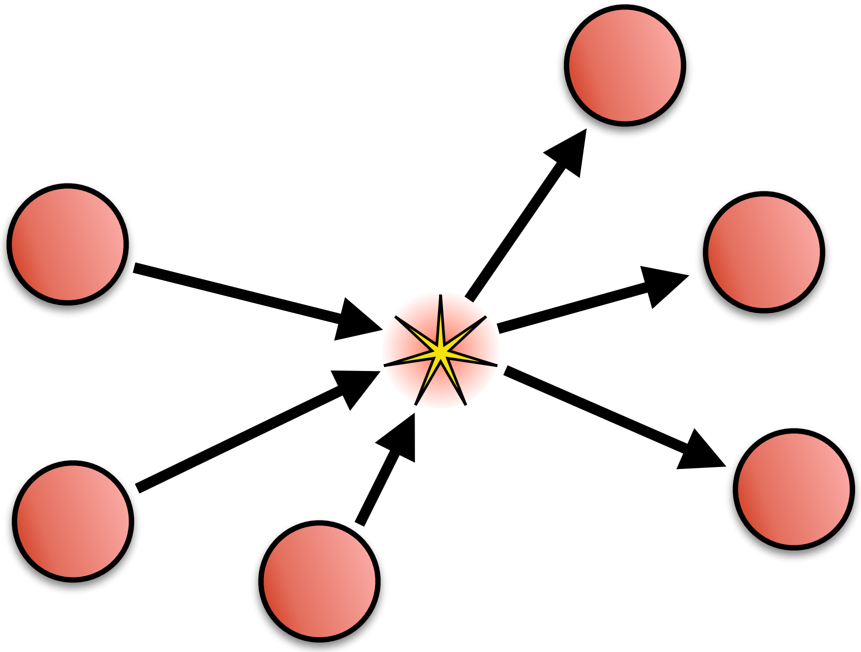
Towards an Analytical Description of Three Particle Scattering



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Joint Physics Analysis Center (JPAC)

8th Workshop of the APS Topical Group on Hadronic Physics
April 10-12 2019

Why 3-Body Scattering?

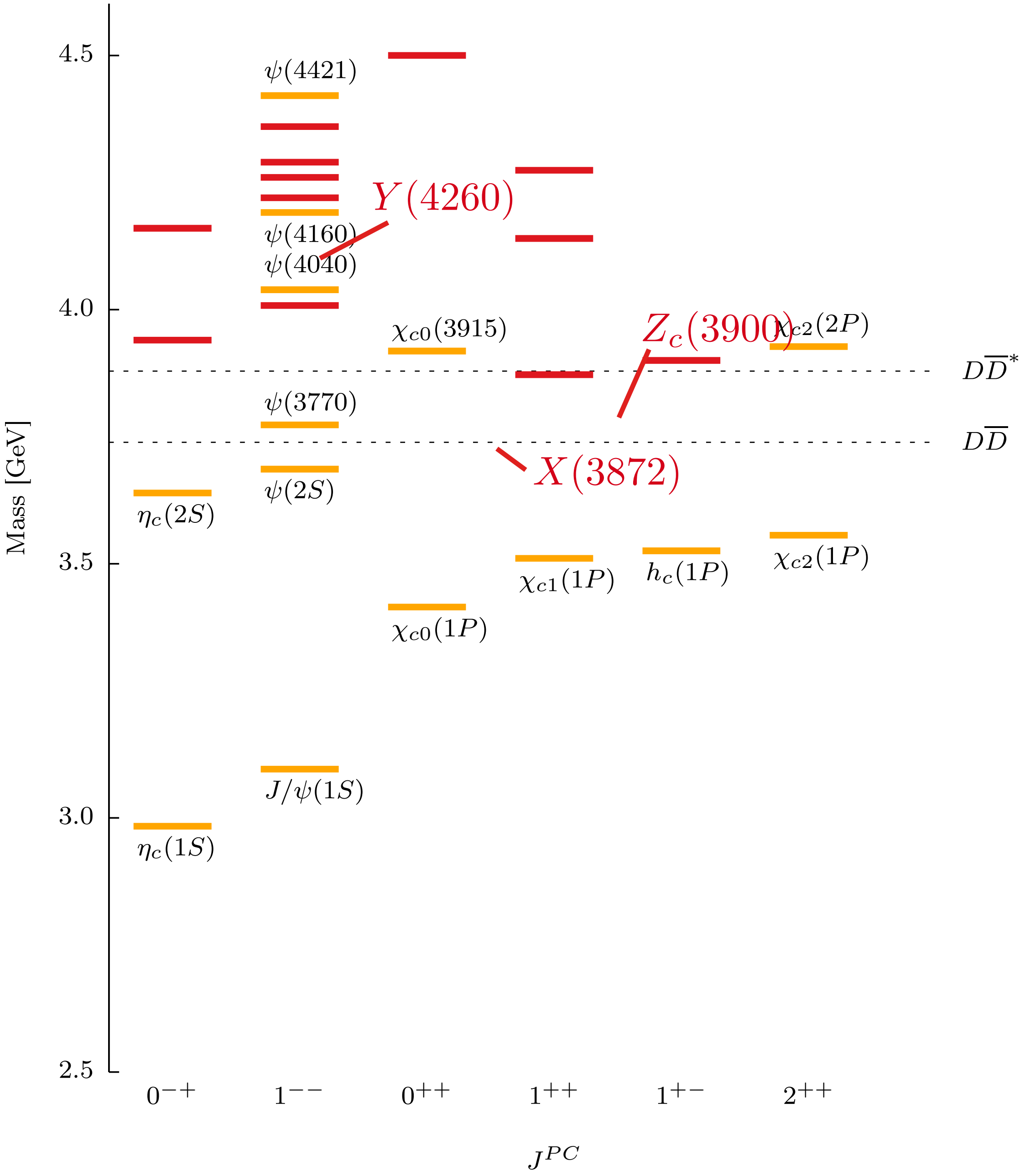
Many hadronic resonances decay into three particles

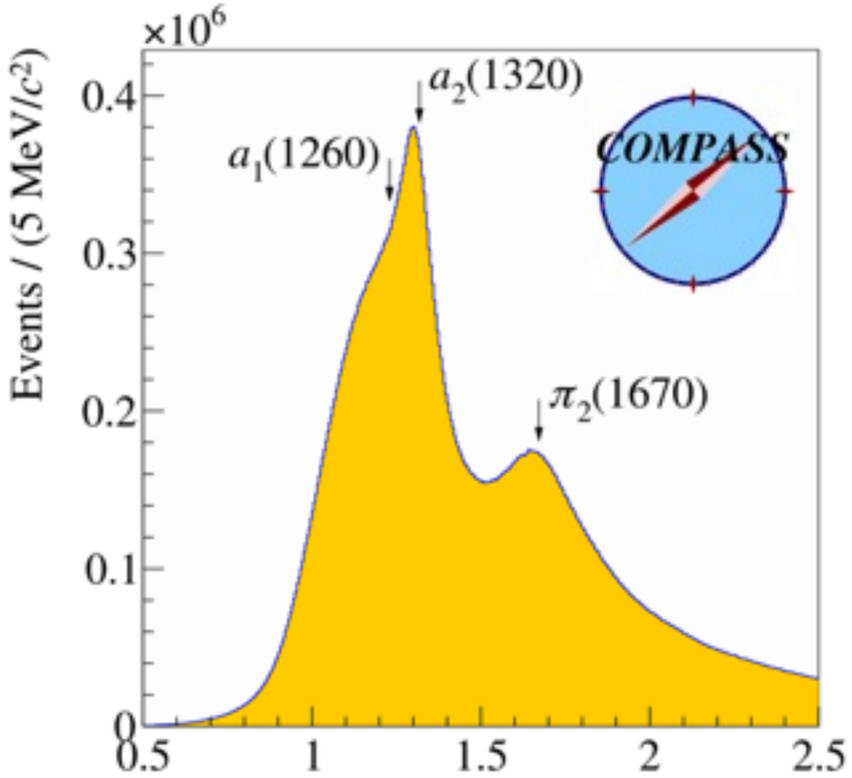
$\Longrightarrow$ Resonances are pole singularities in complex energy plane

$\Longrightarrow$ Need analytic representations for on-shell **3→3** amplitudes

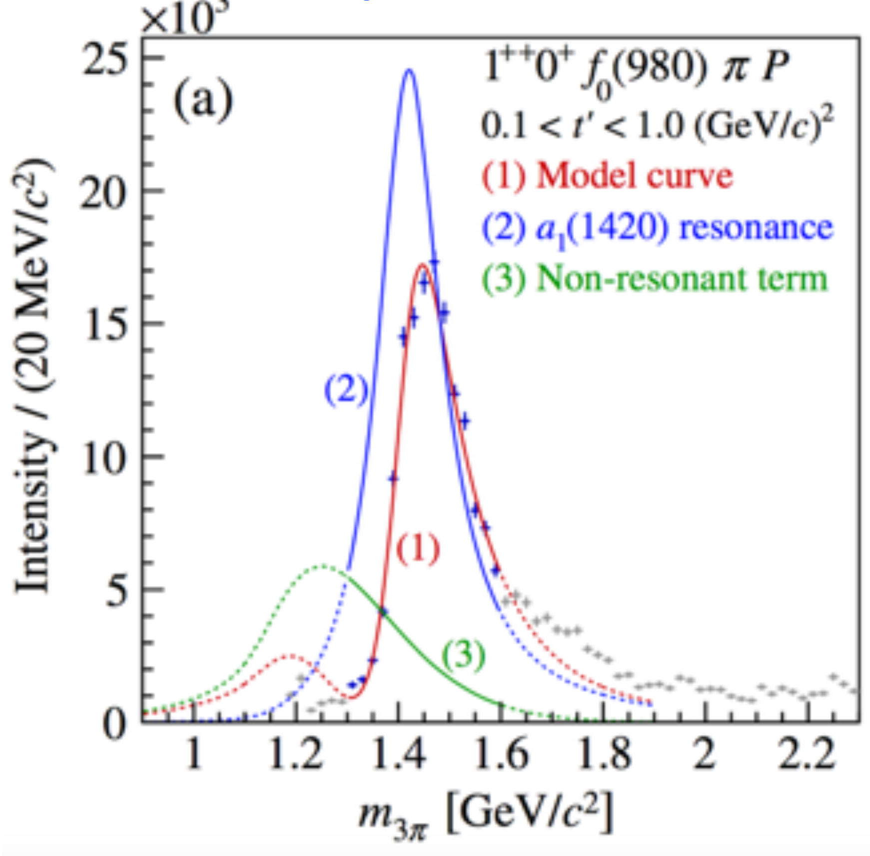
Surprises in light meson spectrum — $a_1(1420)$ in 3π decays

XYZ States — many lie near 3-body thresholds



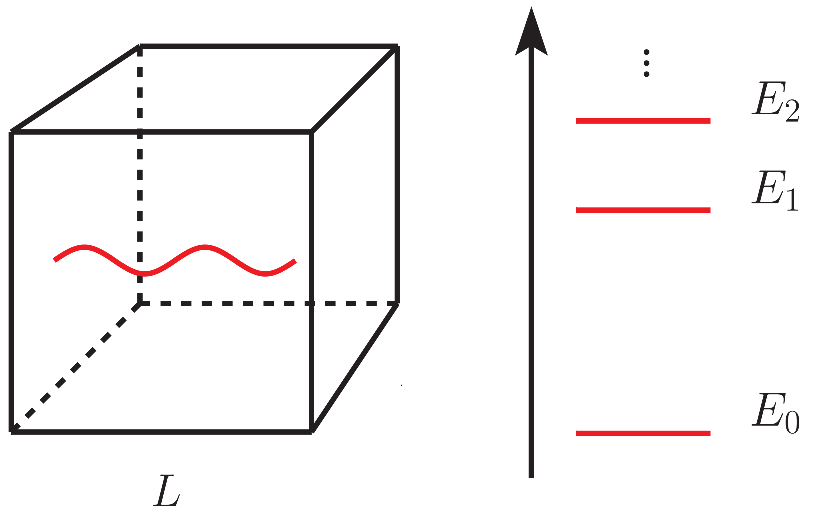


C. Adolph et al. [COMPASS], Phys. Rev. D 95, no. 3, 032004 (2017)



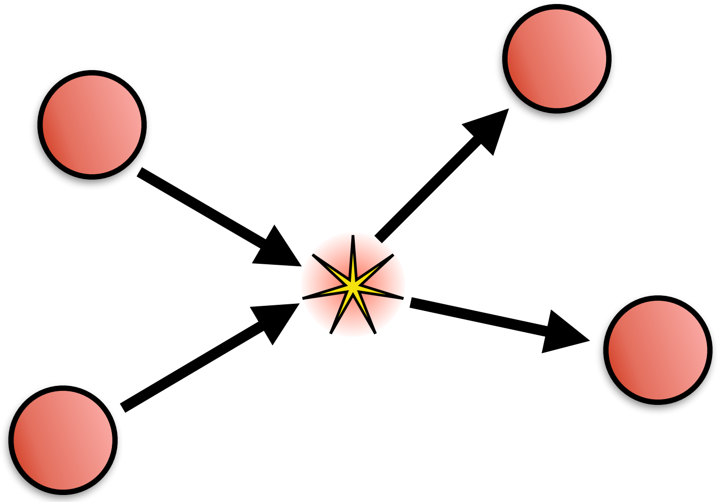
Why 3-Body Scattering?

Advancements in Lattice QCD in extracting hadron resonances



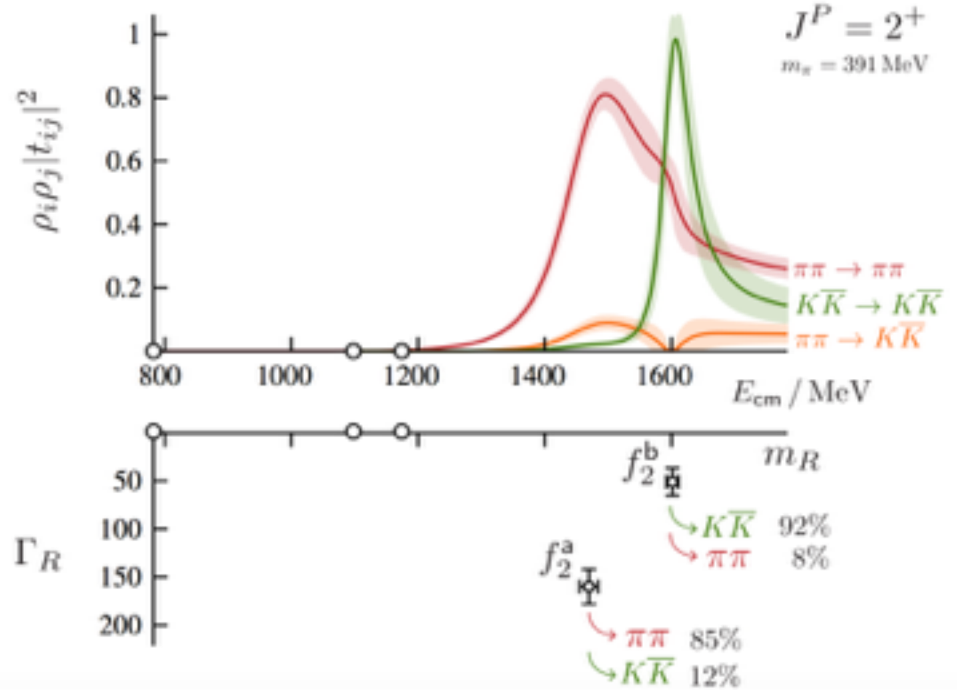
$\Longrightarrow$

$$\det\left[\mathcal{M}(E^{\star}) + F(E, \mathbf{P}; L)\right] = 0$$



Extensions to 3-body systems $\Longrightarrow$ Need Analytic on-shell **3→3** amplitudes

hadspec



R. A. Briceño, M. T. Hansen, and S. R. Sharpe,
Phys. Rev. D95, 074510 (2017)

H. W. Hammer, J. Y. Pang, and A. Rusetsky,
JHEP 09, 109 (2017)

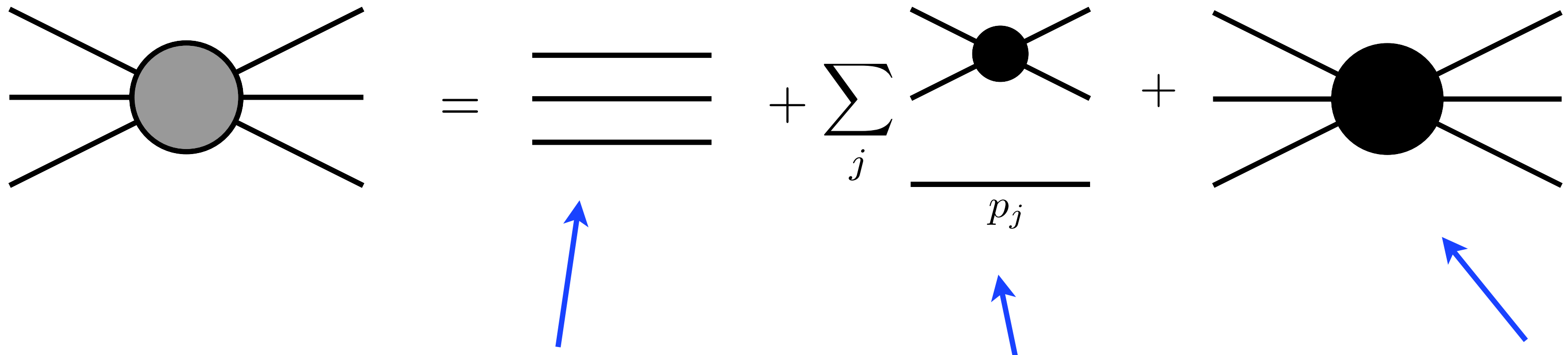
M. Mai and M. Döring,
Eur. Phys. J. A53, 240 (2017) $\Longrightarrow$ See Maxim Mai's talk

P. Guo and V. Gasparian,
Phys. Lett. B 774, 441 (2017) $\Longrightarrow$ See Peng Guo's talk

3→3 Phenomenology

Interested in on-shell amplitudes — Unitarity constrains amplitude $\sum_f \mathrm{Prob}_{i\to f} = 1 \implies S^\dagger S = S S^\dagger = \mathbb{1}$

