# LORENTZ INVARIANCE
# OF
# TWIST-3 QUARK DISTRIBUTIONS

Fatma Aslan & Matthias Burkardt

# Outline

- Discontinuities in twist-3 GPDs
- Singularities in Twist-3 PDFs
- Regularization of the singularities
- Lorentz invariance of twist-3 quark distributions
- Zero Modes

## Why study TWIST-3 ?

❑ **Low $Q^2$: Twist-3 contamination can be significant.**

- **12 GeV @ Jlab** Twist-3 effects may not be negligible in the measurement of DVCS amplitude.
- **Femtography @ UVa**

$$\int_{-1}^{1} dx \frac{GPD}{x \pm \xi + i\varepsilon}$$

Measurement at a fixed $Q^2$ does not allow extracting the underlined GPD unambiguously

Theoretical inputs
- Polynomiality
- PDF limits
- QCD evolution

**Measure the DVCS amplitude**
- At different values of $Q^2$
- At some value of $Q^2$ for a given parameterization ← evolve

**Compare**
- DVCS amplitude measured at different values of $Q^2$
- Parameterization of GPDs evolved over a range $Q^2$

**Constrain the parameterization of GPDs further.**

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**Measure the DVCS amplitude**
- At different values of $Q^2$
- At some value of $Q^2$ for a given parameterization

evolve

**Compare**
- DVCS amplitude measured at different values of $Q^2$
- Parameterization of GPDs evolved over a range $Q^2$

**Subtract the twist-3 contribution**

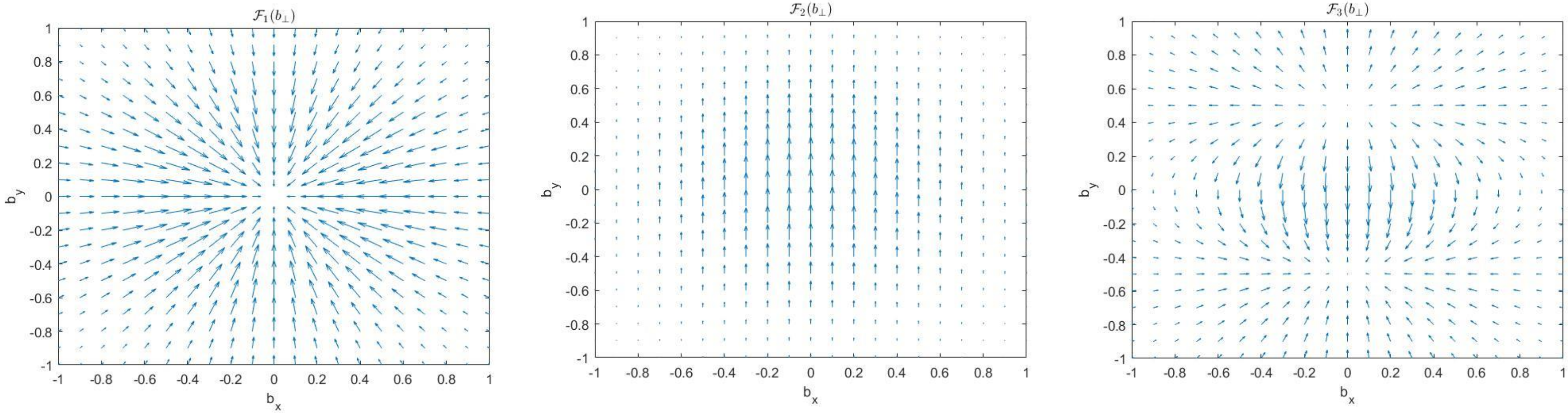
**Constrain the parameterization of GPDs further.**

# Why study TWIST-3 ?

❑ **Information about quark-gluon-quark correlations**.

| Twist-3 PDFs | Twist-3 GPDs |
|---|---|
| Force | Force *Distribution* |

Burkardt
Transverse Force on Quarks in DIS (2008)
Aslan, Burkardt, Schlegel
Transverse Force Tomography (2019)

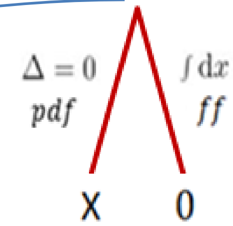



*Next Talk: Matthias Burkardt, Transverse Force Tomography*

# TWIST-3 GPDs

**Twist-3 Vector GPDs** $G_1, G_2, G_3, G_4$

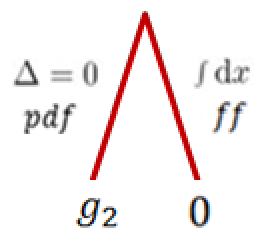
$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle P', S' | \overline{q}(-\frac{z^-}{2})\gamma^j q(\frac{z^-}{2}) | P, S\rangle = \frac{1}{2p^+}\overline{u}(P', S')\Big[\frac{\Delta_\perp^j}{2M}G_1 + \gamma^j(H+E+G_2) + \frac{\Delta_\perp^j}{p^+}\gamma^+ G_3 + \frac{i\epsilon_T^{jk}\Delta_\perp^k}{p^+}\gamma^+\gamma_5 G_4\Big]u(P,S)$$




$$L_{kin}^q = -\int dx\, x G_2^q(x, \xi = 0, t = 0)$$

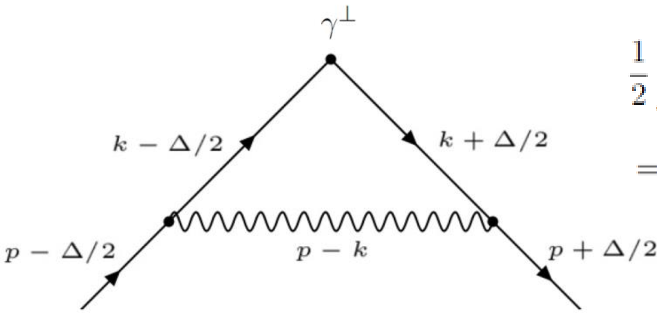
**Twist-3 Axial V. GPDs** $\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4$

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle P', S' | \overline{q}(-\frac{z^-}{2})\gamma^j\gamma_5 q(\frac{z^-}{2}) | P, S\rangle = \frac{1}{2p^+}\overline{u}(P', S')\Big[\frac{\Delta_\perp^j}{2M}\gamma_5(\tilde{E}+\tilde{G}_1) + \gamma^j\gamma_5(\tilde{H}+\tilde{G}_2) + \frac{\Delta_\perp^j}{p^+}\gamma^+\gamma_5\tilde{G}_3 + \frac{i\epsilon_T^{jk}\Delta_\perp^k}{p^+}\gamma^+\tilde{G}_4\Big]\overline{u}(P,S).$$




Kiptily, Polyakov
Genuine twist-3 contributions to the generalized parton distributions from instantons (2003)

# $G_2$ in Quark Target Model

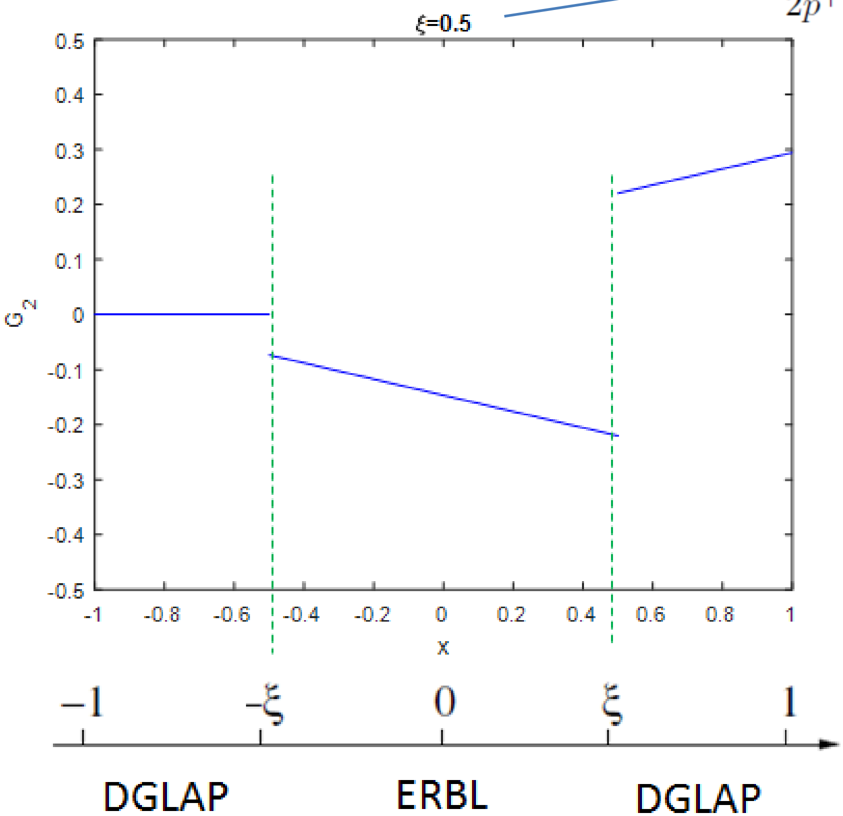



Quark target model in a symmetric frame

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle P', S'|\, \overline{q}(-\frac{z^-}{2})\gamma^j q(\frac{z^-}{2})\, |P,S\rangle$$

$$= \frac{1}{2p^+}\overline{u}(P',S')\Big[\frac{\Delta_\perp^j}{2M}G_1 + \gamma^j\underbrace{(H+E+G_2)} + \frac{\Delta_\perp^j}{p^+}\gamma^+ G_3 + \frac{i\epsilon_T^{jk}\Delta_\perp^k}{p^+}\gamma^+\gamma_5 G_4\Big]\overline{u}(P,S)$$

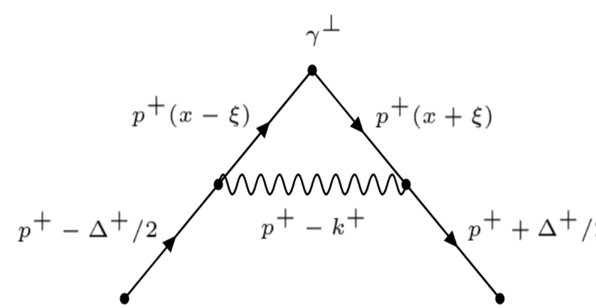
$$\xi = \frac{\Delta^+}{2p^+}$$



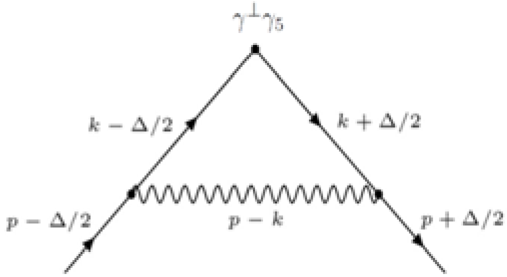

$$x = \frac{k^+}{p^+}$$

$$G_2 = \begin{cases} \frac{g^2}{2\pi^2}\frac{(1+x)}{(1-\xi^2)}\ln\Lambda_\perp & \text{for} \quad \xi < x < 1, \\ -\frac{g^2}{4\pi^2}\frac{(1+x)}{\xi(1+\xi)}\ln\Lambda_\perp & \text{for} \quad -\xi \le x \le \xi, \\ 0 & \text{for} \quad -1 < x < \xi, \end{cases}$$

***$G_2$ has discontinuities***

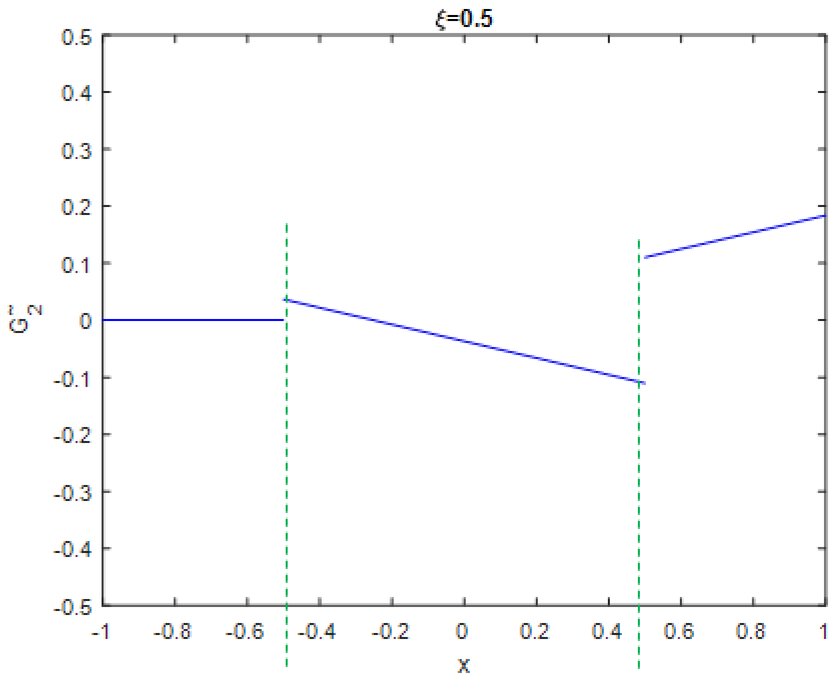


# $\widetilde{G}_2$ in Quark Target Model



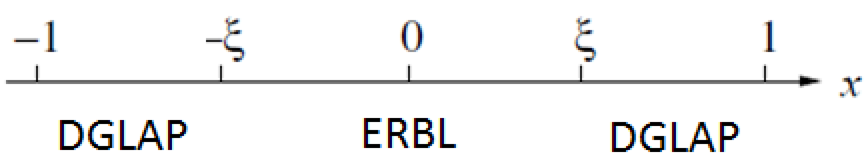

Quark target model in a symmetric frame

$$\frac{1}{2}\int\frac{dz^-}{2\pi}e^{ixp^+z^-}\langle P',S'|\,\overline{q}(-\frac{z^-}{2})\gamma^j\gamma_5 q(\frac{z^-}{2})\,|P,S\rangle$$
$$=\frac{1}{2p^+}\overline{u}(P',S')\Big[\frac{\Delta_\perp^j}{2M}\gamma_5(\widetilde{E}+\widetilde{G}_1)+\gamma^j\gamma_5\underbrace{(\widetilde{H}+\widetilde{G}_2)}+\frac{\Delta_\perp^j}{p^+}\gamma^+\gamma_5\widetilde{G}_3+\frac{i\epsilon_T^{jk}\Delta_\perp^k}{p^+}\gamma^+\widetilde{G}_4\Big]\overline{u}(P,S).$$

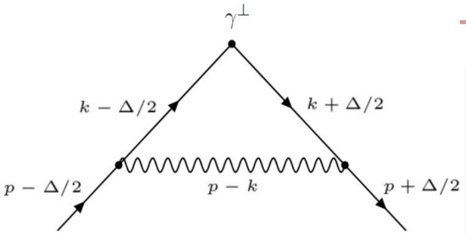



$$\widetilde{G}_2=\begin{cases}\dfrac{g^2}{2\pi^2}\dfrac{(x+\xi^2)}{(1-\xi^2)}\ln\Lambda_\perp & \text{for} \quad \xi<x<1,\\ -\dfrac{g^2}{4\pi^2}\dfrac{(x+\xi^2)}{\xi(1+\xi)}\ln\Lambda_\perp & \text{for} \quad -\xi\le x\le\xi,\\ 0 & \text{for} \quad -1<x<\xi,\end{cases}$$

**$\widetilde{G}_2$ too has discontinuities**



# How do the discontinuities arise?




Quark target model in a symmetric frame

**The parameterization**

$$\frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle P', S'|\, \overline{q}(-\frac{z^-}{2})\gamma^j q(\frac{z^-}{2})\, |P, S\rangle$$

$$= \frac{1}{2p^+}\overline{u}(P', S')\Big[\frac{\Delta_\perp^j}{2M}G_1 + \gamma^j \underbrace{(H + E + G_2)} + \frac{\Delta_\perp^j}{p^+}\gamma^+ G_3 + \frac{i\epsilon_T^{jk}\Delta_\perp^k}{p^+}\gamma^+\gamma_5 G_4\Big]\overline{u}(P, S)$$

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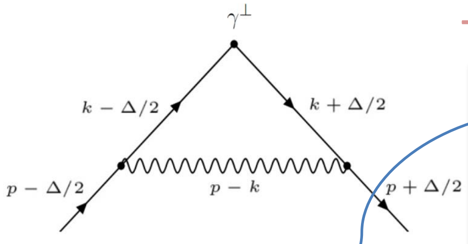


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**The model**

$$-\frac{ig^2}{2}\int \frac{d^4k}{(2\pi)^4}\delta(k^+ - xp^+)\overline{u}(P', S')\gamma^\mu \frac{(\not{k} + \frac{\not{\Delta}}{2} + m)}{[(k + \frac{\Delta}{2})^2 - m^2 + i\epsilon]}\gamma^\perp \frac{(\not{k} - \frac{\not{\Delta}}{2} + m)}{[(k - \frac{\Delta}{2})^2 - m^2 + i\epsilon]}\gamma^\nu \times \Big[g_{\mu\nu} - \frac{n_\nu(p_\mu - k_\mu)}{p^+ - k^+} - \frac{n_\mu(p_\nu - k_\nu)}{p^+ - k^+}\Big]\frac{1}{[(p-k)^2 - \lambda^2 + i\epsilon]}u(P, S)$$

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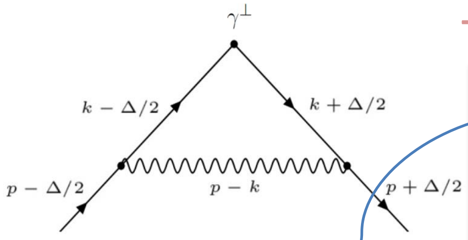


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The divergent part of $G_2$ is calculated as, $-ig^2 \int \frac{d^2k_\perp dk^-}{(2\pi)^4} \frac{k^- 8(p^+)^2 (1+x)}{[(k + \frac{\Delta}{2})^2 - m^2 + i\epsilon] [(k - \frac{\Delta}{2})^2 - m^2 + i\epsilon] [(p-k)^2 - \lambda^2 + i\epsilon]}$.

