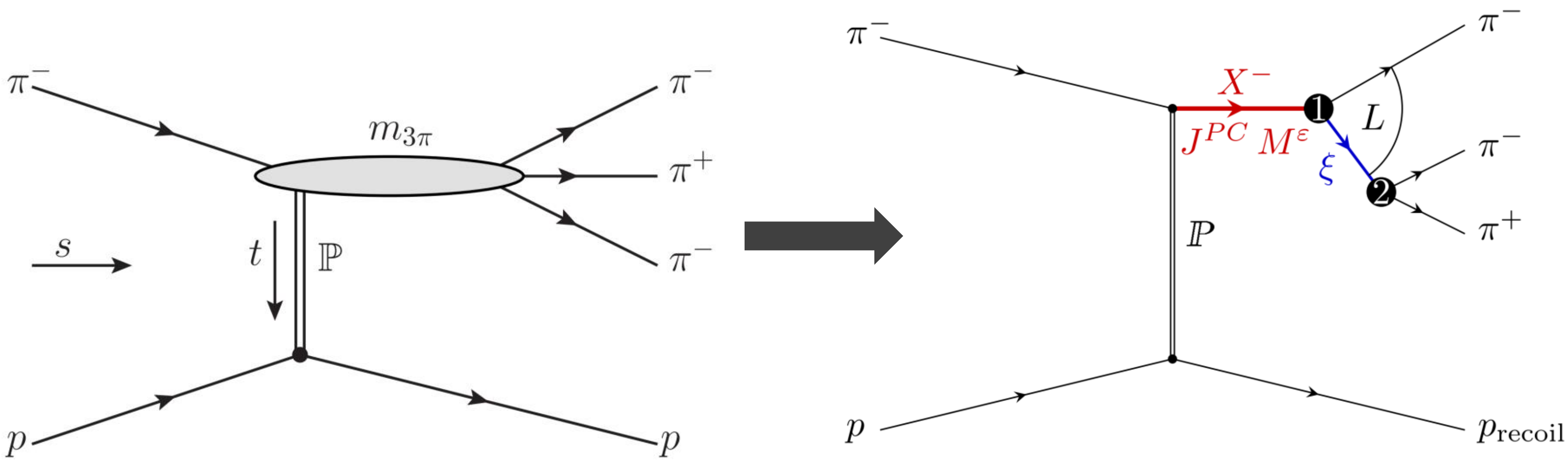
Extensions of Khuri-Treiman Equations and Rescattering Effects