Connectedness structure



Hadron d.o.f., e.g. $\pi, K, \ldots$

No interaction

Pair-wise interactions (disconnected term) $\mathcal{F}_{\mathbf{p}_j}$

Full 3-body interaction (connected term) $\mathcal{A}$

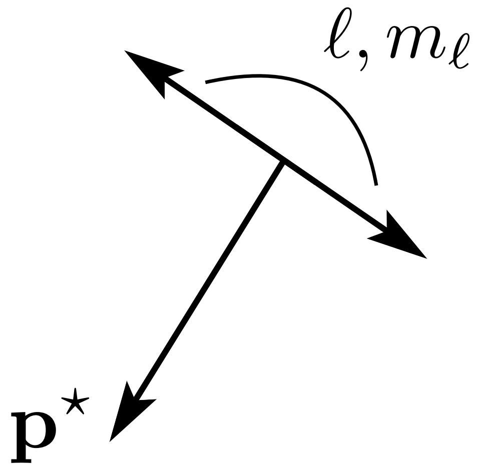
Introduce isobar representation for amplitude

$$\mathcal{A} = \mathcal{S}\left\{ 4\pi \sum_{\substack{\ell', m'_\ell \\ \ell, m_\ell}} Y_{\ell' m'_\ell}(\hat{\mathbf{a}}'^\star)\,[\,\mathcal{A}_{\mathbf{p}'\mathbf{p}}\,]_{\ell' m'_\ell;\ell m_\ell}\, Y^*_{\ell m_\ell}(\hat{\mathbf{a}}^\star) \right\}$$

Sum over all combos

Spectator momentum

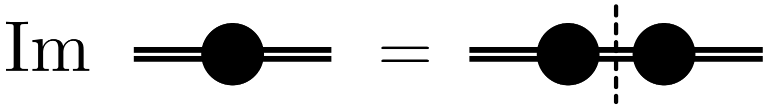
Isobar spin



3→3 Phenomenology

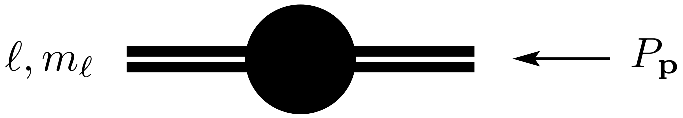
Unitarity for **2→2** on-shell amplitudes

$$\operatorname{Im}\mathcal{F}_{\mathbf{p}} = \mathcal{F}_{\mathbf{p}}^{\dagger}\,\rho_{\mathbf{p}}\,\mathcal{F}_{\mathbf{p}}$$



$$[\,\mathcal{F}_{\mathbf{p}}\,]_{\ell' m'_\ell;\ell m_\ell} = \delta_{\ell'\ell}\delta_{m'_\ell m_\ell}\,\mathcal{F}_\ell(\sigma_{\mathbf{p}})$$

$$=$$



K-matrix representation

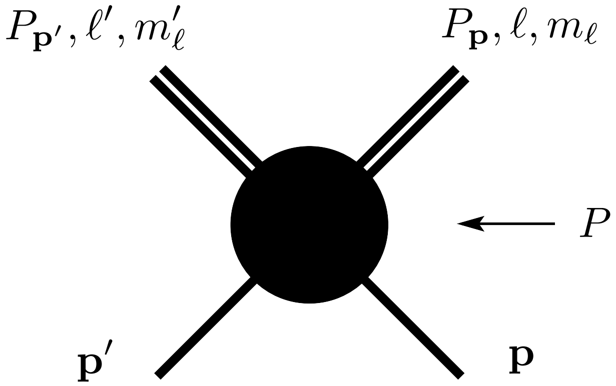
$$\Longrightarrow \quad \mathcal{F}_{\mathbf{p}} = \mathcal{K}_{\mathbf{p}} + \mathcal{K}_{\mathbf{p}}\,i\rho_{\mathbf{p}}\,\mathcal{F}_{\mathbf{p}}$$
$$= [\,1 - \mathcal{K}_{\mathbf{p}}\,i\rho_{\mathbf{p}}\,]^{-1}\,\mathcal{K}_{\mathbf{p}}$$

phase space for on-shell production

Short range physics

$$[\,\mathcal{A}_{\mathbf{p}'\mathbf{p}}\,]_{\ell' m'_\ell;\ell m_\ell} = \mathcal{A}_{\ell' m'_\ell;\ell m_\ell}(\mathbf{p}', s, \mathbf{p})$$

$$=$$



Fix K-matrix from data (e.g. experiment, lattice) or from some model/ EFT

Analytic parameterization for K-matrix

$\Longrightarrow$ Analytic amplitude

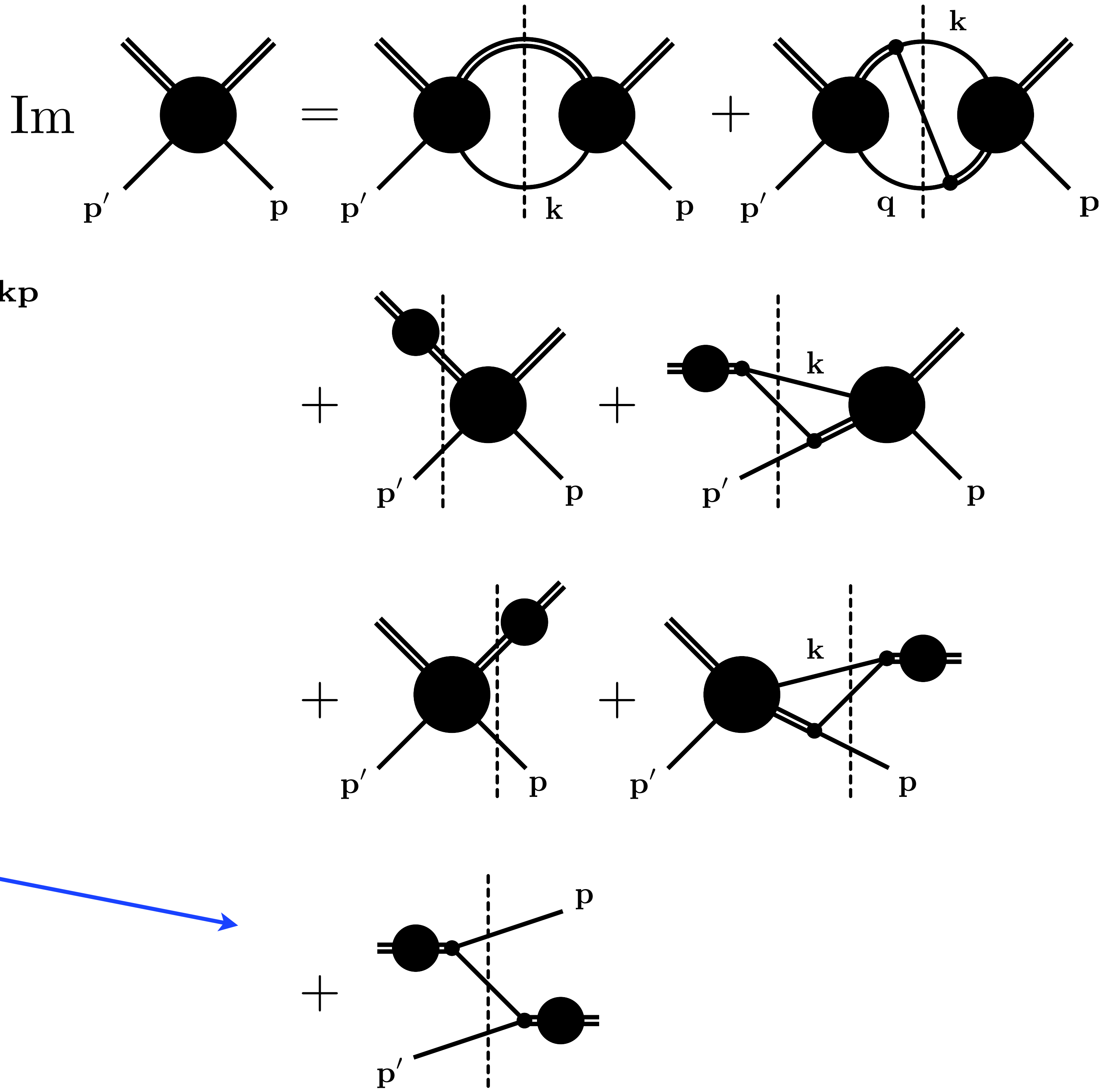
$\Longrightarrow$ Analytically continue in complex energy plane, search for singularities

3→3 Phenomenology

Given the **2→2** on-shell amplitudes, can now look at **3→3** amplitudes

Many new terms not in **2→2** case

$\Longrightarrow$ Includes rescattering effects



$$\begin{aligned}\mathrm{Im}\,\mathcal{A}_{\mathbf{p}'\mathbf{p}} = & \int_{\mathbf{k}} \mathcal{A}^{\dagger}_{\mathbf{p}'\mathbf{k}}\,\rho_{\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}} + \int_{\mathbf{k}}\int_{\mathbf{q}} \mathcal{A}^{\dagger}_{\mathbf{p}'\mathbf{q}}\,\mathcal{C}_{\mathbf{q}\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}} \\ & + \mathcal{F}^{\dagger}_{\mathbf{p}'}\,\rho_{\mathbf{p}'}\,\mathcal{A}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}^{\dagger}_{\mathbf{p}'}\,\mathcal{C}_{\mathbf{p}'\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}} \\ & + \mathcal{A}^{\dagger}_{\mathbf{p}'\mathbf{p}}\,\rho_{\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{A}^{\dagger}_{\mathbf{p}'\mathbf{k}}\,\mathcal{C}_{\mathbf{k}\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} \\ & + \mathcal{F}^{\dagger}_{\mathbf{p}'}\,\mathcal{C}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}}\end{aligned}$$

Recoupling coefficient

Generate One-Particle Exchange (OPE)

$$[\,\mathcal{C}_{\mathbf{p}'\mathbf{p}}\,]_{\ell' m_\ell' ; \ell m_\ell} = \pi\,\delta\left((P_{\mathbf{p}} - p')^2 - m^2\right) 4\pi\, Y^{*}_{\ell' m_\ell'}(\hat{\mathbf{p}}^{\star}) Y_{\ell m_\ell}(\hat{\mathbf{p}}'^{\star})$$

3→3 Phenomenology

Use constraint to find on-shell representation of **3→3** amplitude

Postulate integral equation — *B*-matrix equation

$$\mathrm{Im}\,\mathcal{F}_{\mathbf{p}} = \mathcal{F}^{\dagger}_{\mathbf{p}}\,\rho_{\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} \quad\Longrightarrow\quad \mathcal{F}_{\mathbf{p}} = \mathcal{K}_{\mathbf{p}} + \mathcal{K}_{\mathbf{p}}\,i\rho_{\mathbf{p}}\,\mathcal{F}_{\mathbf{p}}$$

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}}$$

$$\int_{\mathbf{k}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}$$

B-Matrix contains two terms $\mathcal{B}_{\mathbf{p}'\mathbf{p}} = \mathcal{R}_{\mathbf{p}'\mathbf{p}} + \mathcal{G}_{\mathbf{p}'\mathbf{p}}$

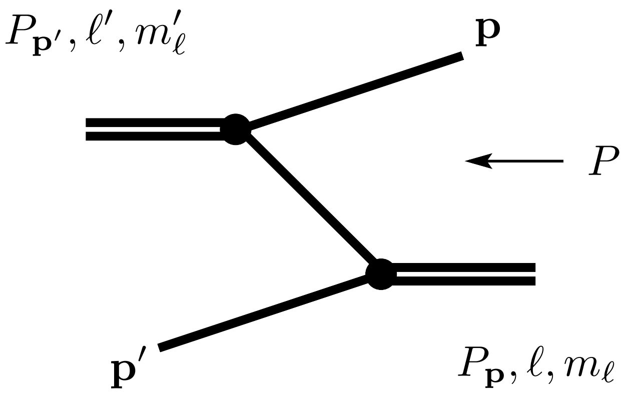
One-Particle Exchange (OPE)

Short range 3-body dynamics

Can parameterize like K-matrix,
fit to data

$$[\,\mathcal{G}_{\mathbf{p}'\mathbf{p}}\,]_{\ell' m_\ell';\ell m_\ell} = -\frac{4\pi\,Y^{*}_{\ell' m_\ell'}(\hat{\mathbf{p}}^{\star})Y_{\ell m_\ell}(\hat{\mathbf{p}}'^{\star})}{(P_{\mathbf{p}} - p')^2 - m^2 + i\epsilon}$$

$=$



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak,
Eur. Phys. J. A53, 177 (2017)

$\Longrightarrow$ Full integration range

AJ et al. [JPAC],
Eur. Phys. J. C 79, no. 1, 56 (2019)

$\Longrightarrow$ Cut off to physical region

Aspects of Analytic Properties

Resonances have poles in total partial wave amplitudes

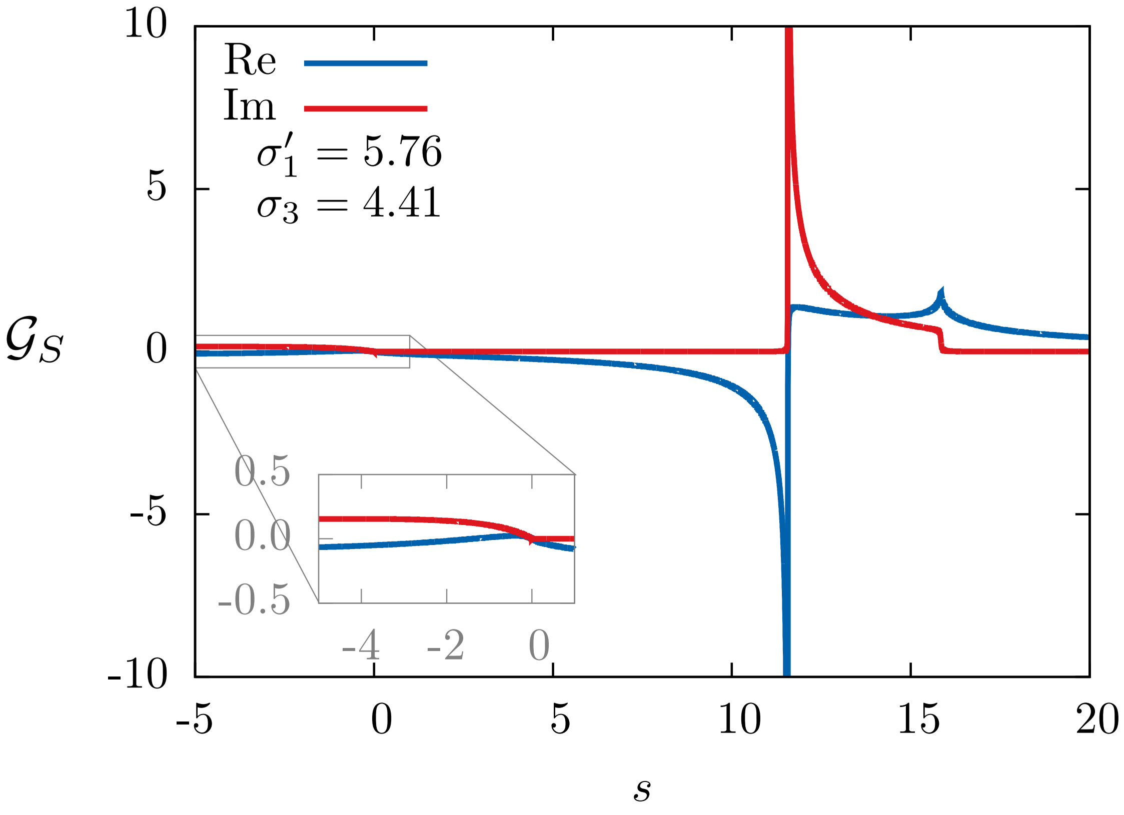
$$[\mathcal{A}_{\mathbf{p}'\mathbf{p}}]_{\ell' m_\ell' ; \ell m_\ell} = \sum_{J,M} (2J+1) D^{(J)\,*}_{M m_\ell'}(\hat{\mathbf{p}}')\, \mathcal{A}^{J}_{\ell' m_\ell' ; \ell m_\ell}(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}})\, D^{(J)}_{M m_\ell}(\hat{\mathbf{p}})$$

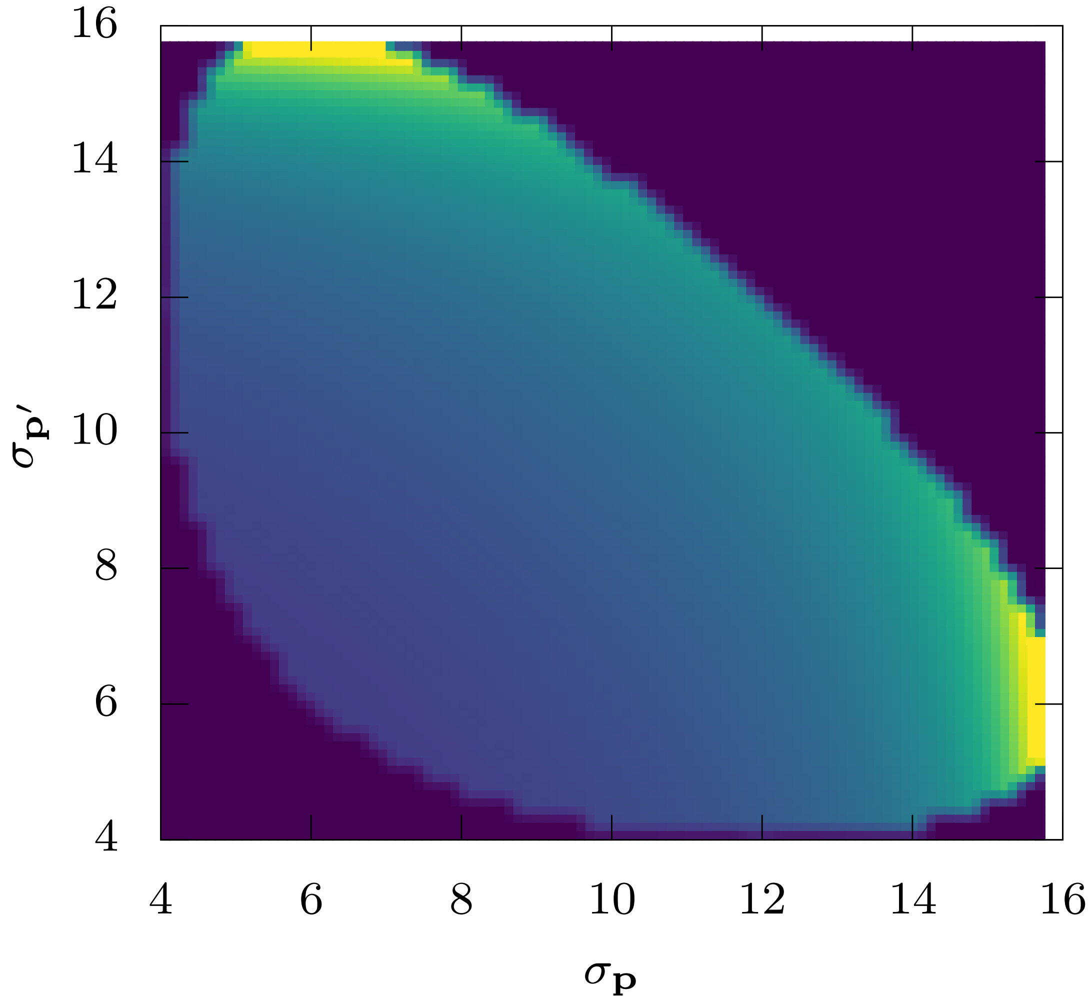
First look at PW projection of OPE — *S*-Wave Quantum Numbers

$$\mathcal{G}_S(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) = \frac{1}{4|\mathbf{p}'||\mathbf{p}|} \log\left(\frac{z_0(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) - 1}{z_0(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) + 1} \right)$$

