Hadron structure in small boxes

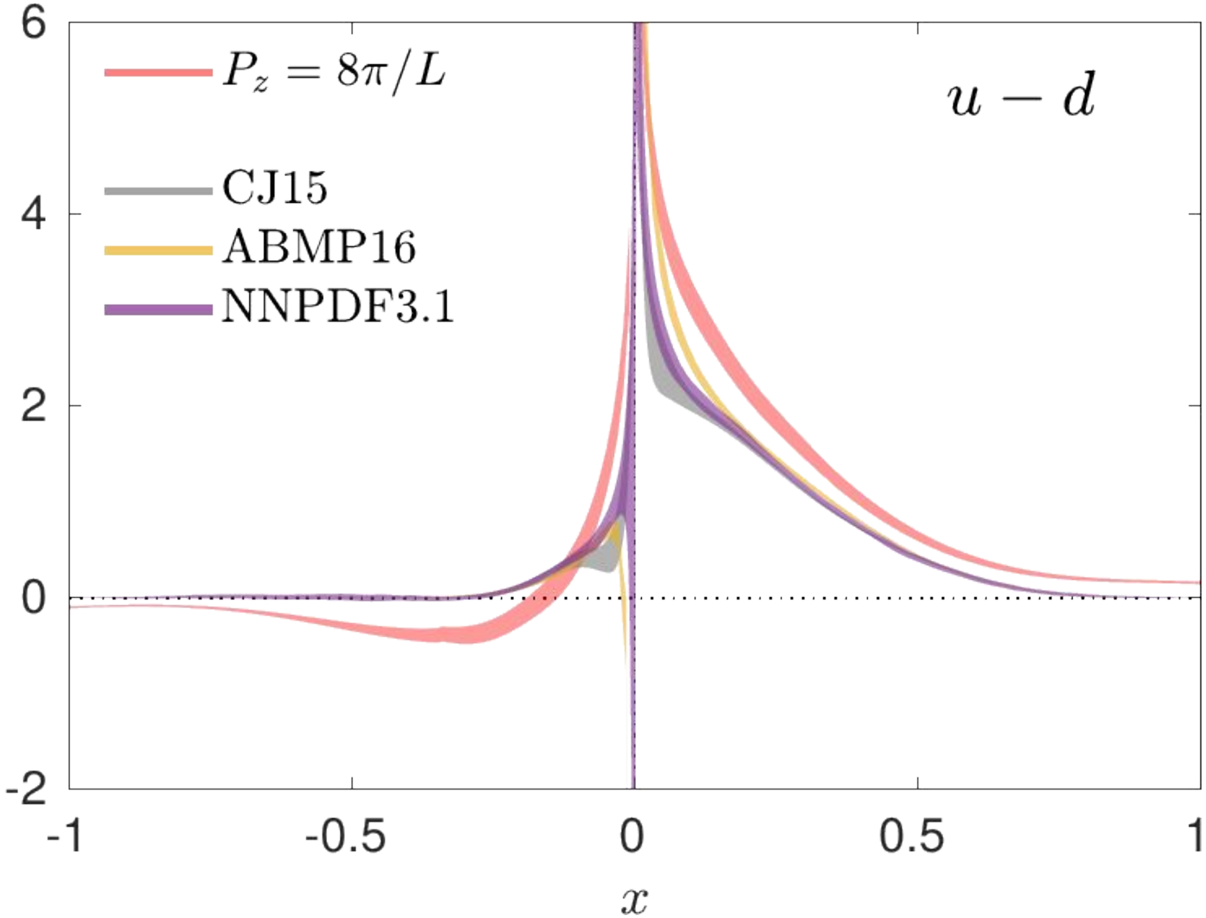
Chris Monahan
Institute for Nuclear Theory
University of Washington

with Raul Briceno, Juan Guerrero and Max Hansen

PRD 98 (2018) 014511

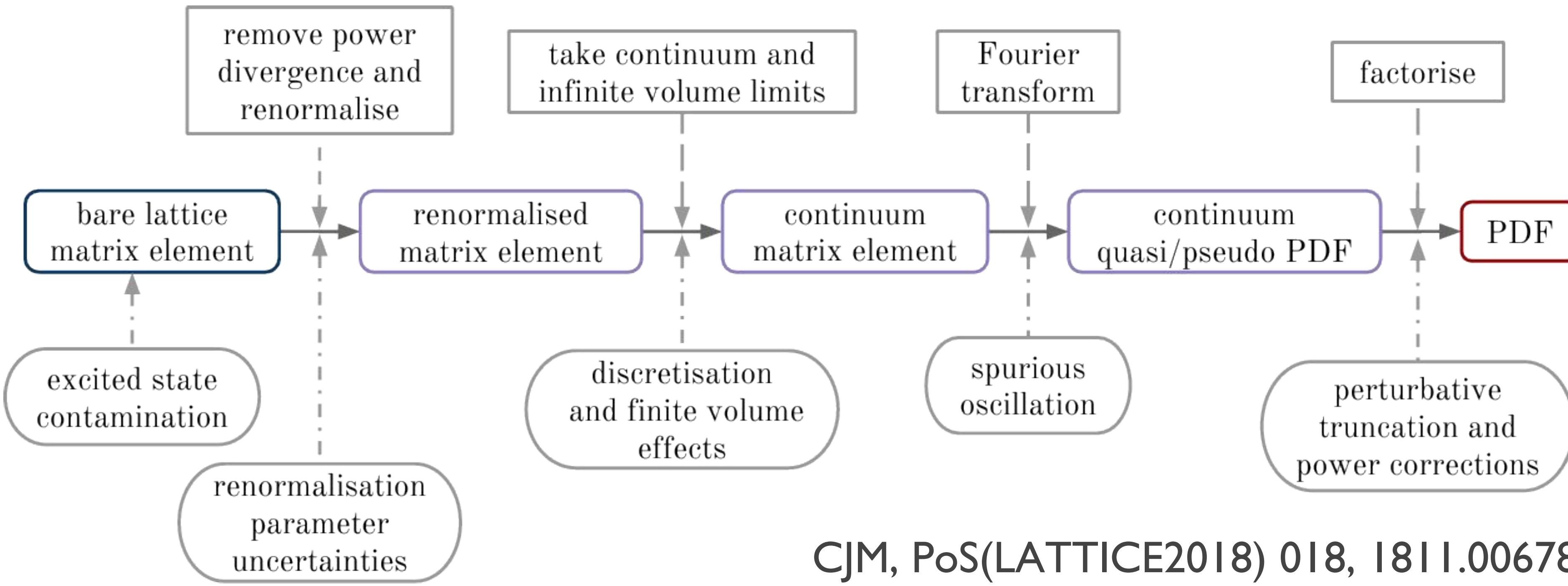
Ab initio nucleon structure

See: M. Constantinou, Mon 11:15
A. Metz, Tues 09:30
C. Alexandrou, Tues 11:15
F. Steffens, Tues 15:00
Y. Zhao, Wed 11:45
D. Richards, Wed 16:00
M. Ebert, Wed 16:30



