big]$$

Double-log region: $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ and $\sigma x_B s \gg p_\perp^2 \gg (x-y)_\perp^{-2}$

$$\Rightarrow \; \frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) \; = \; -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big]$$

Evolution equation at arbitrary x_B and $k_\perp$

Limits are nice but the equation itself is not

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \qquad (4.1)\\
&= -\alpha_s \mathrm{Tr}\Big\{\int d\!\!\!{}^-{}^2k_\perp (x_\perp|\Big\{U^\dagger \frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i} - 2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2}\\
&\quad - 2k^\perp_\mu g_{ik}U^\dagger \frac{1}{\sigma\beta_B s+p_\perp^2}U - 2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U + \frac{2k^\perp_\mu}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times (k_\perp|\mathcal{F}^l\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)\Big\{\frac{\sigma\beta_B s\delta^\mu_j - 2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U\\
&\quad - 2k^\mu_\perp g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U - 2\delta^\mu_l U^\dagger \frac{p_j}{\sigma\beta_B s+p_\perp^2}U + 2g_{jl}\frac{k^\mu_\perp}{k^2_\perp}\Big\}|y_\perp)\\
&\quad + 2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l + U_l)(2\delta^k_m\delta^l_j - g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U\\
&\quad + \mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp)\\
&\quad + 2(x_\perp| -U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta^k_i\delta^l_m - g_{im}g^{kl})(i\partial_k - U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}\\
&\quad + \tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\} + O(\alpha_s^2)
\end{aligned}
$$

Limits are nice but the equation itself is not

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \qquad (4.1)\\
&= -\alpha_s \mathrm{Tr}\bigg\{\int \bar{d}^2k_\perp (x_\perp|\bigg\{U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}(Uk_k+p_kU)\frac{\sigma\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2}\\
&\quad -2k^\perp_\mu g_{ik}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2g_{\mu k}U^\dagger\frac{p_i}{\sigma\beta_B s+p_\perp^2}U+\frac{2k^\perp_\mu}{k_\perp^2}g_{ik}\bigg\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times(k_\perp|\mathcal{F}^l\big(\beta_B+\frac{k_\perp^2}{\sigma s}\big)\bigg\{\frac{\sigma\beta_B s\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_lU^\dagger+U^\dagger p_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U\\
&\quad -2k^\mu_\perp g_{jl}U^\dagger\frac{1}{\sigma\beta_B s+p_\perp^2}U-2\delta^\mu_l U^\dagger\frac{p_j}{\sigma\beta_B s+p_\perp^2}U+2g_{jl}\frac{k^\mu_\perp}{k_\perp^2}\bigg\}|y_\perp)\\
&\quad +2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2+i\epsilon}U\\
&\quad +\mathcal{F}_j(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2+i\epsilon)}|y_\perp)\\
&\quad +2(x_\perp|-U^\dagger\frac{1}{\sigma\beta_B s-p_\perp^2-i\epsilon}U(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}\\
&\quad +\tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_B s}{p_\perp^2(\sigma\beta_B s-p_\perp^2-i\epsilon)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\bigg\}+O(\alpha_s^2)
\end{aligned}
$$

Conformal invariance may help!

TMD factorization in the coordinate space

TMD factorization in the coordinate space (ignoring indices of $\mathcal{F}_i$)

$$
\begin{aligned}
&\langle \mathcal{S}(z_1)\mathcal{S}(z_2)\mathcal{S}(z_3)\mathcal{S}(z_4)\mathcal{F}^2(x)\mathcal{F}^2(y)\rangle \\
&\quad = \langle \mathcal{S}(z_{1-}, z_{1_\perp})\mathcal{S}(z_{2-}, z_{2_\perp})\mathcal{F}(x_+, x_\perp)\mathcal{F}(y_+, y_\perp)\rangle \\
&\qquad \times \langle \mathcal{S}(z_{3+}, z_{3_\perp})\mathcal{S}(z_{4+}, z_{4_\perp})\mathcal{F}(x_-, x_\perp)\mathcal{F}(y_-, y_\perp)\rangle
\end{aligned}
$$