Physically possible for exchanged particle to go on-shell

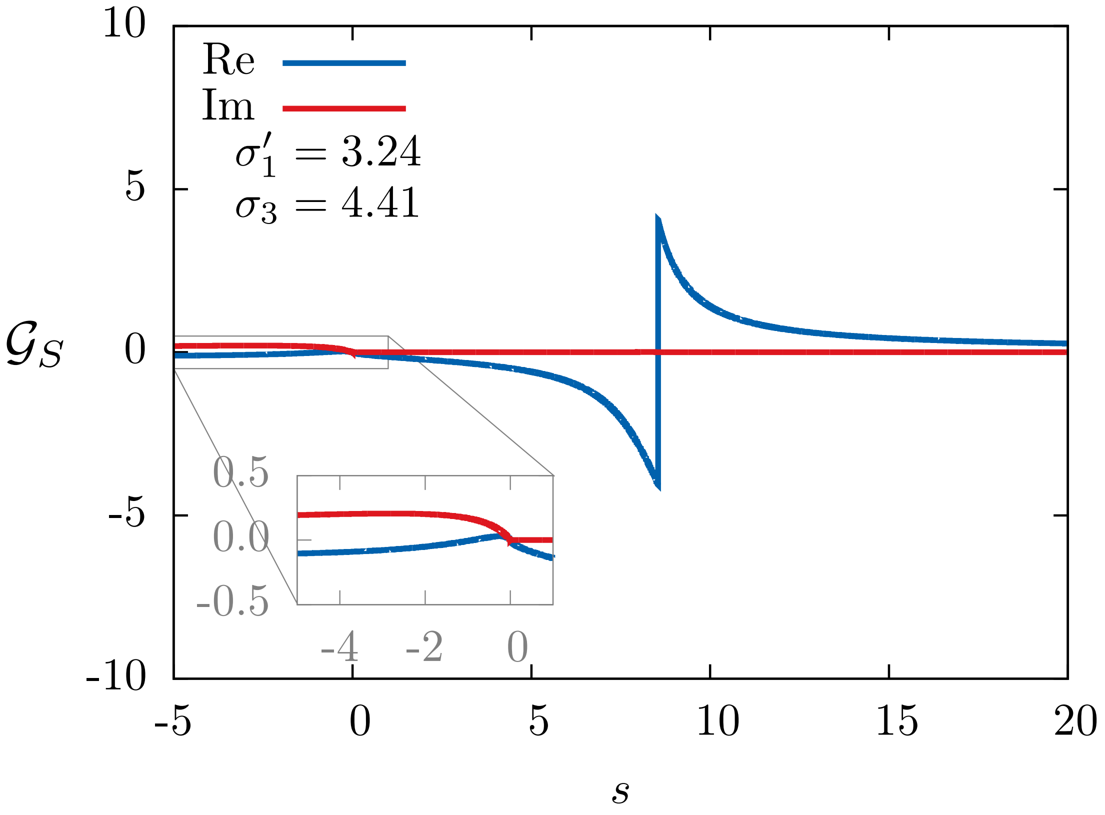
on-shell particles $\Longrightarrow$ singularities in physical region

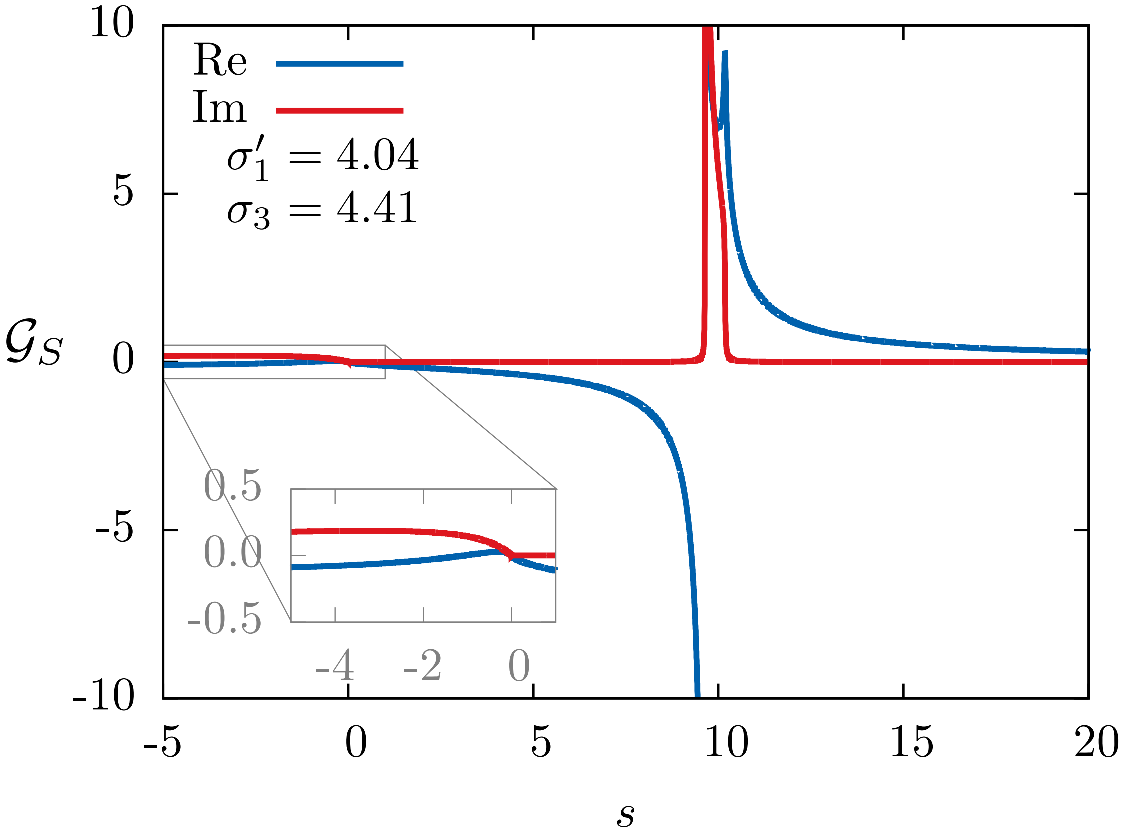


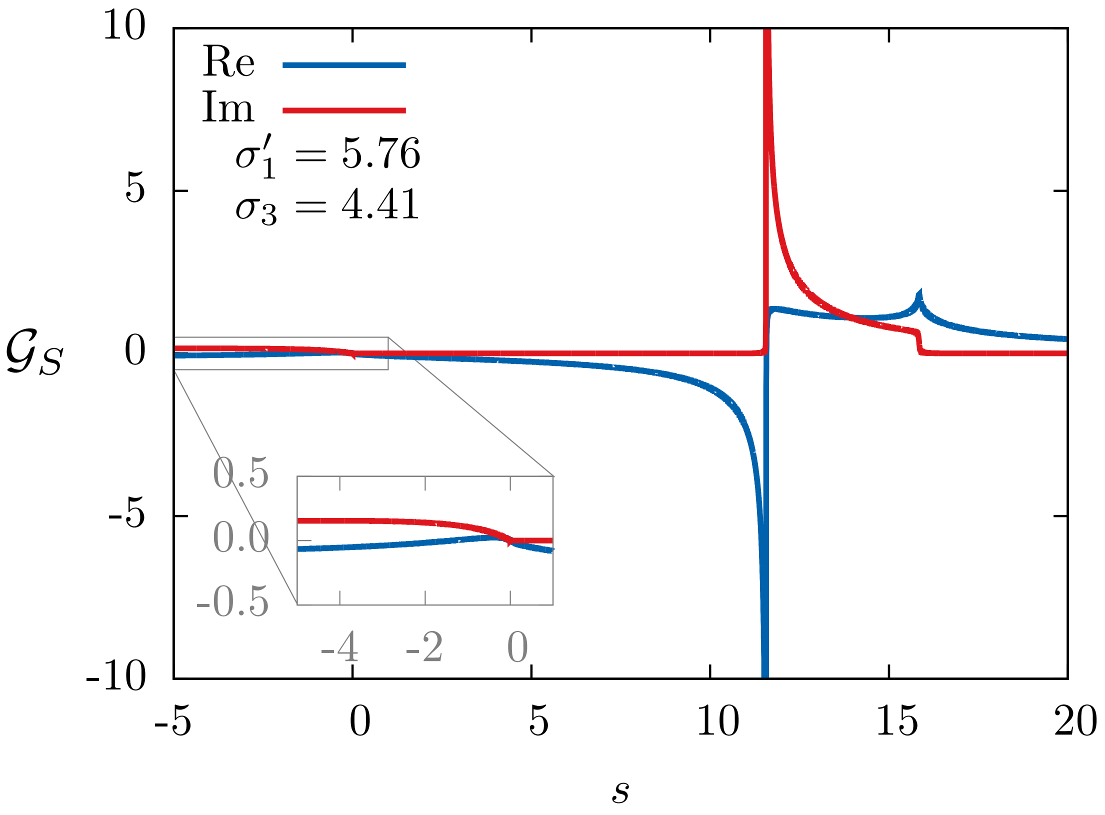


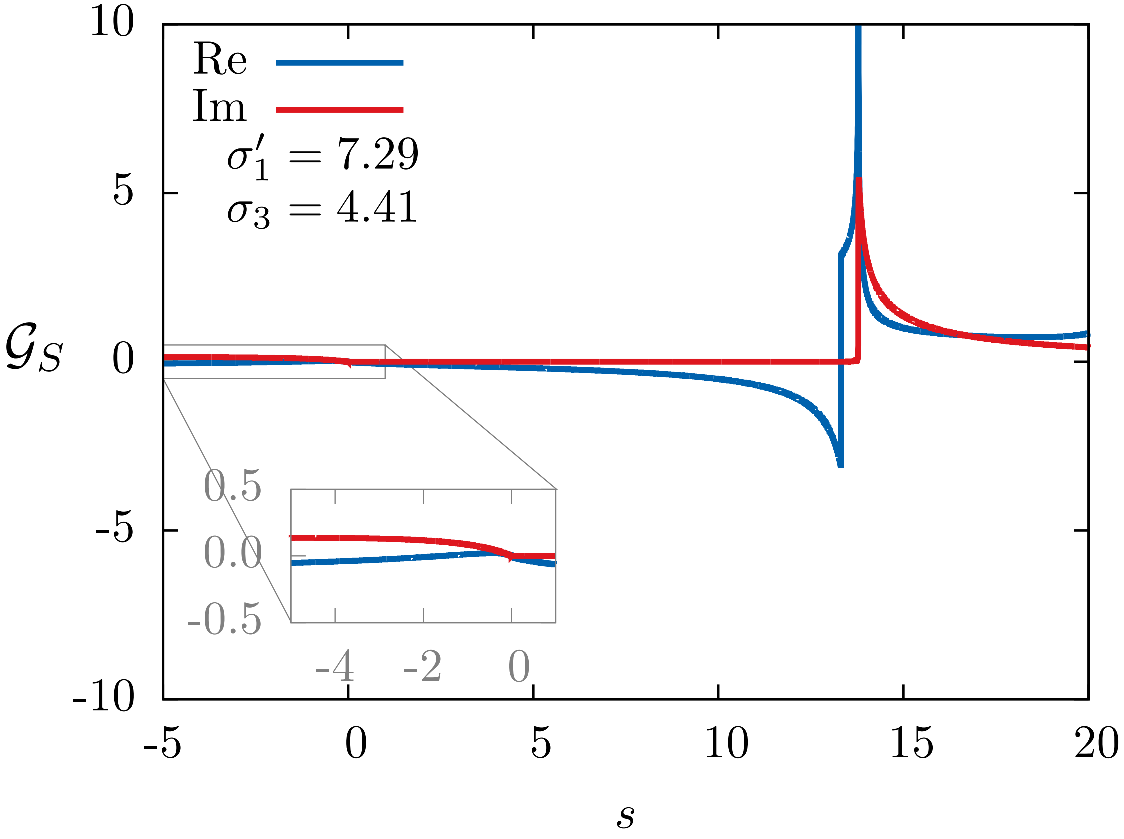
Aspects of Analytic Properties

AJ et al. [JPAC], Eur. Phys. J. C 79, no. 1, 56 (2019)









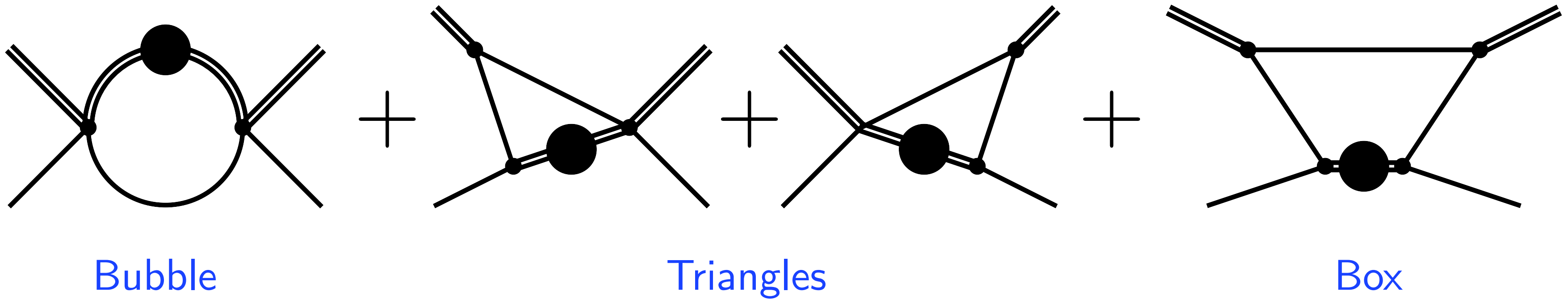
Aspects of Analytic Properties

Can write B-matrix in terms of other kernels

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{k}}\,\mathcal{F}_{\mathbf{k}}\,\mathcal{B}_{\mathbf{k}\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{k}'}\,\mathcal{F}_{\mathbf{k}'}\,\mathcal{B}_{\mathbf{k}'\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}}$$

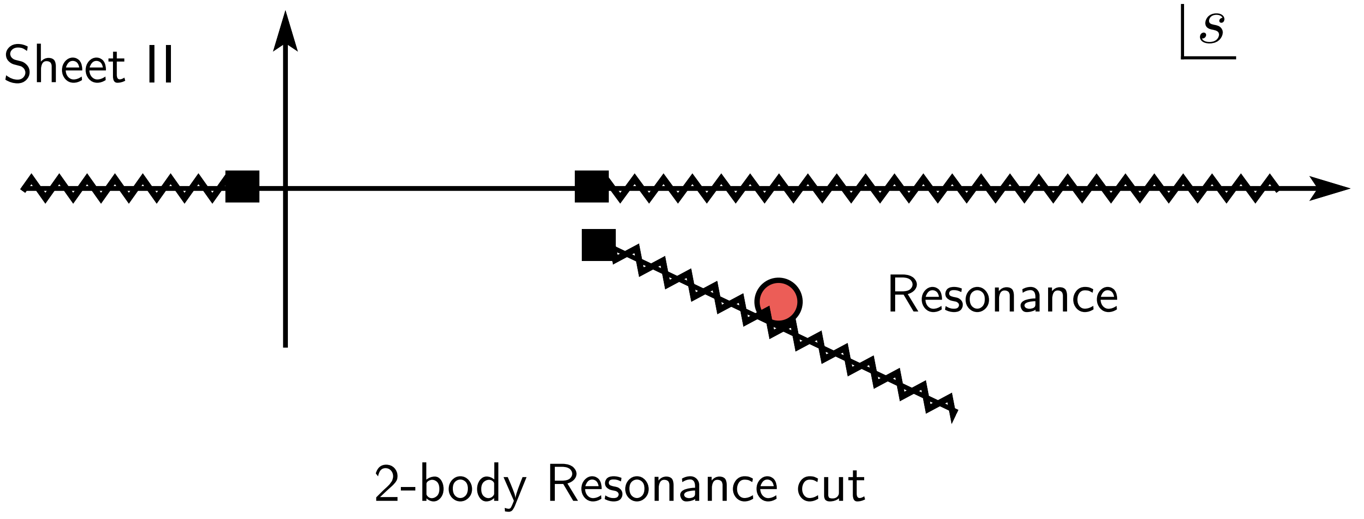
Contains rescattering physics

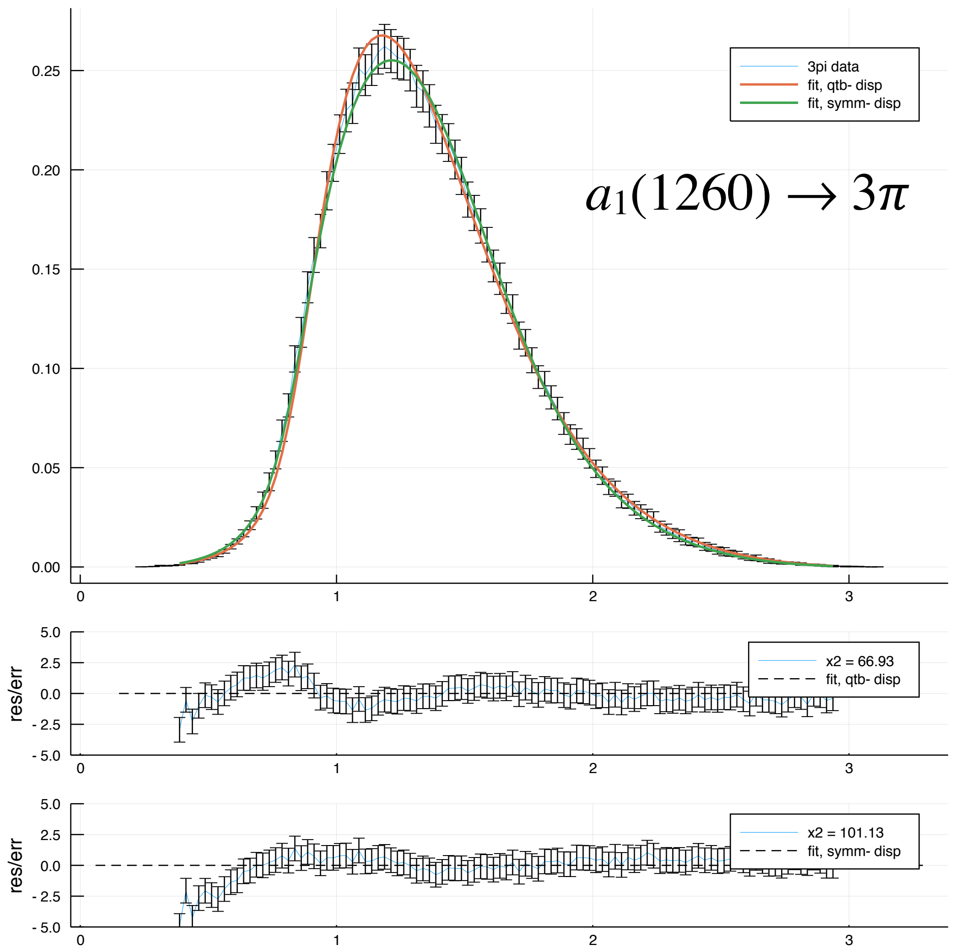
$$\int_{\mathbf{k}} \mathcal{B}_{\mathbf{p}'\mathbf{k}}\,\mathcal{F}_{\mathbf{k}}\,\mathcal{B}_{\mathbf{k}\mathbf{p}} =$$