C. Alexandrou et al. (ETMC), PoS(LATTICE2018) 1811.07248

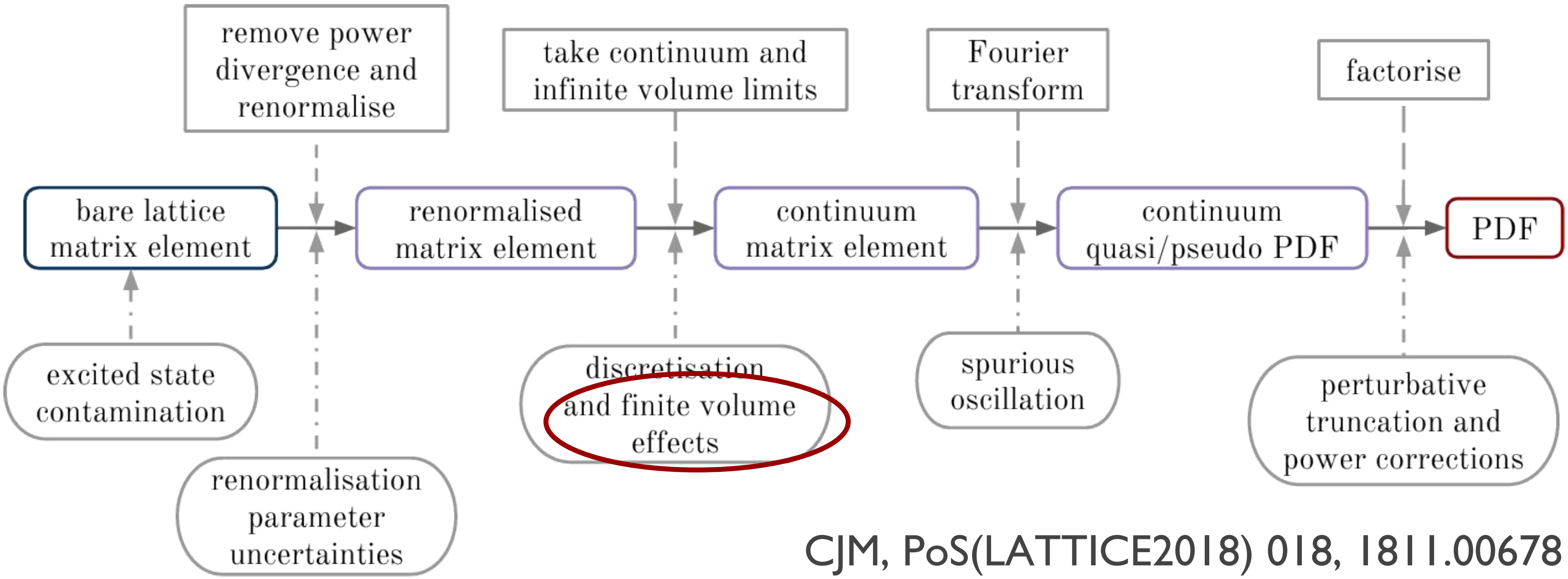
Ab initio nucleon structure in practice



CJM, PoS(LATTICE2018) 018, 1811.00678

See also: Cichy & Constantinou, 1811.07248
Y. Zhao, IJMPA 33 (2019) 1830033

Ab initio nucleon structure in practice



CJM, PoS(LATTICE2018) 018, 1811.00678

See also: Cichy & Constantinou, 1811.07248
Y. Zhao, IJMPA 33 (2019) 1830033

Nonlocal operators and nucleon structure

Wilson-line operators

- Quasi PDFs
- Pseudo PDFs

Currents

- Light quarks
- Auxiliary scalars
- Auxiliary heavy quarks

Smeared local operators

Nonlocal operators and nucleon structure

Wilson-line operators

- Quasi PDFs
- Pseudo PDFs

Currents

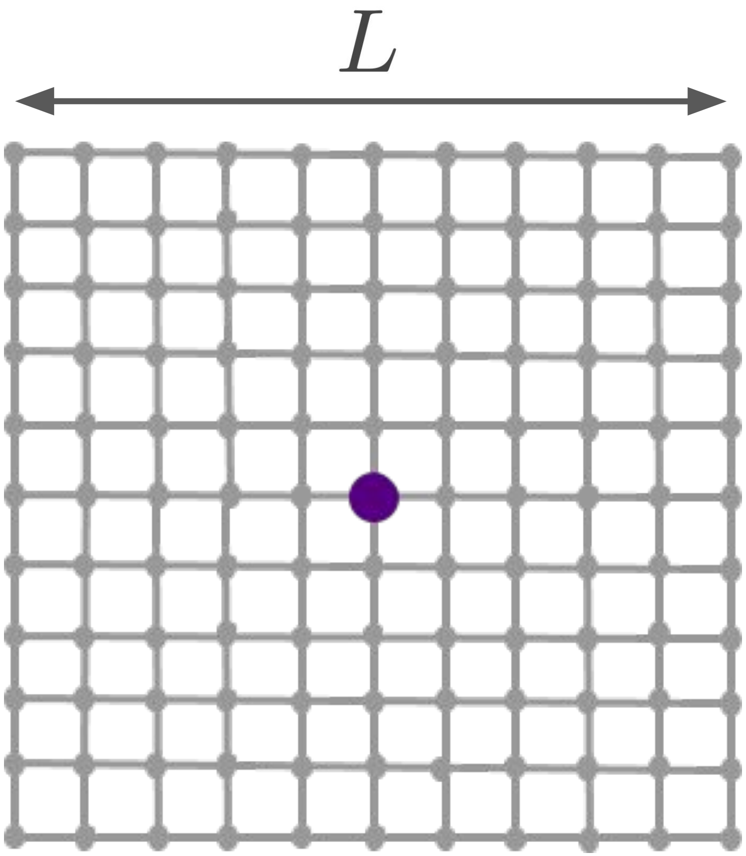
- Light quarks
- Auxiliary scalars
- Auxiliary heavy quarks

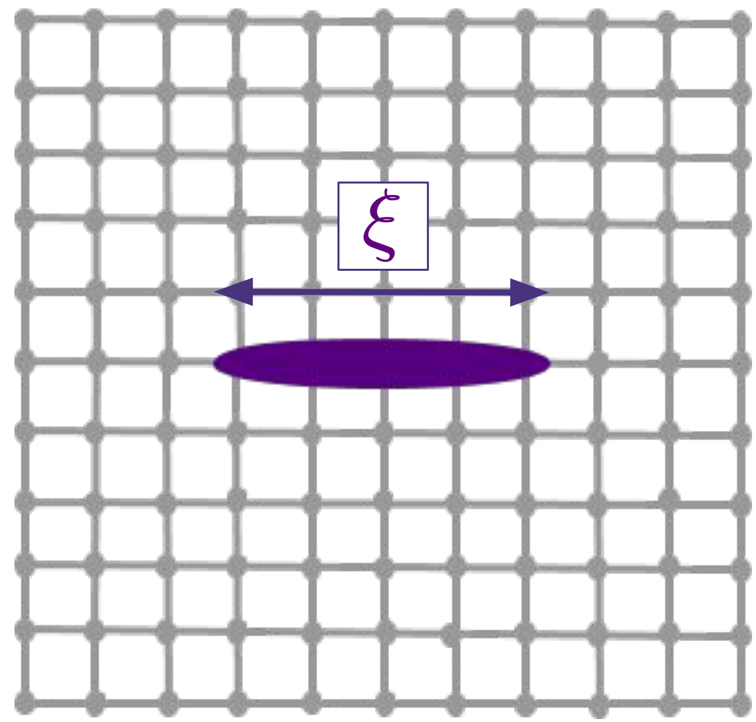
Braun & Mller, EPJC 55 (2008) 349
Ma & Qiu, PRD (2018) 074021
Ma & Qiu, PRL 120 (2018) 022003