For simplicity one can consider scalar operators $\mathcal{S}$ (like TrZ^2 in $\mathcal{N}$=4 SYM)

Small-x region

$$
s \gg Q^2 \sim q_\perp^2 \quad \Leftrightarrow \quad z_{1-}, z_{3+} \to \infty, z_{2-}, z_{4+} \to -\infty, \quad (x-y)_+(x-y)_- \sim (x-y)_\perp^2
$$

“Sudakov” region

$$
Q^2 \gg q_\perp^2 \quad \Leftrightarrow \quad (x-y)_+(x-y)_- \ll (x-y)_\perp^2
$$

“Small-x + Sudakov” region

$$
s \gg Q^2 \gg q_\perp^2 \quad \Leftrightarrow \quad z_{1-}, z_{3+} \to \infty, z_{2-}, z_{4+} \to -\infty, \quad (x-y)_+(x-y)_- \ll (x-y)_\perp^2
$$

Conformal properties of TMD factorization G.A.C. and I.B., in works

Conformal properties of correlators the Regge limit are well studied

$$\langle \mathcal{S}(z_{1-}, z_{1_\perp})\mathcal{S}(z_{2-}, z_{2_\perp})\mathcal{S}(x_+, x_\perp)\mathcal{S}(y_+, y_\perp)\rangle \;=\; \int d\nu\, f(\nu, \alpha_s)\Phi(r, \nu)R^{\frac{\omega(\nu,\alpha_s)}{2}}$$

where $\omega(\nu, \alpha_s)$ = pomeron intercept, $f(\nu, \alpha_s)$ = "pomeron residue",

$$\begin{aligned} R \;&=\; \frac{(z_1-x)(z_2-y)}{z_{12}^2(x-y)^2} \overset{\text{Regge limit}}{\rightarrow} \frac{z_{1-}z_{2-}x_+y_+}{z_{12_\perp}^2(x-y)_\perp^2} && \rightarrow\; \infty \\ r \;&=\; \frac{(x_+-y_+)^2}{x_+|y_+|(x-y)_\perp^2}\frac{[z_{1-}(x-z_2)_\perp^2 - z_{2-}(x-z_1)_\perp^2]^2}{z_{1-}|z_{2-}|z_{12_\perp}^2} && \sim\; 1 \;\; (\text{fixed}), \end{aligned}$$

and $\Phi(r, \nu)$ is some function (hypergeometric).

By analogy, one may assume

$$\langle \mathcal{S}(z_{1-}, z_{1_\perp})\mathcal{S}(z_{2-}, z_{2_\perp})\mathcal{F}(x_+, x_\perp)\mathcal{F}(y_+, y_\perp)\rangle \;=\; \int d\nu\; F(\nu, \alpha_s)\Phi(r, \nu)R^{\frac{\omega(\nu,\alpha_s)}{2}}$$

Motivation: at small $(x-y)_\perp^2 \ll z_{12_\perp}^2$ ($\Leftrightarrow q_\perp^2 \gg m_N^2$) the small-$x$ behavior of "Wilson frames" defining $\mathcal{F}(x_+, x_\perp)\mathcal{F}(y_+, y_\perp)$ is the same as for correlators $\mathcal{S}(x_+, x_\perp)\mathcal{S}(y_+, y_\perp)$

V. Kazakov, E. Sobko, I.B., 2013-2018

Conformal properties of TMD factorization

The conformal ratios R_1 and r_1 are invariant under the inversion

$$z_{1-} \rightarrow \frac{z_{1-}}{z_{1_\perp}^2}, \quad z_{2-} \rightarrow \frac{z_{2-}}{z_{2_\perp}^2}, \quad x_+ \rightarrow \frac{x_+}{x_\perp^2}, \quad y_+ \rightarrow \frac{y_+}{y_\perp^2} \qquad (*)$$