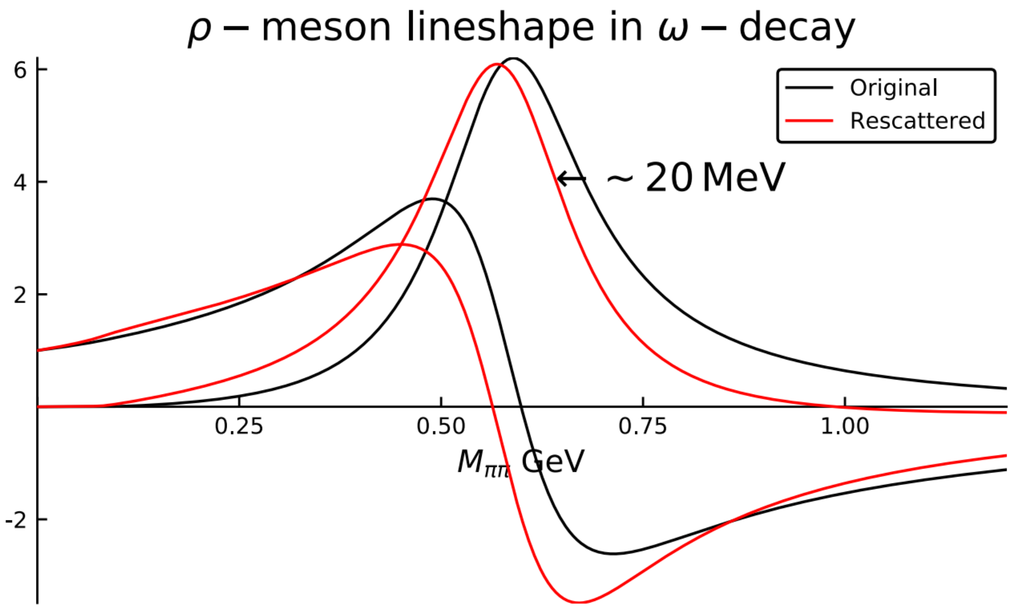
Daniel Winney

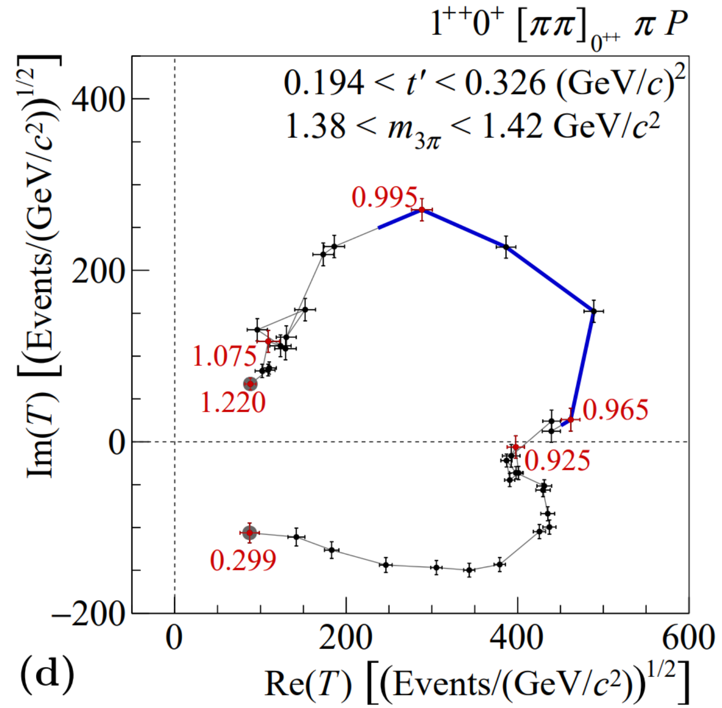
COMPASS Freed-Isobar PWA



Diffractive pion production off proton target

COMPASS Freed-Isobar PWA





Khuri-Treiman Equations for the reaction $J^{PC} \to 3\,\pi$

Helicity Amplitude Formalism

Using helicity amplitudes allows a systematic construction of amplitudes of arbitrary spin.

Easily identify kinematic singularities and threshold behavior:

$$\mathcal{A}_\lambda(s, z_s) = K_\lambda(s) \sum_{j=|\lambda|}^{\infty} (2j+1)\,(k(s)q(s))^{j-|\lambda|}\; \hat{d}^{j}_{\lambda 0}(\theta_s)\; \hat{A}_{j\lambda}(s)$$

Not easily done in covariant tensor formalism which is process-dependent.

$$\mathcal{A}_{\omega\to 3\pi}(s,t,u) = \epsilon_{\mu\alpha\beta\gamma}\; p_1^\alpha\; p_2^\beta\; p_3^\gamma\; \varepsilon^\mu(p_M,\lambda) \times A(s,t,u)$$

$$\mathcal{A}_{a_1\to 3\pi}(s,t,u) = \varepsilon^\mu(p_M,\lambda) \times \Big[(p_1+p_2)_\mu\; F(s,t,u) + (p_1-p_2)_\mu\; G(s,t,u) \Big]$$

See M. Mikhasenko / A. Pilloni and JPAC: What is the right formalism to search for resonances? I & I arXiv:1712.02815 [hep-ph] & arXiv:1805.02113 [hep-ph]

Helicity Amplitude Formalism

Additional Kinematic constraints between helicities at (pseudo)-threshold also readily derived from crossing matrix

$$\mathcal{A}_\lambda^{(s)}(s,t,u) = (-1)^\lambda \, \mathcal{A}_\lambda^{(t)}(t,s,u)$$