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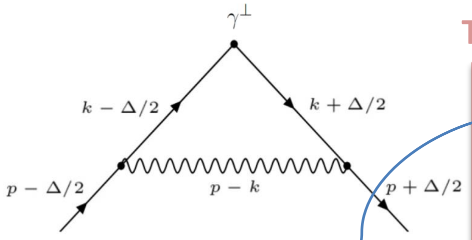


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$$-\frac{ig^2}{2}\int\frac{d^4k}{(2\pi)^4}\delta(k^+-xp^+)\overline{u}(P',S')\gamma^\mu\frac{(\not{k}+\frac{\not\Delta}{2}+m)}{[(k+\frac{\Delta}{2})^2-m^2+i\epsilon]}\gamma^\perp\frac{(\not{k}-\frac{\not\Delta}{2}+m)}{[(k-\frac{\Delta}{2})^2-m^2+i\epsilon]}\gamma^\nu\times\Big[g_{\mu\nu}-\frac{n_\nu(p_\mu-k_\mu)}{p^+-k^+}-\frac{n_\mu(p_\nu-k_\nu)}{p^+-k^+}\Big]\frac{1}{[(p-k)^2-\lambda^2+i\epsilon]}u(P,S)$$

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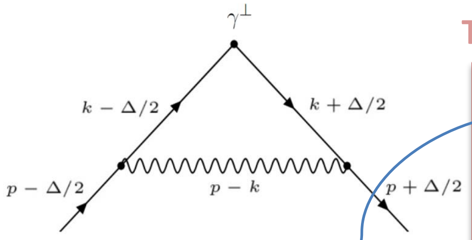
Using $(P-k)^2-\lambda^2 = 2(P^+-k^+)(P^- - k^-) - k_\perp^2 - \lambda^2$, $k^-$ in the numerator can be replaced by the following expression

$$k^- = \frac{M^2}{2p^+} - \frac{[(p-k)^2-\lambda^2]}{2(p^+-k^+)} - \frac{(k_\perp^2+\lambda^2)}{2(p^+-k^+)}$$

# How do the discontinuities arise?




Quark target model in a symmetric frame

## The parameterization

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## The model

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The divergent part of $G_2$ is calculated as, $-ig^2\int\frac{d^2k_\perp dk^-}{(2\pi)^4}\frac{k^-8(p^+)^2(1+x)}{[(k+\frac{\Delta}{2})^2-m^2+i\epsilon]\,[(k-\frac{\Delta}{2})^2-m^2+i\epsilon]\,[(p-k)^2-\lambda^2+i\epsilon]}.$

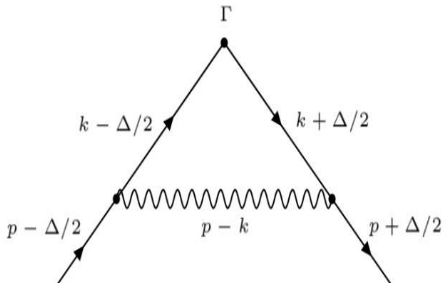
Using $(P-k)^2-\lambda^2=2(P^+-k^+)(P^--k^-)-k_\perp^2-\lambda^2$, $k^-$ in the numerator can be replaced by the following expression

$$k^-=\frac{M^2}{2p^+}-\frac{[(p-k)^2-\lambda^2]}{2(p^+-k^+)}-\frac{(k_\perp^2+\lambda^2)}{2(p^+-k^+)}$$

The second term cancels the propagator in the denominator leading to the following contribution which is nonzero only in the ERBL region, $-\xi<x<\xi$.

$$ig^2 4p^+\frac{(1+x)}{(1-x)}\int\frac{d^2k_\perp dk^-}{(2\pi)^4}\frac{1}{[(k+\frac{\Delta}{2})^2-m^2+i\epsilon]\,[(k-\frac{\Delta}{2})^2-m^2+i\epsilon]}.$$

# Discontinuities of Twist-3 GPDs in Quark Target Model




| *Twist-3 Vector GPDs* | Quark Target Model |
|---|---|
| $G_1$ | ✓ |
| $G_2$ | ✗ |
| $G_3$ | ✗ |
| $G_4$ | ✗ |

| *Twist-3 Axial V. GPDs* | Quark Target Model |
|---|---|
| $\tilde{G}_1$ | ✓ |
| $\tilde{G}_2$ | ✗ |
| $\tilde{G}_3$ | ✗ |
| $\tilde{G}_4$ | ✗ |

✓: Continuous
✗: Discontinuous

Aslan, Burkardt, Lorce, Metz, Pasquini, Twist-3 GPDs in DVCS, 2018

# Discontinuities and DVCS Factorization

$$\int_{-1}^{1} dx \frac{GPD}{x \pm \xi + i\varepsilon}$$ **Discontinuities → Divergent scattering amplitudes → Factorization ?**

**Example:**

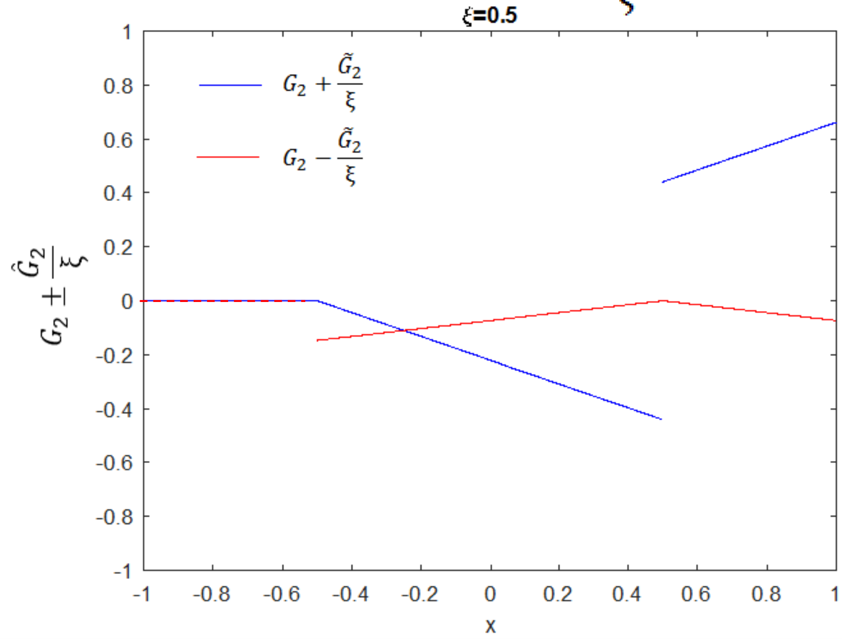
The relevant DVCS amplitude involves $G_2 \pm \frac{\tilde{G}_2}{\xi}$

$$\int_{-1}^{1} dx \frac{G_2 + \frac{1}{\xi}\tilde{G}_2}{x + \xi + i\varepsilon}$$

- $G_2 + \frac{1}{\xi}\tilde{G}_2$ continuous at $x = -\xi$

$$\int_{-1}^{1} dx \frac{G_2 - \frac{1}{\xi}\tilde{G}_2}{x - \xi + i\varepsilon}$$

- $G_2 - \frac{1}{\xi}\tilde{G}_2$ continuous at $x = \xi$




***Twist-3 GPDs are discontinuous → Linear combinations of twist-3 GPDs that enter the DVCS amplitude are well-behaved → Twist -3 DVCS factorization is safe***


Aslan, Burkardt, Lorce, Metz, Pasquini, Twist-3 GPDs in DVCS, 2018

# Discontinuities and DVCS Factorization

**twist-2** ⟹ $$\int_{-1}^{1} dx \frac{GPD}{x \pm \xi + i\varepsilon}$$ **Discontinuities → Divergent scattering amplitudes → Factorization ?**

**twist-3** ⟹ $$\int_{-1}^{1} dx \left[ (F_\perp^\nu - i\varepsilon_{\perp\alpha}^\nu \tilde{F}_\perp^\alpha) \frac{1}{x - \xi + i\varepsilon} + (F_\perp^\nu + i\varepsilon_{\perp\alpha}^\nu \tilde{F}_\perp^\alpha) \frac{1}{x + \xi - i\varepsilon} \right]$$

**Example:**

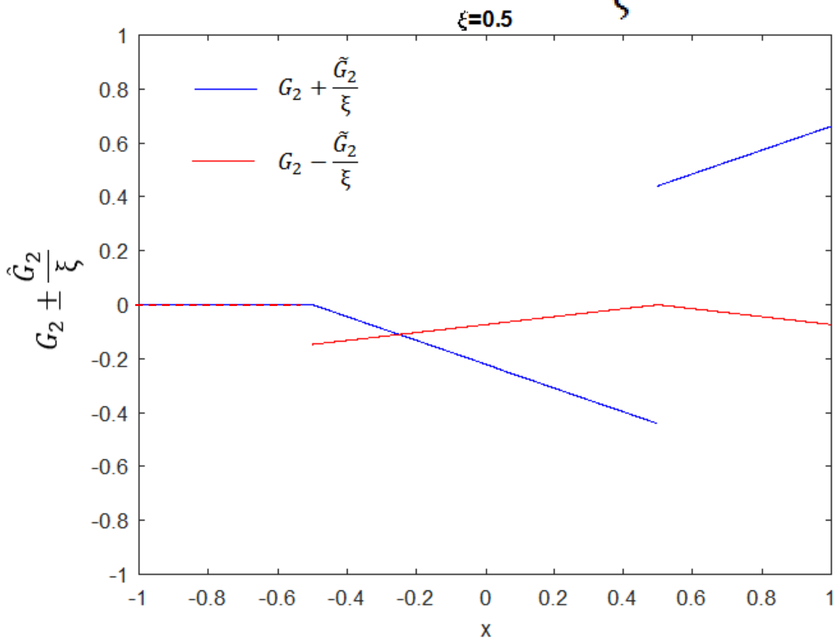
The relevant DVCS amplitude involves $G_2 \pm \frac{\tilde{G}_2}{\xi}$

$$\int_{-1}^{1} dx \frac{G_2 + \frac{1}{\xi}\tilde{G}_2}{x + \xi + i\varepsilon}$$

- $G_2 + \frac{1}{\xi}\tilde{G}_2$ continuous at $x = -\xi$

$$\int_{-1}^{1} dx \frac{G_2 - \frac{1}{\xi}\tilde{G}_2}{x - \xi + i\varepsilon}$$

- $G_2 - \frac{1}{\xi}\tilde{G}_2$ continuous at $x = \xi$



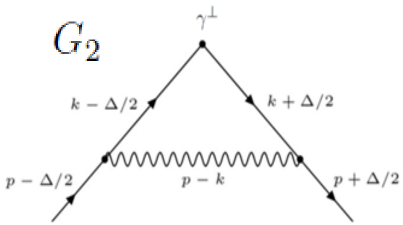

***Twist-3 GPDs are discontinuous → Linear combinations of twist-3 GPDs that enter the DVCS amplitude are well-behaved → Twist -3 DVCS factorization is safe***

Aslan, Burkardt, Lorce, Metz, Pasquini, Twist-3 GPDs in DVCS, 2018

# How do the discontinuities behave as ξ→0 ?

## $G_2$ and $\widetilde{G}_2$ in Quark Target Model

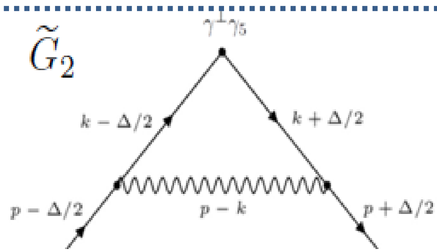
$G_2$




$$ig^2 4p^+ \frac{(1+x)}{(1-x)} \int \frac{d^2k_\perp dk^-}{(2\pi)^4} \frac{1}{[(k+\frac{\Delta}{2})^2 - m^2 + i\epsilon]\,[(k-\frac{\Delta}{2})^2 - m^2 + i\epsilon]}$$

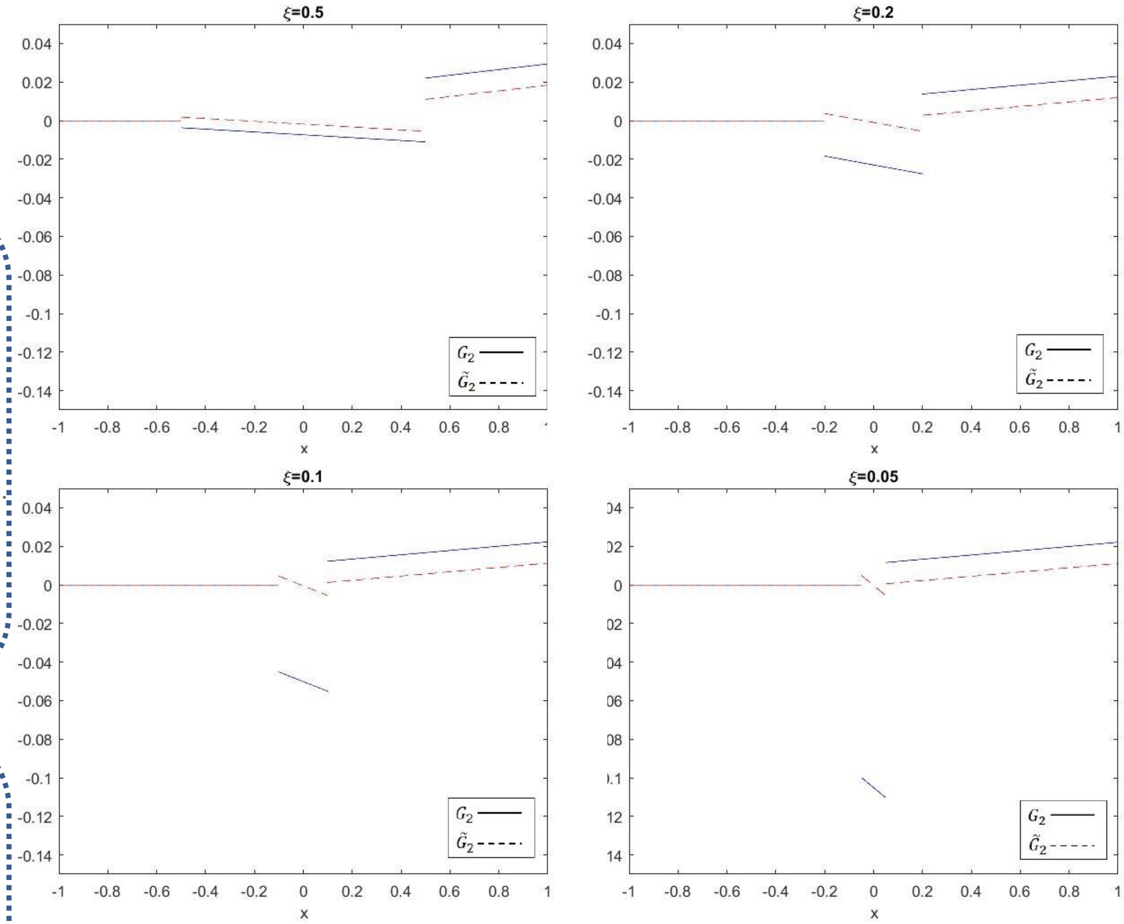
$$\lim_{\xi\to 0} \frac{-g^2}{(2\pi)^2} \frac{(1+x)}{\xi(1-x)} \ln \Lambda_\perp \to \delta(x)$$

$\widetilde{G}_2$




$$ig^2 4p^+ \frac{(x+\xi^2)}{(1-x)} \int \frac{d^2k_\perp dk^-}{(2\pi)^4} \frac{1}{[(k+\frac{\Delta}{2})^2 - m^2 + i\epsilon]\,[(k-\frac{\Delta}{2})^2 - m^2 + i\epsilon]}$$

$$-\frac{g^2}{(2\pi)^2} \frac{(x+\xi^2)}{\xi(1-x)} \ln \Lambda_\perp .$$

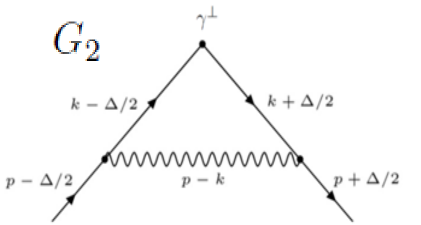



| Twist-3 GPD | Discontinuities as $\xi \to 0$ |
|---|---|
| $G_2$ | Divergent |
| $\widetilde{G}_2$ | Finite |

# How do the discontinuities behave as ξ→0 ?

## $G_2$ and $\widetilde{G}_2$ in Quark Target Model

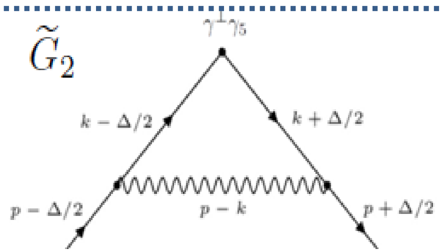
$G_2$




$$ig^2 4p^+ \frac{(1+x)}{(1-x)} \int \frac{d^2k_\perp dk^-}{(2\pi)^4} \frac{1}{[(k+\frac{\Delta}{2})^2 - m^2 + i\epsilon]\,[(k-\frac{\Delta}{2})^2 - m^2 + i\epsilon]}$$

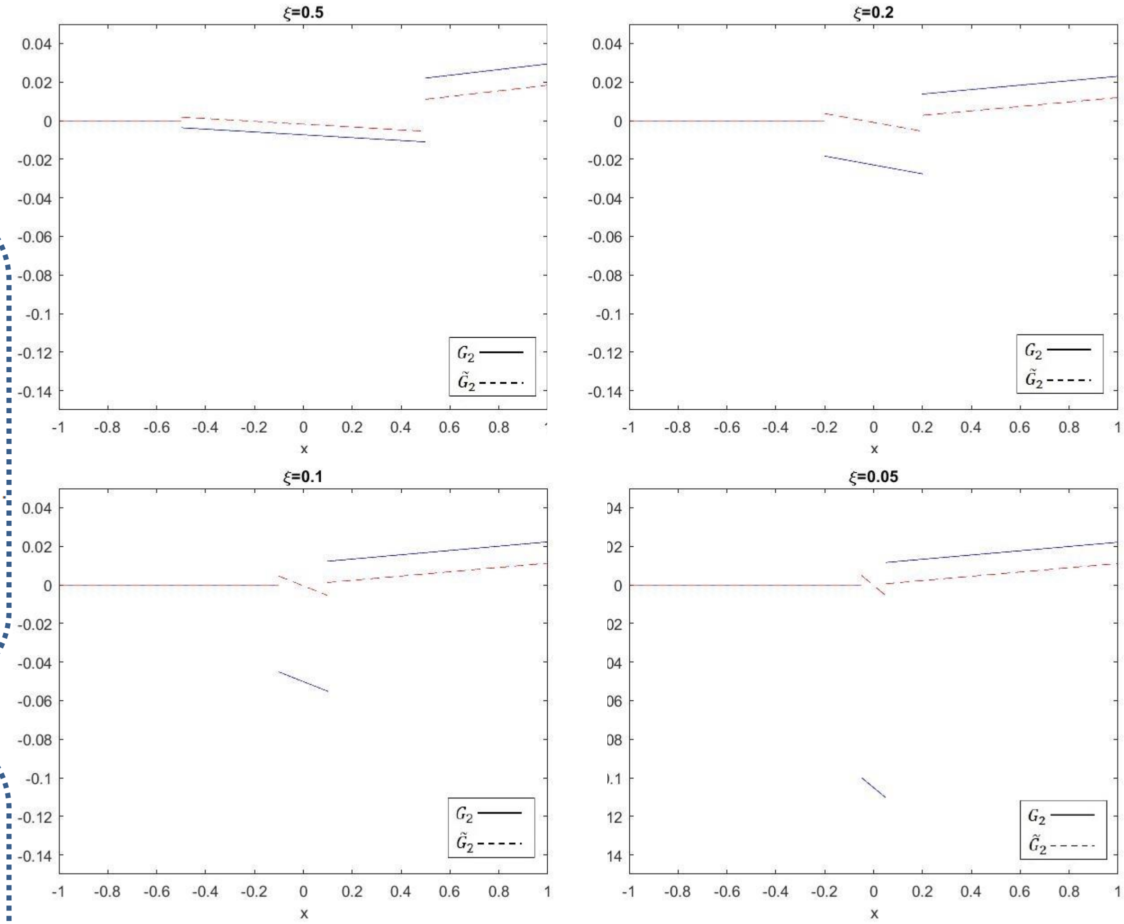
$$\lim_{\xi\to 0} \frac{-g^2}{(2\pi)^2} \frac{(1+x)}{\xi(1-x)} \ln \Lambda_\perp \to \delta(x)$$

$\widetilde{G}_2$




$$ig^2 4p^+ \frac{(x+\xi^2)}{(1-x)} \int \frac{d^2k_\perp dk^-}{(2\pi)^4} \frac{1}{[(k+\frac{\Delta}{2})^2 - m^2 + i\epsilon]\,[(k-\frac{\Delta}{2})^2 - m^2 + i\epsilon]}$$

$$-\frac{g^2}{(2\pi)^2} \frac{(x+\xi^2)}{\xi(1-x)} \ln \Lambda_\perp .$$



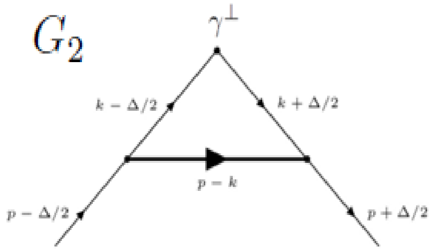

| Twist-3 GPD | Discontinuities as $\xi \to 0$ |
|---|---|
| $G_2$ | Divergent |
| $\widetilde{G}_2$ | Finite |

***The ERBL region becomes a representation of δ(x)***

# What happens in different models ?

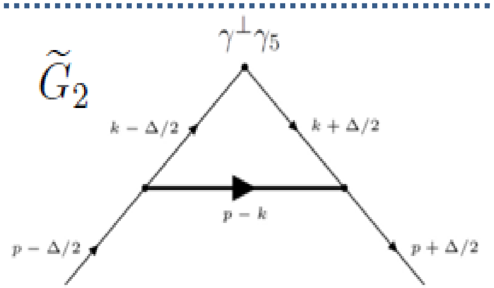
## $G_2$ and $\widetilde{G}_2$ in Scalar Diquark Model