Similarly, for the bottom correlator

$$\langle \mathcal{S}(z_{3+}, z_{3_\perp})\mathcal{S}(z_{4+}, z_{4_\perp})\mathcal{F}(x_-, x_\perp)\mathcal{F}(y_-, y_\perp)\rangle = \int d\nu\ F(\nu, \alpha_s)\Phi(r', \nu)R'^{\frac{\omega(\nu,\alpha_s)}{2}}$$

where

$$R' = \frac{(z_3-x)(z_4-y)}{z_{34}^2(x-y)^2} \stackrel{\text{Regge limit}}{\rightarrow} \frac{z_{3+}z_{4+}x_-y_-}{z_{34_\perp}^2(x-y)_\perp^2} \Rightarrow \infty$$

$$r' = \frac{(x_- - y_-)^2}{x_-|y_-|(x-y)_\perp^2}\frac{[z_{3+}(x-z_4)_\perp^2 - z_{4+}(x-z_3)_\perp^2]^2}{z_{3+}|z_{4+}|z_{34_\perp}^2} \sim 1 \ \ (\text{fixed})$$

It looks like the factorazed correlator $\langle \mathcal{S}(z_1)\mathcal{S}(z_2)\mathcal{S}(z_3)\mathcal{S}(z_4)\mathcal{F}^2(x)\mathcal{F}^2(y)\rangle$ is a function of four conformal ratios R, R', r, r' instead of 9
(In general, n-point correlator is a function of $4n-15$ conformal ratios)

Conformal properties of TMD factorization

However, there are more "conformal ratios" invariant under inversion ($*$)
Example: correlator of two currents and conformal dipole with rapidity cutoff $a \sim \alpha_{\max} \sim e^{\eta_{\max}}$

$$\langle \mathcal{S}(z_{1-}, z_{1_\perp})\mathcal{S}(z_{2-}, z_{2_\perp})\mathcal{U}^a_{\rm conf}(x_\perp, y_\perp)\rangle \;=\; \int d\nu\; F(\nu, \alpha_s)\Phi(r, \nu)\Big(\frac{z_{1-}z_{2-}}{z^2_{12_\perp}}\Big)^{\frac{\omega(\nu,\alpha_s)}{2}} a^{\frac{\omega(\nu)}{2}}$$

$$r \;=\; \frac{z^2_{12_\perp}(x-y)^2_\perp}{z_{1-}z_{2-}\Big[\frac{(x-z_1)^2_\perp}{z_{1-}} - \frac{(x-z_2)^2_\perp}{z_{2-}}\Big]\Big[\frac{(y-z_1)^2_\perp}{z_{1-}} - \frac{(y-z_2)^2_\perp}{z_{2-}}\Big]} \qquad \sim 1 \;\; \text{in the Regge limit}$$

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A: Presumably four...

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Q: How many such “conformal ratios” for TMD correlators?
A: Presumably four...

First, we need to learn more about the group of symmetry of TMDs

TMDs are invariant under the inversion $(*)$:

$$\begin{aligned}\mathcal{F}^m_i(z_\perp, z_+) \;&=\; [\infty_+ + z_\perp, z_+ + z_\perp]^{mn}_z F^n_{\bullet i}(z_+, z_\perp)\\ \rightarrow\;\; &\Big[\infty_+ + \frac{z_\perp}{z^2_\perp}, \frac{z_+}{z^2_\perp} + \frac{z_\perp}{z^2_\perp}\Big]^{mn} F^n_{\bullet i}\Big(\frac{z_+}{z^2_\perp}, \frac{z_\perp}{z^2_\perp}\Big) \;=\; \mathcal{F}^m_i(z'_\perp, z'_+)\end{aligned}$$

but what about the full group of TMD transformations?