Generalize the pion scattering isobar decomposition to arbitrary spin

$$\mathcal{A}_\lambda(s,t,u) = \sum_{j=0}^{j_{\max}} (2j+1)\, d^j_{\lambda 0}(\theta_s)\, a^{(s)}_{j\lambda}(s) + \sum_m \sum_{j=0}^{j_{\max}} (2j+1)\, d^J_{\lambda m}(\hat{\theta}_1)\, d^j_{m0}(\theta_t)\, a^{(t)}_{jm}(t) \cdot$$

$$+ \sum_m \sum_{j=0}^{j_{\max}} (-1)^{j+\lambda+m}\, (2j+1)\, d^J_{\lambda m}(\hat{\theta}_2)\, d^j_{m0}(\theta_u)\, a^{(u)}_{jm}(u)$$

Helicity Amplitude Formalism

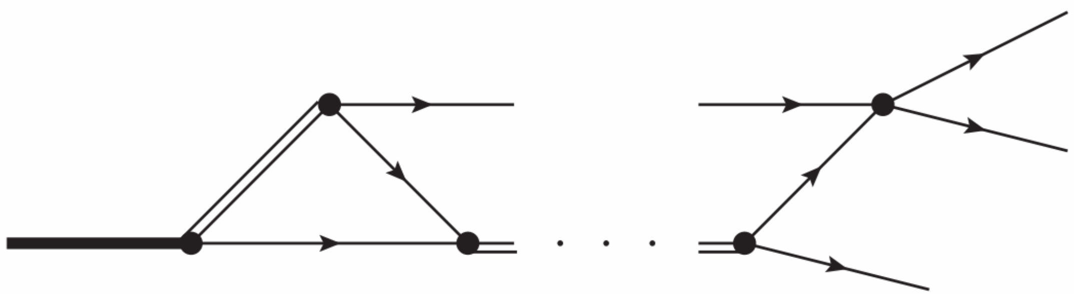
$$\mathcal{A}_\lambda(s,t,u) = \sum_{j=0}^{j_{\max}} (2j+1)\, d^j_{\lambda 0}(\theta_s)\, a^{(s)}_{j\lambda}(s) + \sum_m \sum_{j=0}^{j_{\max}} (2j+1)\, d^J_{\lambda m}(\hat{\theta}_1)\, d^j_{m0}(\theta_t)\, a^{(t)}_{jm}(t) + \sum_m \sum_{j=0}^{j_{\max}} (-1)^{j+\lambda+m}\, (2j+1)\, d^J_{\lambda m}(\hat{\theta}_2)\, d^j_{m0}(\theta_u)\, a^{(u)}_{jm}(u)$$

The isobar functions are directly comparable with COMPASS analysis.

$1^{++}0^{+}\,[\pi\pi]_{0^{++}}\,\pi\mathrm{P}$ $\quad 1^{++}0^{+}\,[\pi\pi]_{1^{--}}\,\pi\mathrm{S}$ $\quad 1^{++}1^{+}\,[\pi\pi]_{1^{--}}\,\pi\mathrm{S}$

Khuri-Treiman Equations

$$\mathrm{Disc}\ \hat{a}^j_\lambda(s) = \rho(s)\ t^*_j(s) \left[\hat{a}^j_\lambda(s)\ +\ 2\,(2j+1)\,\frac{(j-\lambda)!}{(j+\lambda)!} \right.$$
$$\left. \times \left[\sum_{j'm'I'} (2j'+1)\,\frac{C_{II'}}{K_\lambda(s)} \int dz'_s\ P^\lambda_j(z'_s) \times d^J_{\lambda m'}(\hat{\theta}'_1)\ K_{m'}(t')\ (k(t')q(t'))^{j'-m'}\,\hat{d}^{j'}_{m'0}(z'_t)\ \hat{a}^{(I')}_{j'm'}(t') \right] \right]$$



Solution by iteration incorporates the rescattering ladder.

Limitations and Extension for a Regge-behaved KT model.

High-Energy Behavior

The biggest limitation is the polynomial behavior of a truncated sum of partial waves. Requires subtractions for dispersion relations to converge when more than one partial wave contributes.

