Analysis of three body decays

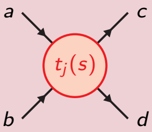
JPAC

Miguel Albaladejo (Jefferson Lab – Theory Center)

April 12, 2019

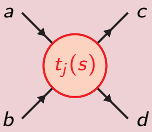
Outline

Introduction: Khuri-Treiman equations in a nutshell

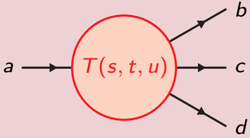


- In **some** scattering problems (LHC negligible, small s-region,...), $T(s,t,u) \to t_j(s)$
- Analytical solution ($\simeq$ BS equation, K-matrix,...)
- Or dispersion relations for one-variable function (neglect LHC)

Introduction: Khuri-Treiman equations in a nutshell

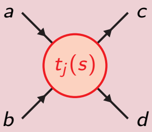


- In **some** scattering problems (LHC negligible, small s-region,...), $T(s,t,u) \to t_j(s)$
- Analytical solution ($\simeq$ BS equation, K-matrix,...)
- Or dispersion relations for one-variable function (neglect LHC)

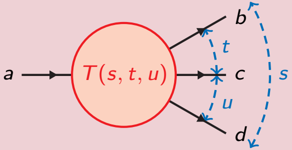


- Certainly not true in most decay process,
- where one usually wants to take into account unitarity/FSI interactions in the three possible channels.

Introduction: Khuri-Treiman equations in a nutshell

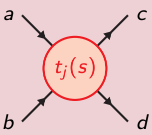


- In **some** scattering problems (LHC negligible, small s-region,...), $T(s,t,u) \to t_j(s)$
- Analytical solution ($\simeq$ BS equation, K-matrix,...)
- Or dispersion relations for one-variable function (neglect LHC)

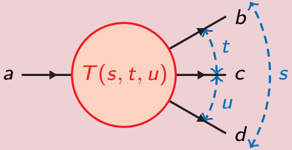


- Certainly not true in most decay process,
- where one usually wants to take into account unitarity/FSI interactions in the three possible channels.

Introduction: Khuri-Treiman equations in a nutshell



- In **some** scattering problems (LHC negligible, small s-region,...), $T(s,t,u) \to t_j(s)$
- Analytical solution ($\simeq$ BS equation, K-matrix,...)
- Or dispersion relations for one-variable function (neglect LHC)



- Certainly not true in most decay process,
- where one usually wants to take into account unitarity/FSI interactions in the three possible channels.

- Khuri-Treiman equations are a tool to achieve this **two-body unitarity** in the **three channels**

$$T(s,t,u) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_s) t_\ell(s)$$

$$T(s,t,u) = \sum_{\ell=0}^{n_s} (2\ell+1) P_\ell(z_s) t_\ell^{(s)}(s) + \sum_{\ell=0}^{n_t} (2\ell+1) P_\ell(z_t) t_\ell^{(t)}(t) + \sum_{\ell=0}^{n_u} (2\ell+1) P_\ell(z_u) t_\ell^{(u)}(u)$$

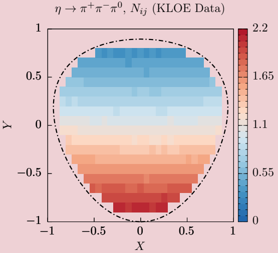
Generalities about $\eta \to 3\pi$

- In **QCD** isospin-breaking phenomena are driven by

$$H_{IB} = -(m_u - m_d)\bar{\psi}\frac{\lambda_3}{2}\psi$$

- Isospin-breaking induced by EM & strong interactions are **similar** in size, but
- **$\eta \to 3\pi$ is special**, since EM effects are smaller
- $\Gamma_{\eta\to 3\pi} \propto Q^4$, with

$$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u + m_d)^2/2}$$



- **Experimental situation:** Several high-statistics studies; $|T|^2$ well known across the Dalitz plot $\Rightarrow$ stringent tests for the amplitudes (before getting Q!)

$\eta \to 3\pi^0$

Crys. Ball, PRL**87**,192001('01)
Crys. Ball@MAMI, A2, PRC**79**,035204('09)
Crys. Ball@MAMI, TAPS, A2, EPJA**39**,169('09)
WASA-at-COSY, PLB**677**,24('09)
KLOE, PLB**694**,16('11)

$\eta \to \pi^+\pi^-\pi^0$

KLOE, JHEP**0805**,006('08)
WASA-at-COSY, PRC**90**,045207('14)
BESIII, PRD**92**,012014('15)
KLOE-2, JHEP**1605**,019('16)

Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** effects.