$G_2$

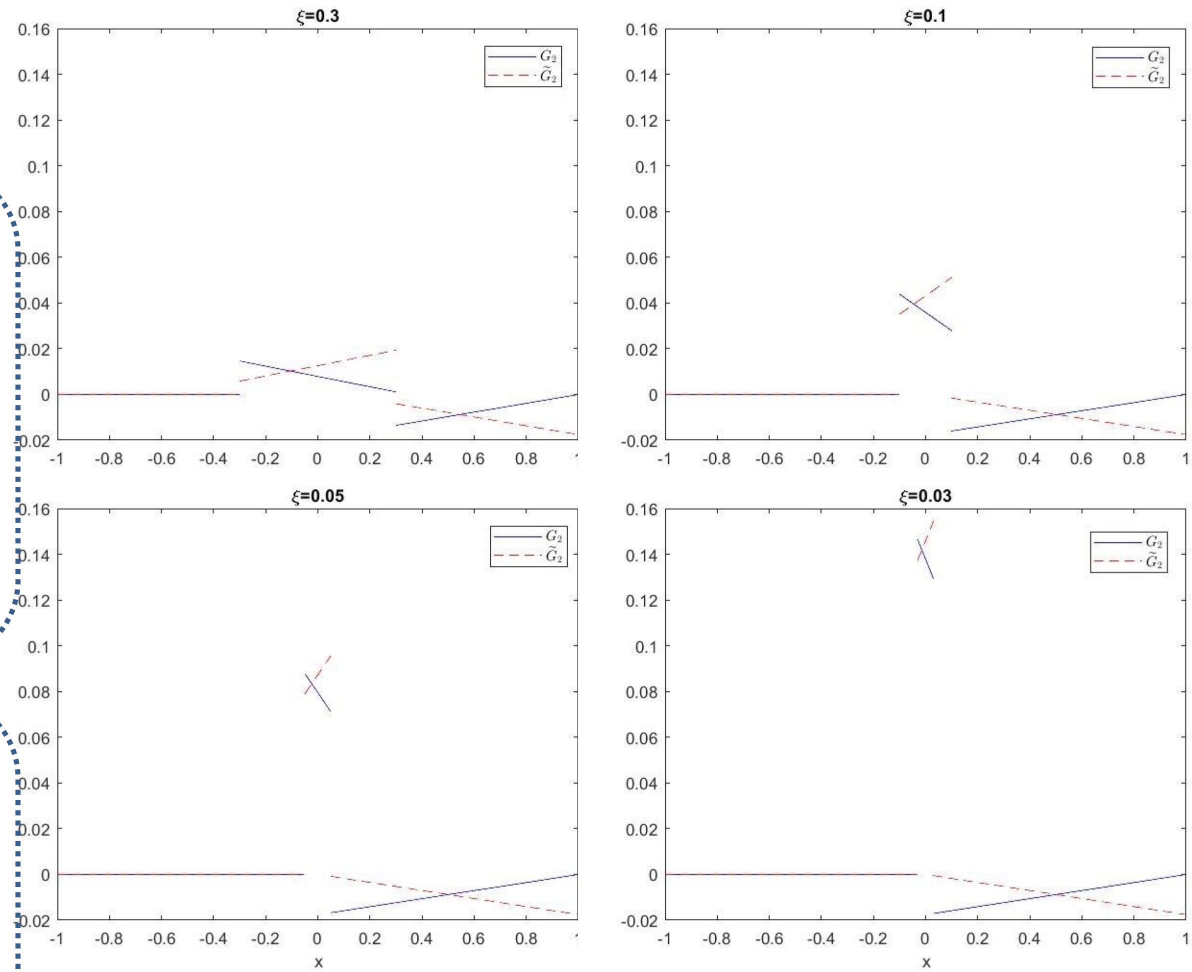



$$G_{2,Div}^{Singular} = \frac{g^2}{8\pi^2}\frac{1}{\xi}\ln\Lambda_\perp$$

$\widetilde{G}_2$




$$\widetilde{G}_{2,Div.}^{Singular} = \frac{g^2}{8\pi^2}\frac{x - 2\xi^2 + 1}{\xi(1-x)}\ln\Lambda_\perp$$



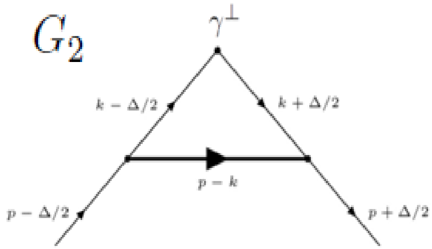

| Twist-3 GPD | Discontinuities as $\xi \to 0$ |
|---|---|
| $G_2$ | Divergent |
| $\widetilde{G}_2$ | Divergent |

***The ERBL region becomes a representation of δ(x)***

# What happens in different models ?

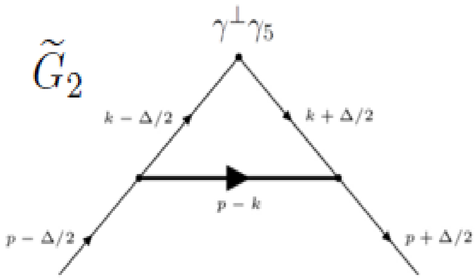
$G_2$ and $\widetilde{G}_2$ in Scalar Diquark Model

$G_2$

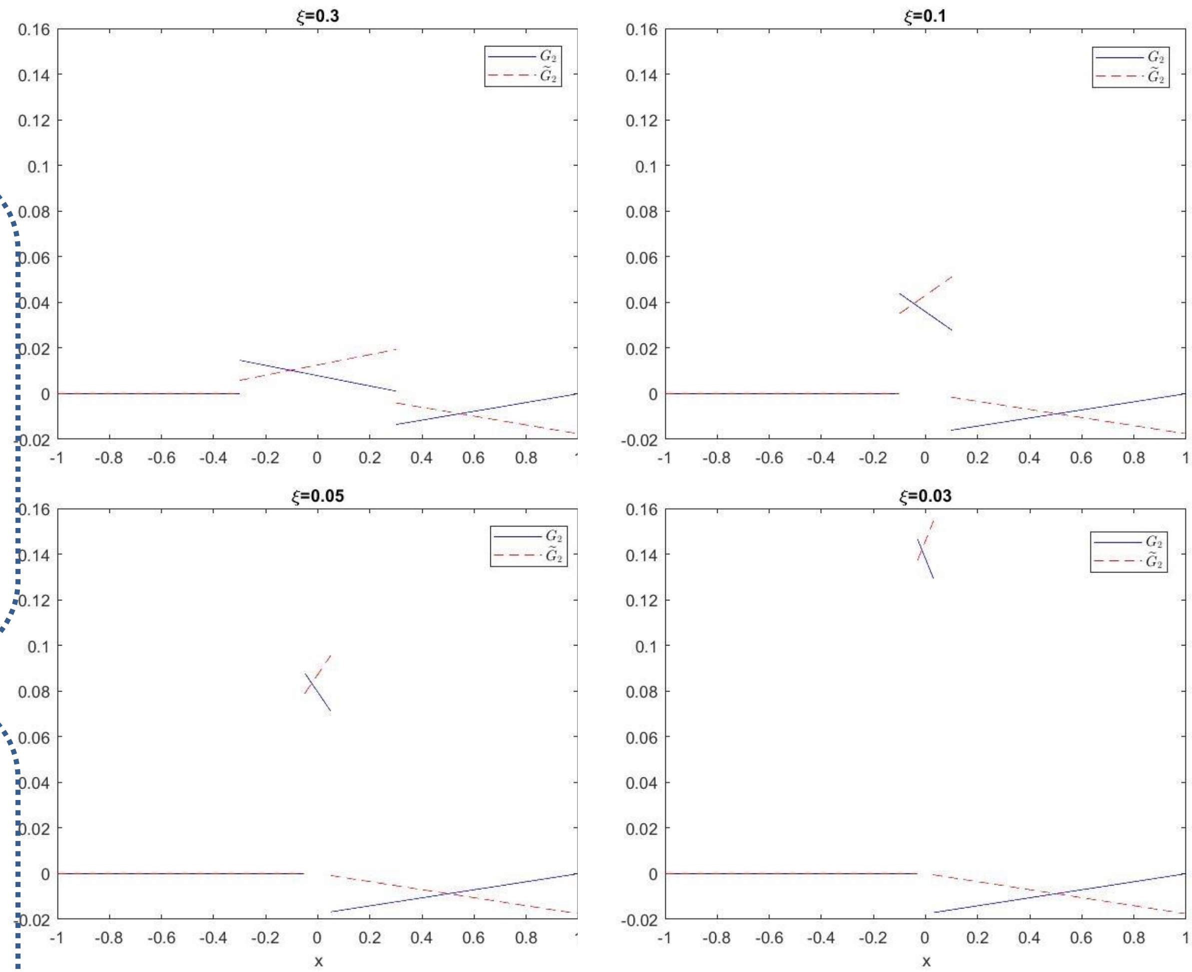



$$G_{2,Div}^{Singular} = \frac{g^2}{8\pi^2}\frac{1}{\xi}\ln\Lambda_\perp$$

$\widetilde{G}_2$




$$\widetilde{G}_{2,Div.}^{Singular} = \frac{g^2}{8\pi^2}\frac{x-2\xi^2+1}{\xi(1-x)}\ln\Lambda_\perp$$



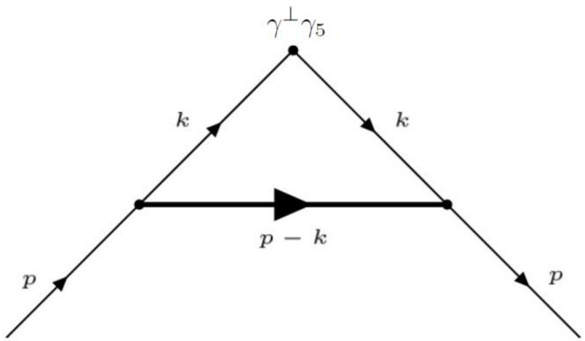

| Twist-3 GPD | Discontinuities as $\xi \to 0$ |
|---|---|
| $G_2$ | Divergent |
| $\widetilde{G}_2$ | Divergent |

$\widetilde{G}_2$

$\Delta = 0$

$g_2$

# The forward limit: $g_2$ and $g_2^{Quasi}$ in SDM




**The parameterization**

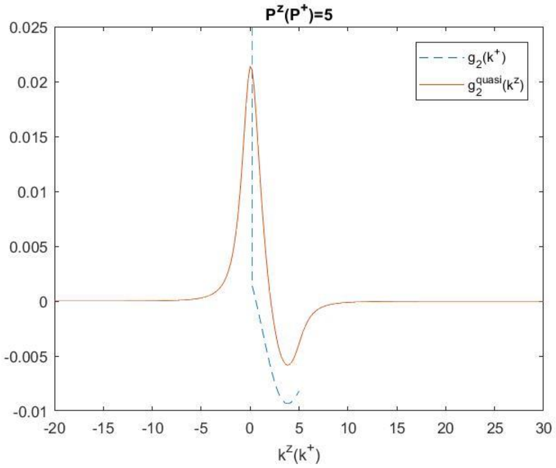
$$\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle P, S | \overline{q}(0) \gamma^{\mu} \gamma_5 q(\lambda n) | P, S \rangle$$
$$= 2\{g_1(x) \hat{p}^{\mu} (S \cdot \hat{n}) + g_T(x) S_{\perp}^{\mu} + M^2 g_3(x) \hat{n}^{\mu} (S \cdot \hat{n})\}$$

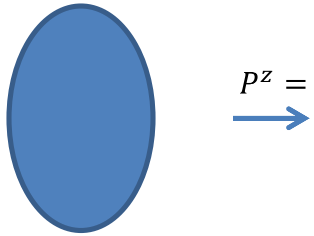
$g_T(x) \rightarrow g_1(x) + g_2(x)$

$$g_2^{quasi} \xrightarrow{P^z \to \infty} g_2$$

Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

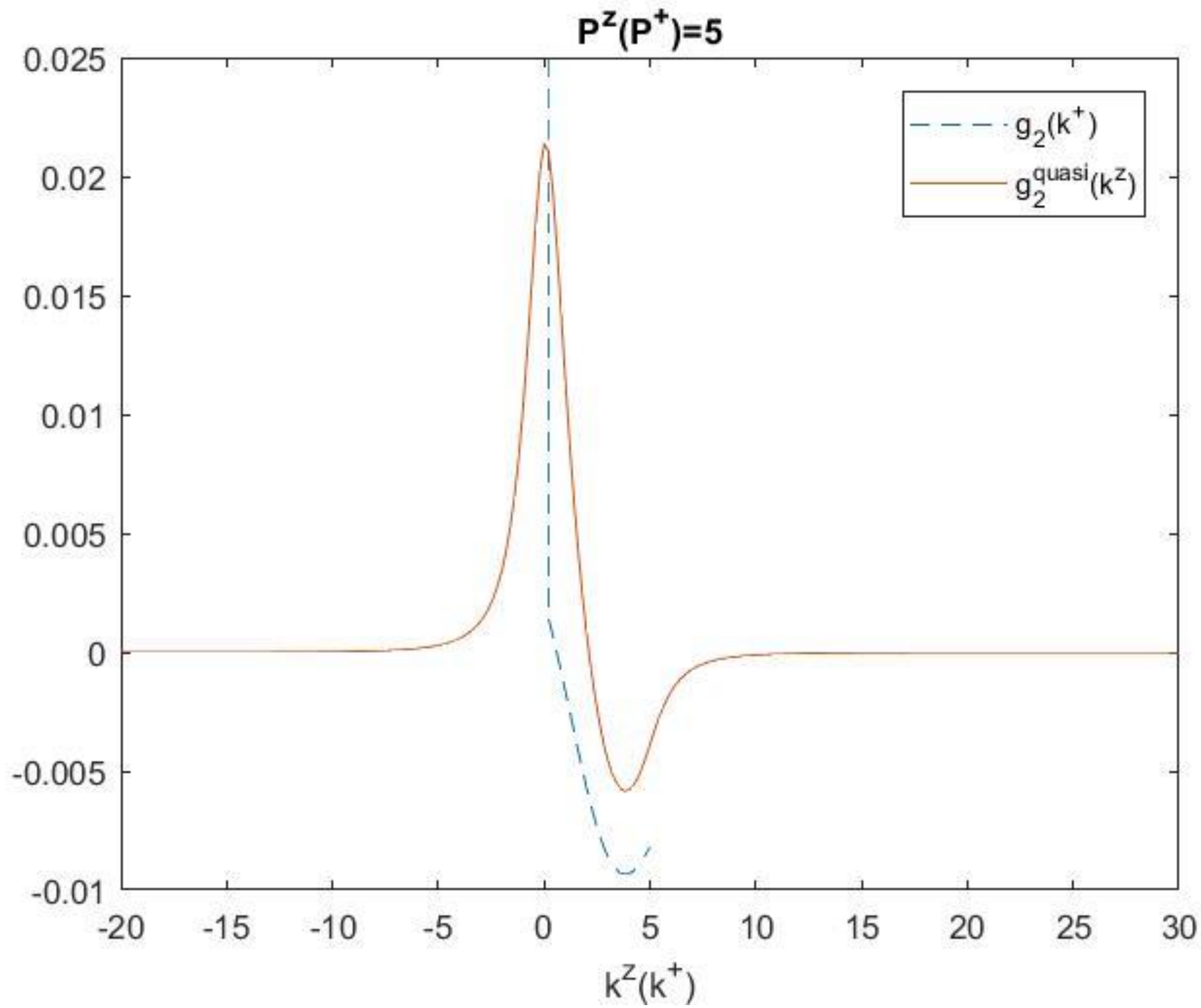
| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |





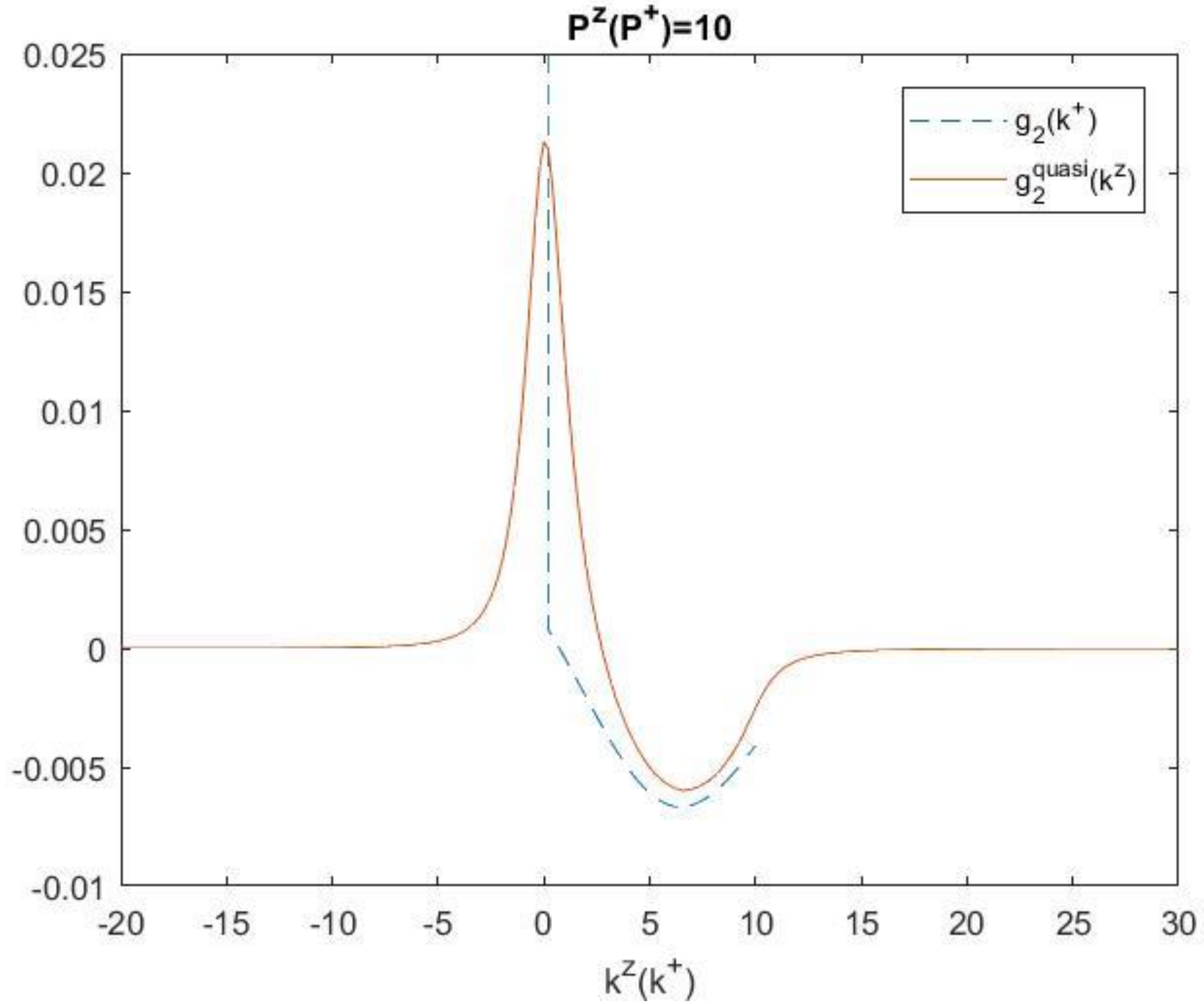
$P^z = P^+ = 5$

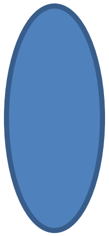
$g_2(\boldsymbol{k^+}), \;\; g_2{}^{quasi}(\boldsymbol{k^z})$



$P^z = P^+ = 10$

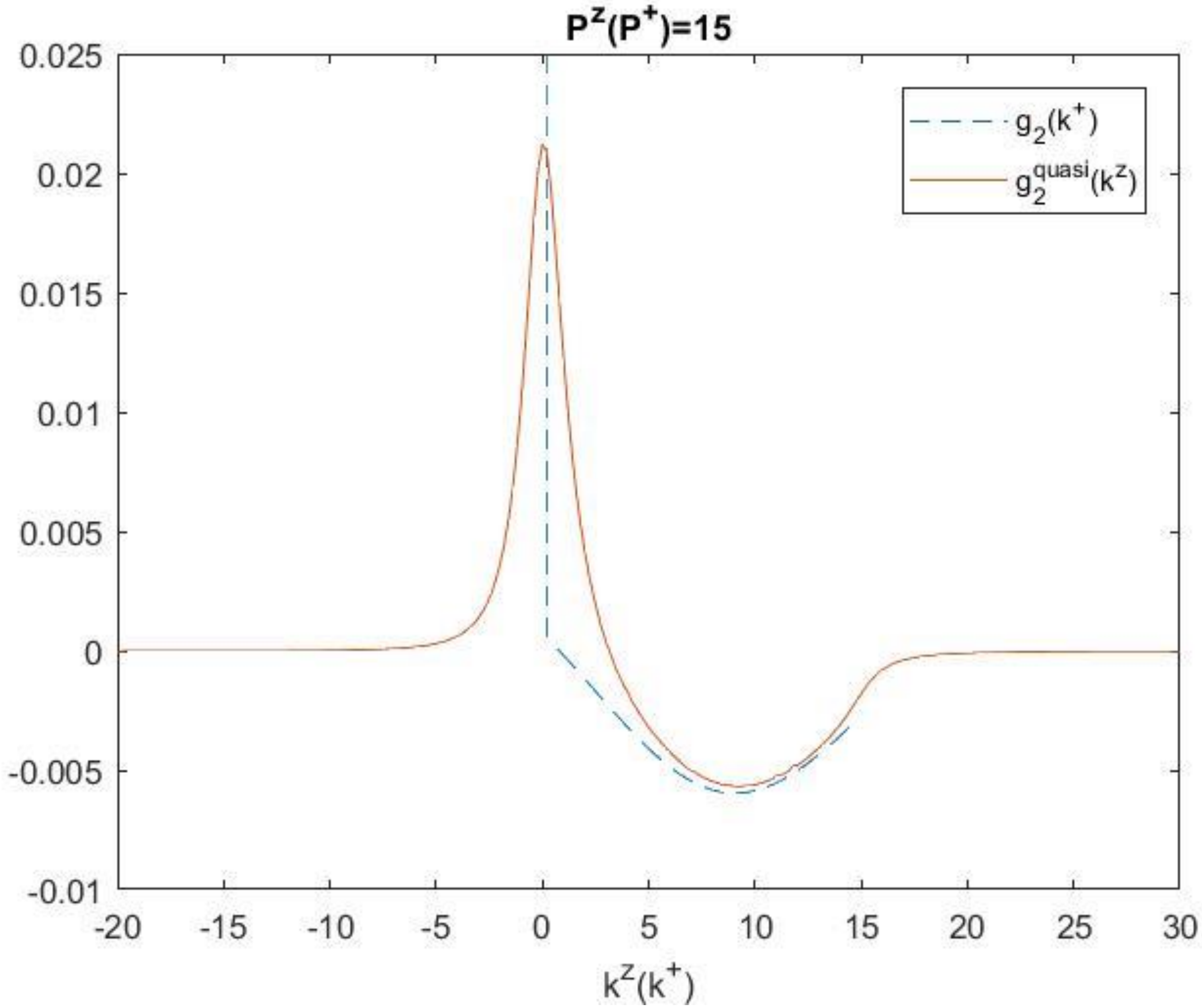
$g_2(\boldsymbol{k^+}),\ \ g_2{}^{quasi}(\boldsymbol{k^z})$