Consider pion-pion scattering
(no complications of spin and nontrivial isospin structure)

$$A(s,t,u) = \sum_{\ell=0}^{\ell_{\max}} (2\ell+1) P_\ell(z_s) p^{2\ell}(s) a_\ell^s(s) + (s \to t) + (s \to u)$$

$$\lim_{s,-u\to\infty} A(s,t,u) \propto z_t^{\ell_{\max}} \sim s^{\ell_{\max}}$$

Limited to low-energies and small number of partial waves

arXiv:1803.06027 [hep-ph]

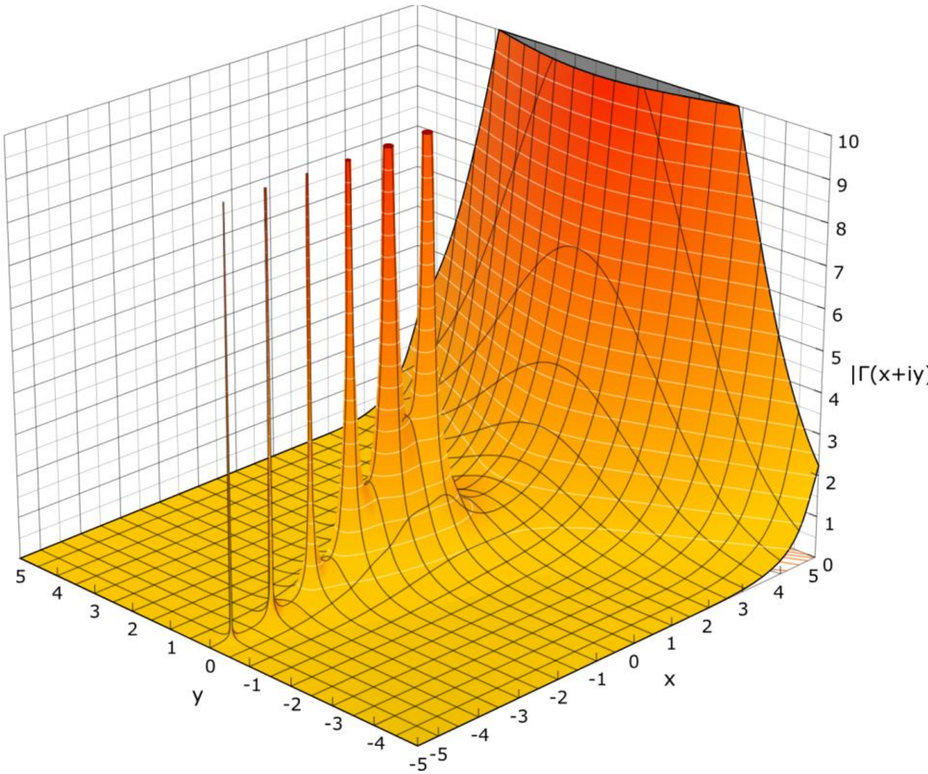
Veneziano Amplitude and Regge Behavior

$$A_{n,m}(s,t) \equiv \frac{\Gamma(n-\alpha_s)\Gamma(n-\alpha_t)}{\Gamma(n+m-\alpha_s-\alpha_t)}$$

The **FULL** scattering amplitudes must incorporate an infinite number of (dual) resonances.

The KT framework adds a finite number of partial waves in each channel separately (interference model).

$$\lim_{s\to\infty} A(s,t,u) \sim s^{\alpha(t)}$$



Models with an Asymptotic Background Function

Possible to split the low-energy parameterization to the level of individual partial waves while retaining the Regge-behaved asymptotic behavior at high-energies.

A better constrained full amplitude incorporates more dynamical constraints than a simple sum of partial waves.