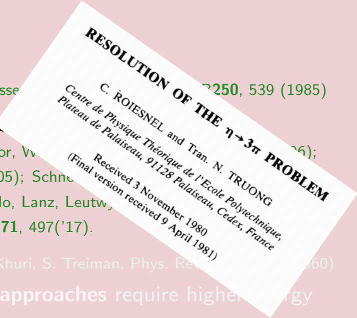
  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, Wiesendanger, Wyler, NPB**465**, 215('96); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schneider, Kubis, Ditsche, JHEP**1102**, 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL**118**, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

Dispersive approaches to $\eta \to 3\pi$


RESOLUTION OF THE $\eta \to 3\pi$ PROBLEM

C. ROIESNEL and Tran. N. TRUONG

Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, 91128 Palaiseau, Cedex, France

Received 3 November 1980

(Final version received 9 April 1981)

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasse ... B250, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** eff...

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, W... 6); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schne... 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutw... 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Re... 960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require highe... rgy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

3 / 18

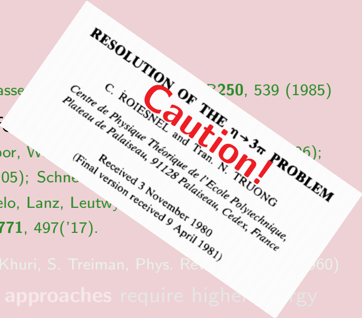
Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasse... B**250**, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** eff...

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, W... ...6); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schne... 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutw... 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).


RESOLUTION OF THE $\eta \to 3\pi$ PROBLEM

C. ROIESNEL and Tran. N. TRUONG

Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, 91128 Palaiseau, Cedex, France

Received 3 November 1980 (Final version received 9 April 1981)

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Re... ...960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require highe... ...rgy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

3 / 18

Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** effects.

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, Wiesendanger, Wyler, NPB**465**, 215('96); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schneider, Kubis, Ditsche, JHEP**1102**, 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL**118**, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** effects.

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, Wiesendanger, Wyler, NPB**465**, 215('96); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schneider, Kubis, Ditsche, JHEP**1102**, 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL**118**, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy T-matrix inputs:
  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

  Abdel-Rehim, Black, Fariborz, Schechter, PRD**67**, 054001('03)

We propose a **generalization to coupled channels** $[(K\bar{K})_{0,1},\ \eta\pi,\ (\pi\pi)_{0,1,2}]$ of the KT equations, extending their validity up to the physical $\eta\pi \to \pi\pi$ region. Allows for the study of the influence of a_0, f_0 into the decay region.

Isospin amplitudes

- Start with well-defined **isospin amplitudes:**

$$\mathcal{M}^{I,I_z}(s,t,u) = \langle \eta\pi; 1, I_z | \hat{T}_0^{(1)} | \pi\pi; I, I_z \rangle = \langle I, I_z; 1, 0 | \, 10 \rangle \langle \eta\pi \| \hat{T}^{(1)} \| \pi\pi; I \rangle$$

- They can be written in terms of a **single amplitude** ($\eta\pi^0 \to \pi^+\pi^-$), $A(s,t,u)$ (like in $\pi\pi$ scattering):

$$\begin{bmatrix} -\sqrt{3}\mathcal{M}^0(s,t,u) \\ \sqrt{2}\mathcal{M}^1(s,t,u) \\ \sqrt{2}\mathcal{M}^2(s,t,u) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\mathcal{M}^{0,0}(s,t,u) \\ \sqrt{2}\mathcal{M}^{1,1}(s,t,u) \\ \sqrt{2}\mathcal{M}^{2,1}(s,t,u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s,t,u) \\ A(t,s,u) \\ A(u,t,s) \end{bmatrix}$$

- Reconstruction theorem (for Goldstone bosons):

J. Stern, H. Sazdjian, N. Fuchs, Phys. Rev. D**47**, 3814 (1993)

$$A(s,t,u) = -\epsilon_L \Big[M_0(s) - \frac{2}{3} M_2(s) + M_2(t) + M_2(u) + (s-u)M_1(t) + (s-t)M_1(u) \Big] \qquad \epsilon_L = \frac{1}{\mathbf{Q^2}} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2}$$

- Or in general, "the" KT approximation:

Infinite sum of s-channel PW $\to$ Truncated sums of s-, t-, and u-channels PWs

- **Single variable functions:** amenable for dispersion relations.