Conformal group

Poincare group (Lorentz transformations+shifts), dilatations, and special conformal transformations

$$x_\mu \Rightarrow \frac{x_\mu + a_\mu x^2}{1 + 2a\cdot x + a^2x^2} \quad \Leftrightarrow \quad \text{inversion} + \text{shift} + \text{inversion}$$

15 generators

$$\begin{aligned}
i[M_{\mu\nu}, M_{\alpha\beta}] &= g_{\mu\alpha}M_{\nu\beta} + g_{\nu\beta}M_{\mu\alpha} - g_{\mu\beta}M_{\nu\alpha} - g_{\nu\alpha}M_{\mu\beta} \\
i[M_{\mu\nu}, P_\alpha] &= g_{\mu\beta}P_\alpha - g_{\mu\alpha}P_\beta, \qquad i[D, P_\mu] = P_\mu, \\
i[D, K_\mu] &= -K_\mu, \quad i[K_\mu, P_\nu] = 2(g_{\mu\nu}D + M_{\mu\nu}) \\
i[K_\mu, M_{\alpha\beta}] &= -g_{\mu\alpha}K_\beta + g_{\mu\beta}K_\alpha
\end{aligned}$$

For scalar fields

$$D\phi(x) = i(x^\xi\partial_\xi + \Delta)\phi(x), \quad K_\mu\phi(x) = i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu + 2x_\mu\Delta)$$

where Δ is the dimension of the field (in the leading order: canonical, in all orders: canonical + anomalous)

Note that Poincare generators + dilatation form an 11-parameter subgroup

Generators of conformal transformations of TMDs

In the leading order 11 generators transform TMD operators covariantly

$$-iP_i\mathcal{F}_{-j}(x_+,x_\perp) = \partial_i\mathcal{F}_{-j}(x_+,x_\perp), \quad -iP_-\mathcal{F}_{-j}(x_+,x_\perp) = \frac{\partial}{\partial x_+}\mathcal{F}_{-j}(x_+,x_\perp)$$

$$-iM_{-i}\mathcal{F}_{-j}(up_1+x_\perp) = -x_i\frac{d}{dx_+}\hat{\mathcal{F}}_{-j}(x_+,x_\perp),$$

$$-iD\mathcal{F}_{-i}(x_+,x_\perp) = \big(x_+\frac{\partial}{\partial x_+}+x_k\frac{\partial}{\partial x_k}+2\big)\mathcal{F}_{-i}(x_+,x_\perp)$$

$$-iM_{ij}\mathcal{F}_{-k}(x_+,x_\perp)$$
$$= (x_i\partial_j - x_j\partial_i)\mathcal{F}_{-k}(x_+,x_\perp) + g_{ik}\mathcal{F}_{-j}(x_+,x_\perp) - g_{jk}\mathcal{F}_{-i}(x_+,x_\perp),$$

$$-iK_i\hat{\mathcal{F}}_{-j}(x_+,x_\perp) = 2x_i(x^k\frac{\partial}{\partial x^k}+x_+\frac{d}{dx_+}+2)\mathcal{F}_{-j}(x_+,x_\perp)$$
$$+ x_\perp^2\frac{\partial}{\partial x^i}\mathcal{F}_{-j}(x_+,x_\perp) - 2x_j\mathcal{F}_{-i}(x_+,x_\perp) + 2g_{ij}\mathcal{F}_{-x}(x_+,x_\perp),$$

$$-iK_-\mathcal{F}_{-j}(x_+,x_\perp) = x_\perp^2\frac{d}{dx_+}\mathcal{F}_{-j}(x_+,x_\perp),$$

$$-iM_{+-}\mathcal{F}_{-i}(x_+,x_\perp)] = \big(x_+\frac{d}{dx_+}+1\big)\mathcal{F}_{-i}(up_1+x_\perp)$$

The remaining operators P_+, K_+, M_{+i} do not preserve the form of $\mathcal{F}_{-i}$

Subgroup of conformal transformations of TMDs

The structure of this subgroup is very transparent in term of "embedding formalism" in 6-dim space

Conformal transformations are Lorentz transformations of light-rays $\left(\frac{1-x^2}{2}, \frac{1+x^2}{2}, x_\mu\right)$ in 6-dim space with metric (1,-1,1,-1,-1,-1).