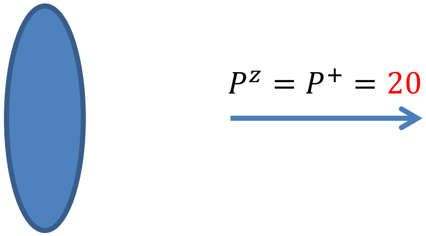




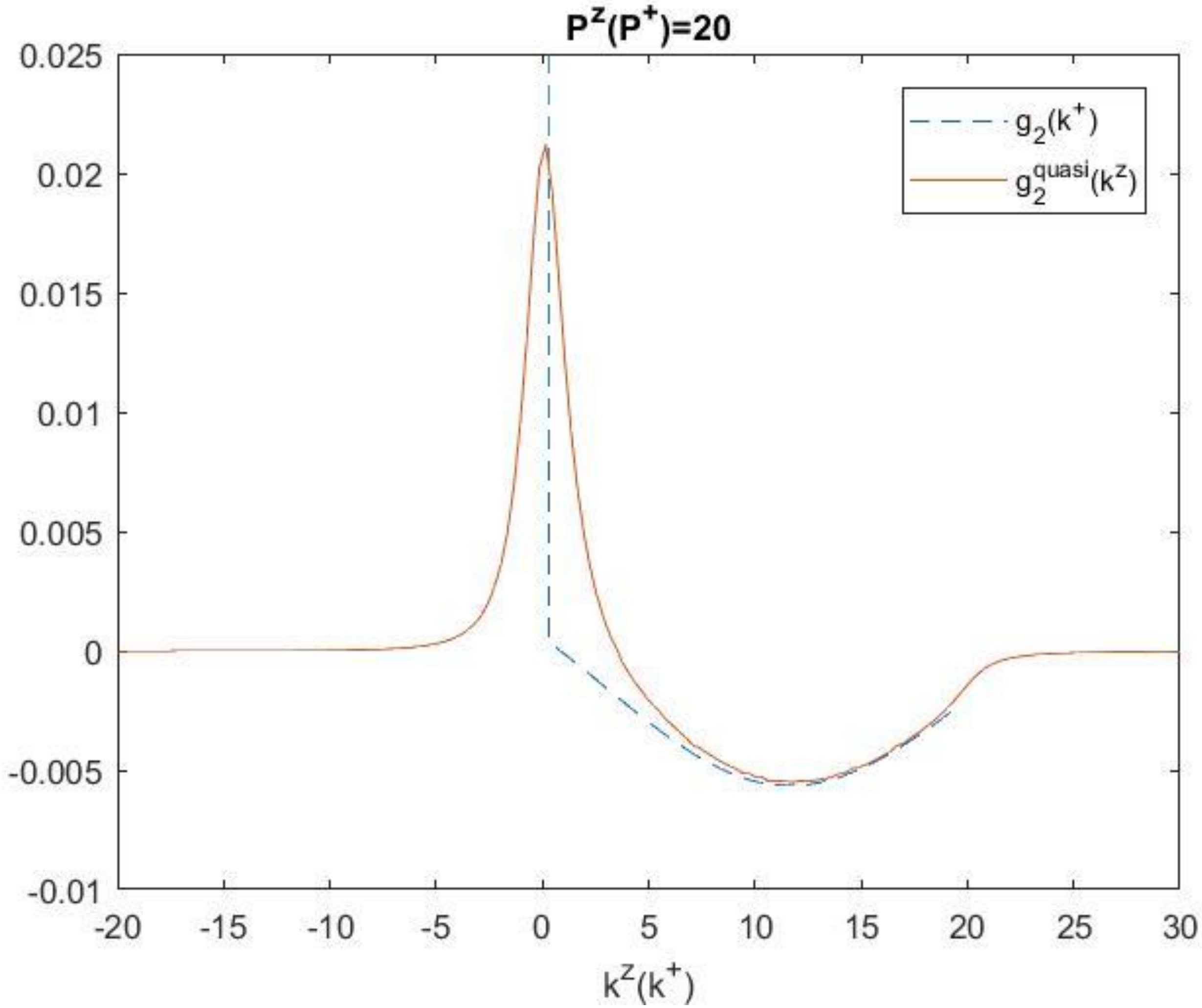
$P^z = P^+ = 15$

$g_2(\boldsymbol{k^+}),\ \ g_2{}^{quasi}(\boldsymbol{k^z})$



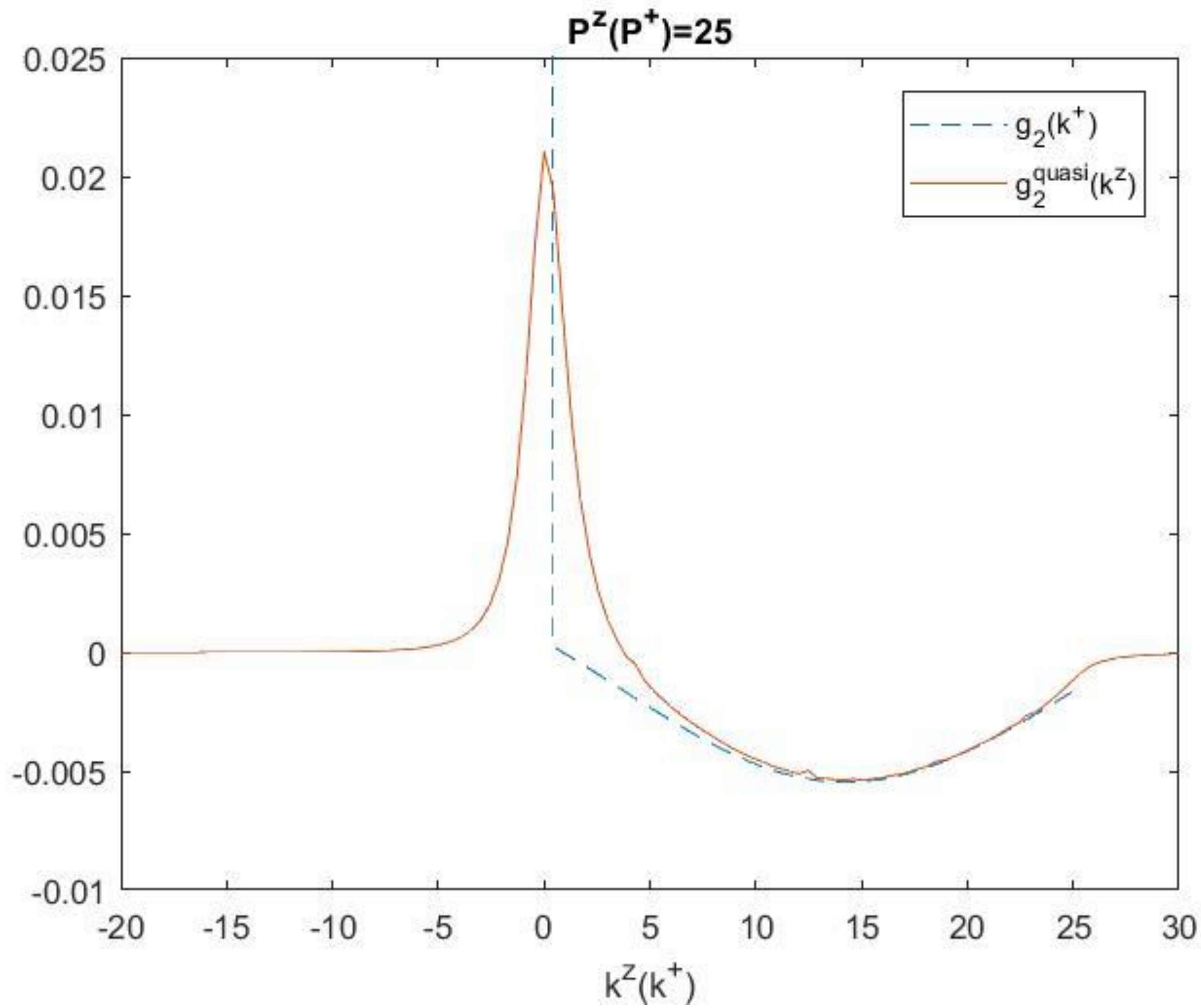



$$g_2(\boldsymbol{k^+}),\ \ g_2{}^{quasi}(\boldsymbol{k^z})$$

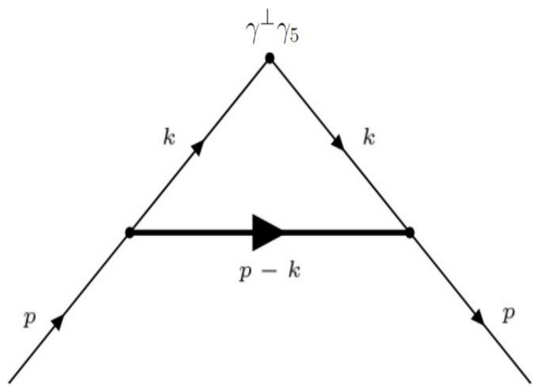


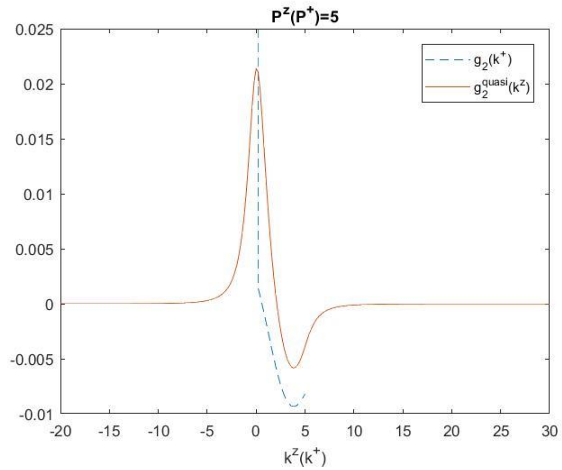
$P^z = P^+ = 25$

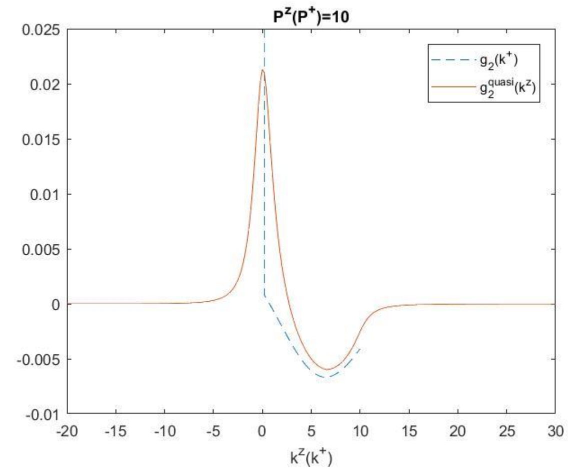
$g_2(\boldsymbol{k^+}),\ \ g_2{}^{quasi}(\boldsymbol{k^z})$

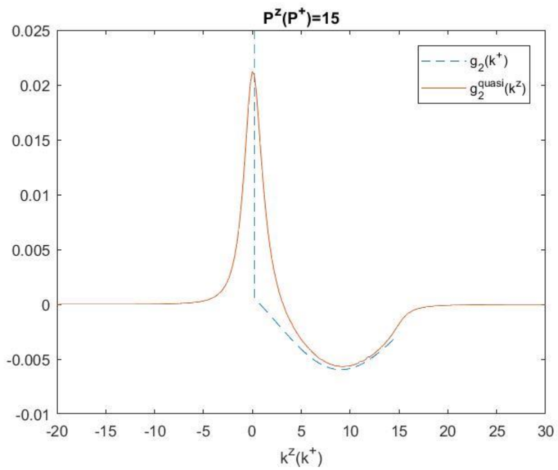


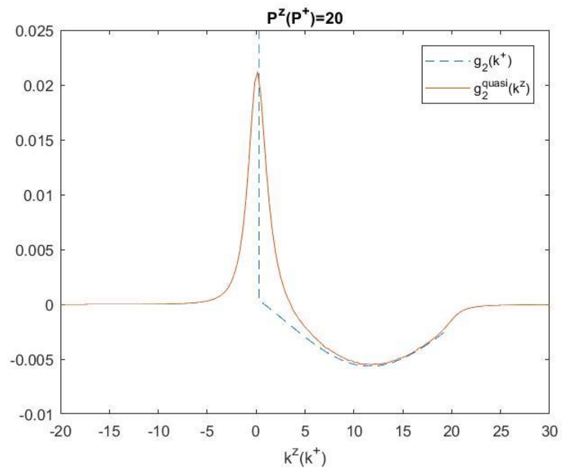
# $g_2$ ($k^+$) and $g_2^{Quasi}$ ($k^z$) in SDM

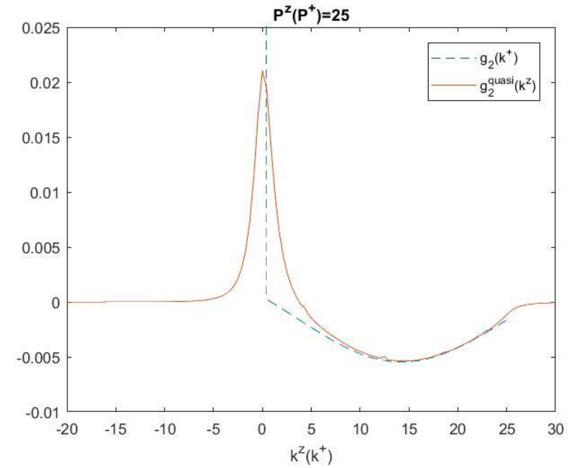


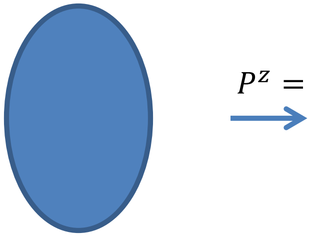



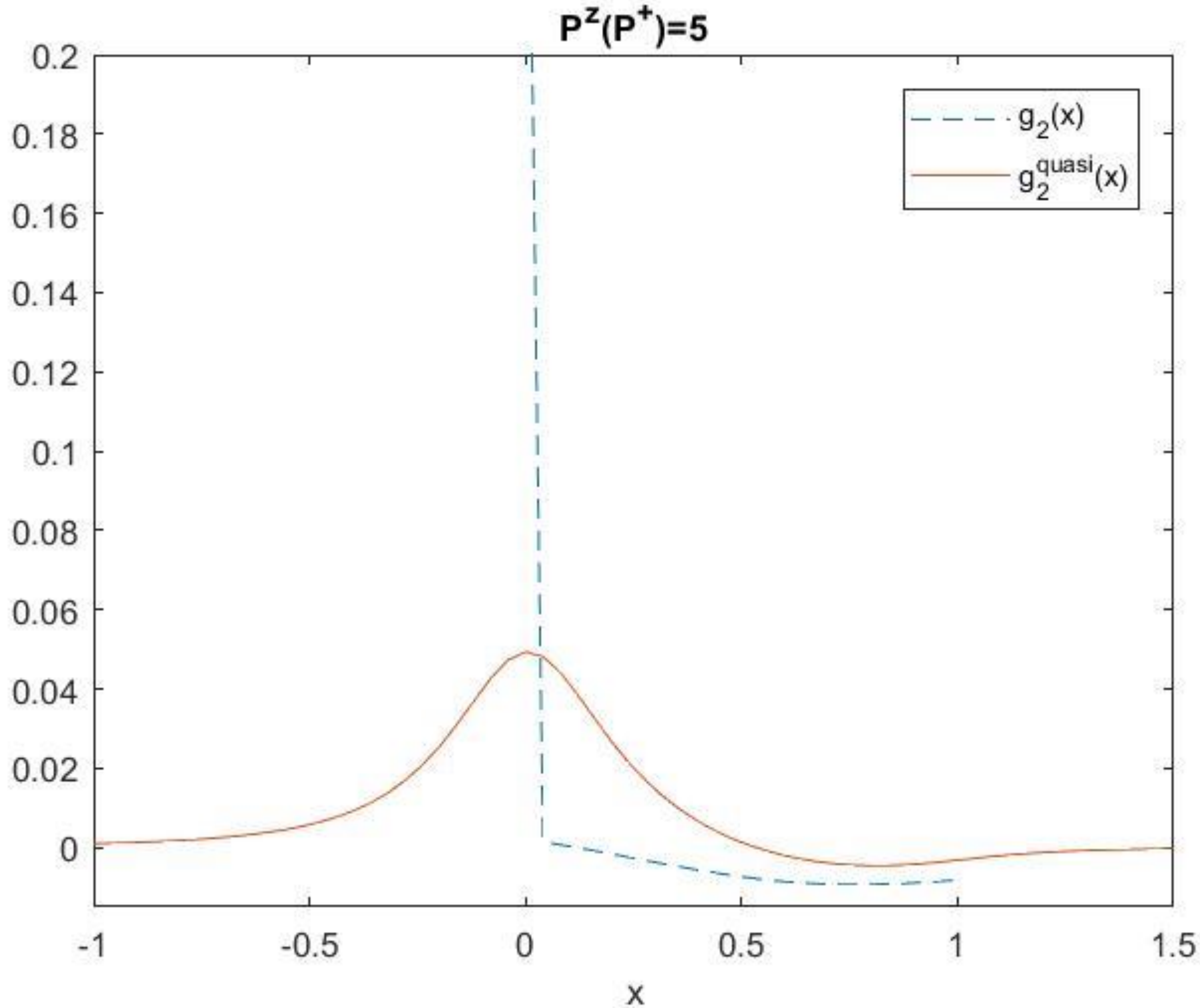



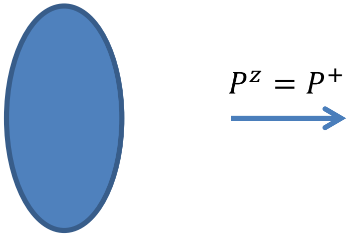

There is a momentum component in the nucleon state which does not scale as the nucleon is boosted to the infinite momentum frame.