Bubble unitarization gives quasi-two-body phase space

M. Mikhasenko et al. [JPAC],
Phys. Rev. D 98, no. 9, 096021 (2018)



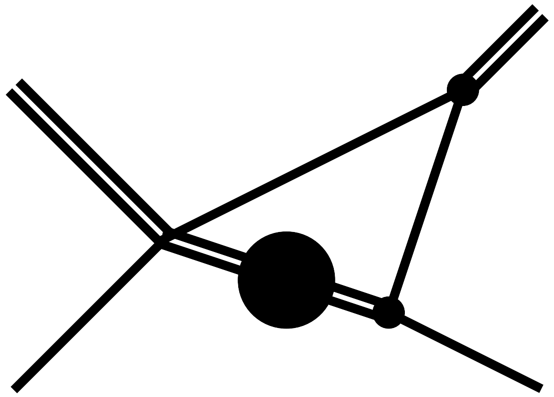


Aspects of Analytic Properties

Look at triangle amplitude in kernel with constant *R*-matrix $\mathcal{R} = 1$

S-wave — assume narrow width approx. for isobar

$$\mathcal{T}_B(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) =$$



Phase space between spectator and isobar

OPE

$$\mathcal{T}_B(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) = \int_{4m^2}^{(\sqrt{s}-m)^2} \mathrm{d}\sigma \, \frac{\tau(s,\sigma)\, \mathcal{G}_S(\sigma, s, \sigma_{\mathbf{p}})}{M^2 - \sigma - i\epsilon}$$

Isobar amplitude

Integration over physical region

Compare analytic structure with Mai et al. triangle
and with Feynman triangle

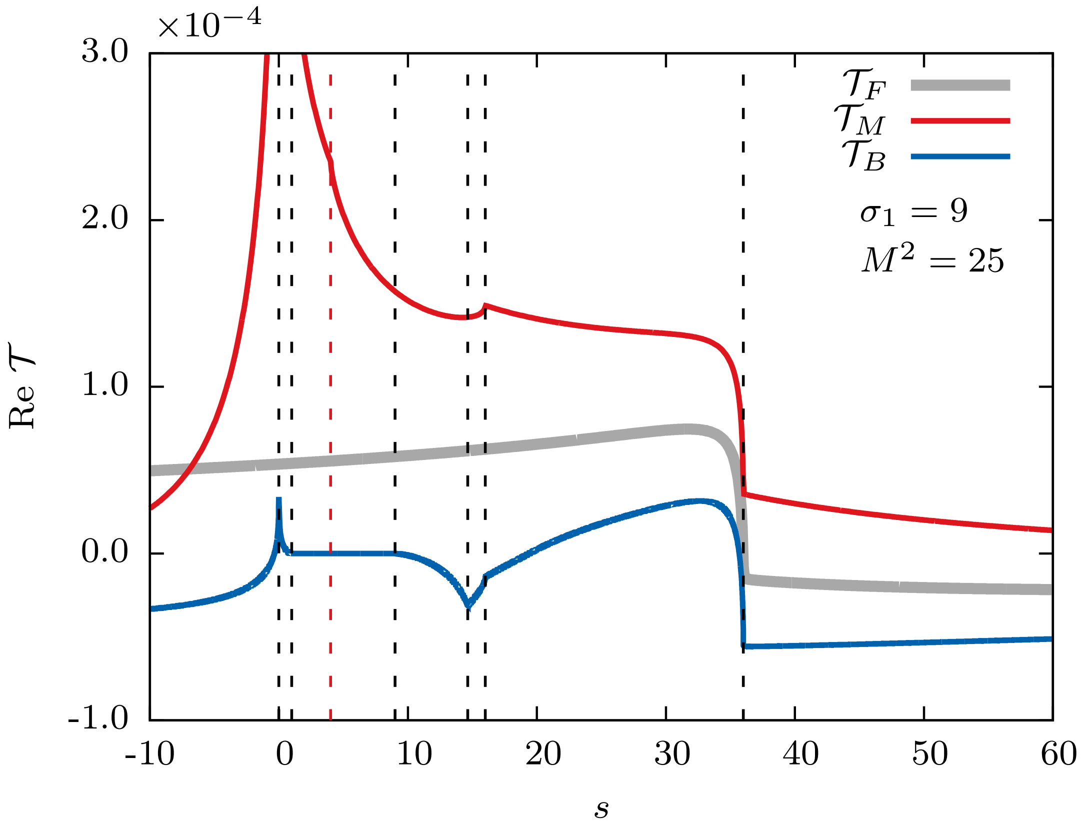
$$\mathcal{T}_M = \int_{-\infty}^{(\sqrt{s}-m)^2} \mathrm{d}\sigma \, \frac{\tau(s,\sigma)\, \mathcal{G}_S(\sigma, s, \sigma_{\mathbf{p}})}{M^2 - \sigma - i\epsilon}$$

Includes off-shell physics

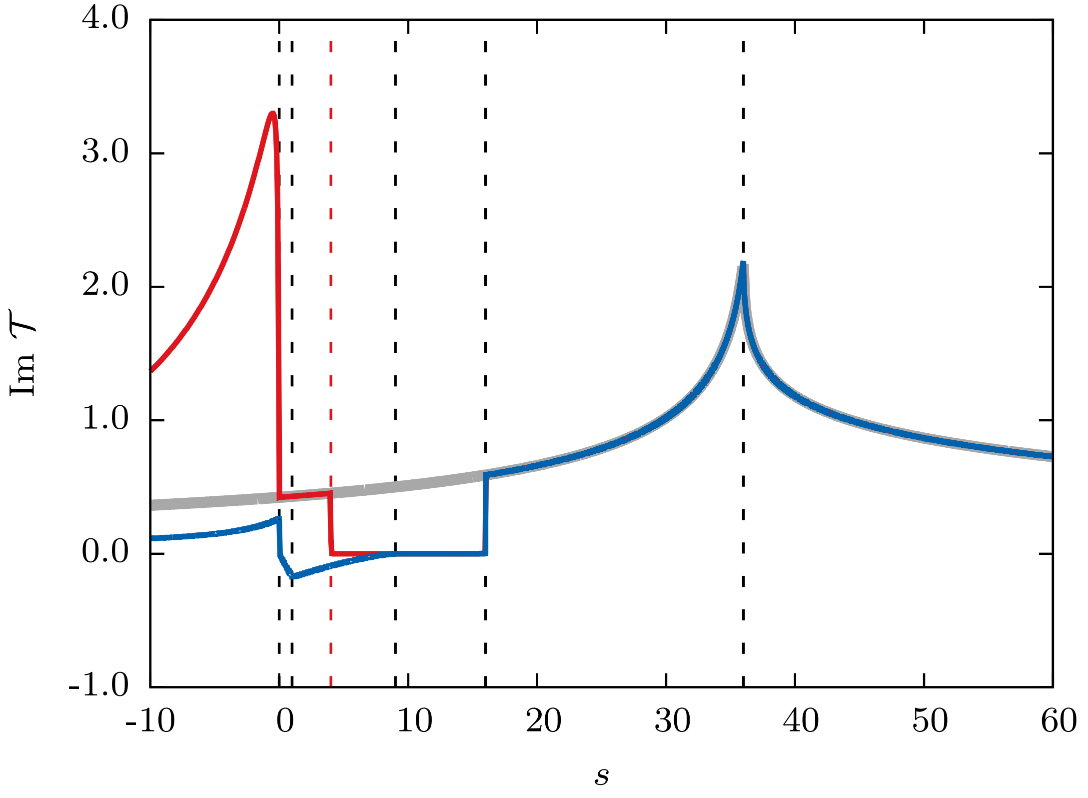
$$\mathcal{T}_F = \int \mathrm{d}s' \, \frac{\tau(s, M^2)\, \mathcal{G}_S(M^2, s, \sigma_{\mathbf{p}})}{s' - s - i\epsilon}$$

Aspects of Analytic Properties

AJ et al. [JPAC], Eur. Phys. J. C 79, no. 1, 56 (2019)



Real parts differ everywhere



Imaginary parts identical above production threshold

Aitchison & Pasquier resolve the differences to Feynman — Different *R*-matrices!

I. J. R. Aitchison and R. Pasquier, Phys. Rev. 152, 1274 (1966)

All are unitary amplitudes, by construction

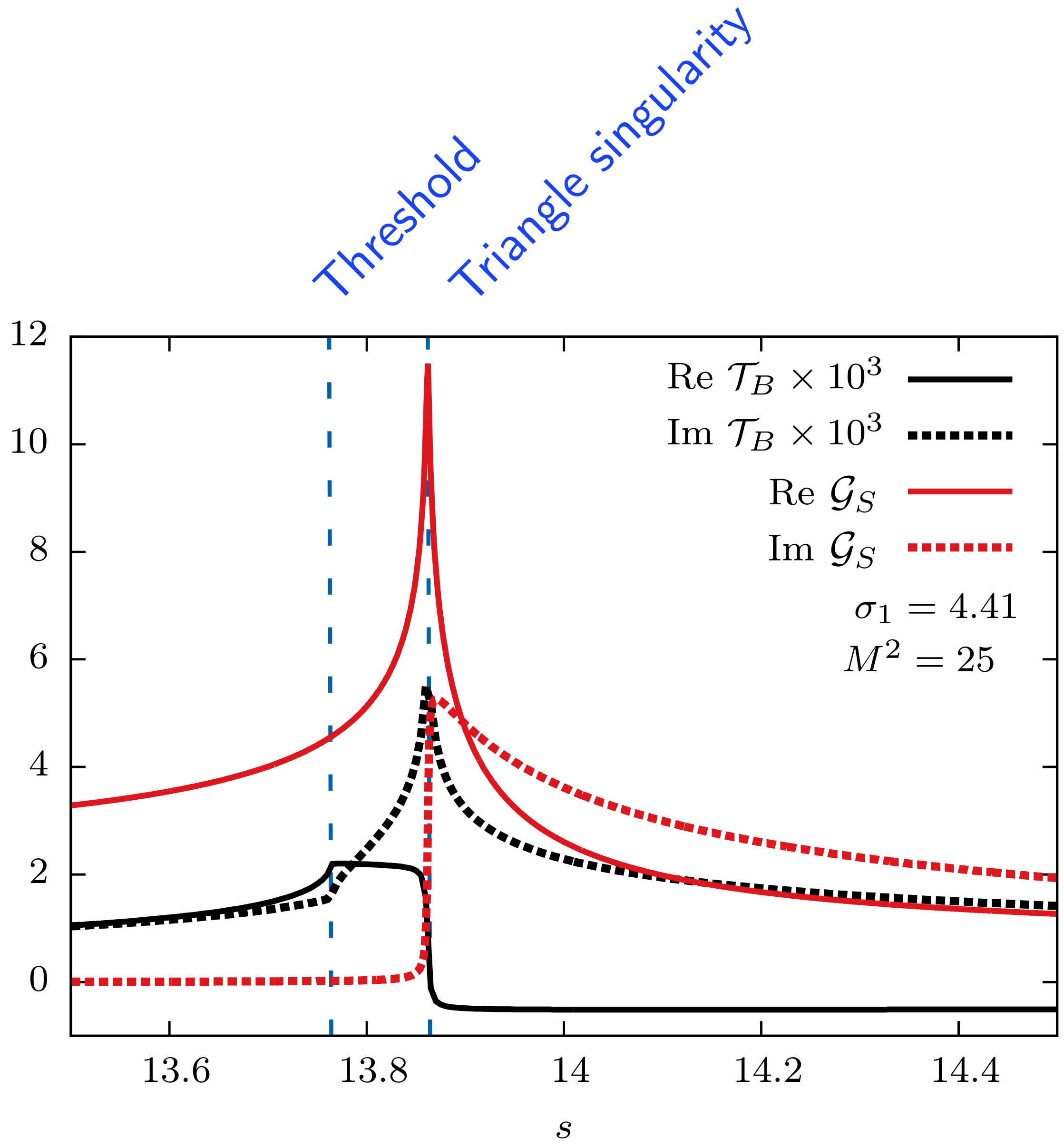
Some below threshold singularities removed via finite width isobar shape — Better analyticity constraints?

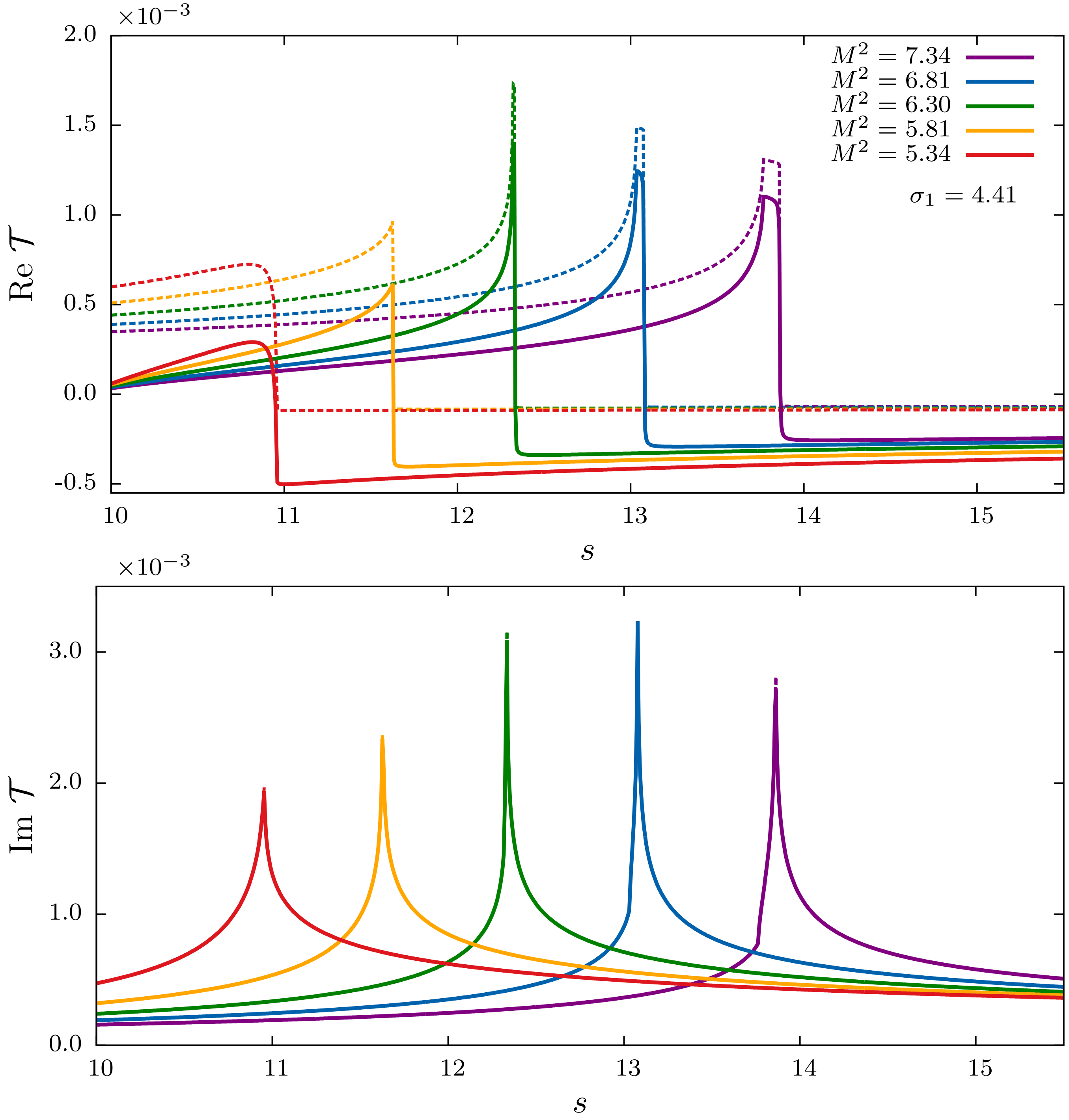
Aspects of Analytic Properties

Triangle amplitude develops singularity in physical region $\Longrightarrow$ OPE becomes physical

Rescattering effect — can mimic resonance signatures

$\Longrightarrow$ Explanation for XYZ states?