Generators

$$L_{\mu\nu} \equiv M_{\mu\nu}, \quad L_{-2,\mu} \equiv \frac{1}{2}(P_\mu - K_\mu), \quad L_{-1,\mu} \equiv \frac{1}{2}(P_\mu + K_\mu), \quad L_{-2,-1} \equiv D$$

where $g^{-2,-2} = -1$, $g^{-1,-1} = 1$, then

$$-i[L_{mn}, L_{ab}] = g_{ma}L_{nb} + g_{nb}L_{ma} - g_{mb}L_{na} - g_{na}L_{mb}$$

If we now define

$$L_{mn} \equiv \mathbb{M}_{mn}, \quad L_{-n} \equiv \mathbb{P}_n, \quad L_{+-} \equiv \mathbb{D}$$

with indices $m, n, l = -2, -1, 1, 2$,

we get usual commutation relations for Poincare generators $\mathbb{M}_{mn}, \mathbb{P}_n$ and "dilatation" $\mathbb{D}$ in the 4-dim subspace orthogonal to our physical "+" and "-" directions

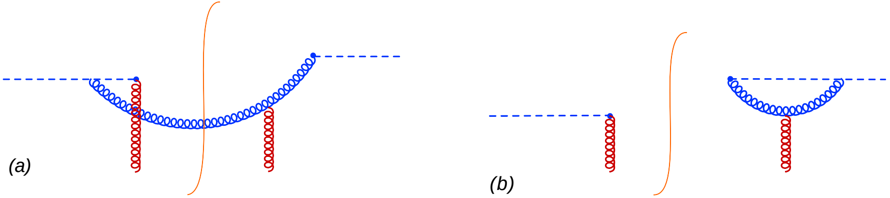
Evolution equation in the Sudakov regime in the coordinate space

In Sudakov regime $\alpha_{\max}\beta_B s \gg (x-y)^{-2}$ and $\beta_B \sim \frac{\sqrt{2/s}}{x_+} \sim \frac{\sqrt{2/s}}{y_+}$

$$\mathcal{O}(x_+,y_+;\lambda) \;\equiv\; F^m_{-i}(x_+,x_\perp)[x,\infty]_x^{ml}[\infty,y_+]_y^{ln}F^n_{-j}(y_+,y_\perp), \qquad \lambda \equiv \frac{(x-y)_\perp^2\,\alpha_{\max}s}{4}$$

$\alpha_{\max}$ is the longitudinal rapidity cutoff $\Leftrightarrow$ α of gluons emitted by Wilson lines is restricted by $\alpha < \alpha_{\max}$

One-loop evolution: same diagrams



Evolution equation in the coordinate space can be obtained by Fourier transformation of momentum-space result, see slide 15

Evolution equation in the Sudakov regime in the coordinate space

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$\Rightarrow$

$$\alpha_{\max}\beta_B s \gg (x-y)^{-2} \;\Leftrightarrow\; \lambda \gg \sqrt{\frac{s}{2}}x_+, \sqrt{\frac{s}{2}}y_+$$

Evolution equation

$$\lambda\frac{d}{d\lambda}\mathcal{O}(x_+,y_+;\lambda)$$
$$= \Big[\int_{x_+}^{\infty} dx'_+ \frac{1}{x'_+ - y_+} e^{i\frac{\lambda\sqrt{2/s}}{x'_+ - y_+}} \mathcal{O}(x_+,y_+;\lambda) - \int_{y_+}^{\infty} dy'_+ \frac{\mathcal{O}(x_+,y_+;\lambda) - \mathcal{O}(x_+t,y'_+;\lambda)}{y'_+ - y_+}$$
$$+ \int_{y_+}^{\infty} dy'_+ \frac{1}{y'_+ - x_+} e^{i\frac{\lambda\sqrt{2/s}}{y'_+ - x_+}} \mathcal{O}(x_+,y_+;\lambda) - \int_{x_+}^{\infty} dx'_+ \frac{\mathcal{O}(x_+,y_+;\lambda) - \mathcal{O}(x'_+,y_+;\lambda)}{x'_+ - x_+}\Big]$$