G. Bali et al., EPJC 78 (2018) 217
R. Sufian et al., PRD 99 (2019) 074507

Smeared local operators

Finite-volume effects





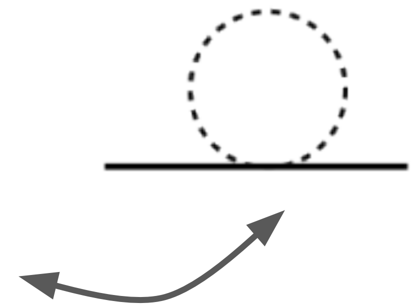
$$e^{-m_\pi L}$$

$$e^{-m_\pi |L-\xi|}$$

$$e^{-m_\pi \xi}$$

Local operators

Finite-volume effects - infrared degrees of freedom



- pions and nucleons
- chiral perturbation theory

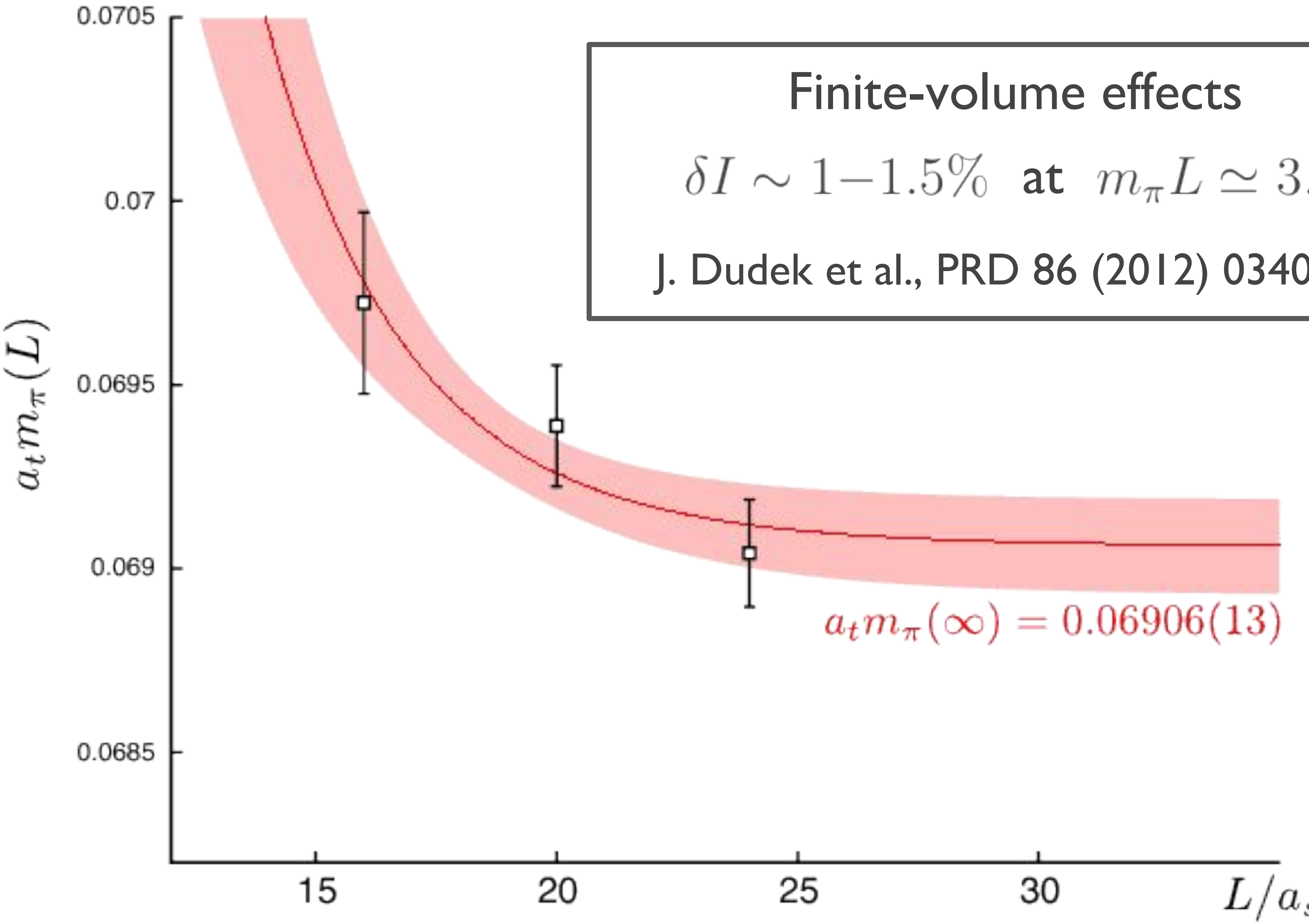
Consider pion tadpole correction to nucleon mass

- infinite volume $I_\infty = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 + m_\pi^2}$
- finite volume $I_{\mathrm{FV}} = \frac{1}{L^3} \sum_{\mathbf{k}} \int \frac{\mathrm{d}k_4}{2\pi} \frac{1}{k^2 + m_\pi^2}$
- finite-volume effects $\delta I = I_\infty - I_{\mathrm{FV}} = \sum_{\mathbf{n} \neq 0} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{i \mathbf{n} \cdot \mathbf{k} L}}{k^2 + m_\pi^2}$

$$\delta I = \frac{1}{(4\pi)^2} \sum_{\mathbf{n}} \left(\frac{4 m_\pi}{|\mathbf{n}| L} \right) K_1(|\mathbf{n}| L) \sim e^{-m_\pi L}$$

M. Lüscher, 1983

Local operators: pion mass



Finite-volume effects

$\delta I \sim 1\text{–}1.5\%$ at $m_\pi L \simeq 3.8$

J. Dudek et al., PRD 86 (2012) 034031

Also S. Beane et al., PRD 85 (2011) 034505

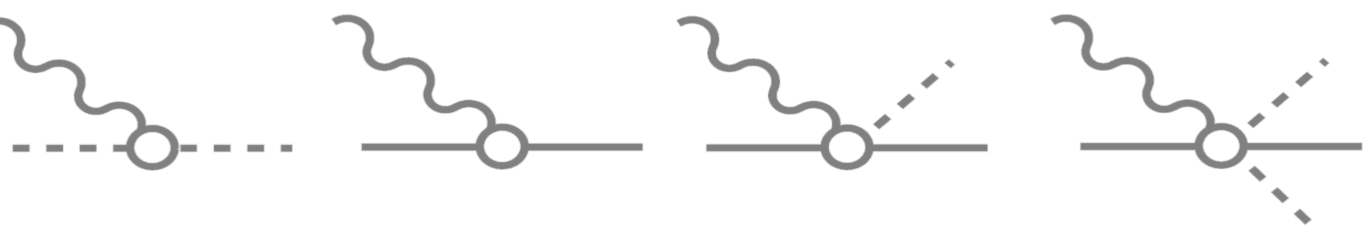
Our model

Couple two scalar particles $m_\varphi \ll m_\chi$:

- φ plays the role of the pion
- χ plays the role of the nucleon

Introduce external current

$$\mathcal{J}(x) = \frac{1}{2} Z_\varphi g_\varphi \varphi^2 + \frac{1}{2} Z_\chi g_\chi \chi^2 + \frac{1}{2} Z_{\chi\varphi} g_{\chi\varphi} \chi^2 \varphi + \frac{1}{4} Z_{\chi\varphi\varphi} g_{\chi\varphi\varphi} \chi^2 \varphi^2 + \cdots$$



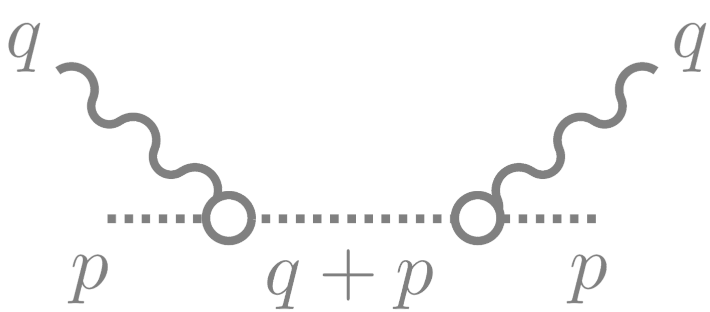
Consider

$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) \equiv \langle \mathbf{p} | \mathcal{J}(0, \boldsymbol{\xi}) \mathcal{J}(0) | \mathbf{p} \rangle$$

$$\delta \mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) \equiv \mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) - \mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p})$$

Light particles

Leading order



$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} \int_q \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E + q_E)^2 + m_\varphi^2}$$