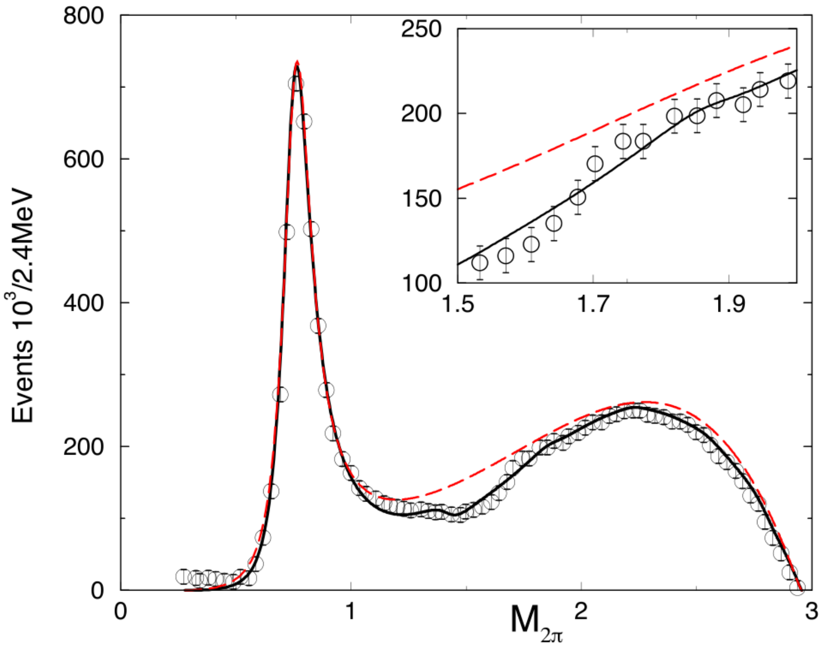
$$\mathcal{A}_n(s,t;N) = \frac{2n-\alpha_s-\alpha_t}{(n-\alpha_s)(n-\alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s-\alpha_t)^{i-1} \times \frac{\Gamma(N+1-\alpha_s)\Gamma(N+1-\alpha_t)}{\Gamma(N+1-n)\Gamma(N+n+1-\alpha_s-\alpha_t)}$$

Application of the Veneziano Model in Charmonium Dalitz Plot Analysis

Adam P. Szczepaniak[1,2,3] and M.R. Pennington[2]

arXiv:1403.5782 [hep-ph]

Models with an Asymptotic Background Function



Resonance region:

Sum of simple poles with polynomial residues.

Regge region:

Veneziano model behavior

Veneziano term serves as background from higher resonances.

arXiv:1403.5782 [hep-ph]

Khuri-Treiman with a Regge Background

Parameterizations of dispersive approaches matched with Regge-term considered before for J/ψ to 3π and KKπ.

$$t_{\pi\pi}(s) = \begin{cases} t_{\pi\pi}^{Kmatrix}(s), s < s_{low} \\ t_{\pi\pi}^{Regge}(s), s > s_{high} \end{cases} \qquad t_{\pi\pi}^{\mathrm{Regge}}(s) \sim \frac{1 \pm e^{i\pi\alpha(t)}}{\sin \pi\alpha(t)} \times \left(\frac{s}{\hat{s}}\right)^{\alpha(t)}$$

However, goal is to incorporate Regge-behavior smoothly in the full amplitude which is more sensitive to dynamics and further constrains partial waves.

Peng Guo et. Al, Phys. Rev. D **85**, 056003

Isobar model as a “Dual Resonance Model”

We can use symmetry relations to constrain the possible structure of a model coupling channels together.

$$A(s,t,u) = \mathcal{A}(s,t) + \mathcal{A}(s,u) - \mathcal{A}(t,u)$$

For processes like pion scattering, KT equations are imposed on partial waves of definite isospin. We need to determine the general dual resonance decomposition for isospin definite functions.

$$\begin{bmatrix} A^{(0)}(s,t,u) \\ A^{(1)}(s,t,u) \\ A^{(2)}(s,t,u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s,t,u) \\ A(t,s,u) \\ A(u,t,s) \end{bmatrix}$$

See for example Sivers & Yellin; Rev.Mod.Phys. 43 (1971) 125-188

Isobar model as a “Dual Resonance Model”

Bose symmetry: $$A^{(I)}(s,t,u) = (-1)^I \, A^{(I)}(s,u,t)$$

Each symmetric scalar function describes resonances of specific isobar in both channels simultaneously.

$$A^{(I)}(s,t,u) = \sum_{I'} \zeta_{II'} \left[\mathcal{A}^{(I')}(s,t) + (-1)^I \mathcal{A}^{(I')}(s,u) \right] + \sum_{I'} \eta_{II'} \, \mathcal{A}^{(I')}(t,u)$$

$$\zeta = \begin{bmatrix} 1 & 1 & 0 \\ \frac{2}{3} & -1 & -\frac{5}{3} \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} -\frac{1}{3} & 3 & \frac{10}{3} \\ 0 & 0 & 0 \\ \frac{2}{3} & 0 & -\frac{2}{3} \end{bmatrix}$$