Comparison with Lattice Formalism

Have seen can use different *B*-matrix representations (Mai et al.)

$\Longrightarrow$ known differences is what we put into real part

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}} \qquad \mathcal{B}_{\mathbf{p}'\mathbf{p}} = \mathcal{R}_{\mathbf{p}'\mathbf{p}} + \mathcal{G}_{\mathbf{p}'\mathbf{p}}$$

Lattice formalism by Briceño, Hansen, and Sharpe (BHS) yield different infinite volume amplitudes in $L\to\infty$ limit

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{L}_{\mathbf{p}'\mathbf{k}'}\,\mathcal{T}(\mathbf{k}',\mathbf{k})\,\mathcal{L}^{\top}_{\mathbf{k}'\mathbf{p}}$$

3-body *K*-matrix

$$\mathcal{T}(\mathbf{p}',\mathbf{p}) = \mathcal{K}(\mathbf{p}',\mathbf{p}) + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{K}(\mathbf{p}',\mathbf{k}')\,i\rho_{\mathbf{k}'}\,\mathcal{L}_{\mathbf{k}'\mathbf{k}}\,\mathcal{T}(\mathbf{k},\mathbf{p})$$

Ladder series

$$\mathcal{D}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{k}}\,\mathcal{D}_{\mathbf{k}\mathbf{p}}$$

$$\mathcal{L}_{\mathbf{p}'\mathbf{p}} = \left(\frac{1}{3} + \mathcal{F}_{\mathbf{p}'}\,i\rho_{\mathbf{p}'}\right)\delta_{\mathbf{p}'\mathbf{p}} + \mathcal{D}_{\mathbf{p}'\mathbf{p}}\,i\rho_{\mathbf{p}}.$$

R. A. Briceño, M. T. Hansen, and S. R. Sharpe, Phys. Rev. D95, 074510 (2017)

Are these equivalent? — YES!

Comparison with Lattice Formalism

Can check equivalency

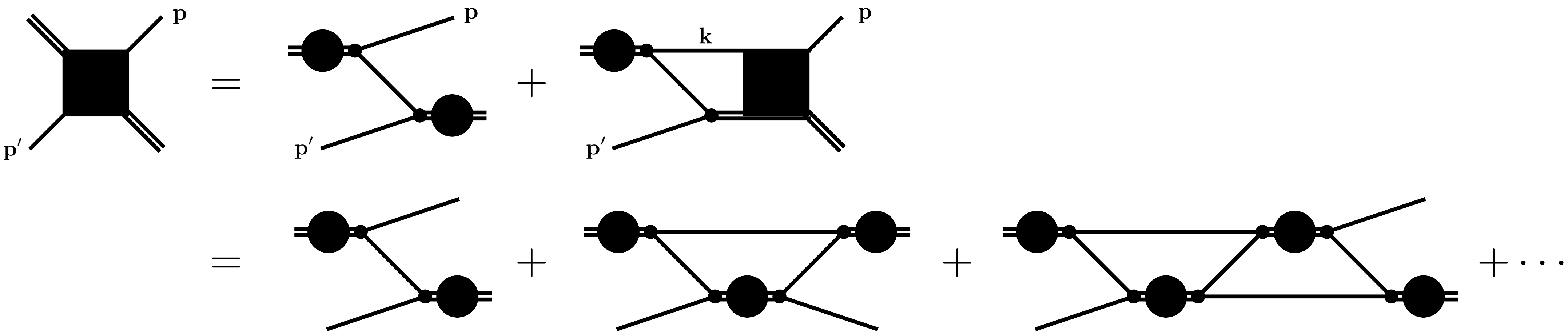
AJ et al. [JPAC], In Preparation

First consider $\mathcal{R}_{\mathbf{p}'\mathbf{p}} = 0$ limit

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{B}_{\mathbf{p}'\mathbf{k}}\,\mathcal{A}_{\mathbf{k}\mathbf{p}}$$

$$\rightarrow \qquad \mathcal{D}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{k}}\,\mathcal{D}_{\mathbf{k}\mathbf{p}}$$

Ladder series



Same Ladder series as BHS

Comparison with Lattice Formalism

Then, yield relation

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{L}'_{\mathbf{p}'\mathbf{k}'}\,\mathcal{T}'_{\mathbf{k}'\mathbf{k}}\,\mathcal{L}'_{\mathbf{k}\mathbf{p}}$$

$$\mathcal{T}'_{\mathbf{p}'\mathbf{p}} = \mathcal{R}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{R}_{\mathbf{p}'\mathbf{k}}\,\mathcal{L}'_{\mathbf{k}'\mathbf{k}}\,\mathcal{T}'_{\mathbf{k}\mathbf{p}}$$

$$\mathcal{L}'_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}}\,\delta_{\mathbf{p}'\mathbf{p}} + \mathcal{D}_{\mathbf{p}'\mathbf{p}}$$

BHS

$$\mathcal{A}_{\mathbf{p}'\mathbf{p}} = \mathcal{D}_{\mathbf{p}'\mathbf{p}} + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{L}_{\mathbf{p}'\mathbf{k}'}\,\mathcal{T}(\mathbf{k}',\mathbf{k})\,\mathcal{L}^{\top}_{\mathbf{k}'\mathbf{p}}$$

$$\mathcal{T}(\mathbf{p}',\mathbf{p}) = \mathcal{K}(\mathbf{p}',\mathbf{p}) + \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{K}(\mathbf{p}',\mathbf{k}')\,i\rho_{\mathbf{k}'}\,\mathcal{L}_{\mathbf{k}'\mathbf{k}}\,\mathcal{T}(\mathbf{k},\mathbf{p})$$

$$\mathcal{L}_{\mathbf{p}'\mathbf{p}} = \left(\frac{1}{3} + \mathcal{F}_{\mathbf{p}'}\,i\rho_{\mathbf{p}'}\right)\delta_{\mathbf{p}'\mathbf{p}} + \mathcal{D}_{\mathbf{p}'\mathbf{p}}\,i\rho_{\mathbf{p}}.$$

Finally, there is a relation between *R*-matrix and *K*-matrix

$$\mathcal{R}_{\mathbf{p}'\mathbf{p}} = \int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\,\mathcal{K}(\mathbf{k}',\mathbf{k})\,\mathcal{U}_{\mathbf{k}\mathbf{p}} - \frac{1}{3}\int_{\mathbf{k}'}\int_{\mathbf{k}} \mathcal{U}_{\mathbf{p}'\mathbf{k}'}\,\mathcal{K}(\mathbf{k}',\mathbf{k})\,\mathcal{R}_{\mathbf{k}\mathbf{p}}$$

Relationship is real, both representations are equivalent

$$\begin{aligned}\mathcal{U}_{\mathbf{p}'\mathbf{p}} &= i\rho_{\mathbf{p}'}\,\delta_{\mathbf{p}'\mathbf{p}} + \frac{1}{3}\mathcal{L}'^{\,-1}_{\mathbf{p}'\mathbf{p}} \\ &= i\rho_{\mathbf{p}'}\,\delta_{\mathbf{p}'\mathbf{p}} + \frac{1}{3}\mathcal{F}^{-1}_{\mathbf{p}'}\delta_{\mathbf{p}'\mathbf{p}} - \frac{1}{3}\mathcal{G}_{\mathbf{p}'\mathbf{p}}\end{aligned}$$

AJ et al. [JPAC],
In Preparation

Summary

Have constructed an on-shell representation for **3→3** amplitude

- Satisfies elastic three-body unitarity
- Integral equation in terms of OPE and short-range dynamics

Investigated some of its analytic properties

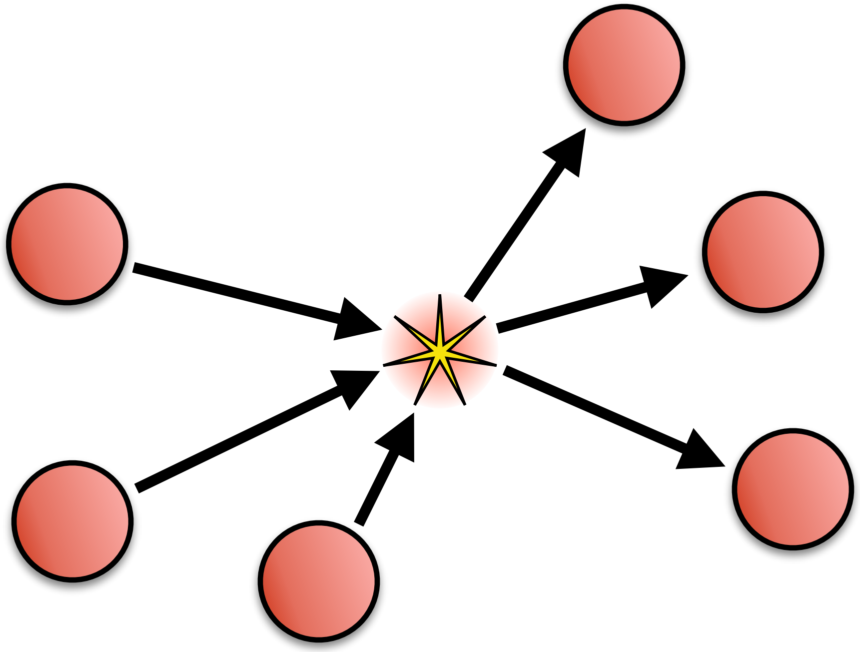
- OPE singularities in physical region — Triangle singularities

All formalisms on the market appear to be equivalent — Is there an alternative representation?

Future directions:

- Can OPE generate resonance signatures?
- How to systematically continue to complex plane?
- What singularities does the box diagram contribute?
- Synthesis with other techniques?
- External particles with spin?
- Numerical infrastructure for computing amplitudes — Many integral equations with singularities
- ...

Back up

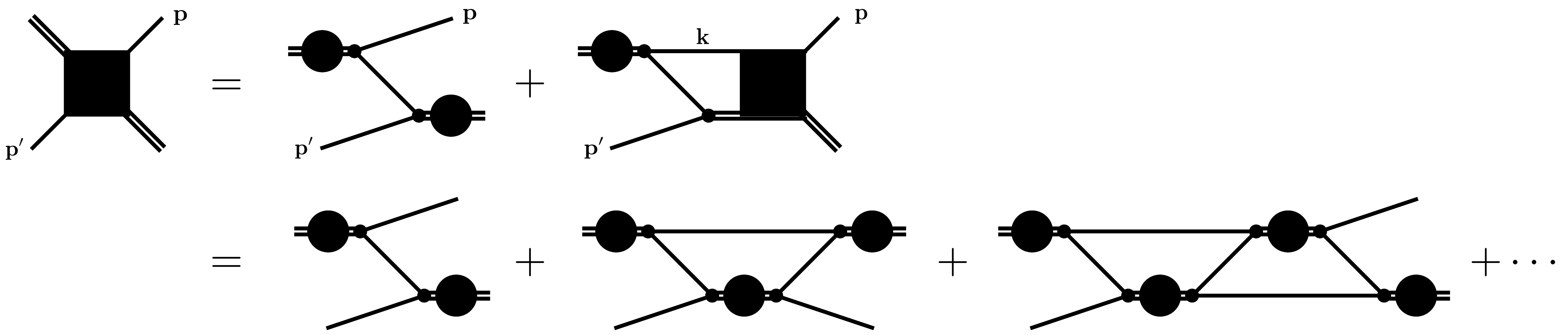


Ladder Series

$$\mathcal{D}_{\mathbf{p}'\mathbf{p}} = \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{p}}\,\mathcal{F}_{\mathbf{p}} + \int_{\mathbf{k}} \mathcal{F}_{\mathbf{p}'}\,\mathcal{G}_{\mathbf{p}'\mathbf{k}}\,\mathcal{D}_{\mathbf{k}\mathbf{p}}$$

$$\begin{aligned}\mathcal{D}_{\ell' m_\ell';\ell m_\ell}(\mathbf{p}', s, \mathbf{p}) &= \mathcal{F}_{\ell'}(\sigma_{\mathbf{p}'})\,\mathcal{G}_{\ell' m_\ell';\ell m_\ell}(\mathbf{p}', s, \mathbf{p})\mathcal{F}_{\ell}(\sigma_{\mathbf{p}}) \\ &+ \sum_{\ell'', m_\ell''} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3\, 2\omega(\mathbf{k})}\,\mathcal{F}_{\ell'}(\sigma_{\mathbf{p}'})\,\mathcal{G}_{\ell' m_\ell';\ell'' m_\ell''}(\mathbf{p}', s, \mathbf{k})\,\mathcal{D}_{\ell'' m_\ell'';\ell m_\ell}(\mathbf{k}, s, \mathbf{p})\end{aligned}$$

$$\begin{aligned}\mathcal{D}^J_{\ell' m_\ell';\ell m_\ell}(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}}) &= \mathcal{F}_{\ell'}(\sigma_{\mathbf{p}'})\,\mathcal{G}^J_{\ell' m_\ell';\ell m_\ell}(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{p}})\mathcal{F}_{\ell}(\sigma_{\mathbf{p}}) \\ &+ \sum_{\ell'', m_\ell''} \int \frac{\mathrm{d}\sigma_{\mathbf{k}}}{2\pi}\,\mathcal{F}_{\ell'}(\sigma_{\mathbf{p}'})\,\mathcal{G}^J_{\ell' m_\ell';\ell'' m_\ell''}(\sigma_{\mathbf{p}'}, s, \sigma_{\mathbf{k}})\,\tau(s, \sigma_{\mathbf{k}})\,\mathcal{D}^J_{\ell'' m_\ell'';\ell m_\ell}(\sigma_{\mathbf{k}}, s, \sigma_{\mathbf{p}})\end{aligned}$$

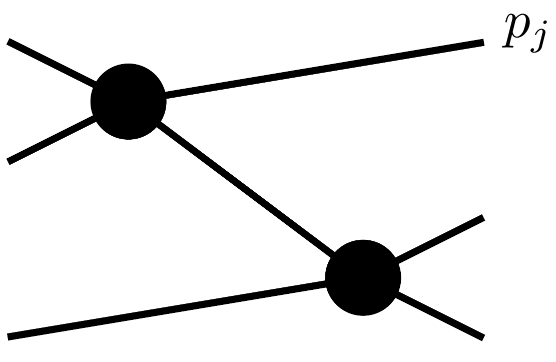


One-Particle Exchange

An important feature of **3→3** scattering is the one-particle exchange (OPE)

Physically possible for exchanged particle to go on-shell

on-shell particles $\Longrightarrow$ singularities in physical region $\mathcal{E}_{kj}(\mathbf{p}';\mathbf{p}) =$



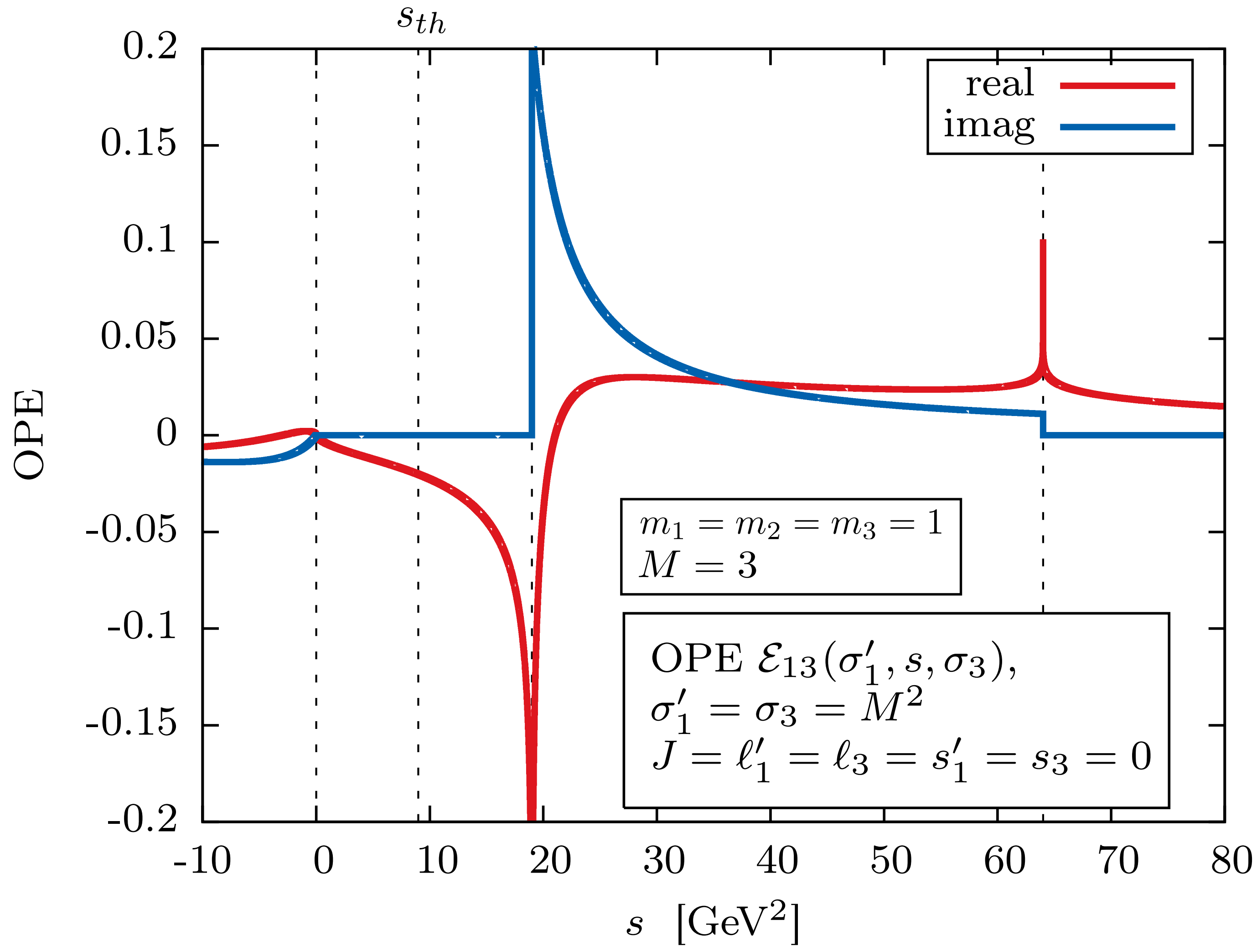
$$\mathcal{E}_{kj}(\mathbf{p}';\mathbf{p}) = \mathcal{F}_k(\mathbf{p}';\mathbf{p}) \frac{1}{\mu_{jk}^2 - u_{jk} - i\epsilon} \mathcal{F}_j(\mathbf{p}';\mathbf{p})$$