R. Briceno, M Hansen & CJM, PRD 96 (2017) 014502

Finite-volume scaling

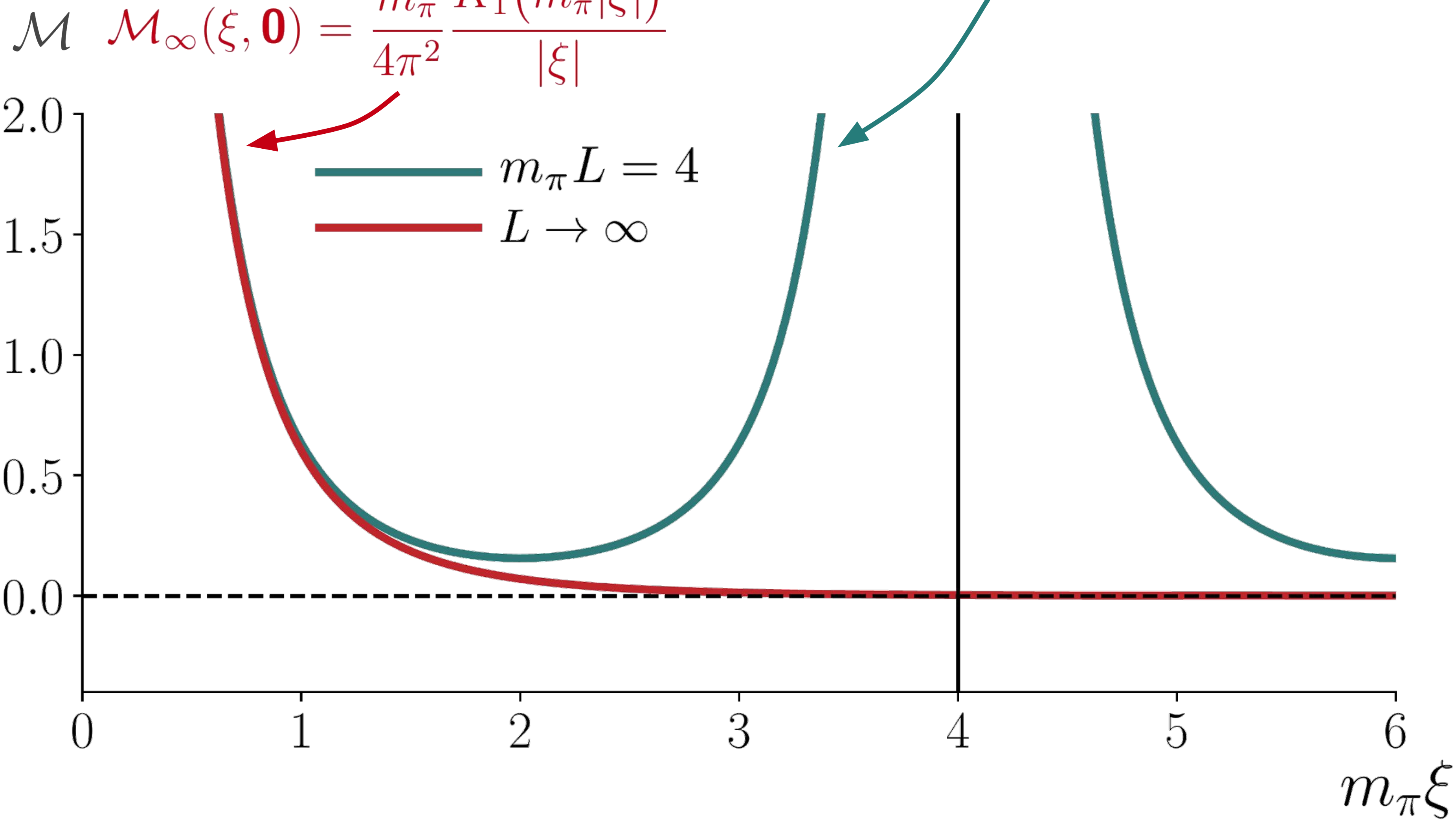
$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L-\xi)^{3/2}}$$

A. Cherman et al, PRD 95 (2017) 074512

Light particles

$$\delta\mathcal{M}_L(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}}^{\mathbf{N}} \frac{K_1\big(m_\pi|\xi+L\mathbf{n}|\big)}{|\xi+L\mathbf{n}|}$$

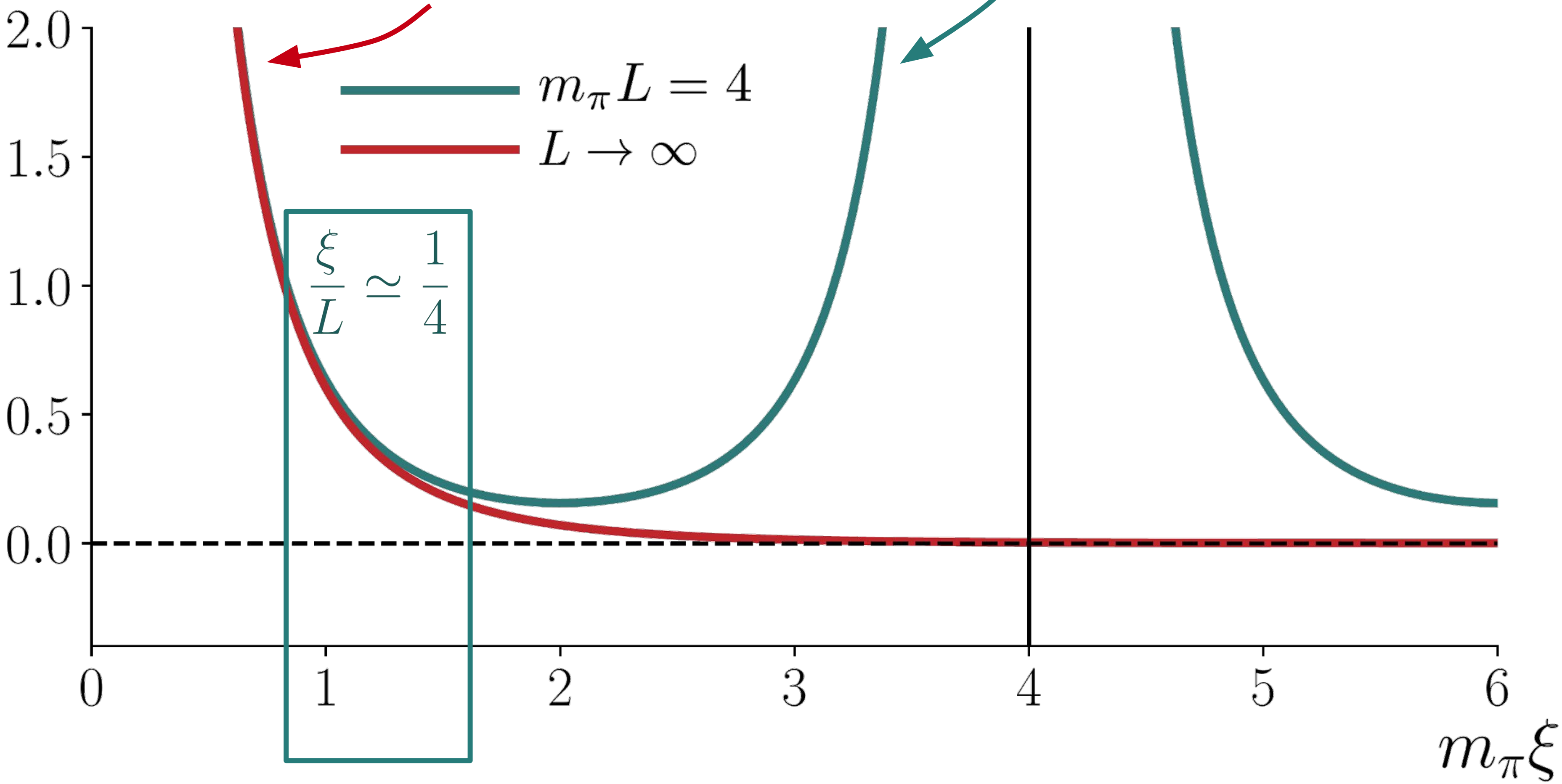
$$\mathcal{M}_\infty(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\frac{K_1\big(m_\pi|\xi|\big)}{|\xi|}$$