Partial wave amplitudes

- Summary of previous slide: $\mathcal{M}^I(s,t,u)$ is written in terms of $A(s,t,u)$ (and permutations), and $A(s,t,u)$ is written in terms of $M_I(w)$.
- Now, define **partial waves:** $\mathcal{M}^I(s,t,u) = 16\pi\sqrt{2}\sum_j (2j+1)\mathcal{M}_j^I(s)P_j(z)$

$$\mathcal{M}_0^0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi}[M_0(s) + \hat{M}_0(s)] \; , \quad \mathcal{M}_0^2(s) = \epsilon_L \frac{-1}{32\pi}[M_2(s) + \hat{M}_2(s)] \; ,$$

$$\mathcal{M}_1^1(s) = \epsilon_L \frac{\kappa(s)}{32\pi}[M_1(s) + \hat{M}_1(s)] \; ,$$

LHC $[\hat{M}_I(s)]$

$\hat{M}_I(s)$ written as angular averages.
Take $M_0(s)$ as an example:

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + \frac{20}{9}\langle M_2\rangle$$

$$+ 2(s-s_0)\langle M_1\rangle + \frac{2}{3}\kappa(s)\langle z M_1\rangle$$

$$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}z \; z^n M_I(t(s,z))$$

$$\kappa(s) = \sqrt{(1-4m_\pi^2/s)\lambda(s, m_\eta^2, m_\pi^2)}$$

RHC $[M_I(s)]$

$\hat{M}(s)$ no discontinuity along the RHC:

$$\text{disc } M_I(s) = \text{disc}\mathcal{M}_j^I(s) =$$

$$= \sigma_\pi(s) t^I(s)^* \mathcal{M}_j^I(s)$$

$$= \sigma_\pi(s) t^I(s)^* \left(M_I(s) + \hat{M}_I(s)\right)$$

$$\sigma_\pi(s) = \sqrt{1-4m_\pi^2/s}$$

$$\sigma_\pi(s) t^I(s) = \sin\delta_I(s)\, e^{i\delta_I(s)}$$

Muskhelisvili-Omnès representation

$$\text{disc}M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) + \hat{M}_I(s) \right]$$

- MO (dispersive) representation of $M_I(s)$:

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT:**

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- **Matching conditions:** fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

Muskhelisvili-Omnès representation

$$\text{disc}M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) \right]$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s) \left[P_0(s) \right] \; ,$$
$$M_1(s) = \Omega_1(s) \left[P_1(s) \right] \; ,$$
$$M_2(s) = \Omega_2(s) \left[P_2(s) \right] \; .$$

$$\Omega_I(s) = \exp \left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \, \frac{\delta_I(s')}{s'(s'-s)} \right] \; \text{(Omnès function/matrix)}$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT:**

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- **Matching conditions:** fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

Muskhelisvili-Omnès representation

$$\text{disc} M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) + \hat{M}_I(s) \right]$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s) \left[P_0(s) + \hat{I}_0(s) s^2 \right] ,$$
$$M_1(s) = \Omega_1(s) \left[P_1(s) + \hat{I}_1(s) s \right] ,$$
$$M_2(s) = \Omega_2(s) \left[P_2(s) + \hat{I}_2(s) s^2 \right] .$$

$$\Omega_I(s) = \exp \left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \, \frac{\delta_I(s')}{s'(s'-s)} \right] \quad \text{(Omnès function/matrix)}$$

$$\hat{I}_I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sin \delta_I(s') \hat{M}_I(s')}{|\Omega_I(s')| \, (s')^{m_I} (s'-s)} , \quad (m_{0,2} = 2, \; m_1 = 1)$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT:**

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$ Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- **Matching conditions:** fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

6 / 18

Muskhelisvili-Omnès representation

$$\text{disc} M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) + \hat{M}_I(s) \right]$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s) \left[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{I}_0(s) s^2 \right] ,$$
$$M_1(s) = \Omega_1(s) \left[\beta_1 s + \hat{I}_1(s) s \right] ,$$
$$M_2(s) = \Omega_2(s) \left[\hat{I}_2(s) s^2 \right] .$$

$$\Omega_I(s) = \exp\left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s)} \right] \quad \text{(Omnès function/matrix)}$$

$$\hat{I}_I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sin \delta_I(s') \hat{M}_I(s')}{|\Omega_I(s')| (s')^{m_I} (s'-s)} , \quad (m_{0,2} = 2,\ m_1 = 1)$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT:**