Solution of the evolution equation in the Sudakov regime

Solution $(\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{4\pi})$

$$
\begin{aligned}
\mathcal{O}(x_+, y_+; \lambda) \; &= \\
= \; e^{-\frac{\bar{\alpha}_s}{2}\left[\ln^2 \frac{2\lambda^2/s}{x_+ y_+} - \ln^2 \frac{2\lambda_0^2/s}{x_+ y_+}\right] + \bar{\alpha}_s[4\gamma_E - \ln 2]\ln\frac{\lambda}{\lambda_0}} & \int dx'_+ dy'_+ \; \mathcal{O}(x'_+, y'_+; \lambda_0)(x_+ y_+)^{-\bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}} \\
\times \left[\frac{i\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}\right)}{(x_+ - x'_+ + i\epsilon)^{1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}}} - \frac{i\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}\right)}{(x_+ - x'_+ - i\epsilon)^{1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}}}\right] \\
\times \left[\frac{i\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}\right)}{(y_+ - y'_+ + i\epsilon)^{1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}}} - \frac{i\Gamma\left(1 - \bar{\alpha}_s \ln\frac{\lambda}{\lambda_0}\right)}{(y_+ - y'_+ - i\epsilon)^{1 - \bar{\alpha}_s \ln\frac{a}{\lambda_0}}}\right]
\end{aligned}
$$

$\ln^2 \frac{2\lambda^2}{s x_+ y_+}$ does not look conformally invariant...

If we use rapidity cutoff at $\alpha_{\max} = \frac{4\sigma}{|x-y|_\perp\sqrt{s}} \quad \Rightarrow \lambda = \sigma|x-y|\sqrt{s}$,
the solution

$$\begin{aligned}
&\mathcal{O}(x_+, y_+; \sigma) \\
&= e^{-\frac{\bar{\alpha}_s}{2}\left(\ln^2 \frac{2(x-y)_\perp^2\sigma^2}{x_+y_+} - \ln^2 \frac{2(x-y)_\perp^2\sigma_0^2}{x_+y_+}\right)} e^{4\bar{\alpha}_s\psi(1)\ln\frac{\sigma}{\sigma_0}} \int dx'_+ dy'_+\ \mathcal{O}(x'_+, y'_+; \sigma_0) \\
&\times (x_+y_+)^{-\bar{\alpha}_s \ln\frac{\sigma}{\sigma_0}} \left[\frac{i\Gamma\left(1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}\right)}{(x_+ - x'_+ + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}} - \frac{i\Gamma\left(1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}\right)}{(x_+ - x'_+ - i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}}\right] \\
&\times \left[\frac{i\Gamma\left(1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}\right)}{(y_+ - y'_+ + i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}} - \frac{i\Gamma\left(1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}\right)}{(y_+ - y'_+ - i\epsilon)^{1-\bar{\alpha}_s\ln\frac{\sigma}{\sigma_0}}}\right]
\end{aligned}$$

is obviously invariant under the inversion $x_+ \to \frac{x_+}{x_\perp^2}$, $y_+ \to \frac{y_+}{y_\perp^2}$.

Conformal invariance of the evolution equation

It is easy to see that the r.h.s. of our equation transforms covariantly under all transformations of the subgroup except Lorentz boost generated by M_{+-}. The reason is that the Lorentz boost in z direction changes cutoffs for the evolution.

Our Sudakov-type evolution is applicable in the region between

$$\sigma_2 = \sigma_B = \frac{|x-y|_\perp}{(x-y)_-} \quad \text{and} \quad \sigma_1 = \frac{(x-y)_+}{|x-y|_\perp} \tag{1}$$

The Lorentz boost $z_+ \to \lambda z_+,\ z_- \to \frac{1}{\lambda} z_-$ changes the value of target matrix element $\langle p_A|\mathcal{O}|p_B\rangle$ by $\exp\{4\lambda\bar{\alpha}_s \ln \frac{(x-y)_\parallel^2}{(x-y)_\perp^2}\}$,
but simultaneously it will change the result of similar evolution for projectile matrix element $\langle p_A|\tilde{\mathcal{O}}|p_A\rangle$ by $\exp\{-4\lambda\bar{\alpha}_s \ln \frac{(x-y)_\parallel^2}{(x-y)_\perp^2}\}$

$\Rightarrow$ the overall result for the amplitude remains intact.