u-channel pole gives
s-channel cuts for
partial waves

S-wave amputated PWIS OPE

$$\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_k', s, \sigma_j\right) = \frac{1}{4\,|\mathbf{p}_3'^{\star}|\,|\mathbf{p}_1^{\star}|} \log\left(\frac{z_{13}-1}{z_{13}+1}\right)$$

function of external kinematics



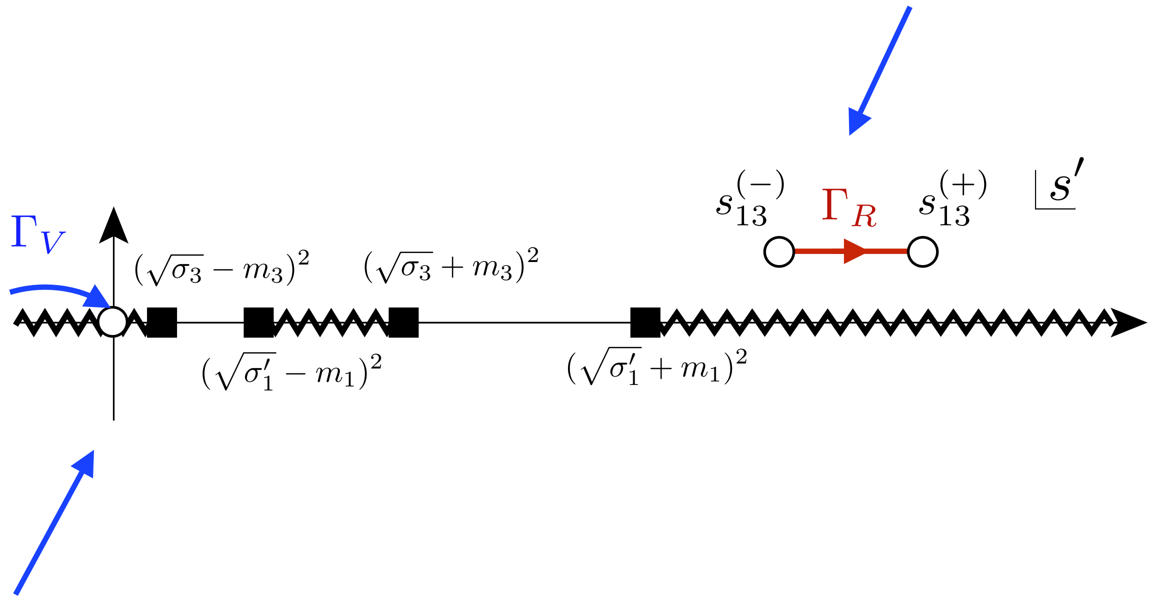
One-Particle Exchange

Consider dispersive representation

$$\widetilde{\mathcal{E}}^{S}_{kj}\left(\sigma'_k, s, \sigma_j\right) = \int_{\Gamma_V} ds' \frac{1}{s' - s - i\epsilon} \frac{1}{4|\mathbf{p}'^{\star}_k||\mathbf{p}^{\star}_j|} + \int_{\Gamma_R} ds' \frac{1}{s' - s - i\epsilon} \frac{1}{4|\mathbf{p}^{\star}_k||\mathbf{p}^{\star}_j|}$$

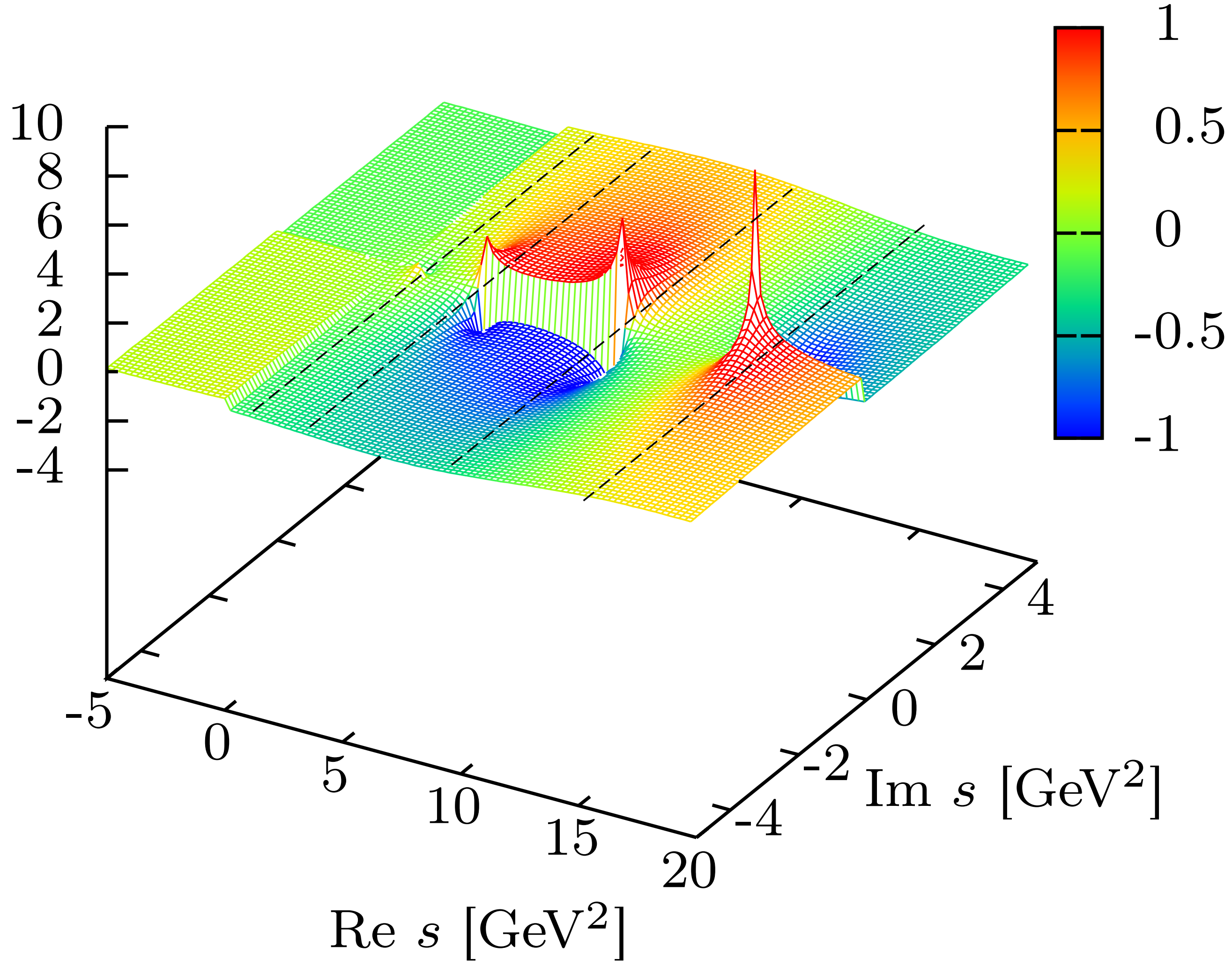
OPE branch points: $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ — depend on kinematics of isobar

$$\mathrm{Im}\,\widetilde{\mathcal{E}}^{S}_{kj}\left(\sigma'_k, s, \sigma_j\right) = \frac{\pi}{4\left|\mathbf{p}^{\star}_j\right|\left|\mathbf{p}'^{\star}_k\right|}$$



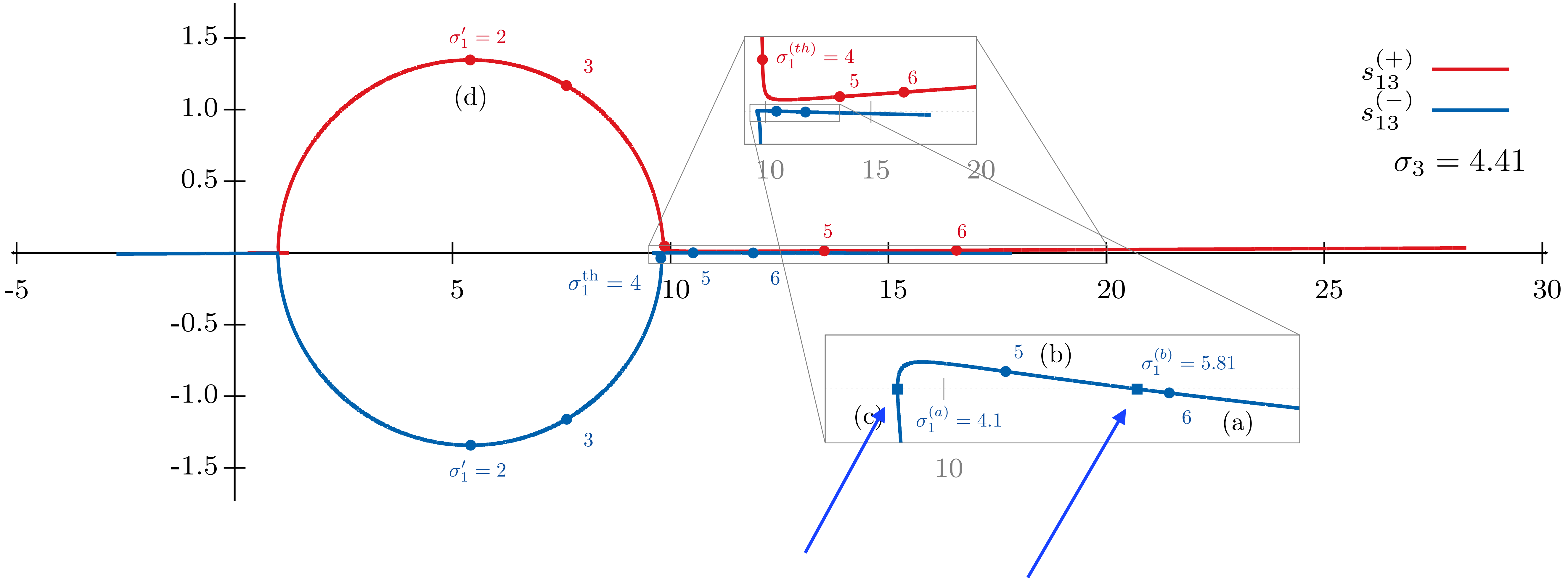
Real particle exchange

Virtual particle exchange



One-Particle Exchange

OPE branch points: $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ — depend on kinematics of isobar

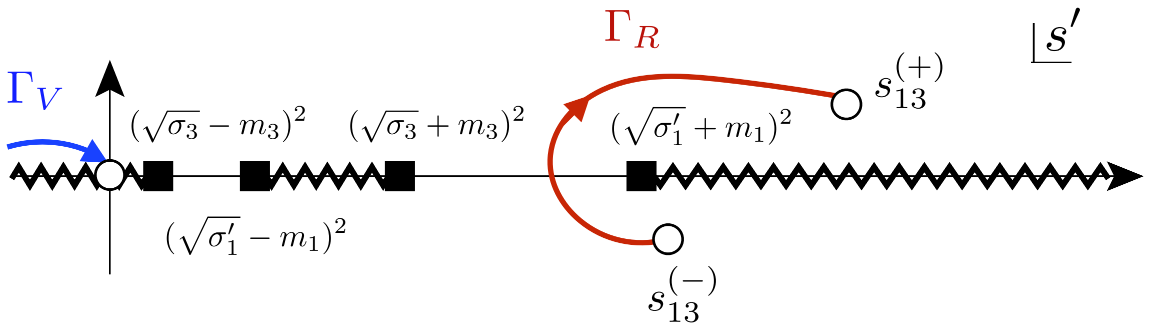


Im = 0 inflection points

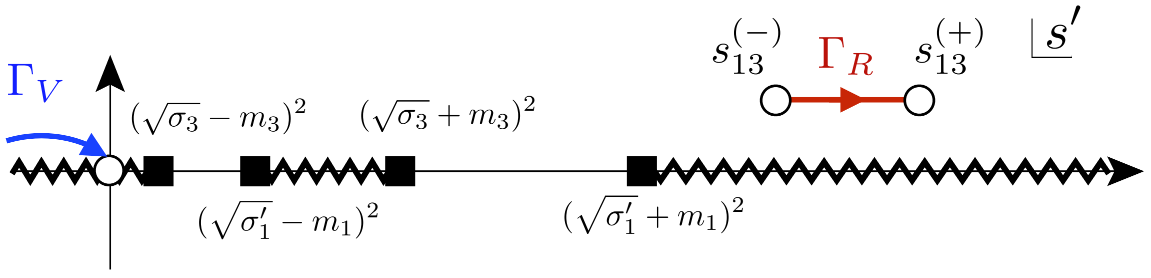
$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2 - \sigma_j\right)\left(m_j^2 - \sigma_k'\right) - \mu_{jk}^4 + \mu_{jk}^2\left(m_k^2 + m_j^2 + \sigma_j + \sigma_k'\right) \pm \lambda^{1/2}\left(\mu_{jk}^2, m_k^2, \sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2, m_j^2, \sigma_k'\right)\right]$$

One-Particle Exchange

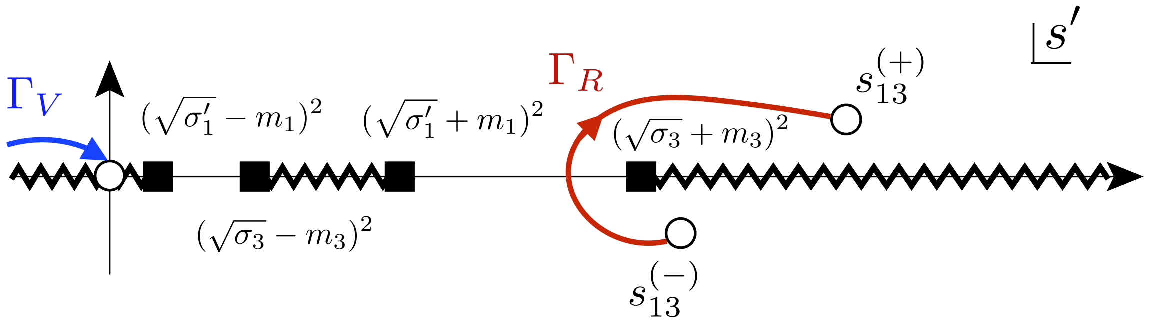
OPE branch points: $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ — depend on kinematics of isobar