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- **Matching conditions:** fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

Coupled channels

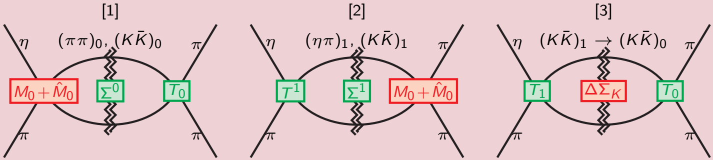
Coupled channels: take into account **intermediate states** other than $(\pi\pi)_I$.

$$\mathbf{M}_0 = \begin{bmatrix} M_0 & G_{10} \\ N_0 & H_{10} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_0 & (K\bar{K})_1 \to (\pi\pi)_0 \\ (\eta\pi)_1 \to (K\bar{K})_0 & (K\bar{K})_1 \to (K\bar{K})_0 \end{bmatrix},$$

$$\mathbf{T}_0 = \begin{bmatrix} t_{(\pi\pi)_0\to(\pi\pi)_0} & t_{(\pi\pi)_0\to(K\bar{K})_0} \\ t_{(\pi\pi)_0\to(K\bar{K})_0} & t_{(K\bar{K})_0\to(K\bar{K})_0} \end{bmatrix}, \mathbf{T}_1 = \begin{bmatrix} t_{(\eta\pi)_1\to(\eta\pi)_1} & t_{(\eta\pi)_1\to(K\bar{K})_1} \\ t_{(\eta\pi)_1\to(K\bar{K})_1} & t_{(K\bar{K})_1\to(K\bar{K})_1} \end{bmatrix}$$

$$\begin{aligned} \text{disc}\, \mathbf{M}_0(s) &= \mathbf{T}^{0*}(s)\Sigma^0(s)\left[\mathbf{M}_0(s+i\epsilon) + \hat{\mathbf{M}}_0(s)\right] && \to [1] \\ &+ \left[(\mathbf{M}_0(s-i\epsilon) + \hat{\mathbf{M}}_0(s)\right]\Sigma^1(s)\,\mathbf{T}^1(s) && \to [2] \\ &+ \mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s) && \to [3] \end{aligned}$$

Schematically:



Coupled channels: MO representations

$$
\begin{aligned}
\text{disc}\, \mathbf{M}_0(s) &= \mathbf{T}^{0*}(s)\Sigma^0(s)\left[\mathbf{M}_0(s+i\epsilon)+\hat{\mathbf{M}}_0(s)\right] && \to [1] \\
&+ \left[(\mathbf{M}_0(s-i\epsilon)+\hat{\mathbf{M}}_0(s)\right]\Sigma^1(s)\,\mathbf{T}^1(s) && \to [2] \\
&+ \mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s) && \to [3]
\end{aligned}
$$

- MO representation for $\mathbf{M}_0(s)$:

$$
\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \mathbf{\Omega}_0(s)\left[\mathbf{P}_0(s) + s^2\left(\hat{\mathbf{I}}_a(s)+\hat{\mathbf{I}}_b(s)\right)\right]{}^t\mathbf{\Omega}_1(s)
$$

- $\mathbf{P}_0(s)$ is a matrix of polynomials.
- The $\hat{\mathbf{I}}(s)$ functions are:

$$
\hat{\mathbf{I}}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-s)}\Delta\mathbf{X}_{a,b}(s')\,,
$$

$$
\Delta\mathbf{X}_a = \mathbf{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{\mathbf{T}^{0*}(s)\,\Sigma^0(s)\,\hat{\mathbf{M}}_0(s)}_{[1]} + \underbrace{\hat{\mathbf{M}}_0(s)\,\Sigma^1(s)\,\mathbf{T}^1(s)}_{[2]}\right]{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)\,,
$$

$$
\Delta\mathbf{X}_b = \underbrace{\mathbf{\Omega}_0^{-1}(s-i\epsilon)\mathbf{T}^{0*}(s)\,\Delta\Sigma_K(s)\,\mathbf{T}^1(s)\,{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}
$$