$P^z = P^+ = 5$

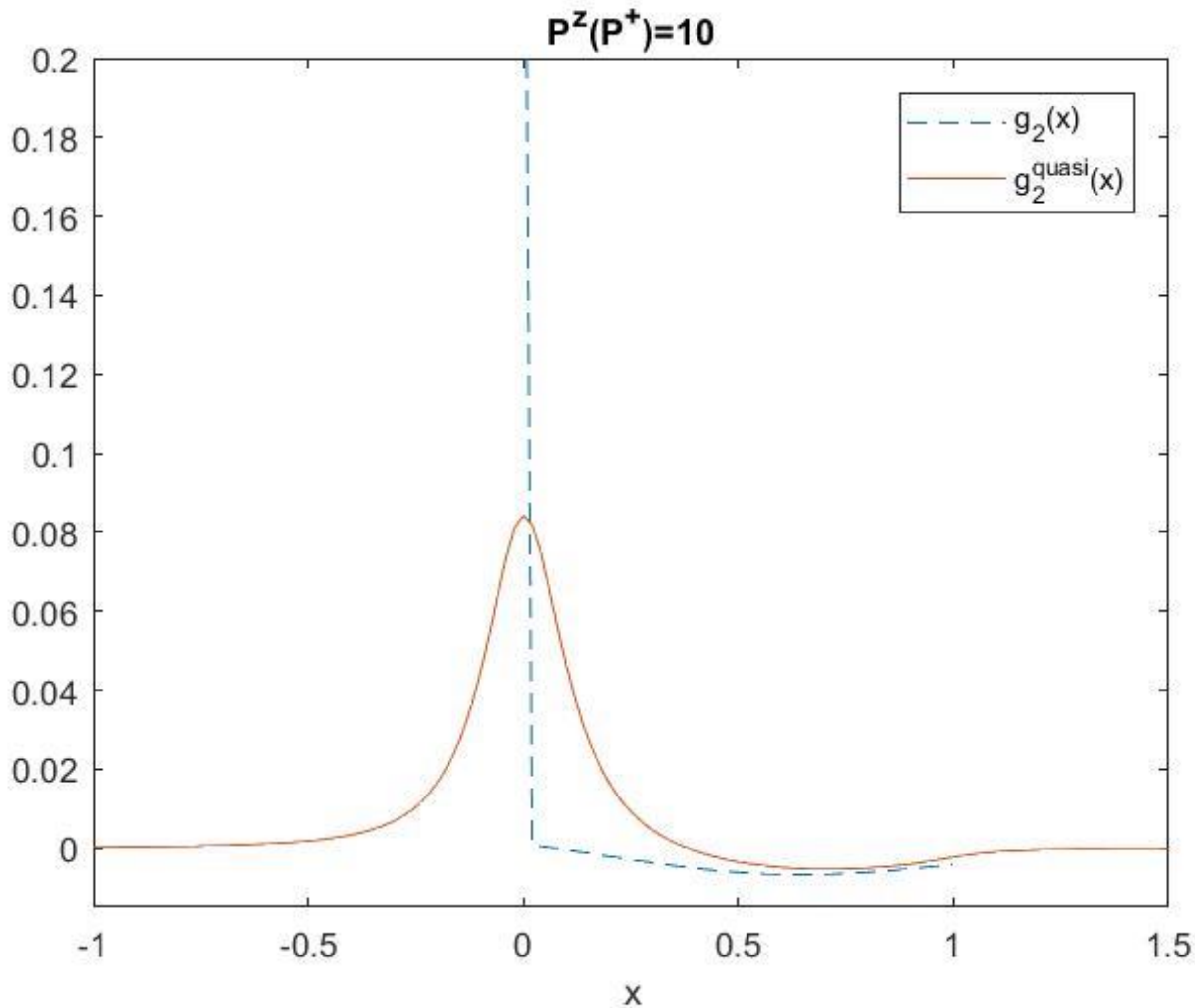
$g_2(x), \ g_2{}^{quasi}(x)$





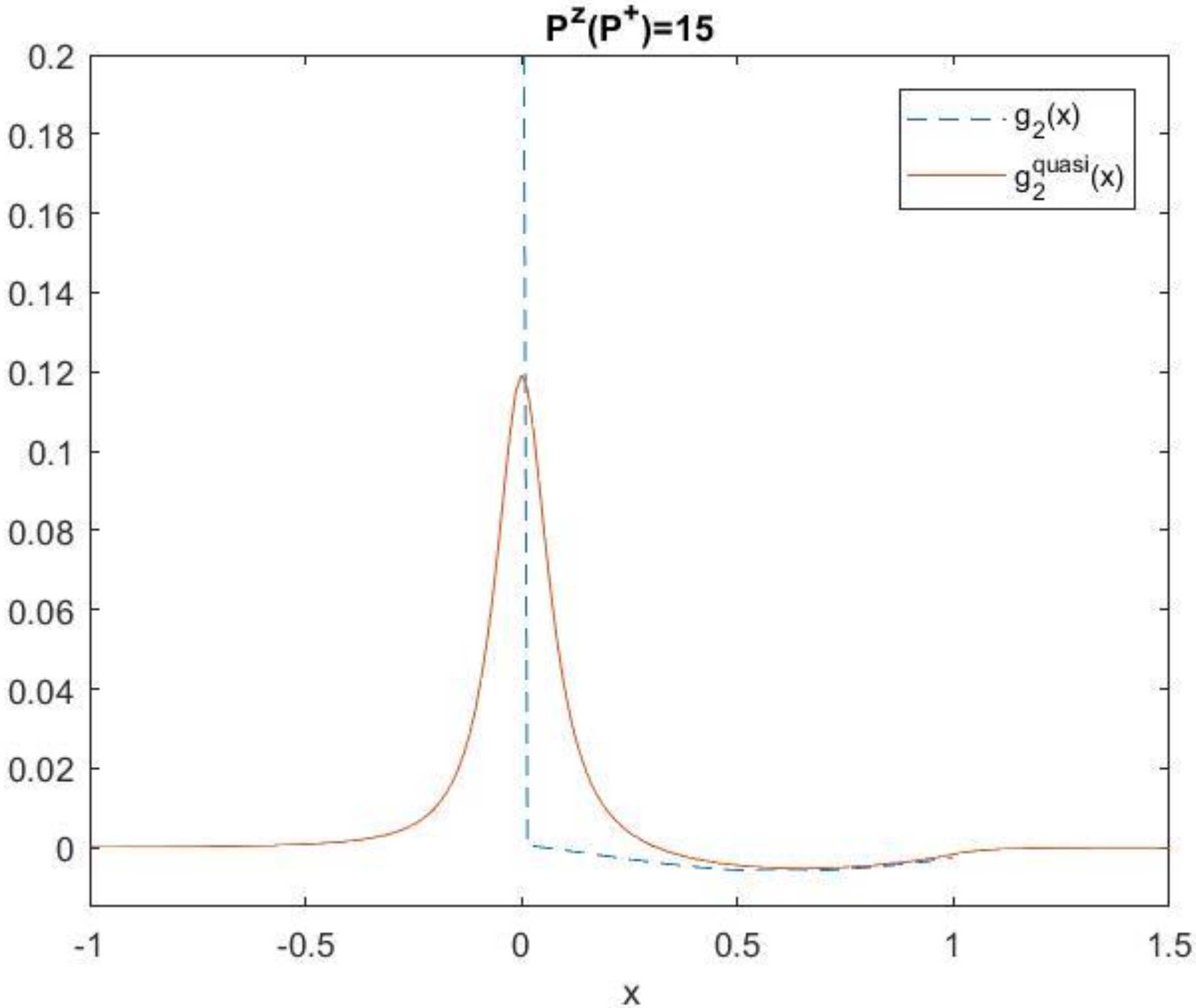
$P^z = P^+ = 10$

$g_2(\boldsymbol{x}),\ \ g_2{}^{quasi}(\boldsymbol{x})$



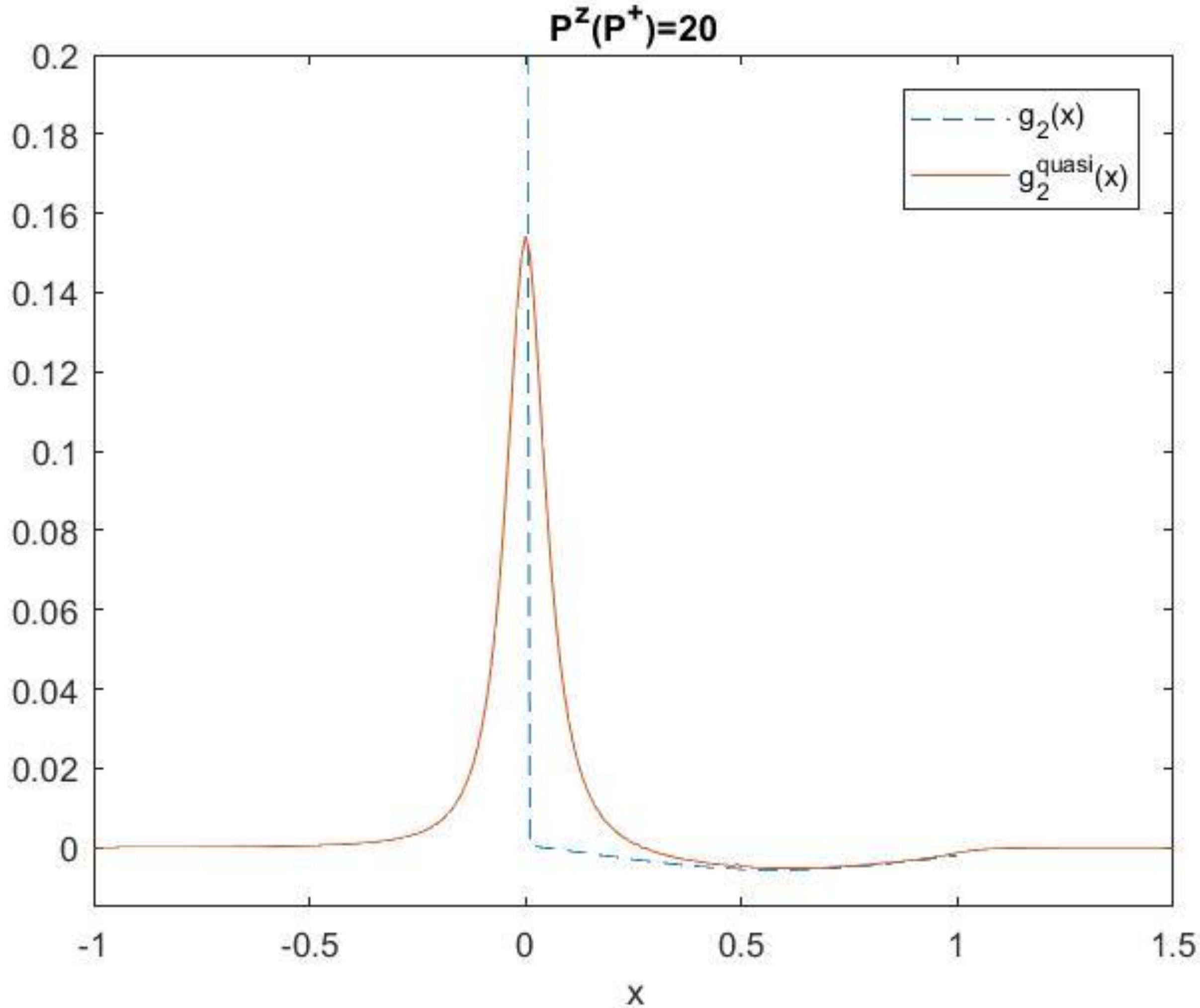
$P^z = P^+ = 15$

$g_2(x), \ g_2{}^{quasi}(x)$



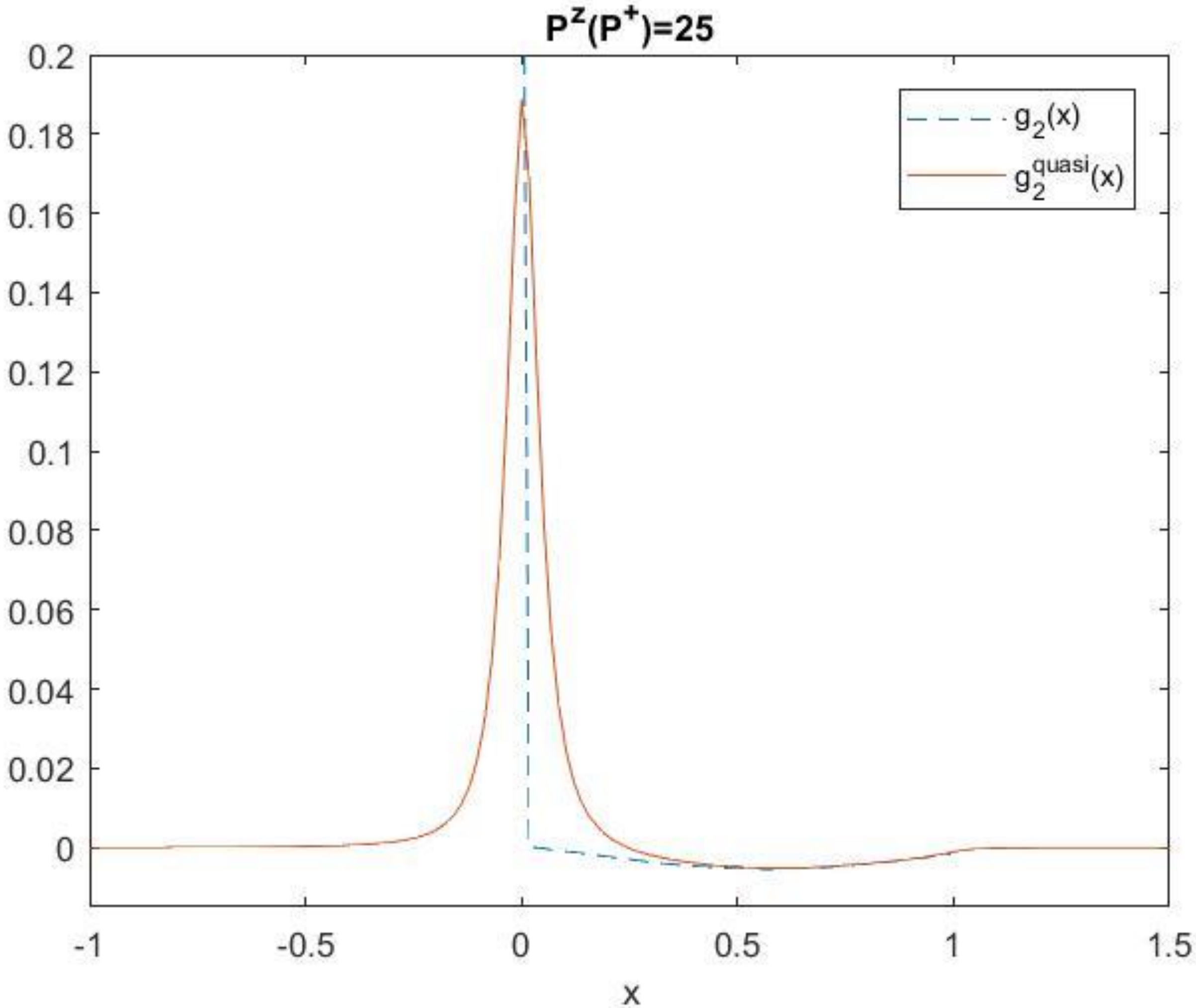
$P^z = P^+ = 20$

$g_2(\boldsymbol{x}),\ \ g_2{}^{quasi}(\boldsymbol{x})$

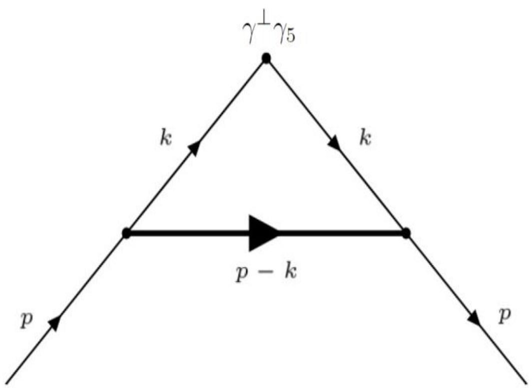


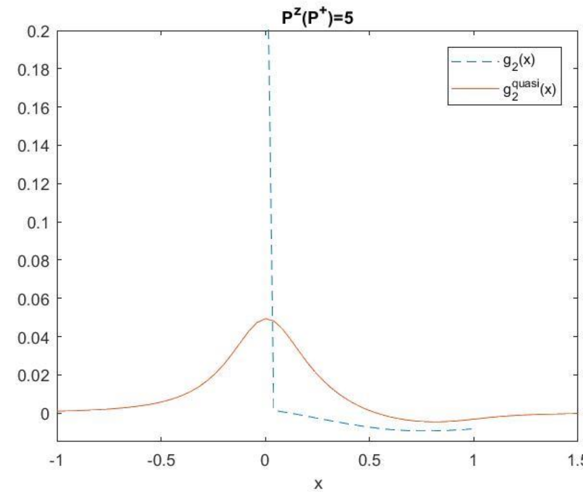
$P^z = P^+ = 25$

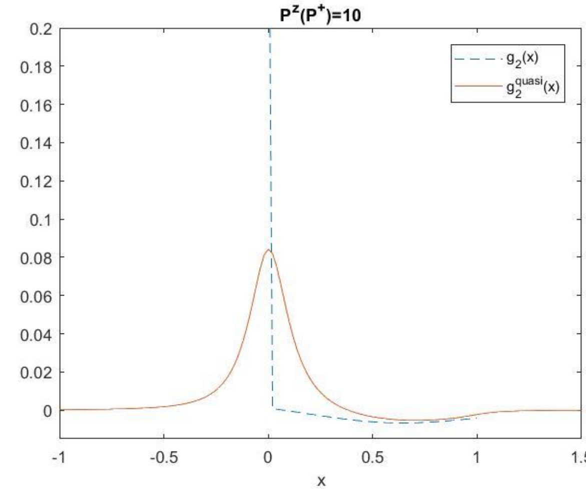
$g_2(x), \ g_2{}^{quasi}(x)$

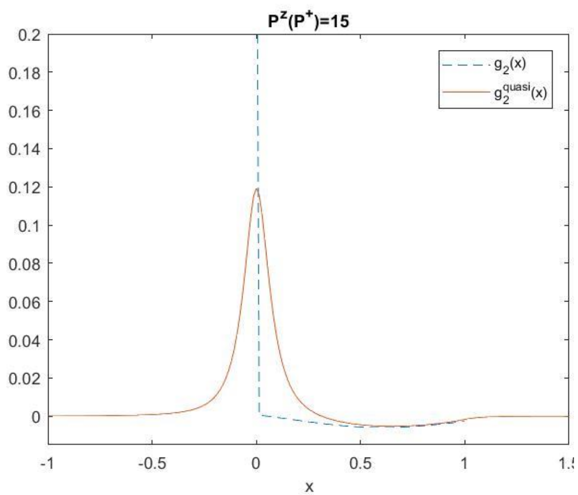


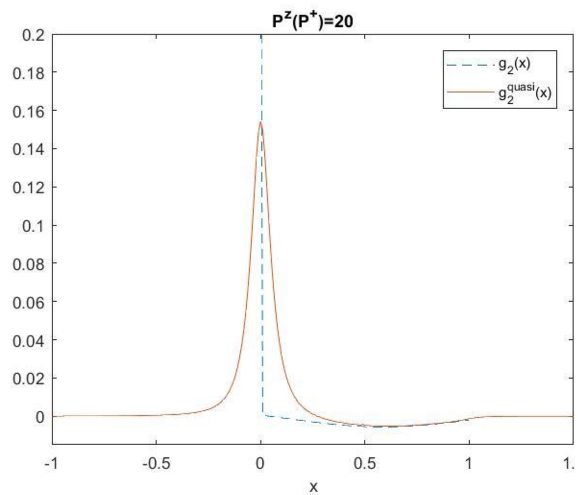
# $g_2(x)$ & $g_2{}^{quasi}(x)$ in scalar di-quark model

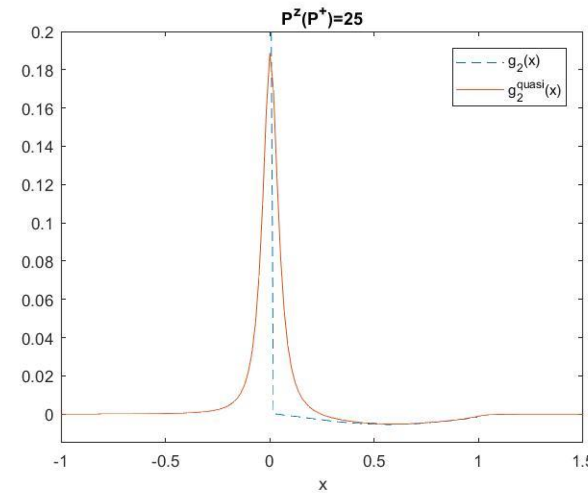


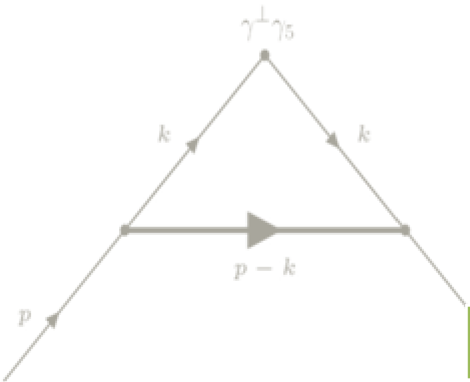



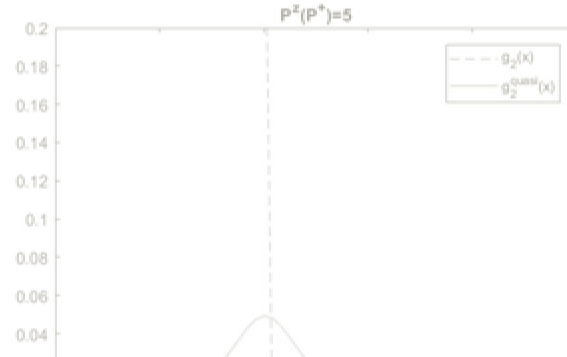



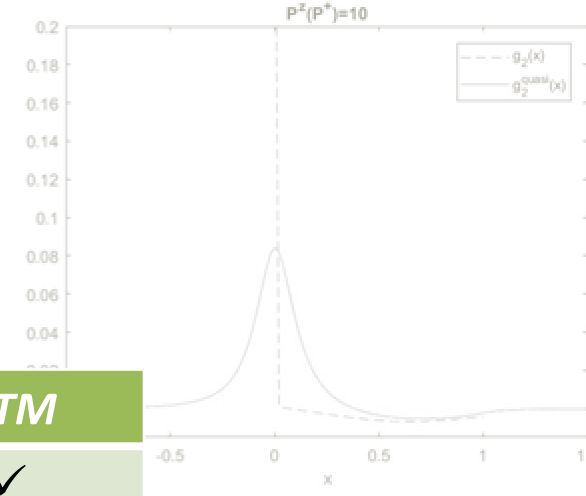



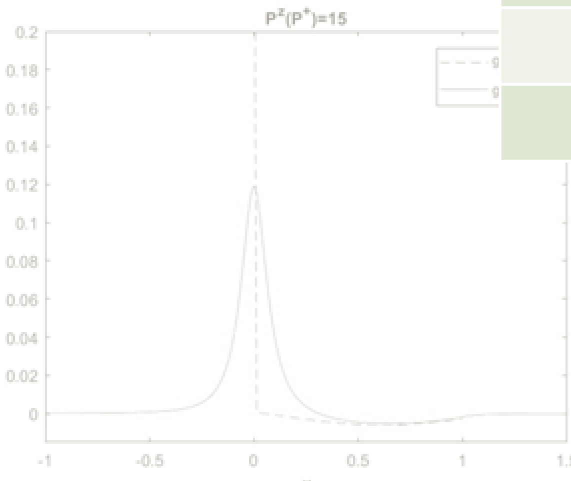

There is a singularity at x=0

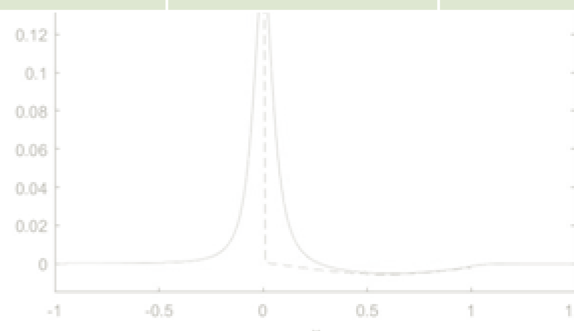
# $g_2(x)$ & $g_2{}^{quasi}(x)$ in scalar di-quark model