Evolution of “generalized TMD’

To compare with conventional TMD analysis let us write down the evolution of “generalized TMD’

$$D^\sigma(x_B,\xi) \ = \ \int dz_+ e^{-ix_Bz_*}\langle p'_B|\mathcal{O}^\sigma\Big(-\frac{z_+}{2},\frac{z_+}{2}\Big)|p_B\rangle, \qquad \xi\equiv -\frac{p'_B-p_B}{\sqrt{2s}}$$

Our result

$$\frac{D^{\sigma_2}(x_B,\xi)}{D^{\sigma_1}(x_B,\xi)} \ = \ e^{-2\bar\alpha_s\ln\frac{\sigma_2}{\sigma_1}[\ln\sigma_2\sigma_1(x_B^2-\xi^2)s(x-y)_\perp^2+4\gamma_E-2\ln 2]}$$

For usual TMD at $\xi=0$ with the limits of Sudakov evolution set by Eq. (1) one obtains

$$\frac{D^{\sigma_2}(x_B,q_\perp)}{D^{\sigma_1}(x_B,q_\perp)} \ = \ e^{-2\bar\alpha_s\ln\frac{Q^2}{q_\perp^2}\Big[\ln\frac{Q^2}{q_\perp^2}+4\gamma_E-2\ln 2\Big]}$$

One-loop result with “usual” cutoff differs by $-2\ln 2\to -4\ln 2$.

$4\gamma_E-2\ln 2$ in our result is “scheme-dependent” (depends on the way to cut α-integration)

Limits of Sudakov evoluton

$q_\perp \ll Q$ in the momentum space $\Leftrightarrow (x-y)_\parallel^2 \ll (x-y)_\perp^2$ in the coord. space.

During Sudakov evolution:

- the transverse separation between gluon operators $\mathcal{F}_i$ and $\mathcal{F}_j$ remains intact
- the longitudinal separation increases.

The Sudakov approximation can be trusted until $k_+ \gg \frac{(x-y)_+}{(x-y)_\perp^2}$.

If $x_B \sim 1$, the relative energy between Wilson-line operators $\mathcal{F}$ and target nucleon at the final point of evolution is $\sim m_N^2$ so one should use phenomenological models of TMDs with this low rapidity cutoff as a starting point of the evolution.

If $x_B \ll 1$, this relative energy is $\frac{q_\perp^2}{x_B} \gg m_N^2$ so one can continue the rapidity evolution in the region $\frac{q_\perp^2}{x_B s} > \sigma > \frac{m_N^2}{s}$ beyond the Sudakov region into the small-x region.

The transition between Sudakov and small-x regimes means the study of operator $\mathcal{O}$ at $(x-y)_\parallel^2 \sim (x-y)_\perp^2$ and we hope that conformal considerations can help us to obtain the nice-looking TMD evolution in that region.

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   - Conformal evolution for all x_B

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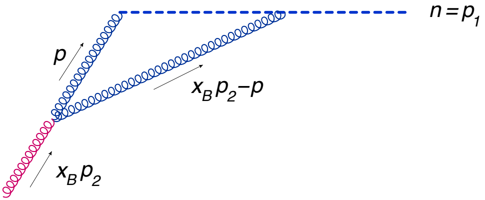
Thank you for attention!

Backup

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

$$I = \int \frac{d^4p}{\pi^2 i}\,\frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)}\,\frac{x_B p_2\cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