Coupled channels: MO representations

$$
\begin{aligned}
\text{disc}\,\mathbf{M}_0(s) &= \mathbf{T}^{0*}(s)\Sigma^0(s)\left[\mathbf{M}_0(s+i\epsilon)+\hat{\mathbf{M}}_0(s)\right] && \to [1]\\
&+ \left[(\mathbf{M}_0(s-i\epsilon)+\hat{\mathbf{M}}_0(s)\right]\Sigma^1(s)\,\mathbf{T}^1(s) && \to [2]\\
&+ \mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s) && \to [3]
\end{aligned}
$$

- MO representation for $\mathbf{M}_0(s)$:

$$
\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \mathbf{\Omega}_0(s)\left[\mathbf{P}_0(s) + s^2\left(\hat{\mathbf{I}}_a(s)+\hat{\mathbf{I}}_b(s)\right)\right]{}^t\mathbf{\Omega}_1(s)
$$

- $\mathbf{P}_0(s)$ is a matrix of polynomials.
- The $\hat{\mathbf{I}}(s)$ functions are:

$$
\hat{\mathbf{I}}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-s)}\Delta\mathbf{X}_{a,b}(s')\,,
$$

$$
\Delta\mathbf{X}_a = \mathbf{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{\mathbf{T}^{0*}(s)\,\Sigma^0(s)\,\hat{\mathbf{M}}_0(s)}_{[1]} + \underbrace{\hat{\mathbf{M}}_0(s)\,\Sigma^1(s)\,\mathbf{T}^1(s)}_{[2]}\right]{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)\,,
$$

$$
\Delta\mathbf{X}_b = \underbrace{\mathbf{\Omega}_0^{-1}(s-i\epsilon)\mathbf{T}^{0*}(s)\,\Delta\Sigma_K(s)\,\mathbf{T}^1(s)\,{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}
$$

Coupled channels: MO representations

$$\begin{aligned} \text{disc}\, \mathbf{M}_0(s) &= \mathbf{T}^{0*}(s)\Sigma^0(s)\left[\mathbf{M}_0(s+i\epsilon)+\hat{\mathbf{M}}_0(s)\right] && \to [1] \\ &+ \left[(\mathbf{M}_0(s-i\epsilon)+\hat{\mathbf{M}}_0(s)\right]\Sigma^1(s)\,\mathbf{T}^1(s) && \to [2] \\ &+ \mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s) && \to [3] \end{aligned}$$

- MO representation for $\mathbf{M}_0(s)$:

$$\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \mathbf{\Omega}_0(s)\left[\mathbf{P}_0(s) + s^2\left(\hat{\mathbf{I}}_a(s)+\hat{\mathbf{I}}_b(s)\right)\right]{}^t\mathbf{\Omega}_1(s)$$

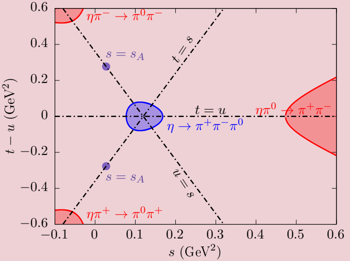
- $\mathbf{P}_0(s)$ is a matrix of polynomials.
- The $\hat{\mathbf{I}}(s)$ functions are:

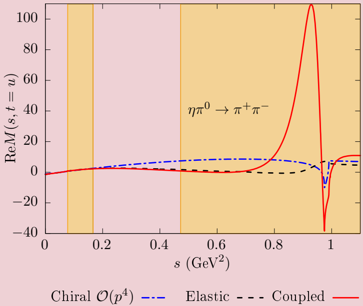
$$\hat{\mathbf{I}}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^2(s'-s)}\,\Delta\mathbf{X}_{a,b}(s')\ ,$$

$$\Delta\mathbf{X}_a = \mathbf{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{\mathbf{T}^{0*}(s)\,\Sigma^0(s)\,\hat{\mathbf{M}}_0(s)}_{[1]} + \underbrace{\hat{\mathbf{M}}_0(s)\,\Sigma^1(s)\,\mathbf{T}^1(s)}_{[2]}\right]{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)\ ,$$

$$\Delta\mathbf{X}_b = \underbrace{\mathbf{\Omega}_0^{-1}(s-i\epsilon)\mathbf{T}^{0*}(s)\,\Delta\Sigma_K(s)\,\mathbf{T}^1(s)\,{}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}$$

Results





Behaviour in different regions:

- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.