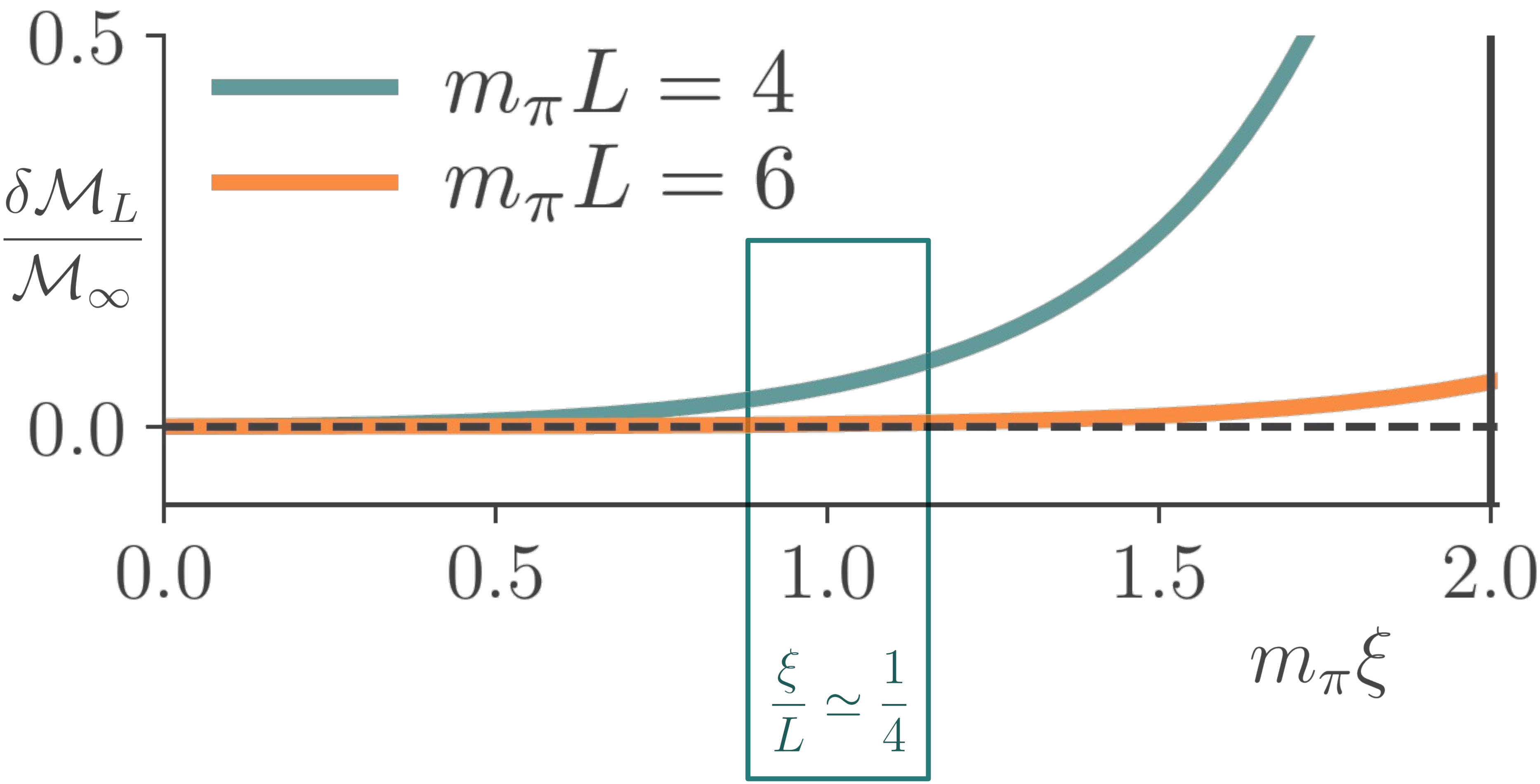
Light particles

$$\delta\mathcal{M}_L(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}}^{\mathbf{N}} \frac{K_1\big(m_\pi|\xi+L\mathbf{n}|\big)}{|\xi+L\mathbf{n}|}$$

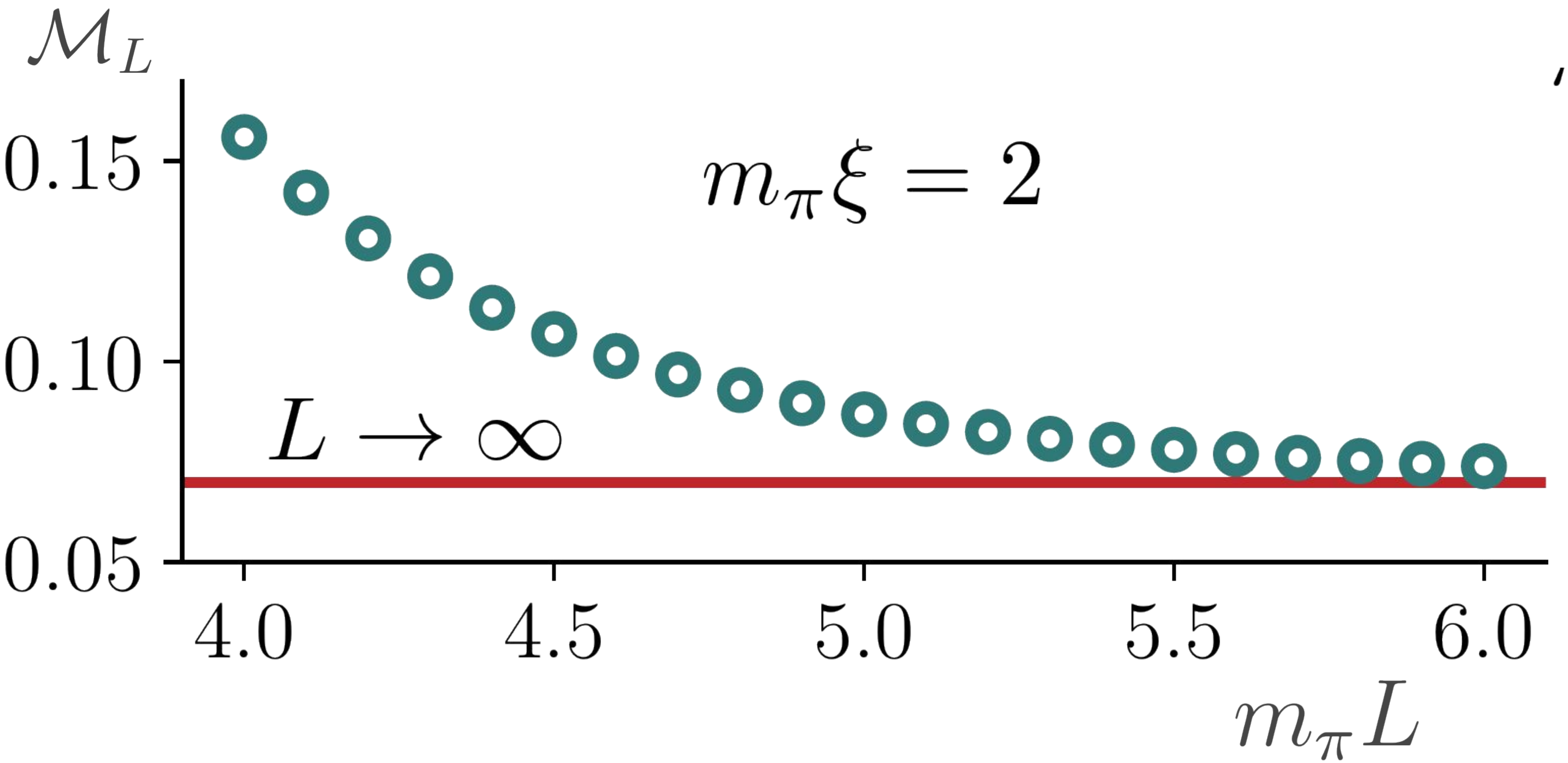
$\mathcal{M}$ $\quad \mathcal{M}_\infty(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\frac{K_1\big(m_\pi|\xi|\big)}{|\xi|}$