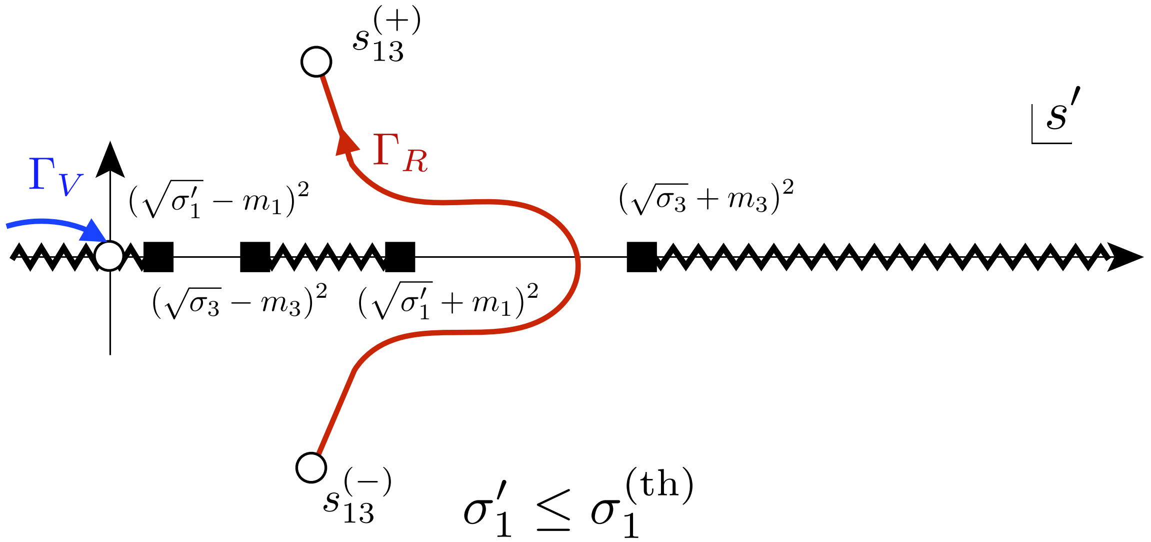
$$\sigma_1' \geq \sigma_1^{(b)}$$



$$\sigma_1^{(a)} \leq \sigma_1' \leq \sigma_1^{(b)}$$

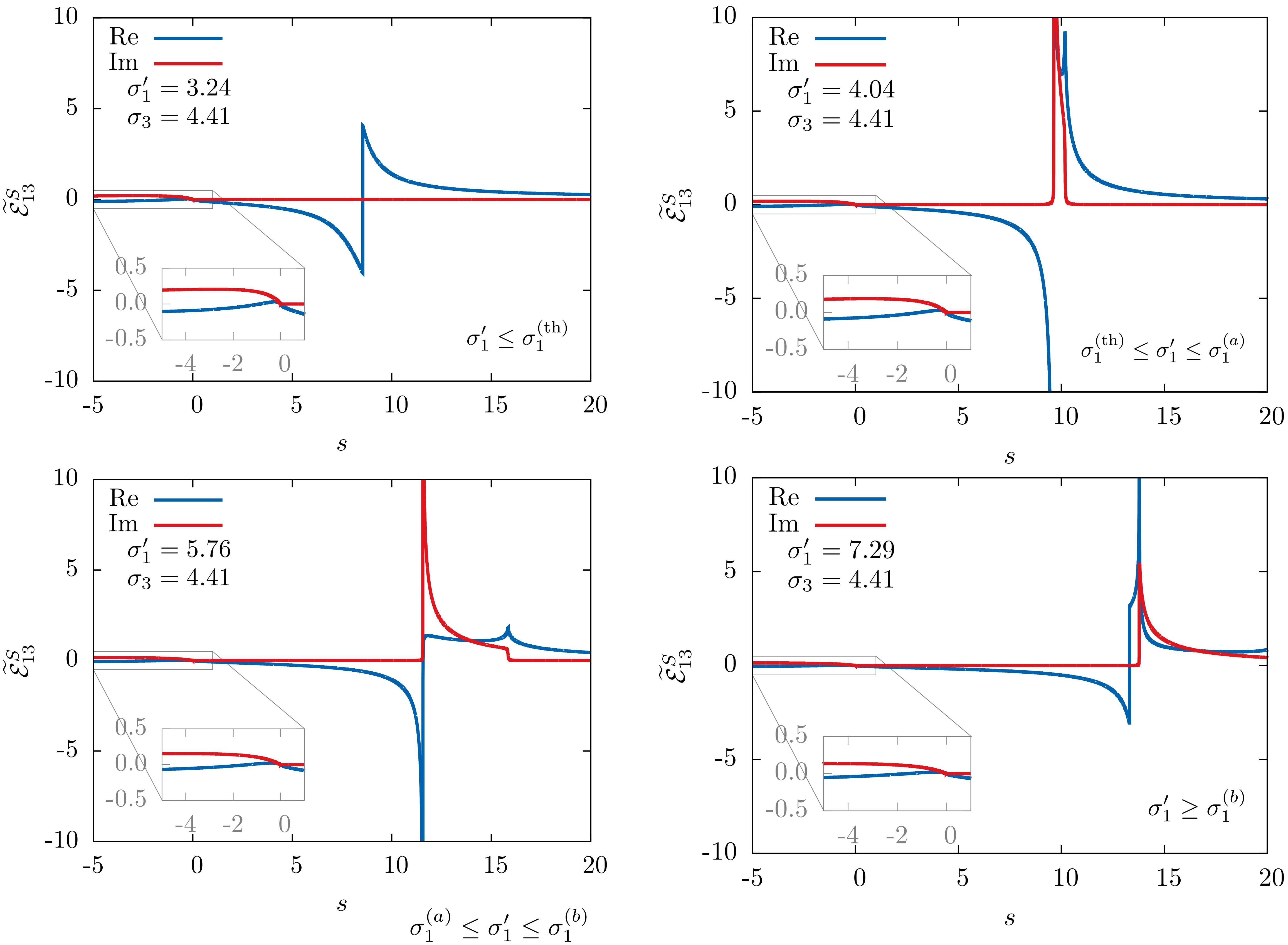


$$\sigma_1^{(\mathrm{th})} \leq \sigma_1' \leq \sigma_1^{(a)}$$



$$\sigma_1' \leq \sigma_1^{(\mathrm{th})}$$

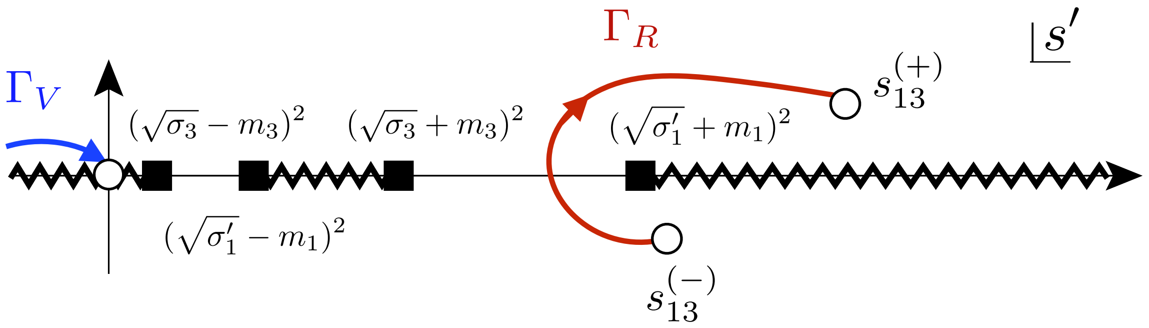
One-Particle Exchange

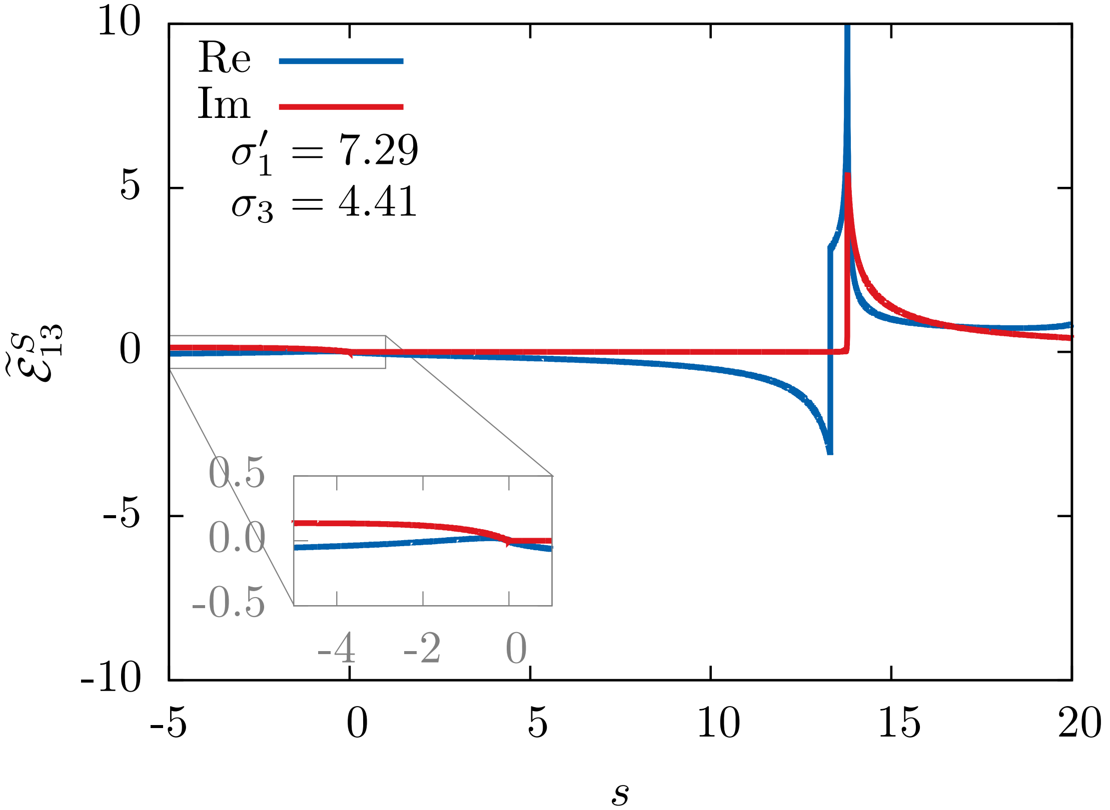


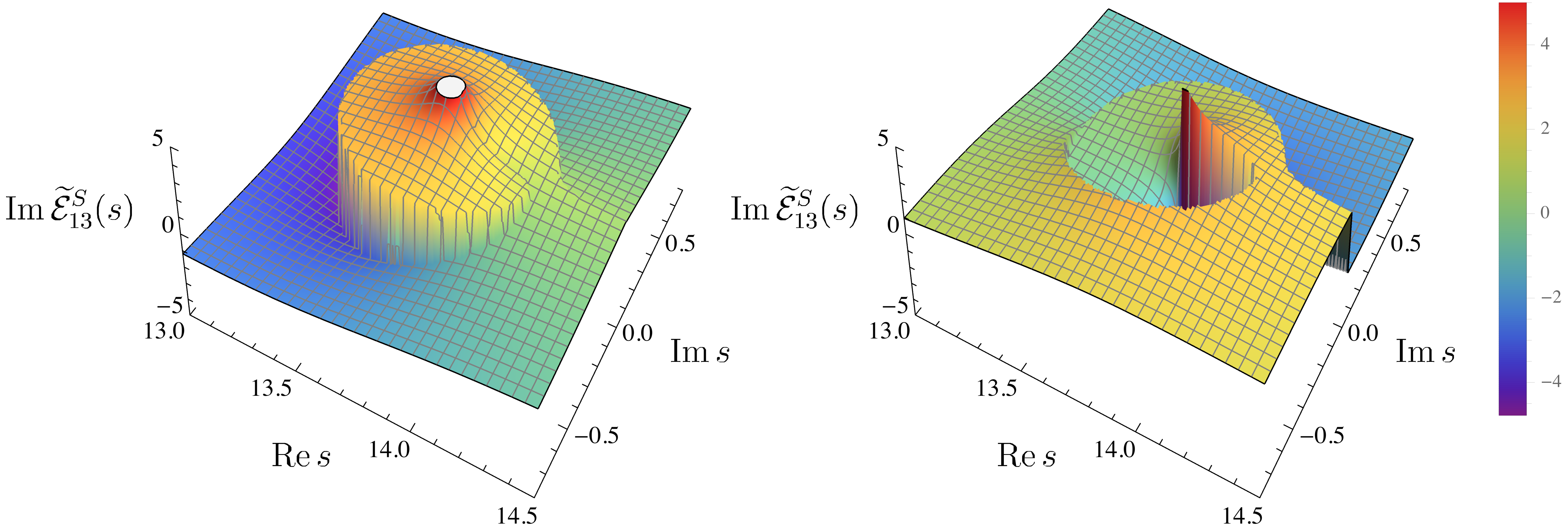
One-Particle Exchange

In some cases with real particle exchange, OPE has circular cut

Triangle singularities occur in physical region



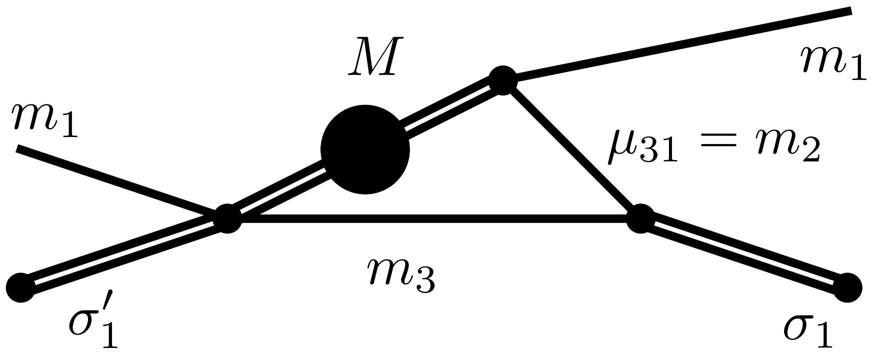




The Triangle Amplitude

Look at triangle amplitude in kernel

S-wave — assume narrow width approx. for isobar



phase space

OPE

$$\mathcal{T}_B(s) = \int_{\sigma_3^{(\text{th})}}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

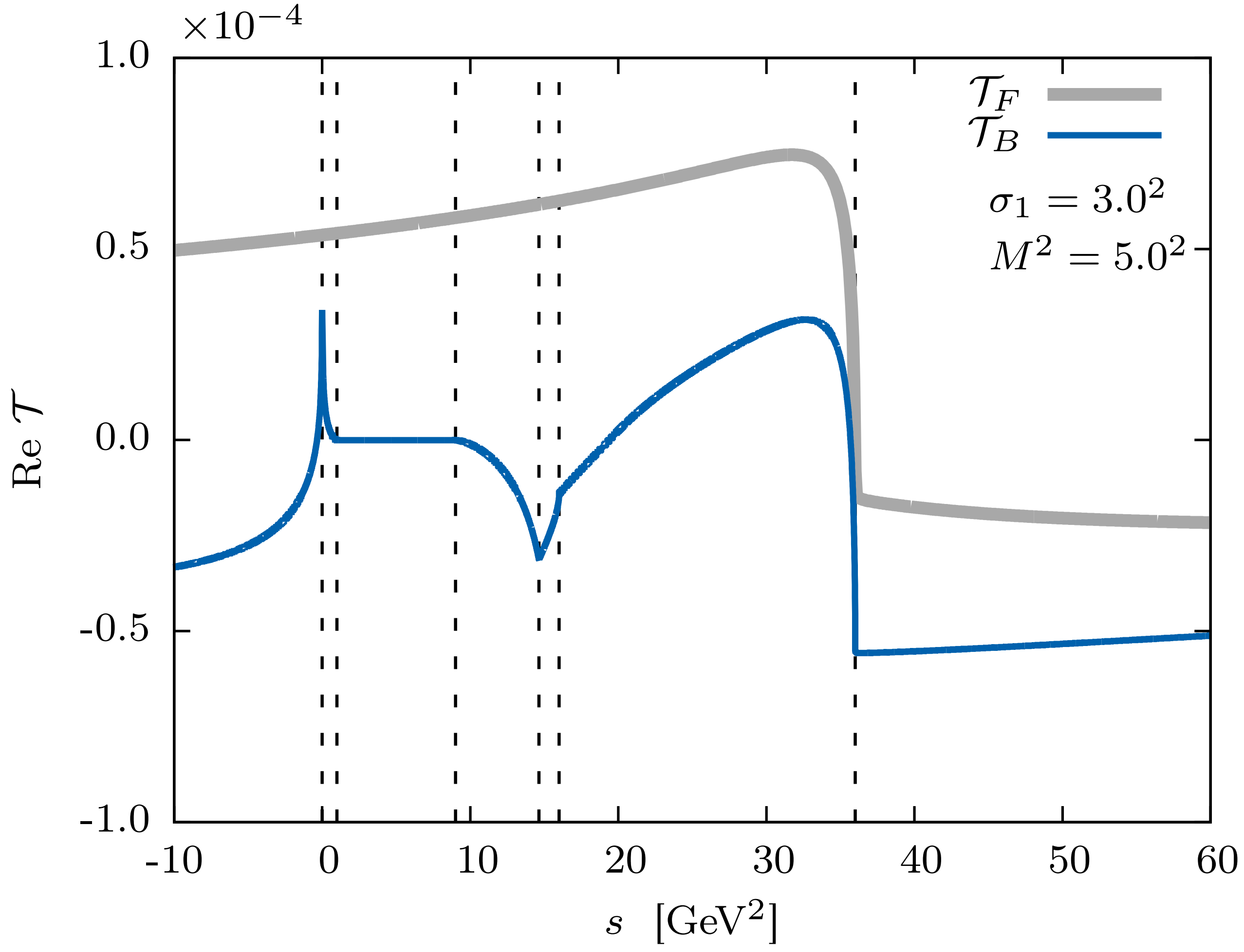
Integration over physical region

Isobar amplitude

Compare analytic structure with Feynman triangle

$$\mathcal{T}_F(s) = \int_{\Gamma_T} ds' \frac{\rho_3\left(s',M^2\right)\widetilde{\mathcal{E}}_{13}^{S}\left(M^2,s',\sigma_1\right)}{s'-s-i\epsilon}$$

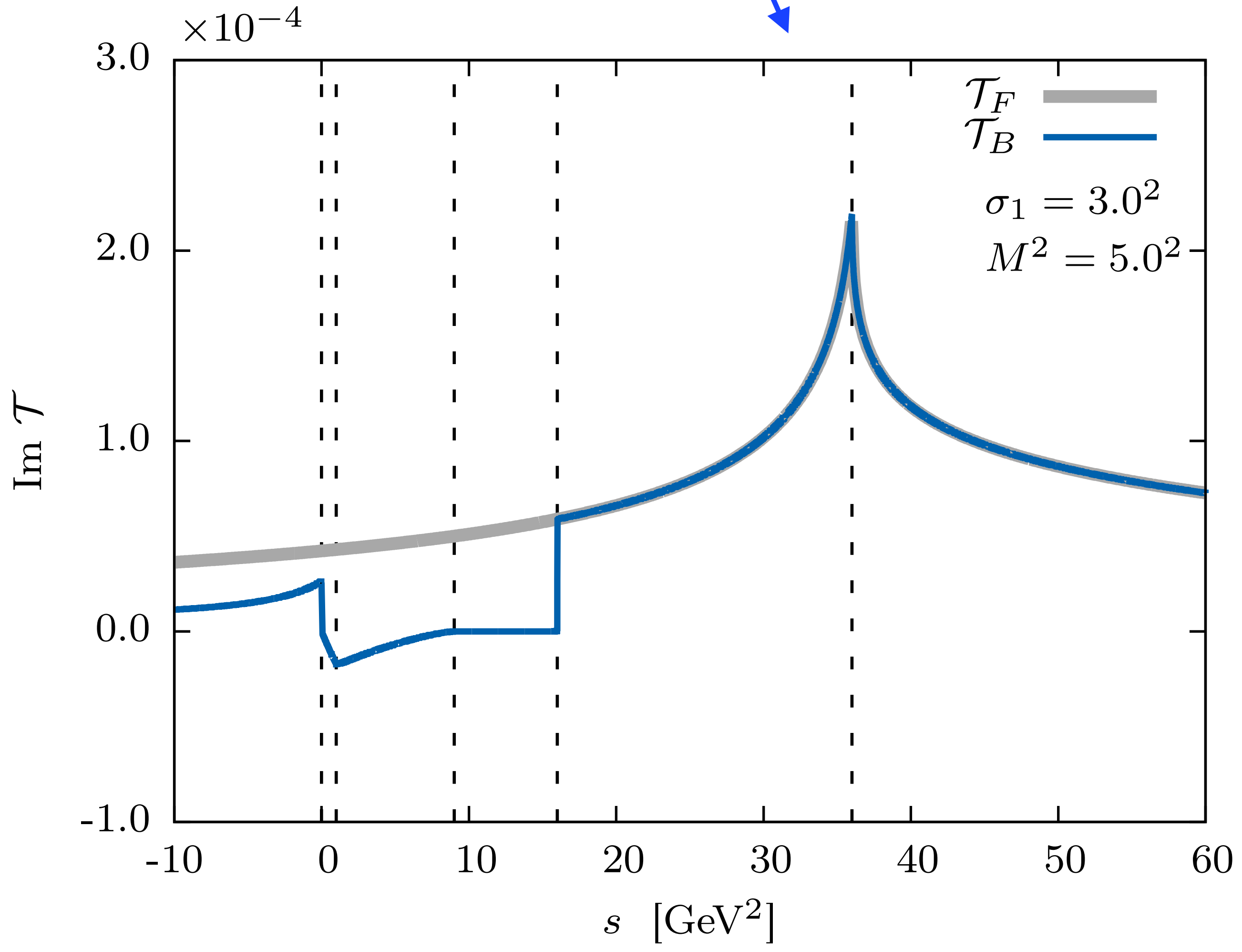
The Triangle Amplitude



$$\mathcal{T}_B(s) = \int_{\sigma_3^{(\mathrm{th})}}^{(\sqrt{s}-m_3)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^S\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

Real parts differ everywhere

Imaginary parts identical above production threshold



Extra kinks due to Endpoint and Pinch singularities of *B*-matrix Triangle

The Triangle Amplitude

Can compare to other formalisms, e.g, Mai et al.