Consistent with conventional KT

By choose each scalar function to be a truncated sum of partial waves, we recover

$$\mathcal{A}^{(I)}(t,u) = \frac{1}{2}\sum_{\ell=0}^{\ell_{\max}}(2\ell+1)\left[P_\ell(z_t)\; a_\ell^{(I)}(t) + P_\ell(z_u)\; a_\ell^{(I)}(u)\right]$$

Isospin crossing matrix naturally emerges from crossing coefficients

$$C = (-1)^{I'}\zeta + \eta = \begin{bmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$$

Briefly Back to Veneziano Amplitudes

Decomposition into symmetric scalar functions with definite isospin allows gives greater flexibility by removing exchange degeneracy.

$$\alpha(s) \to \alpha^{(I)}(s)$$

Different Regge-trajectories in each isospin channel may allow more precise determination of resonance parameters from fits.

Application of Szczepaniak-Pennington-Veneziano amplitudes to decays with isospin.

Background Function for Regge-behaved KT

We may add a background function to “fix” the low-energy polynomial behavior of the isobars and recover the physical Regge-behavior.

$$\mathcal{A}^{(I)}(t,u) = \frac{1}{2}\sum_{\ell=0}^{\ell_{\max}}(2\ell+1)\left[P_\ell(z_t)\, a_\ell^{(I)}(t)\, V_\ell^{(I)}(t,s,u) + P_\ell(z_u)\, a_\ell^{(I)}(u)\, V_\ell^{(I)}(u,t,s)\right]$$

This in general requires coupling the direct and cross-channels.

can show that if : $V_\ell^{(I)}(t,s,u) = V_\ell^{(I)}(t,u,s)$

$$A^{(I)}(s,t,u) = \sum_{\ell=0}^{\ell_{\max}}(2\ell+1)\, p^{2\ell}(s)\, P_\ell(z_s)\, a_\ell^{(I)}(s)\, \frac{1}{2}\left[1+(-1)^{I+\ell}\right] V_\ell^{(I)}(s,t,u)$$
$$+\sum_{I'}\sum_{\ell'=0}^{\ell'_{\max}}(2\ell'+1)\,\frac{1}{2}C_{II'}\left[p^{2\ell'}(t)\, P_{\ell'}(z_t)\, a_{\ell'}^{(I')}(t)\, V_{\ell'}^{(I')}(t,s,u) + (-1)^{I+I'}\, p^{2\ell'}(u)\, P_{\ell'}(z_u)\, a_{\ell'}^{(I')}(u)\, V_{\ell'}^{(I')}(u,t,s)\right]$$

Duality in Isobar Models

The structure of KT models have additive poles in each channel which is a different duality as in Veneziano or “true” dual resonance models.

If the isobar functions obey a dispersion relation and converge in the kinematic range, we avoid double counting in any Regge limit.

A PROOF OF THE VALIDITY OF GENERALIZED INTERFERENCE MODELS

R. JENGO
CERN, Geneva, Switzerland

Received 20 January 1969

Analyticity at the heart of the structure of KT equations

Choices of Background

The background function be symmetric in cross particles whose asymptotic gives rise to Regge behavior.

No poles in these variables to not introduce non-physical poles (i.e. wrong isospin poles).

If $V_\ell(s,t,u) \to t^{\alpha(s)-\ell}$ *as* $t \to \infty$ *, the full amplitude and each partial wave term will have Regge-behavior.*

$$V_\ell(s,t,u) \propto \frac{1}{\Gamma(1-\frac{1}{2}(\alpha(s)+\alpha(t)+I-\ell)\,\Gamma(\frac{1}{2}(\alpha(t)+I))} \times (t \to u)$$

Summary

The Khuri-Treiman formalism allows for a systematic study of rescattering effects and is directly comparable to COMPASS freed-isobar data.

Extending the viability of the KT formalism to higher-energies via a dual background applicable to COMPASS data for $J = 2, 3, 4$ decays with higher phase spaces.

Additionally, applicable to analysis of, for example, J/ψ or B meson decays to three pions. Larger phase spaces which can produce 2-body resonances above the $\rho(770)$.

Nonperturbative, dispersive input / test of perturbative QCD regimes.

Thank you!