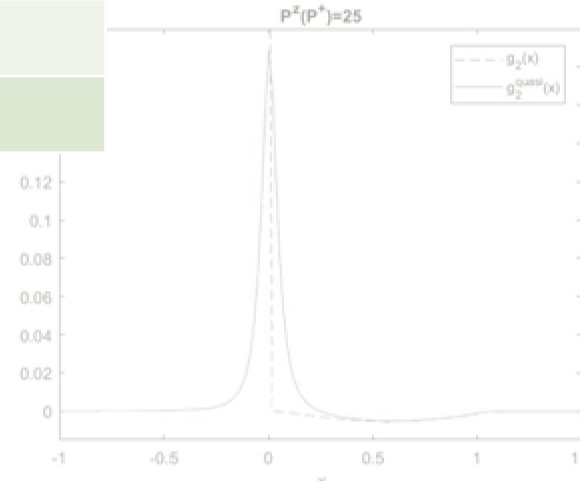


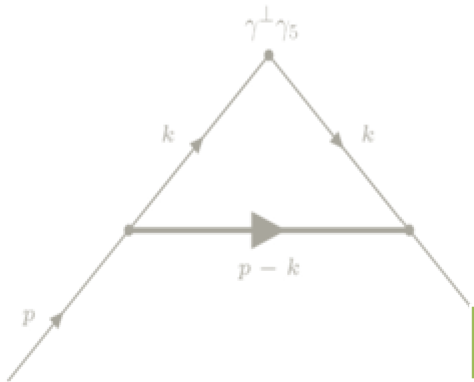



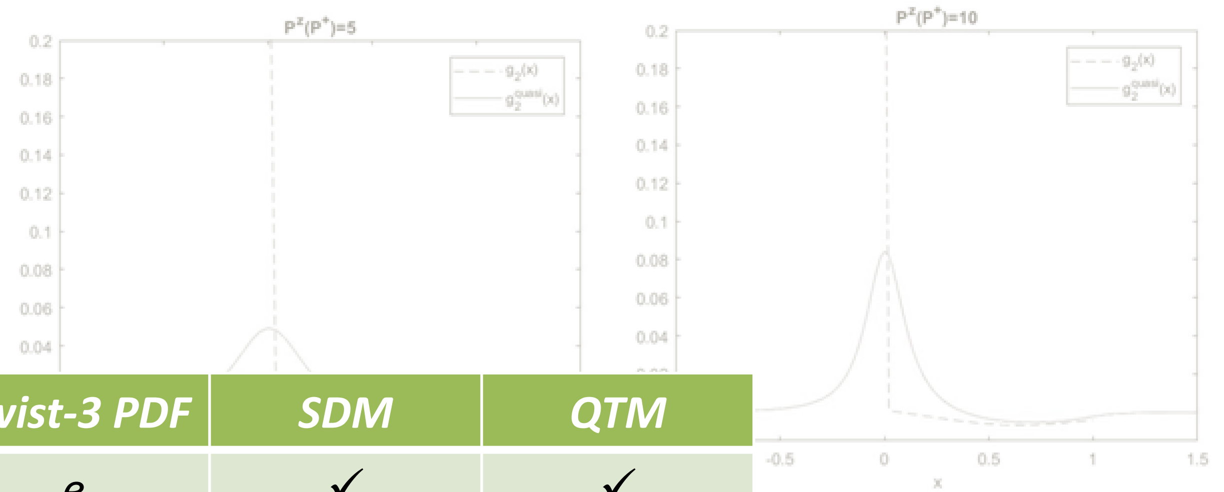



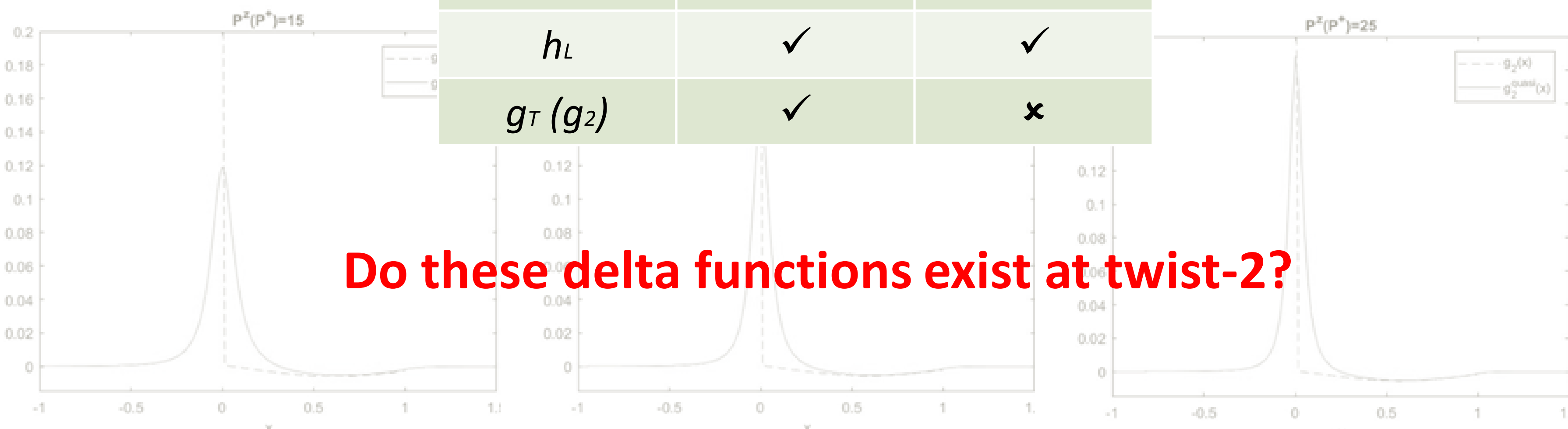

| *Twist-3 PDF* | *SDM* | *QTM* |
|---|---|---|
| *e* | ✓ | ✓ |
| $h_L$ | ✓ | ✓ |
| $g_T$ ($g_2$) | ✓ | ✗ |

There is a singularity at x=0

# $g_2(x)$ & $g_2^{quasi}(x)$ in scalar di-quark model



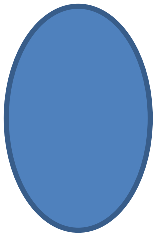




| Twist-3 PDF | SDM | QTM |
|---|---|---|
| $e$ | ✓ | ✓ |
| $h_L$ | ✓ | ✓ |
| $g_T$ ($g_2$) | ✓ | ✗ |

**Do these delta functions exist at twist-2?**

There is a singularity at x=0

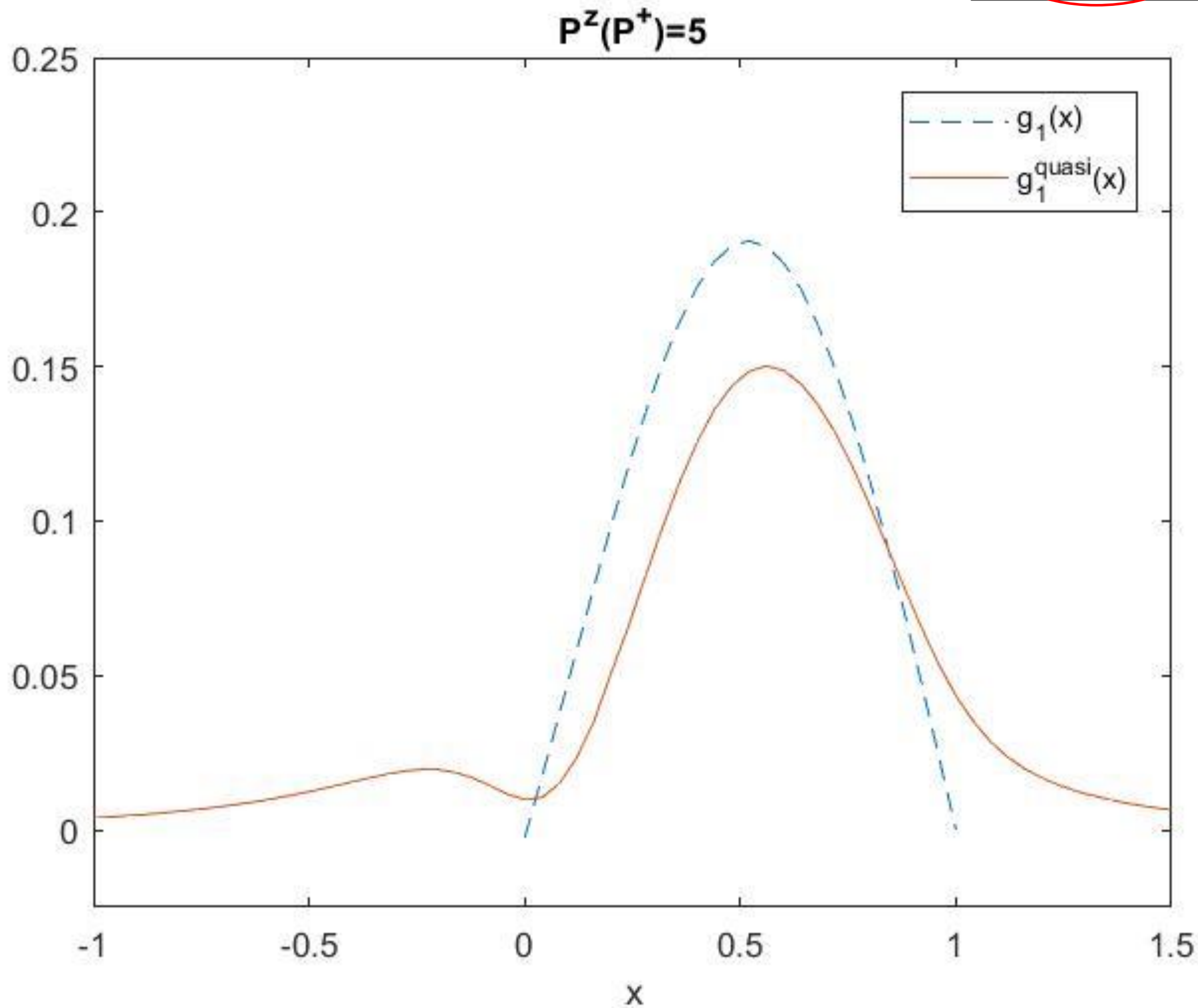


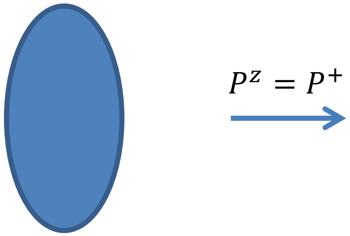
$P^z = P^+ = 5$

$g_1(x),\ g_1{}^{quasi}(x)$

Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |



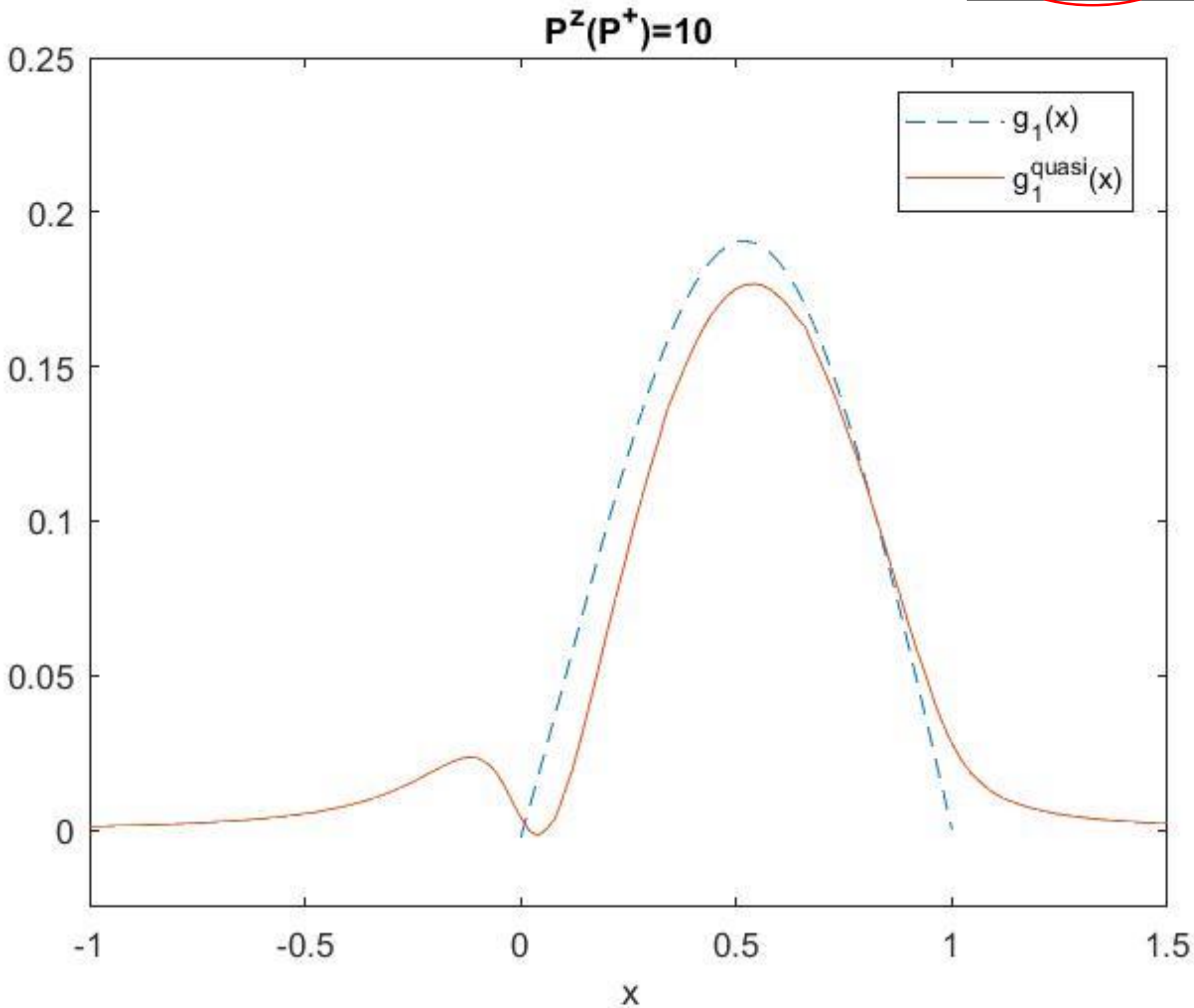


$P^z = P^+ = 10$

$g_1(x),\ g_1{}^{quasi}(x)$

Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |

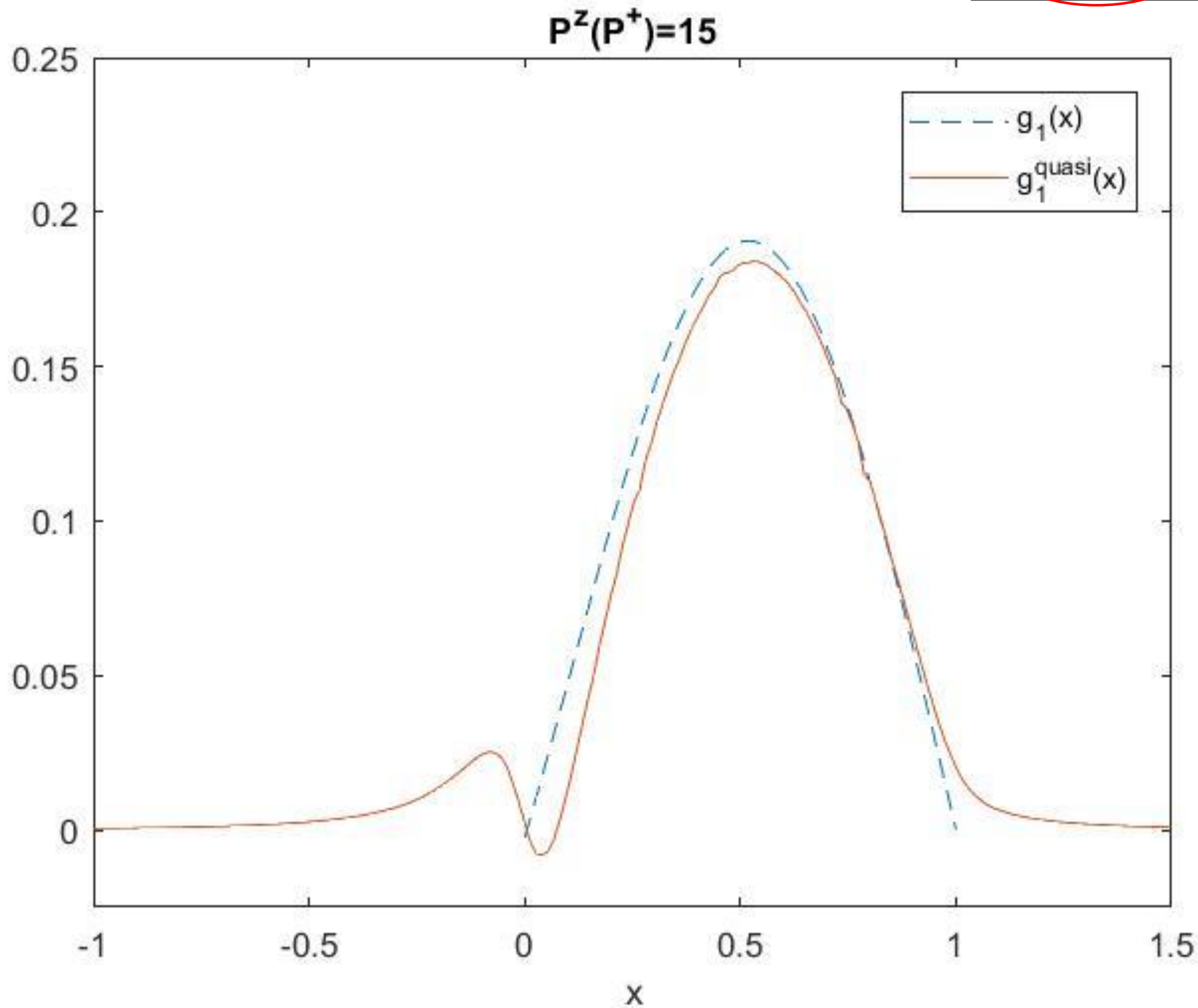


$P^z = P^+ = 15$

$g_1(x), \ g_1{}^{quasi}(x)$

Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |

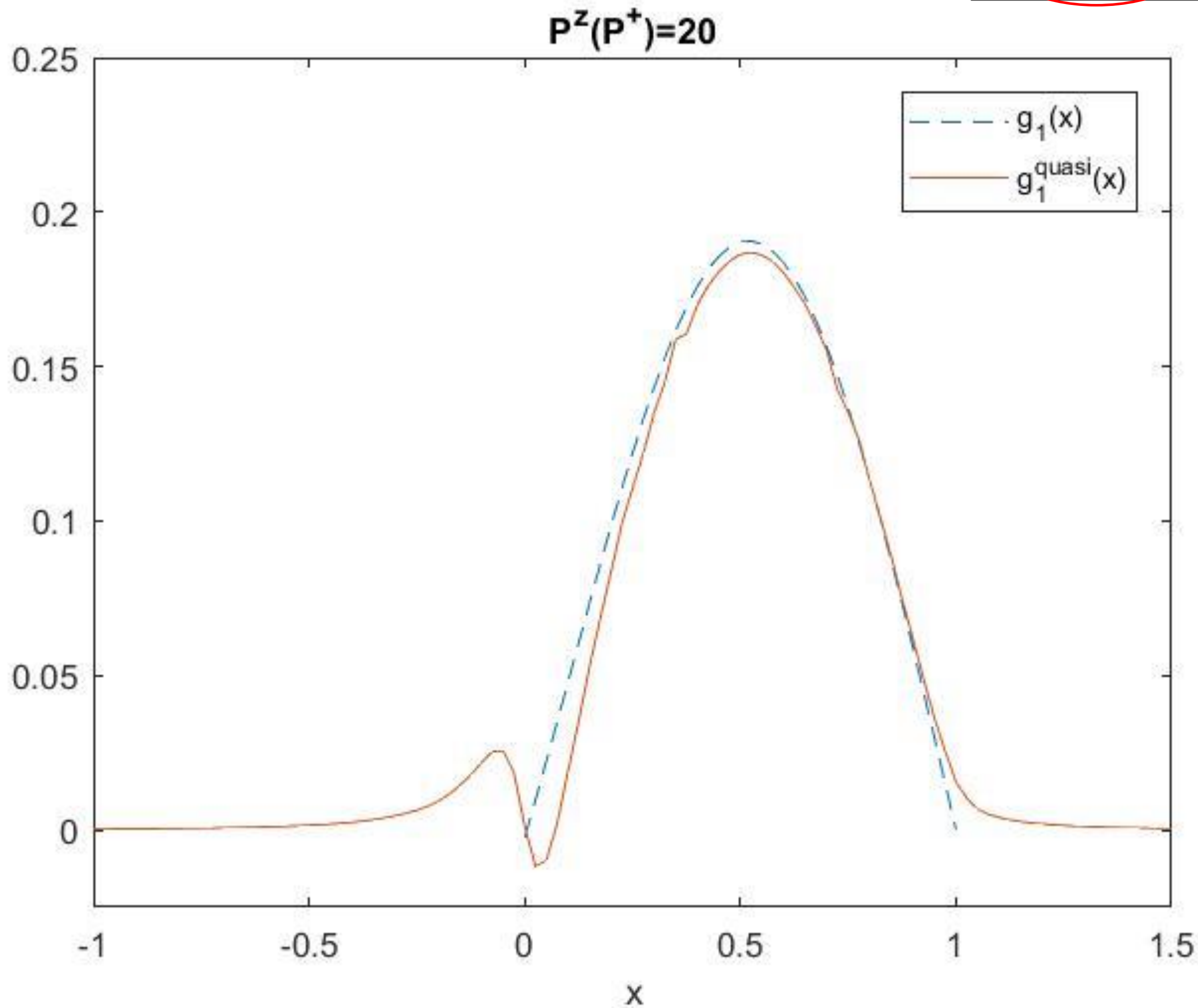


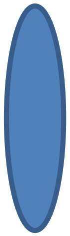
Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |

$P^z = P^+ = 20$

$g_1(x)$, $g_1{}^{quasi}(x)$



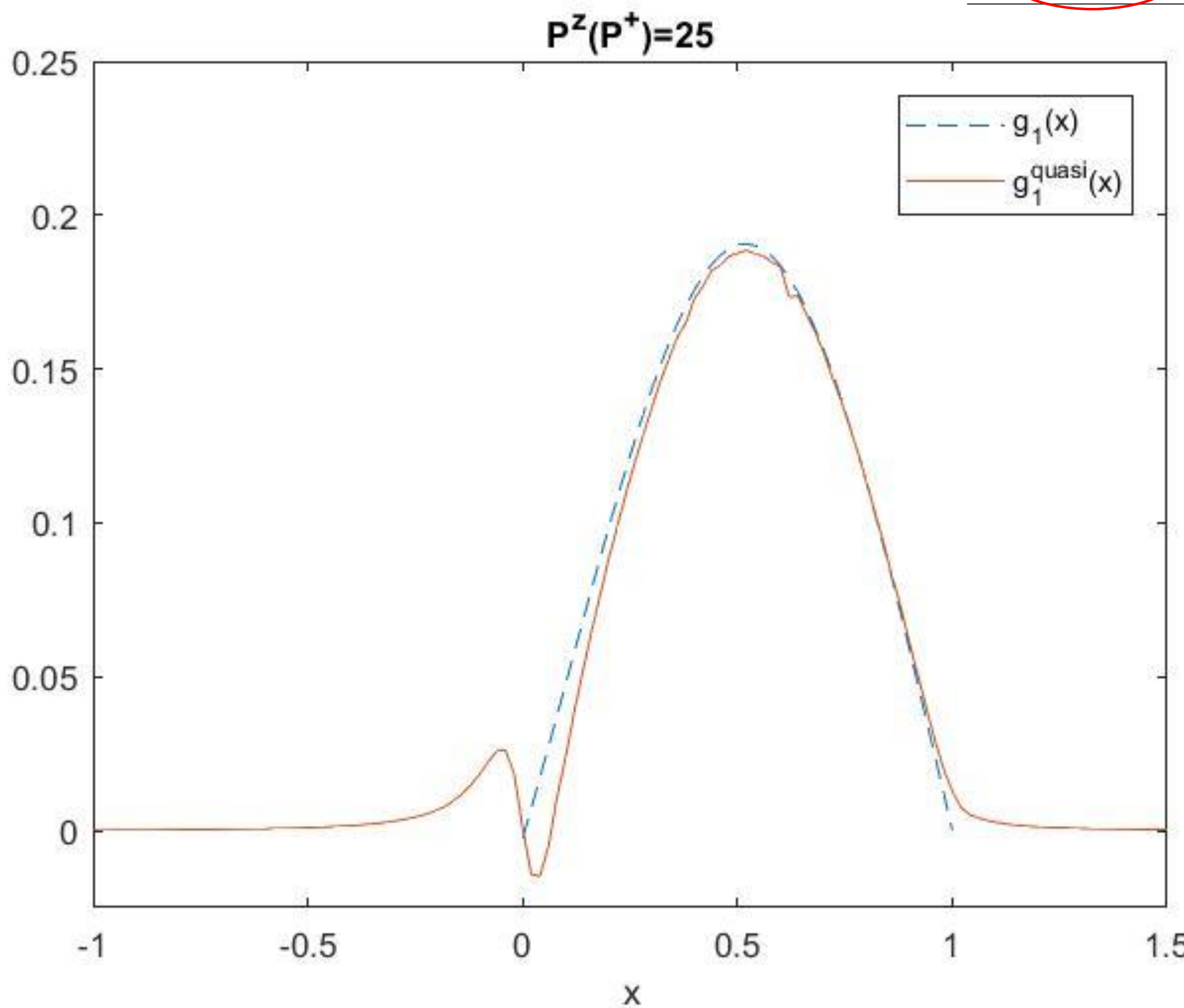


$P^z = P^+ = 20$

$g_1(x), \; g_1{}^{quasi}(x)$

Behavior of twist-2 and twist-3 distributions under longitudinal momentum boost, $P^+$.

| Twist-2 | Twist-3 |
|---|---|
| Independent of $P^+$ | $1/P^+$ |



# Singularities in twist-3 quark distributions

✓ : There is $\delta(x)$

✗ : There is no $\delta(x)$

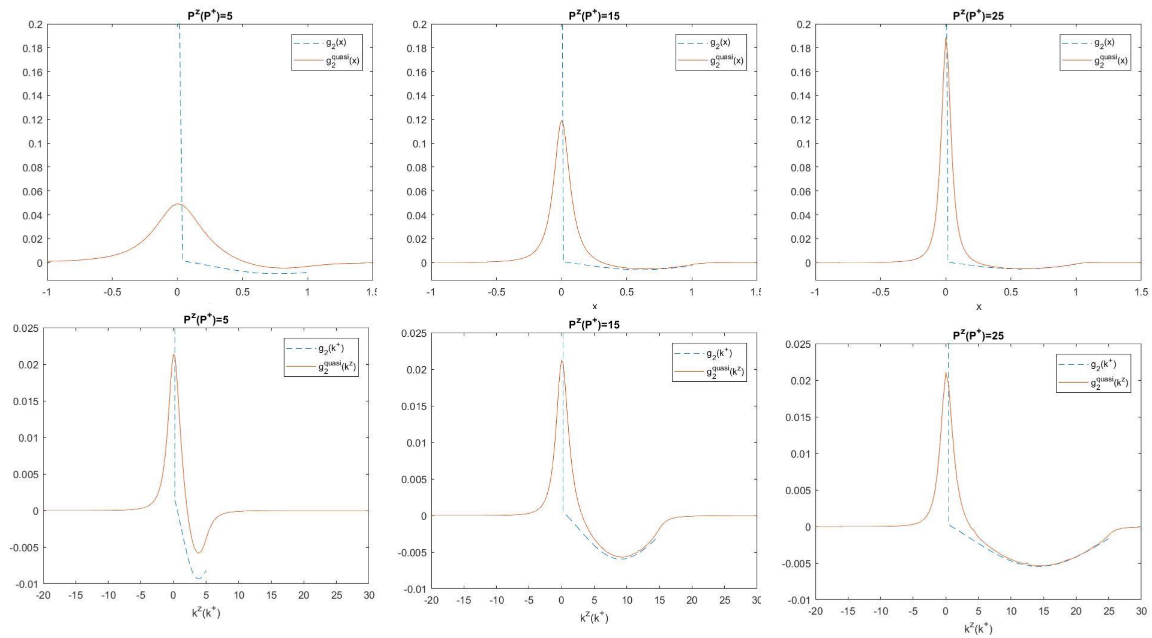
| Twist-2 PDF | SDM | QTM |
|---|---|---|
| $f_1$ | ✗ | ✗ |
| $g_1$ | ✗ | ✗ |
| $h_1$ | ✗ | ✗ |

| Twist-3 PDF | SDM | QTM |
|---|---|---|
| $e$ | ✓ | ✓ |
| $h_L$ | ✓ | ✓ |
| $g_T$ ($g_2$) | ✓ | ✗ |

Aslan, Burkardt,
Singularities in Twist-3 Quark Distributions, 2018.

Burkardt, Koike,
Violation of sum rules for twist three parton distributions in QCD, 2001.

- **At twist-3 there is something that does not exist in twist-2: There are delta functions.**
- **We identify these delta functions with momentum components in the nucleon state that do not scale as the nucleon is boosted to the infinite momentum.**




**Decomposition of twist-3**

$$h_L(x) = h_L^{WW}(x) + h_L^m(x) + h_L^3(x)$$

$\delta(x)$ term appears not only in $h_L^m$ but also in $h_L^3$

Burkardt & Koike,
Violation of Sum Rules for
Twist 3 Parton Distributions in QCD, 2001

# Regularization of the singularities

The effects of different regularization schemes on the δ(x)

| $\delta(x)$ remains | $\delta(x)$ is recovered |
|---|---|
| Transverse momentum cut off | Pauli-Villars regularization |
| Dimensional regularization | |
| Adding form factors | |
| Adding axial diquark contribution | |

# Regularization of the singularities

The effects of different regularization schemes on the δ(x)

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| Dimensional regularization | |
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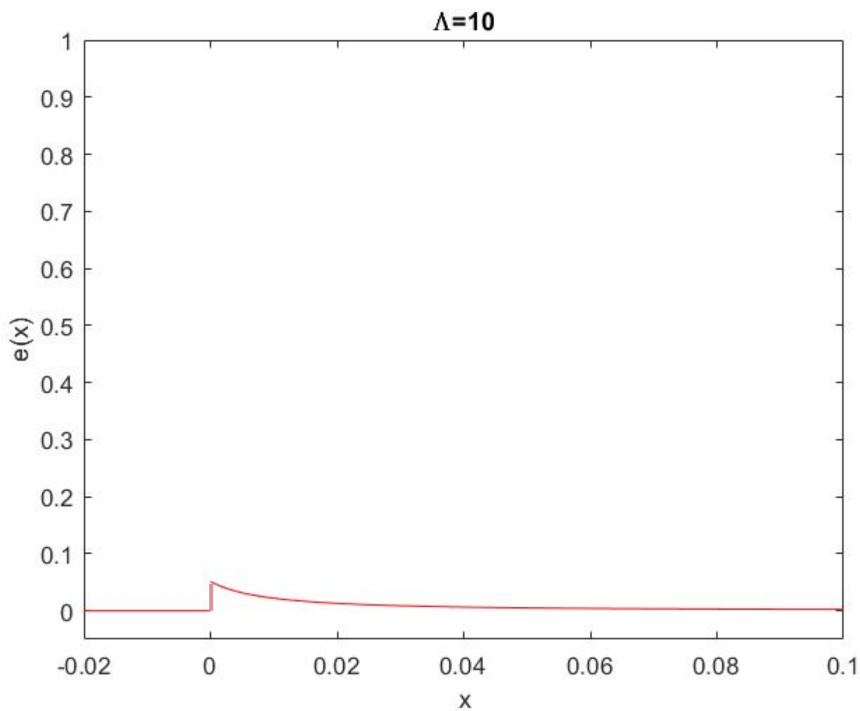

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The effects of different regularization schemes on the δ(x)

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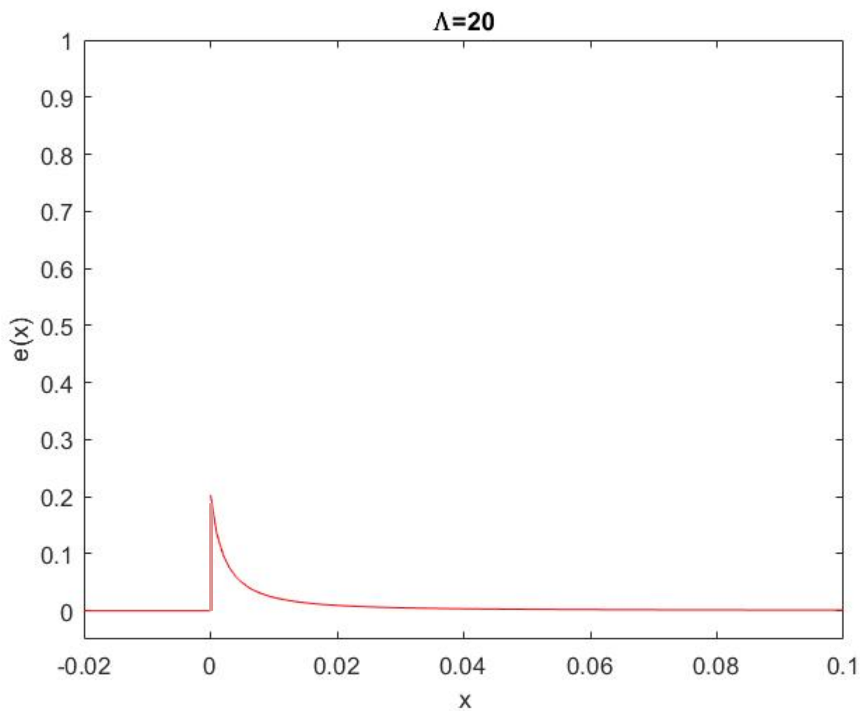

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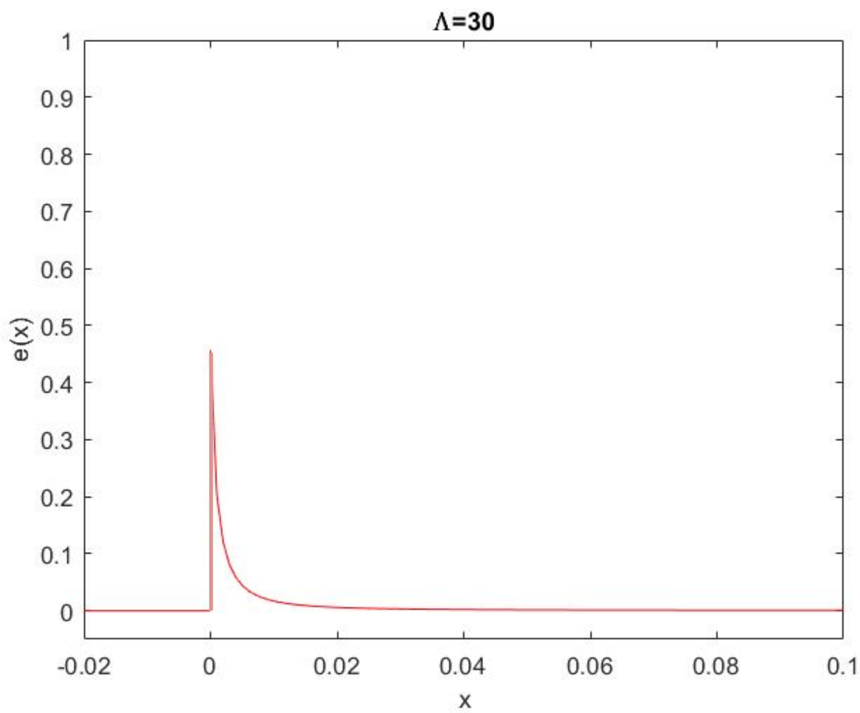

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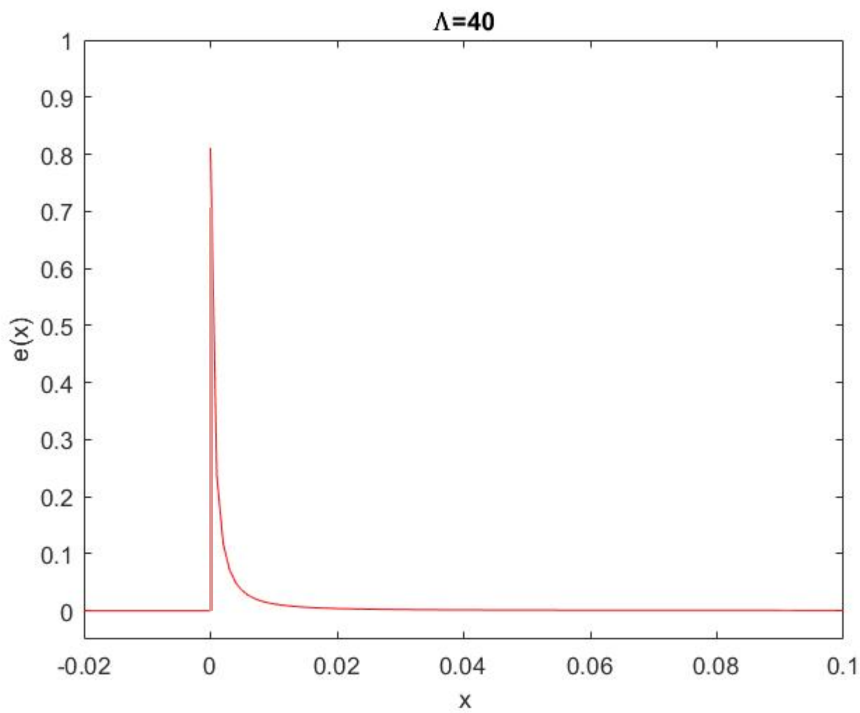

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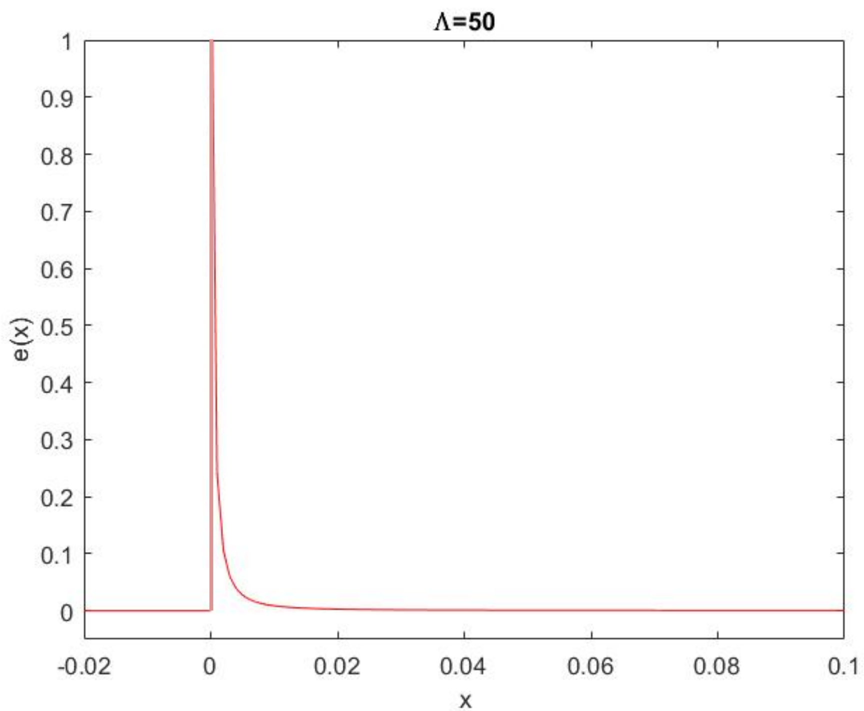


-- What happens if a twist-3 distribution involves a $\delta(x)$?