M. Mai, B. Hu, M. Döring, A. Pilloni,
and A. Szczepaniak,
Eur. Phys. J. A53, 177 (2017)

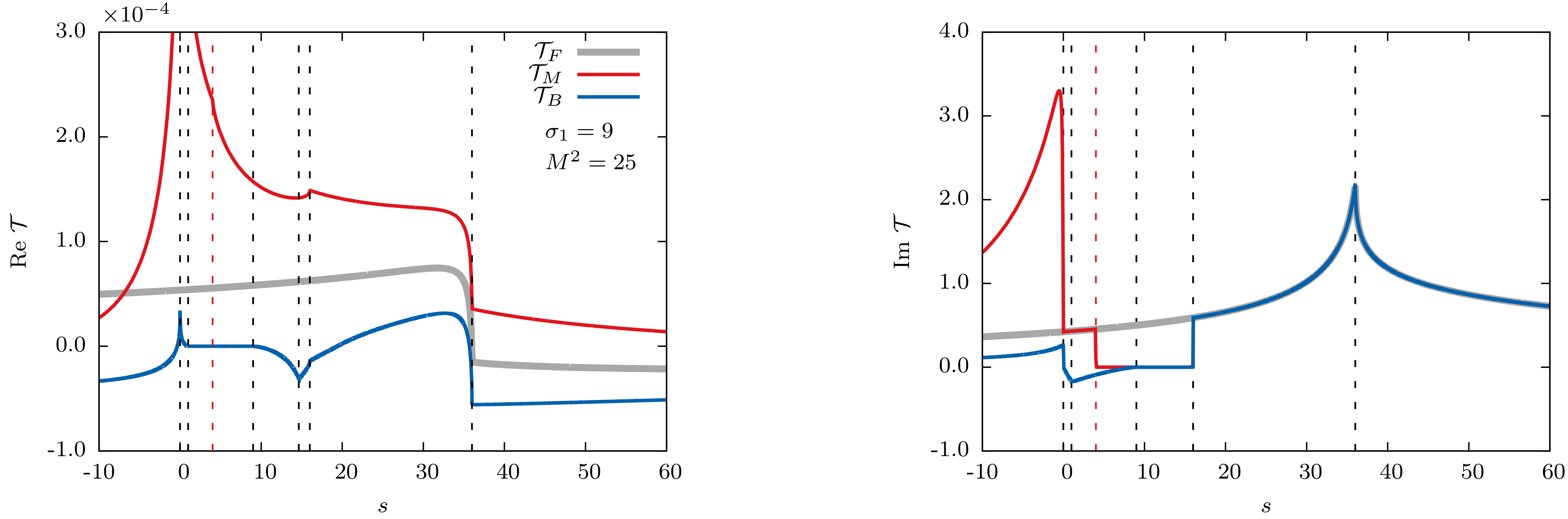
$$\mathcal{T}_M(s) = \int_{-\infty}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

Includes off-shell physics

Aitchison & Pasquier resolve the differences to Feynman — Different *R*-matrices!

I. J. R. Aitchison and R. Pasquier,
Phys. Rev. 152, 1274 (1966)

Imaginary parts identical
above production threshold

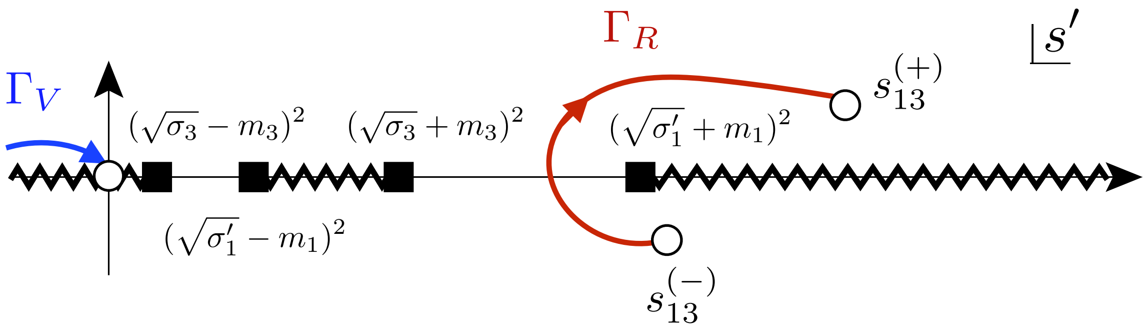


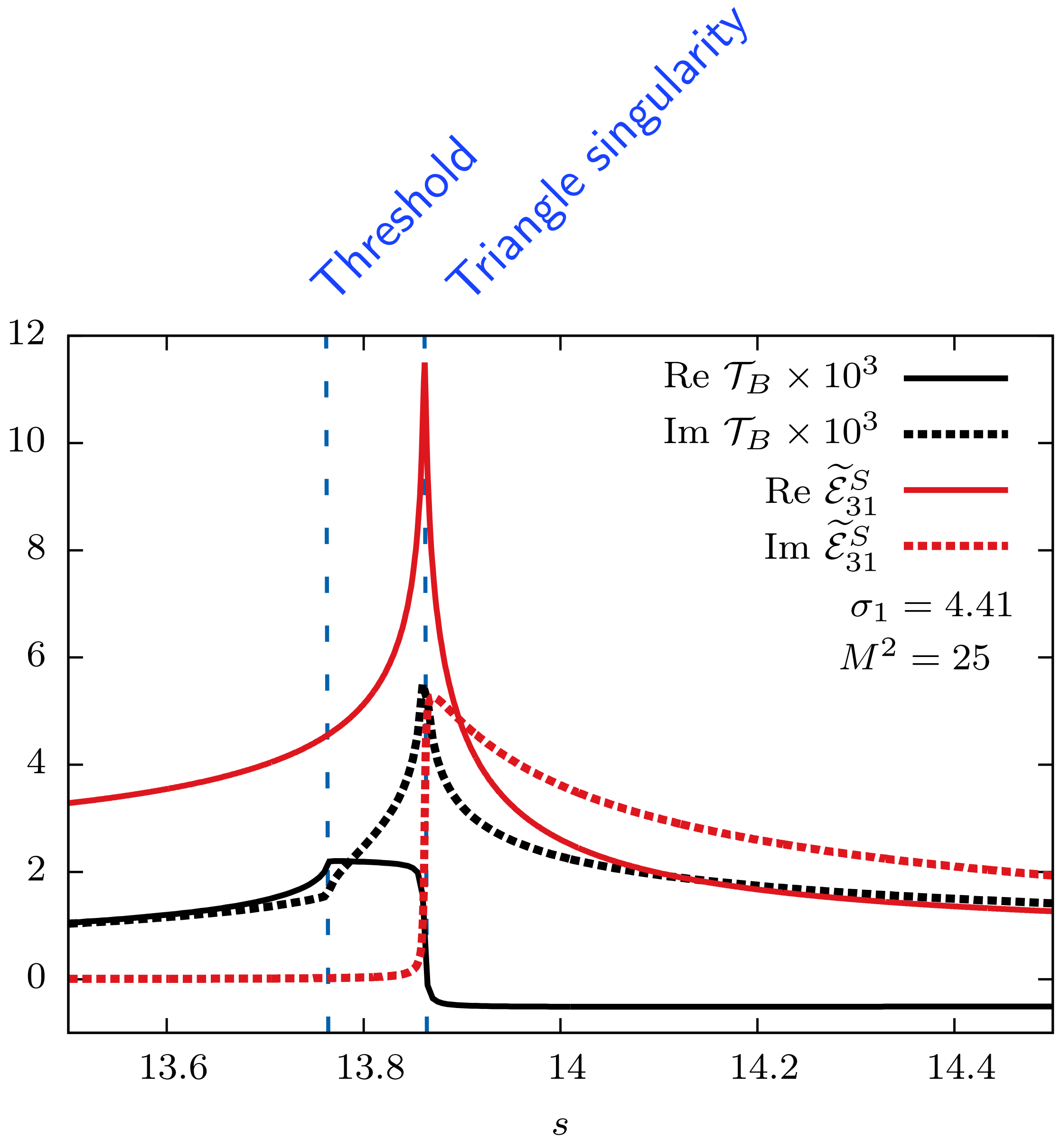
Triangle Singularity

Triangle amplitude develops singularity in physical region

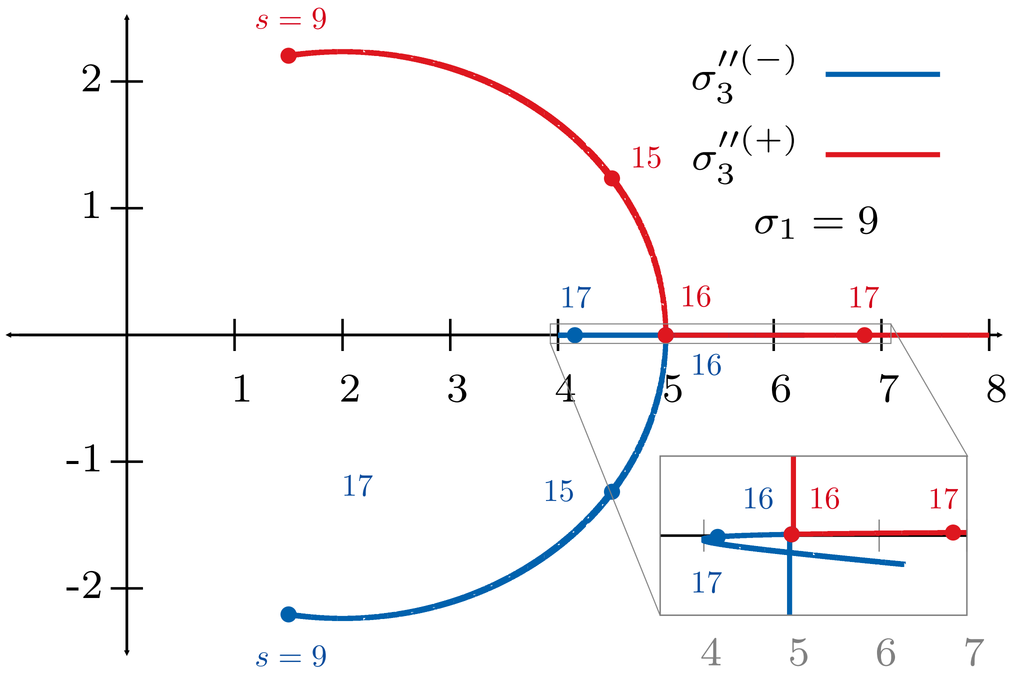
$\Longrightarrow$ OPE becomes physical

Rescattering effect — can mimic resonance signature

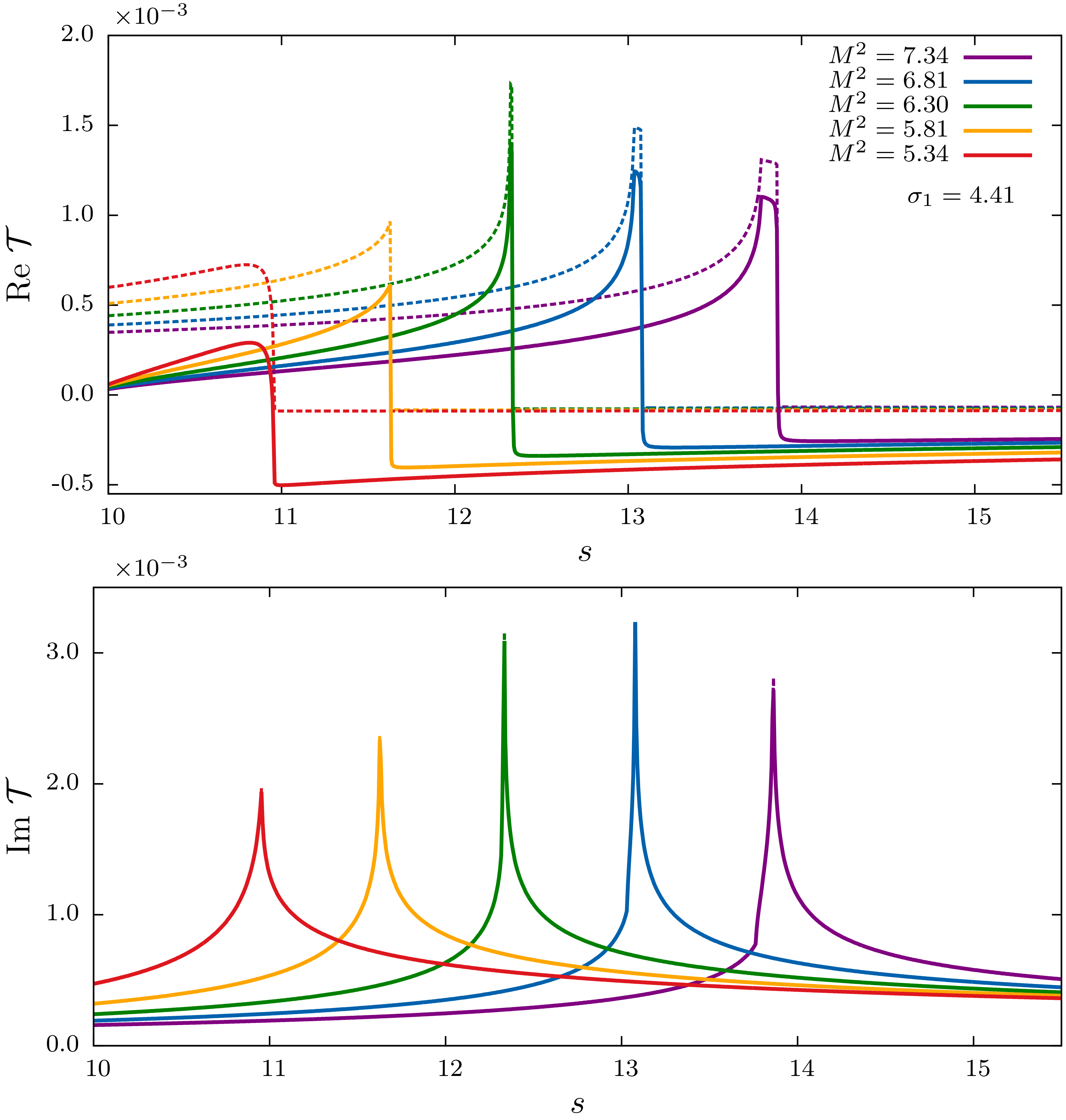




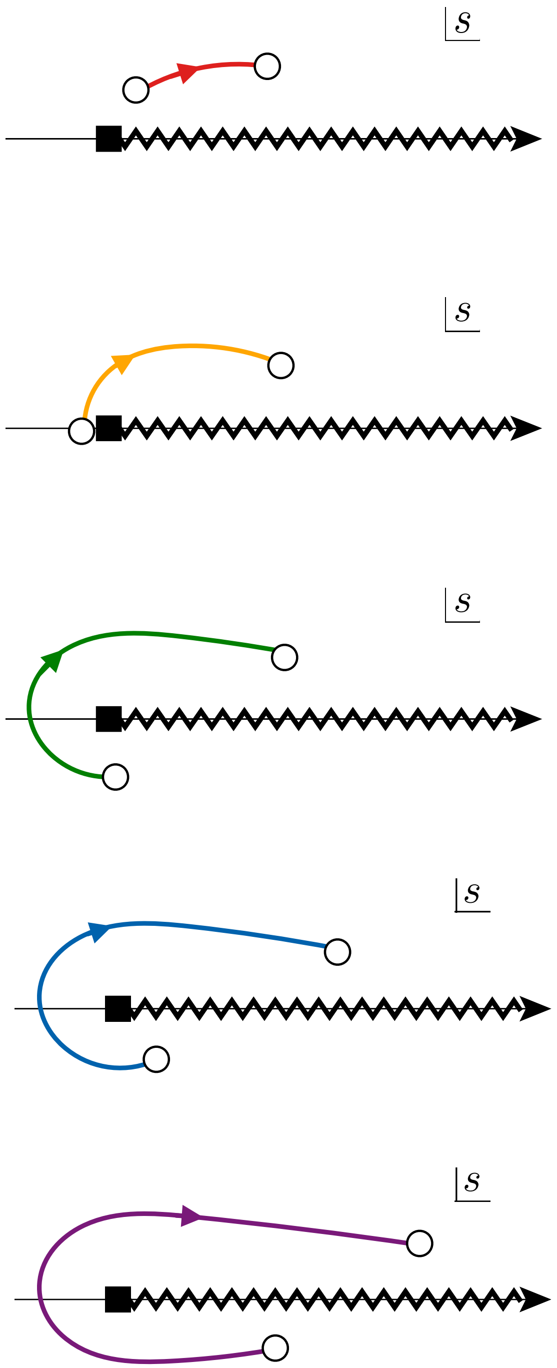
OPE branch points



Triangle Singularity



OPE branch points



The Triangle Amplitude

Saw triangle amplitude has extra singularities below threshold

Integrating over isobar shape cures some of them

Remaining singularities due to phase space

$$\mathcal{T}_B(s) = \frac{1}{\pi}\int_{\sigma_3^{(\mathrm{th})}}^{\infty} d\widehat{\sigma}\, \mathrm{Im}\, f_3(\widehat{\sigma}) \int_{\sigma_3^{(\mathrm{th})}}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}^S_{31}\left(\sigma_3'',s,\sigma_1\right)}{\widehat{\sigma}-\sigma_3''-i\epsilon}$$

Isobar shape

$$f_3\left(\sigma_3''\right) = \frac{1}{\pi}\int_{\sigma_3^{(\mathrm{th})}}^{\infty} d\widehat{\sigma}\, \frac{\mathrm{Im}\, f_3(\widehat{\sigma})}{\widehat{\sigma}-\sigma_3''-i\epsilon}$$

Singularities due to σ integration are moved to second sheet

What about dispersing kernel in s ?

$$\mathcal{K}^F_{33}(s) = \int_{\mathrm{th}}^{\infty} ds' \frac{\widetilde{\mathcal{B}}_{31}\left(s'\right)\rho_3\left(s'\right)\widetilde{\mathcal{B}}_{13}\left(s'\right)}{s'-s-i\epsilon}$$

Persistent singularities due to phase space

$$\rho_3(s) \propto \frac{1}{s}$$

$$\widetilde{\mathcal{A}}_{33} = \mathcal{K}^F_{33}(s) + \mathcal{K}^F_{33}(s)\tau_3(s)\mathcal{K}^F_{33}(s) + \cdots$$