Light particles



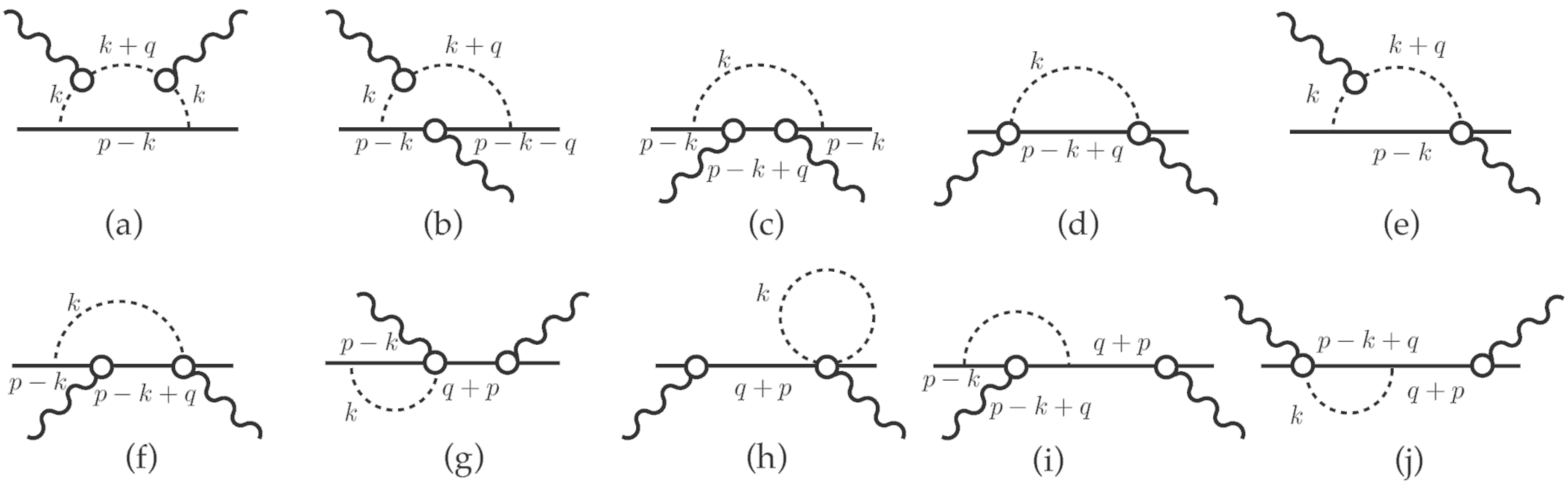
Light particles



Heavy particles

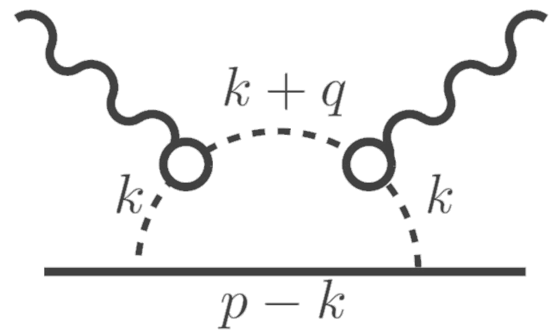
Leading order $$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{e^{-m_\chi(L-\xi)}}{(L-\xi)^{3/2}} \ll e^{-m_\varphi L}$$

One loop



Heavy particles

Dominant diagram



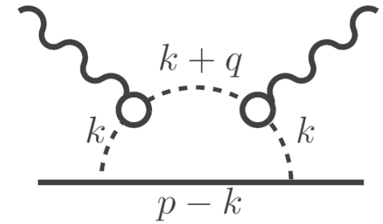
$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L-\xi)\right] e^{-m_\varphi L}$$

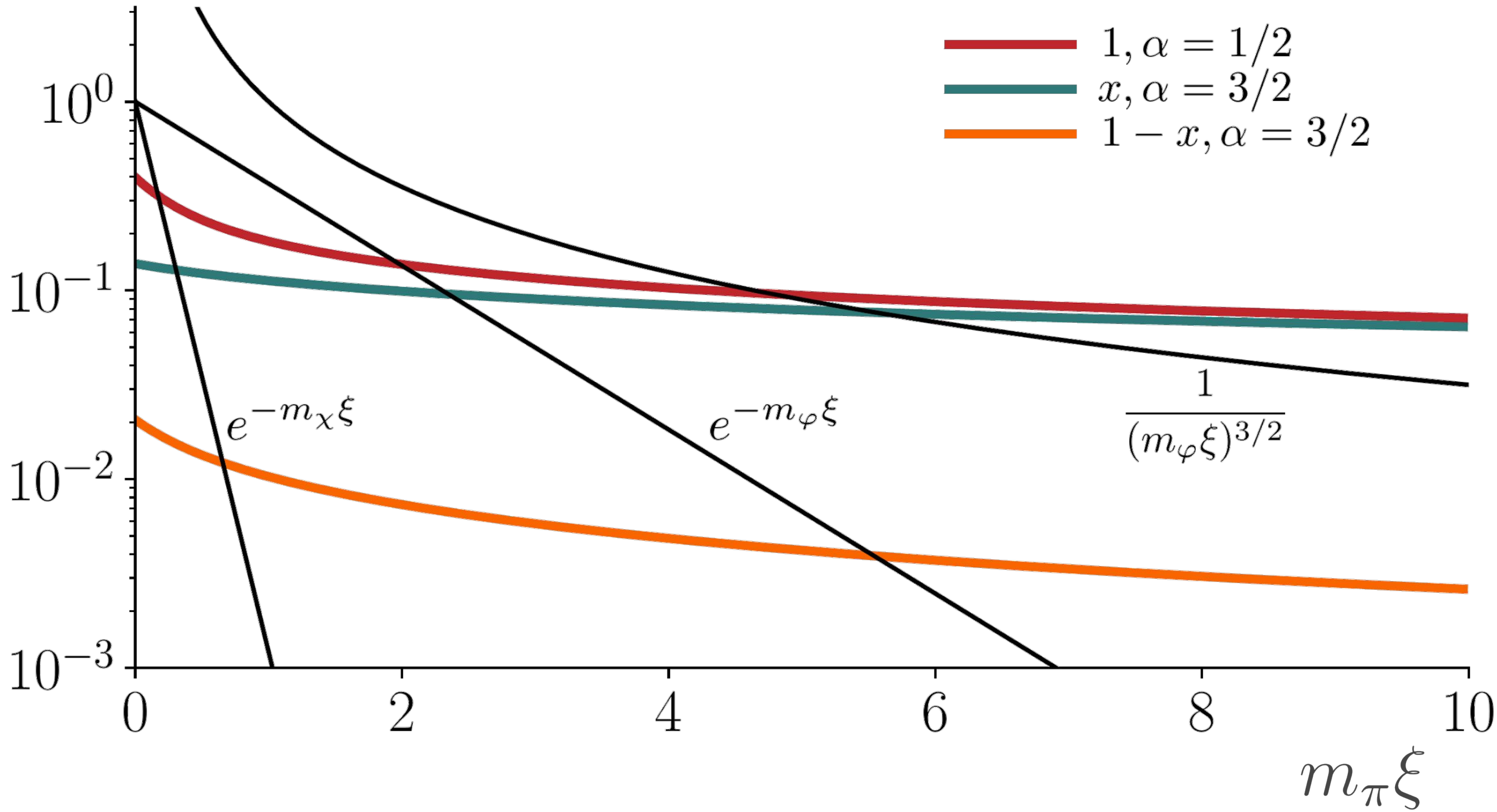
$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x f(x)\, \frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