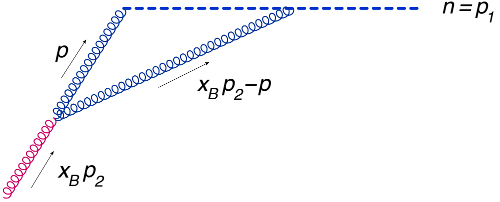
Regularization # 1 (ours): $n = p_1$, $|\alpha| < \sigma$

$$I_1 = -i\frac{s}{2\pi^2}\int_{-\sigma}^{\sigma} d\alpha \int \frac{d\beta}{\beta - i\epsilon}\int d^2p_\perp \frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\,\frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$= \frac{1}{\pi}\int_0^{\sigma} d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2}\,\frac{1}{\alpha + \frac{m^2 + p_\perp^2}{s x_B}} = \int_0^{\sigma}\frac{d\alpha}{\pi\alpha}\ln\left(1 + \frac{\alpha s x_B}{m^2}\right) = \frac{1}{2}\ln^2\frac{\sigma s x_B}{m^2} + \frac{\pi^2}{6}$$

Double log of σ, no UV.

Rapidity *vs* UV cutoff



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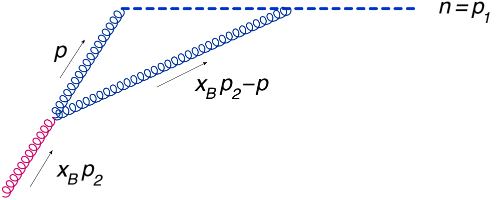
Regularization # 2 (by slope of Wilson line): $n = p_1 + \gamma p_2,\ \gamma \ll 1$

$$I_2 \;=\; -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta + \gamma\alpha - i\epsilon}\,\frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\,\frac{1}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow\ I_2 \;\stackrel{(p_2\cdot n)^2 \gg m^2n^2}{\rightarrow}\; \frac{1}{2}\ln^2\frac{x_Bs^2}{m^2n^2} + \frac{\pi^2}{6} \;=\; \frac{1}{2}\ln^2\frac{x_Bs}{m^2\gamma} + \frac{\pi^2}{6}$$

Double log of σ, no UV.

Rapidity *vs* UV cutoff



Typical integral ($n \equiv p_1$, "gluon mass" m = IR cutoff)

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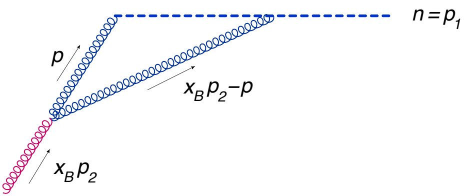
Regularization # 3: $n = p_1,\ \beta > b$

$$I_3 \;=\; -i\frac{s}{2\pi^2}\int d\alpha d\beta \int d^2p_\perp \frac{1}{\beta - i\epsilon}\,\frac{1}{m^2 + p_\perp^2 - \alpha\beta s - i\epsilon}\,\frac{x_B}{m^2 + p_\perp^2 + \alpha(x_B - \beta)s - i\epsilon}$$

$$\Rightarrow\ I_3 \;=\; \frac{1}{\pi}\int_b^{x_B}\frac{d\beta}{\beta}\int \frac{d^2p_\perp}{m^2 + p_\perp^2} \;=\; \ln\frac{x_B}{b}\,\ln\frac{\mu_{\rm UV}^2}{m^2}$$

UV $\times$ single log of the cutoff

Rapidity *vs* UV cutoff



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$$I = \int \frac{d^4p}{\pi^2 i} \frac{1}{(p\cdot n - i\epsilon)(p^2 - m^2 + i\epsilon)} \frac{x_B p_2 \cdot n}{(x_B p_2 - p)^2 - m^2 + i\epsilon}$$

Regularization # 1 $\Rightarrow$ Regularization # 3:

$$\text{change of variables} \quad \beta = \frac{x_B(m^2 + p_\perp^2)}{\alpha s x_B + m^2 + p_\perp^2}$$

$$\int_0^\sigma d\alpha \int d^2p_\perp \frac{1}{m^2 + p_\perp^2} \frac{1}{\alpha + \frac{m^2 + p_\perp^2}{s x_B}} = \int_b^{x_B} \frac{d\beta}{\beta} \int \frac{d^2p_\perp}{m^2 + p_\perp^2}, \qquad b = \frac{x_B(m^2 + p_\perp^2)}{\sigma s x_B + m^2 + p_\perp^2}$$