-- Sum rules are violated if we don't take it into account.

Lorentz invariance of twist-3 GPDs

$$\int_{-1}^{1} dx G_i(x,\xi,\Delta) = 0, \qquad \int_{-1}^{1} dx \widetilde{G}_i(x,\xi,\Delta) = 0.$$

$$\lim_{\epsilon\to 0}\int_{-1}^{\epsilon} dx G_i(x,\xi=0,\Delta) + \lim_{\epsilon\to 0}\int_{\epsilon}^{1} dx G_i(x,\xi=0,\Delta) \neq 0,$$

$$\lim_{\epsilon\to 0}\int_{-1}^{\epsilon} dx \widetilde{G}_i(x,\xi=0,\Delta) + \lim_{\epsilon\to 0}\int_{\epsilon}^{1} dx \widetilde{G}_i(x,\xi=0,\Delta) \neq 0.$$

**Example:**

In SDM the divergent part of $G_2$ was calculated as $G_2 = \begin{cases} -\frac{g^2}{4\pi^2}\frac{(1-x)}{(1-\xi^2)}\ln\Lambda_\perp & \text{for} \quad \xi < x \leq 1, \\ -\frac{g^2}{16\pi^2}\frac{(2x+\xi-1)}{\xi(1+\xi)}\ln\Lambda_\perp & \text{for} \quad -\xi \leq x \leq \xi, \\ 0 & \text{for} \quad -1 < x < \xi. \end{cases}$

$$\int_{-1}^{1} dx\ G_2 = -\frac{g^2}{16\pi^2}\int_{-\xi}^{\xi} dx\ \frac{(2x+\xi-1)}{\xi(1+\xi)}\ln\Lambda_\perp - \frac{g^2}{4\pi^2}\int_{\xi}^{1} dx\ \frac{(1-x)}{(1-\xi^2)}\ln\Lambda_\perp = 0$$

The Lorentz invariance of $G_2$ is satisfied ✓


-- What happens if a twist-3 distribution involves a δ(x)?

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Lorentz invariance of twist-3 GPDs

$$\int_{-1}^{1} dx G_i(x, \xi, \Delta) = 0, \qquad \int_{-1}^{1} dx \widetilde{G}_i(x, \xi, \Delta) = 0.$$

$$\int_{-1}^{1} dx g_1(x) = \int_{-1}^{1} dx g_T(x)$$

$$\int_{-1}^{1} dx h_1(x) = \int_{-1}^{1} dx h_L(x)$$

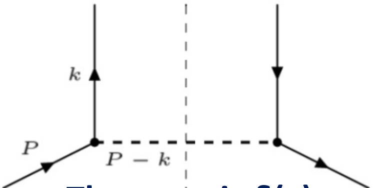
$$\int_{-1}^{1} dx e(x) = \frac{1}{2M} \langle p|\overline{\psi}(0)\psi(0)|p\rangle = \frac{d}{dm} M$$

If one tries to confirm these sum rules experimentally by drawing conclusions from the behavior <u>near x=0</u> about the behavior <u>at x=0</u> they might claim that the sum rules are violated.

# LORENTZ INVARIANCE RELATIONS (LIR): **CUT DIAGRAMS vs UNCUT DIAGRAMS**

For $I^\mu$ the appropriate LIR reads, $\dfrac{I^+}{P^+} = \dfrac{I^-}{P^-}$ ⇦ **Check if holds using cut and uncut diagrams**.

CUT DIAGRAM

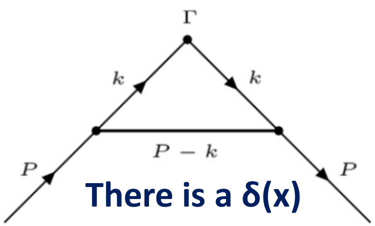



**There no is δ(x)**

$$I^\mu_{cut} \equiv \int d^4k \frac{k^\mu}{(k^2-m^2)^2}\delta[(P-k)^2-\lambda^2]\delta(k^+-xP^+).$$

$$\frac{I^+_{cut}}{P^+} \neq \frac{I^-_{cut}}{P^-}$$ ⇦ LIR is **violated** when **cut** diagrams are used.

UNCUT DIAGRAM




**There is a δ(x)**

$$I^\mu_{uncut} \equiv \int d^4k \frac{k^\mu}{(k^2-m^2+i\epsilon)^2}\frac{i}{[(P-k)^2-\lambda^2+i\epsilon]}\delta(k^+-xP^+)$$

$$\frac{I^+_{uncut}}{P^+} = \frac{I^-_{uncut}}{P^-}$$ ⇦ LIR is **satisfied** when **uncut** diagrams are used.

*-- There is no difference between the two approaches at twist-2 level. Both methods are equivalent and yields identical PDFs. They also agree for 0<x<1,*
***so how can one method result in a violation of LIR and other does not?***

*--The answer is in the appearance of δ(x) term when using the uncut diagrams which is not present in the cut diagrams.*

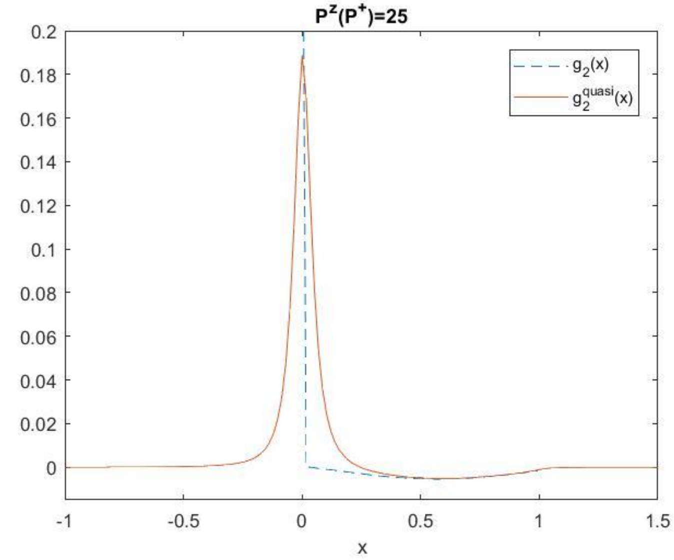
Aslan, Burkardt, Lorentz invariance of twist-3 quark distributions (in preparation)

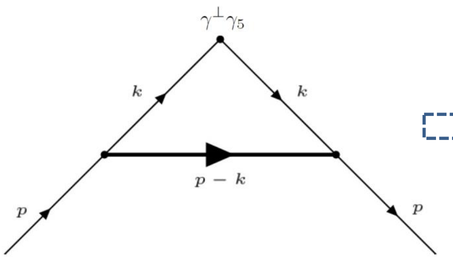
# The origin of δ(x)

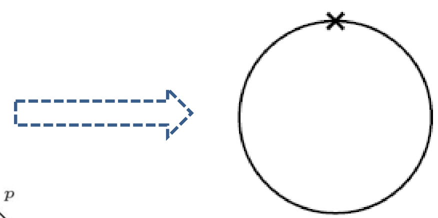
$$g_T(x) = ig^2 \int \frac{d^4k}{(2\pi)^4}\delta(k^+ - xP^+)\frac{(x+\frac{m}{M})(2k^-P^+ + mM)}{(k^2-m^2+i\epsilon^2)[(p-k)^2-\lambda^2+i\epsilon]}$$

$$k^- = \frac{M^2}{2p^+} - \frac{[(p-k)^2-\lambda^2]}{2(p^+-k^+)} - \frac{(k_\perp^2+\lambda^2)}{2(p^+-k^+)}$$

$$\int \frac{dk^-}{(k^2-m^2+i\epsilon)^2}$$



for $k^+ \neq 0$, $$\int \frac{dk^-}{(k^2-m^2+i\epsilon)^2} = \int \frac{dk^-}{\left[2k^+\left(k^- - \frac{(k_\perp^2+m^2)}{2k^+} + \frac{i\epsilon}{2k^+}\right)\right]^2} = 0$$

for all $k^+$ $$\int dk^+dk^- \frac{1}{(k^2-m^2+i\epsilon)^2} = \int dk^+dk^- \frac{1}{(2k^+k^- - k_\perp^2 - m^2+i\epsilon)^2}$$

$$= \int d^2k_L \frac{1}{(k_L^2 - k_\perp^2 - m^2 + i\epsilon)^2} = \frac{i\pi}{k_\perp^2+m^2}$$

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$$g_T(x) = ig^2 \int \frac{d^4k}{(2\pi)^4} \delta(k^+ - xP^+) \frac{(x + \frac{m}{M})(2k^- P^+ + mM)}{(k^2 - m^2 + i\epsilon^2)[(p-k)^2 - \lambda^2 + i\epsilon]}$$

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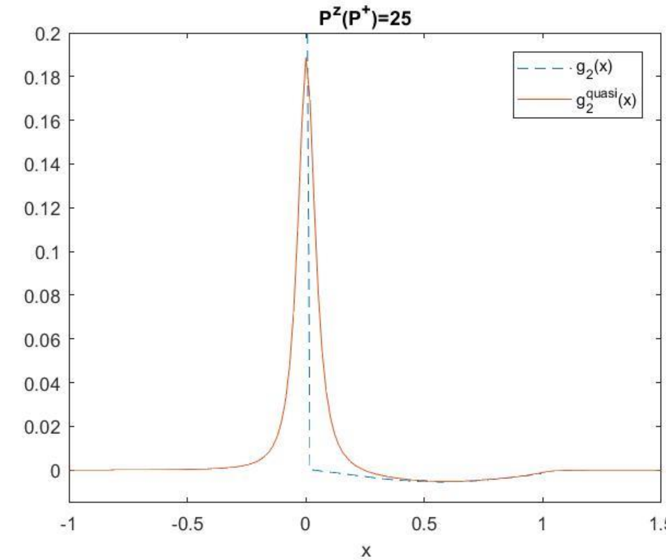


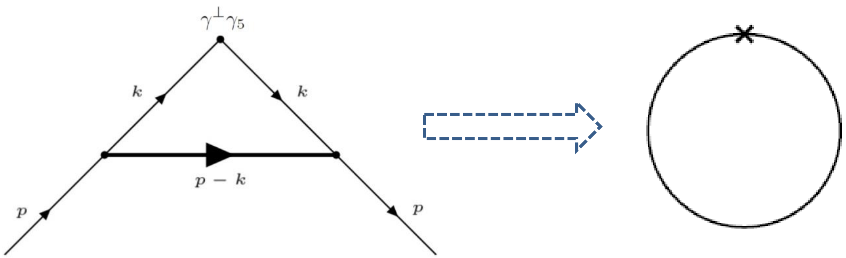


$$\text{for } k^+ \neq 0, \quad \int \frac{dk^-}{(k^2 - m^2 + i\epsilon)^2} = \int \frac{dk^-}{\left[2k^+\left(k^- - \frac{(k_\perp^2 + m^2)}{2k^+} + \frac{i\epsilon}{2k^+}\right)\right]^2} = 0$$

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$$\text{for all } k^+ \quad \int dk^+ dk^- \frac{1}{(k^2 - m^2 + i\epsilon)^2} = \int dk^+ dk^- \frac{1}{(2k^+k^- - k_\perp^2 - m^2 + i\epsilon)^2}$$

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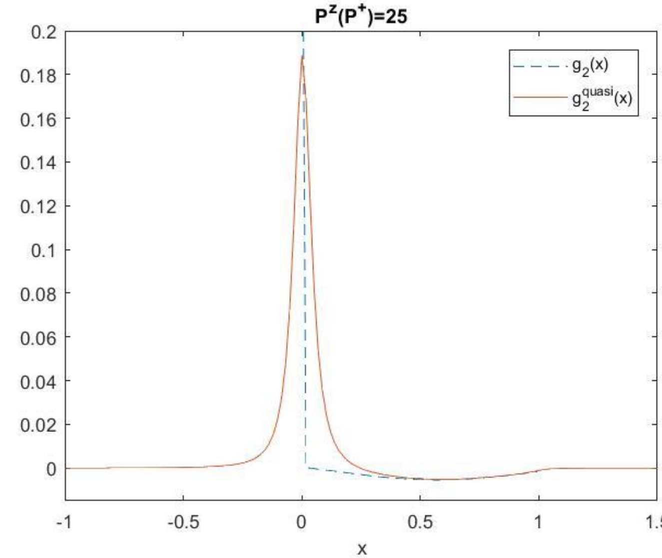


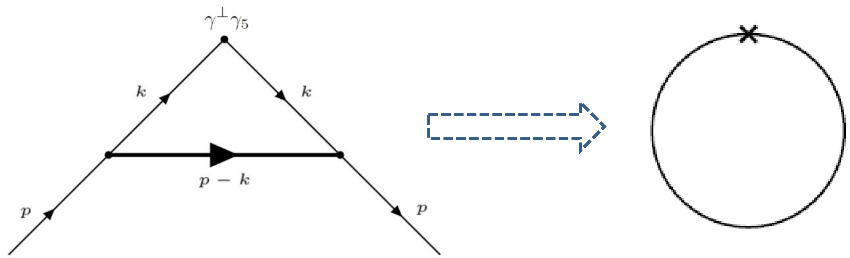


ZERO MODES

$$k^+ = 0$$

$$\text{for } k^+ \neq 0, \quad \int \frac{dk^-}{(k^2-m^2+i\epsilon)^2} = \int \frac{dk^-}{\left[2k^+\left(k^- - \frac{(k_\perp^2+m^2)}{2k^+} + \frac{i\epsilon}{2k^+}\right)\right]^2} = 0$$

$$\int \frac{dk^-}{(k^2-m^2+i\epsilon)^2} = \frac{i\pi}{k_\perp^2+m^2}\delta(k^+).$$

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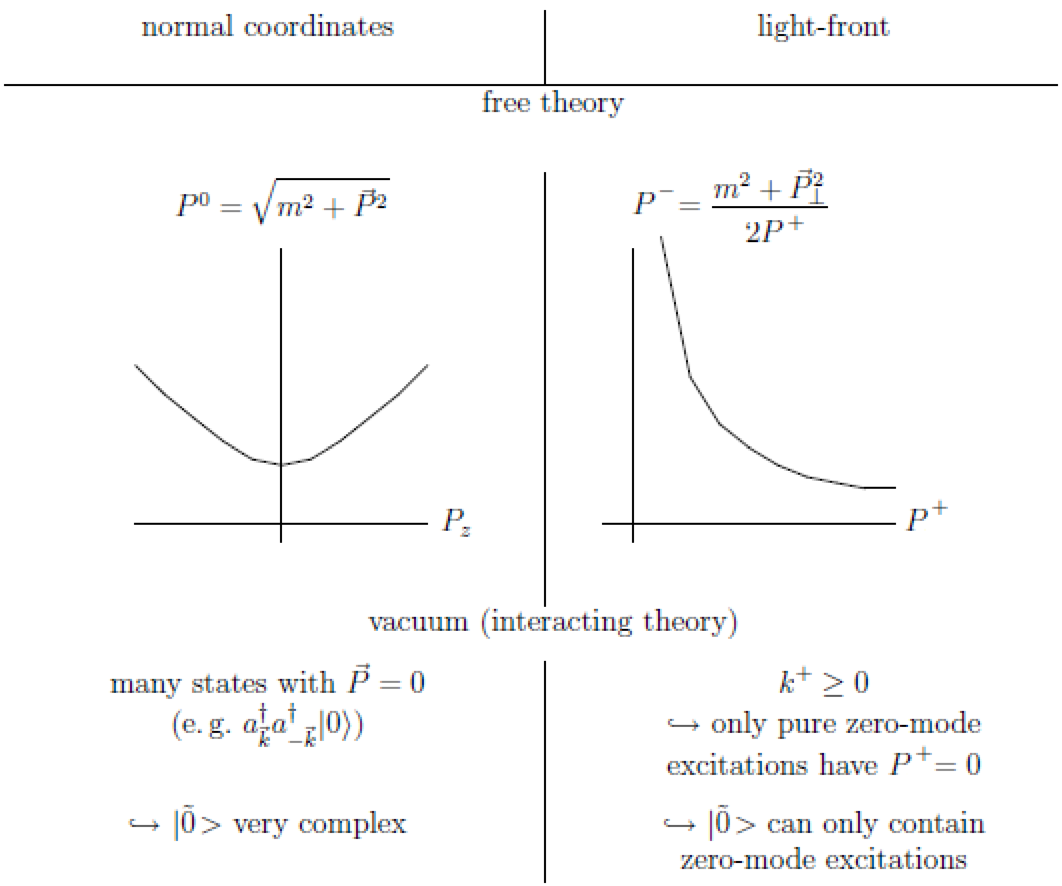
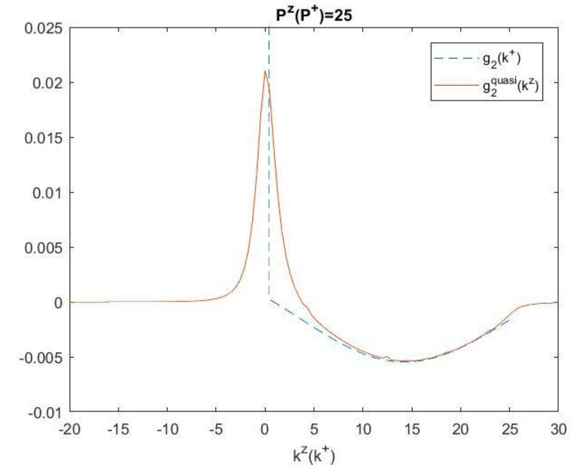
# ZERO MODES and THE VACUUM

In LF framework zero modes are responsible for vacuum condensates.




In a regularized framework, physics of light-cone zero modes is not described correctly by one single mode with $k^+=0$, but by an infinite number of modes in an infinitesimal vicinity of $k^+=0$.

Burkardt, 1995
Light Front Quantization

Burkardt, 1993
LF quantization of the sine-Gordon model

# Related work: Zero modes and vacuum condensates

- Mannheim, Lowdon, Brodsky, 2019
  Structure of light front vacuum sector

  D($x^+ > 0$, instant)= D($x^+ > 0$, LF)

  D($x^+ = 0$, instant)$\neq$ D($x^+ = 0$, LF)

- Collins, 2018
  The non-triviality of the vacuum in light-front quantization: An elementary treatment

  ............ Evidently there is a mathematical error in evaluating the integral in Eq. (5) by first performing the $k^-$ integral and blindly using the zero result. The correct result, as found by Chang and Ma [5] and Yan [6], is that the integral over $k^-$ in Eq. (5) gives a delta function at $k^+$. ............

- Perturbative study of $k^+ \approx 0$, LF quantization of the sine-Gordon model.

  Burkardt, 1993

- Non-perturbative condensates from zero modes in a LF Fock space approach.

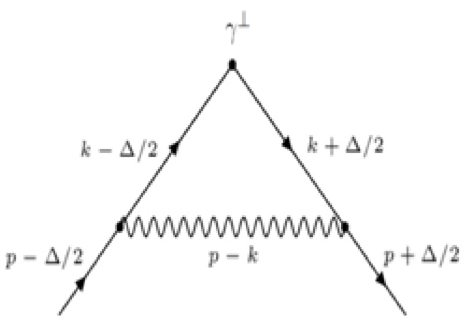
  Burkardt, Lenz, Thies, 2002
  Burkardt, Chabysheva, Hiller 2016

## Conclusions

- Twist-3 GPDs have discontinuities.
- Twist-3 PDFs contain a δ(x).
- δ(x) is related to the zero modes in the LF framework.

| *Twist-2 PDF* | *SDM* | *QTM* | *Twist-3 PDF* | *SDM* | *QTM* |
|---|---|---|---|---|---|
| $f_1$ | ✗ | ✗ | $e$ | ✓ | ✓ |
| $g_1$ | ✗ | ✗ | $h_L$ | ✓ | ✓ |
| $h_1$ | ✗ | ✗ | $g_T$ ($g_2$) | ✓ | ✗ |

- Zero modes are related to the twist-3 distributions.




*Twist -3 evolution*

- Zero modes are generated by twist-3 evolution.

- Lorentz Invariance of twist-3 distributions are violated if δ(x) is not taken into account.

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| $g_1$ | ✗ | ✗ | $h_L$ | ✓ | ✓ |
| $h_1$ | ✗ | ✗ | $g_T$ ($g_2$) | ✓ | ✗ |

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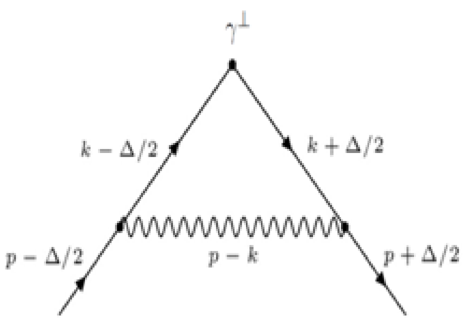


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**THANK YOU**