Finite-volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}} e^{-m_\varphi L}$$

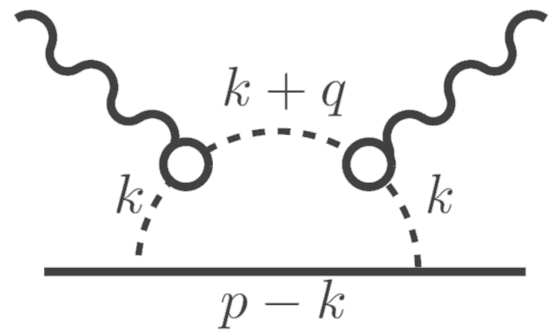
Heavy particles





Heavy particles

Dominant diagram



$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}}H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{x,3/2}(L-\xi)\right]e^{-m_\varphi L}$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x\, f(x)\, \frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

Finite-volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}} e^{-m_\varphi L}$$

Preliminary chiral perturbation theory calculation

Leading order SU(2) meson chiral perturbation theory

Scaling holds for the pion case with, e.g., vector currents $\mathcal{J}_\mu(\xi)\mathcal{J}_\nu(0)$

Preliminary

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{e^{-m_\pi|L-\xi|}}{(L-\xi)^{3/2}}\left[4\delta_{\mu\nu} - m_\pi|L-\xi|\widehat{\xi}_\mu\widehat{\xi}_\nu\right]$$

Preliminary

Suggests careful choice of current could remove leading FV effects

- could conflict with extraction of specific quantities

R. Sufian et al., PRD 99 (2019) 074507
See: D Richards, Wed 16:00

Generalisation to NLO and the nucleon case underway

Quark mass dependence also relevant for chiral extrapolation

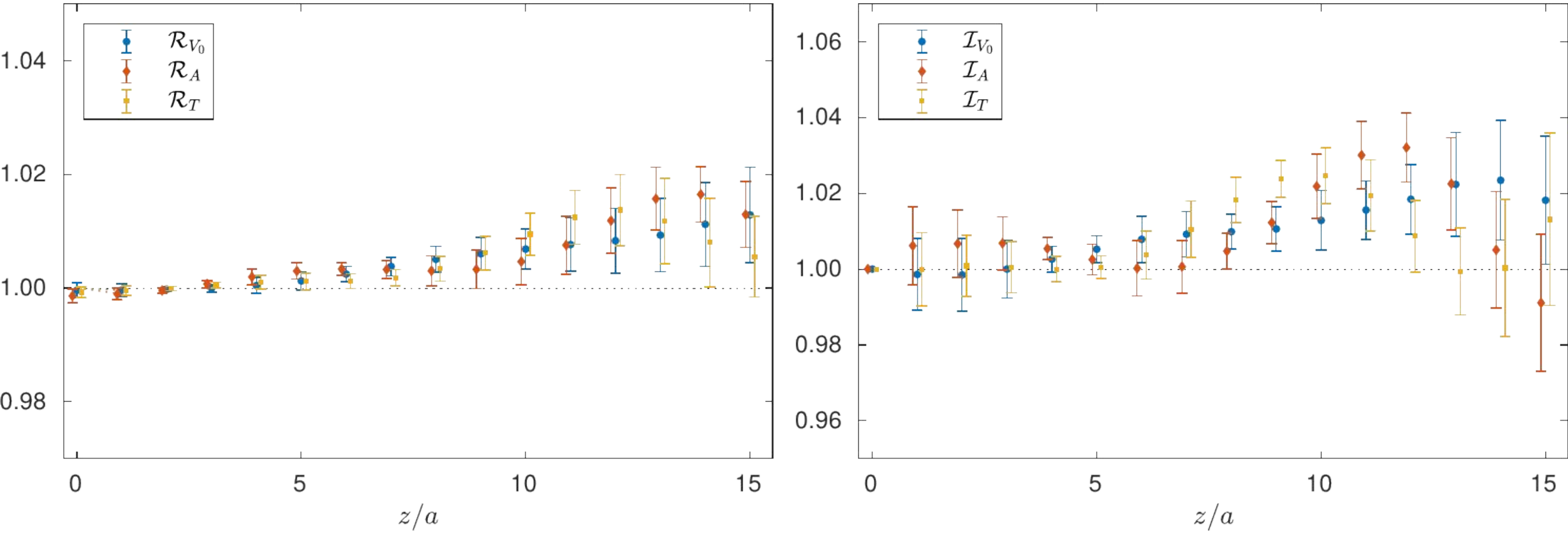
C. Alexandrou et al (ETMC), 1902.00587

Preliminary tests (ETM Collaboration)

First extensive study of systematic uncertainties for quasi PDFs

Study nonlocal renormalisation parameter for Wilson line operator

- pion mass dependence
- finite volume effects



Suggest effects small for renormalized matrix element

Summary

Spatially extended operators

- introduce new IR scales
- potentially modify finite volume effects

Toy model matrix elements

- light particle external states

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L-\xi)^{3/2}}$$

- heavy particle external states

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}}e^{-m_\varphi L}$$

Looking forward

- complete chiral perturbation theory calculation
- several groups studying finite volume effects nonperturbatively
- treatment of Wilson line operators is much trickier

Thank you

cjm373@uw.edu

Light particles

More generally

$$\mathcal{M}_\infty^{(d)}(\boldsymbol{\xi}, \mathbf{p}) = \int_{q_E} e^{i\mathbf{q}\cdot\boldsymbol{\xi}} \int_{k_{1,E}} \cdots \int_{k_{n-1,E}} D_E^{(d)}(p_E, q_E, k_{1,E}, \cdots, k_{n,E})$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi}, \mathbf{p}) \equiv \mathcal{M}_L^{(d)}(\boldsymbol{\xi}, \mathbf{p}) - \mathcal{M}_\infty^{(d)}(\boldsymbol{\xi}, \mathbf{p})$$

$$= \sum_{\mathbf{M}\in\mathbb{Z}^{3n}/\{\mathbf{0}\}} \int_{K_E} e^{i\mathbf{q}\cdot\boldsymbol{\xi} + i\mathbf{K}\cdot L\mathbf{M}} D_E^{(d)}(p_E, K_E)$$

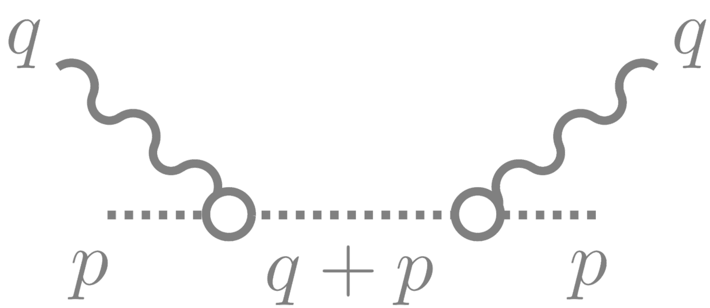
$$K_E \equiv \{q_E, k_{1,E}, \cdots, k_{n-1,E}\}$$

$$\mathbf{M} = \{\mathbf{n}, \mathbf{m}_1, \cdots, \mathbf{m}_{n-1}\}$$

R. Briceno, M Hansen & CJM, PRD 96 (2017) 014502

Light particles

Leading order



$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} \int_q \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E + q_E)^2 + m_\varphi^2}$$

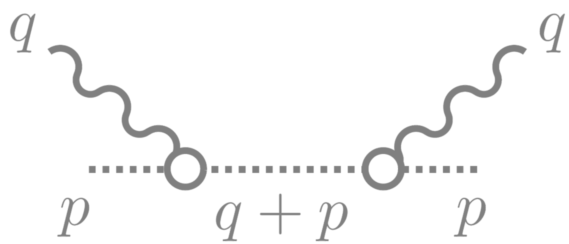
Finite volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = \frac{m_\varphi g_\varphi^2}{4\pi^2} \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} e^{-i\mathbf{p}\cdot(\boldsymbol{\xi}+L\mathbf{n})} \frac{K_1\big(m_\varphi|\boldsymbol{\xi} + L\mathbf{n}|\big)}{|\boldsymbol{\xi} + L\mathbf{n}|} \sim \frac{m_\varphi g_\varphi^2}{4\pi^2} \frac{K_1\big(m_\varphi|L - \xi|\big)}{|L - \xi|}$$

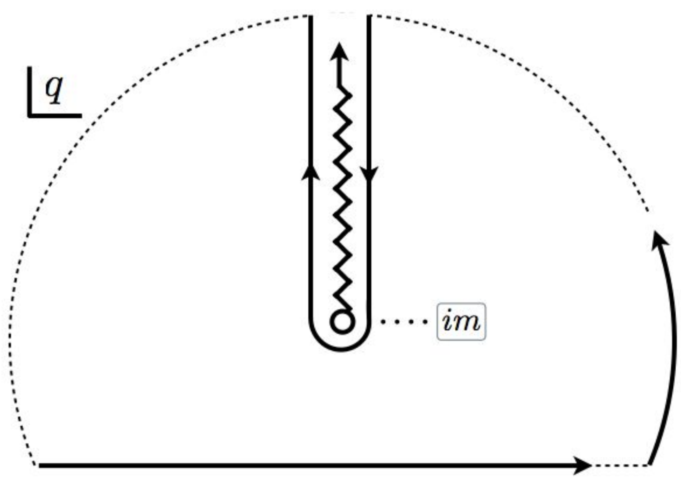
$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L - \xi)^{3/2}}$$

Bessel functions

Recall

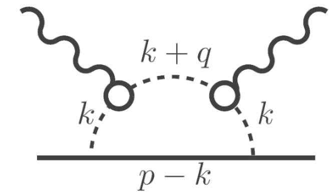


$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$



$$I_{\mathrm{FV}} = \sum_{\mathbf{n}} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{i\mathbf{n}\cdot\mathbf{k}L}}{k^2 + m_\pi^2}$$

Heavy particles



Dominant diagram

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{p}) = g^2 g_\varphi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \int_{q_E,k_E} e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+L\mathbf{n})} e^{iL\mathbf{k}\cdot\mathbf{m}} \frac{1}{[k_E^2+m_\varphi^2]^2} \frac{1}{(k_E+q_E)^2+m_\varphi^2} \frac{1}{(p_E-k_E)^2+m_\chi^2}$$

Shift momentum variables, introduce Feynman parameters

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{p}) = 2g^2 g_\varphi^2 \int_0^1 \mathrm{d}x\, x \sum_{\mathbf{n},\mathbf{m}} e^{i(1-x)\mathbf{p}\cdot[L(\mathbf{m}-\mathbf{n})-\boldsymbol{\xi}]} \int_{q_E} \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+L\mathbf{n})}}{q_E^2+m_\varphi^2} \int_{k_E} \frac{e^{i\mathbf{k}\cdot[L(\mathbf{m}-\mathbf{n})-\boldsymbol{\xi}]}}{[k_E^2+M(x)^2]^3}$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x)m_\chi^2 + x(1-x)p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$

Carry out momentum integrals

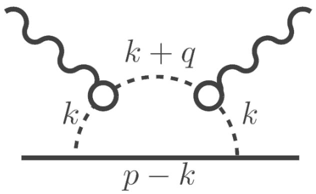
$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\varphi^2 \sum_{\mathbf{n},\mathbf{m}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, x\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right]$$

Asymptotically

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L-\xi)\right] e^{-m_\varphi L}$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x\, f(x)\, \frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

Finite-volume references



M.F.L. Golterman and K.-C. Leung, *Phys. Rev.* **D56**, 2950 (1997); *ibid* **D58**, 097503 (1998)
M. Golterman and E. Pallante, *Nucl. Phys. Proc. Suppl.* **83**, 250 (2000).
C.-J.D. Lin *et al.*, *Nucl. Phys.* **B650**, 301 (2003).
C.-J.D. Lin *et al.*, *Phys. Lett.* **B553**, 229 (2003).
G. Colangelo, S. Dürr and R. Sommer, *Nucl. Phys. Proc. Suppl.* **119**, 254 (2003).
G. Colangelo and S. Dürr, hep-lat/0311023.
D. Becirevic and G. Villadoro, hep-lat/0311028.
D. Arndt and C.-J.D. Lin, hep-lat/0403012.
G. Colangelo and C. Haefeli, hep-lat/0403025.
A. Ali Khan *et al.* [QCDSF Collaboration], *Nucl. Phys. Proc. Suppl.* **119**, 419 (2003).
A. Ali Khan *et al.* [QCDSF Collaboration], hep-lat/0309133.
A. Ali Khan *et al.* [QCDSF and UKQCD Collaborations], hep-lat/0312029.
A.A. Khan *et al.*, hep-lat/0312030.
A.S. Kronfeld, hep-lat/0205021.
S.R. Beane, hep-lat/0403015.

S. Beane & M. Savage, PRD 70 (2004) 074029

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\varphi^2 \sum_{\mathbf{n},\mathbf{m}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, x\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right]$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) = g^2 g_\varphi g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{n}-\boldsymbol{\xi}|;M(x)\right]\right] \left[\int_0^1 \mathrm{d}y\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(y)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\chi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\,(1-x)\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = g_{\chi\varphi}^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}-\boldsymbol{\xi}|;m_\varphi\right] ,$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = g g_\varphi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = g g_\chi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = g g_{\chi\varphi} g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{1}{2} g_\chi g_{\chi\varphi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}|;m_\varphi\right] .$$

$$\mathcal{I}_\gamma\left[|\boldsymbol{\xi}|;m\right] \equiv \int_{k_E} \frac{e^{i\mathbf{k}\cdot\boldsymbol{\xi}}}{[k_E^2+m^2]^\gamma} = \frac{1}{8\pi^2\Gamma(\gamma)} \left(\frac{|\boldsymbol{\xi}|}{2m}\right)^{\gamma-2} K_{\gamma-2}\left(|\boldsymbol{\xi}|m\right)$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x)m_\chi^2 + x(1-x)p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L-\xi)\right] e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi g_\chi}{64\pi^3 m_\varphi}\left[\frac{1}{\xi^{1/2}(L-\xi)^{1/2}} H_{1,1/2}(\xi) H_{1,1/2}(L-\xi)\right] e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g^2 g_\chi^2}{128\pi^3}\frac{m_\chi^{1/2}}{m_\varphi^{3/2}}\left[\frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{1-x,3/2}(L-\xi)\right] e^{-\xi(m_\chi - m_\varphi)} e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_{\chi\varphi}^2 m_\chi^{1/2} m_\varphi^{1/2}}{32\pi^3}\left[\frac{1}{\xi^{3/2}(L-\xi)^{3/2}}\right] e^{-\xi(m_\chi - m_\varphi)} e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\varphi g_{\chi\varphi}}{64\pi^3}\left[\frac{1}{\xi^{1/2}(L-\xi)^{3/2}} H_{1,1/2}(\xi) + \frac{1}{\xi^{3/2}(L-\xi)^{1/2}} H_{1,1/2}(L-\xi)\right] e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\chi g_{\chi\varphi} m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}(L-\xi)^{1/2}} H_{1,1/2}(L-\xi)\right] e^{-\xi(m_\chi - m_\varphi)} e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_{\chi\varphi} g_\chi m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2} L^{1/2}} H_{1,1/2}(L)\right] e^{-\xi m_\chi} e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_\chi g_{\chi\varphi\varphi} m_\varphi^{1/2} m_\chi^{1/2}}{64\pi^3}\left[\frac{1}{\xi^{3/2} L^{3/2}}\right] e^{-m_\chi \xi} e^{-m_\varphi L},$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x f(x)\,\frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x) m_\chi^2 + x(1-x) p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$