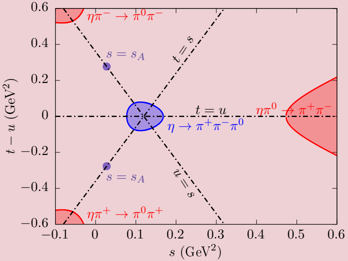
Coupled channel largely **enhanced** compared with elastic amplitude.

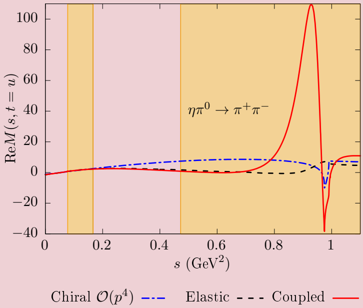
Effect of coupled channels is to reduce the amplitude.

Elastic and inelastic amplitudes indistinguishable.

9 / 18

Results





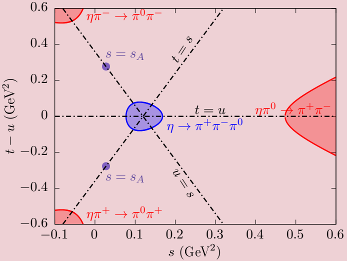
Behaviour in different regions:

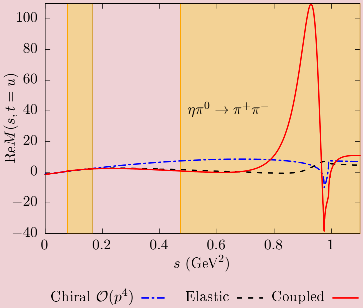
- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.

Effect of coupled channels is to reduce the amplitude.

Elastic and inelastic amplitudes indistinguishable.

Results



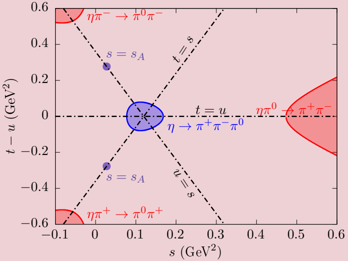


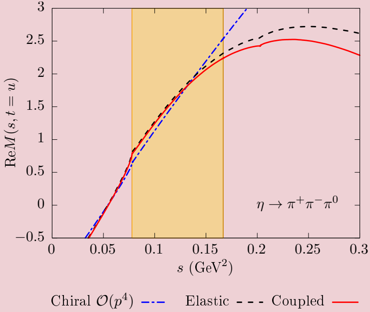
Behaviour in different regions:

- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7$ GeV^2 Effect of coupled channels is to reduce the amplitude.

Elastic and inelastic amplitudes indistinguishable.

Results





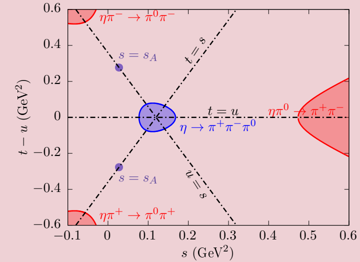
Behaviour in different regions:

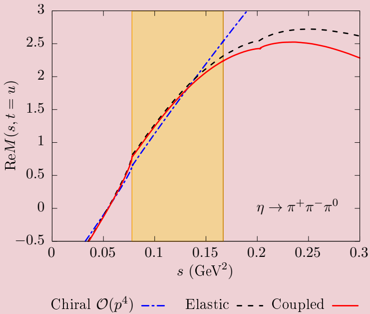
- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7$ GeV^2 Effect of coupled channels is to reduce the amplitude.

Elastic and inelastic amplitudes indistinguishable.

9 / 18

Results





Behaviour in different regions:

- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7$ GeV^2 Effect of coupled channels is to reduce the amplitude.
- $s \lesssim s_{\text{th}}$ Elastic and inelastic amplitudes indistinguishable.

Dalitz plot

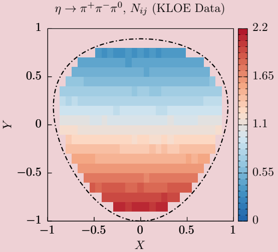
- DP variables X,Y: $X = \frac{\sqrt{3}}{2m_\eta Q_c}(u-t)$, $Y = \frac{3}{2m_\eta Q_c}\left((m_\eta - m_{\pi^0})^2 - s\right) - 1$

- Charged mode amplitude written as:

$$\frac{|M_c(X,Y)|^2}{|M_c(0,0)|^2} = 1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3 + g\,X^2 Y + \cdots$$

- Neutral decay mode amplitude $[Q_c \to Q_n]$:

$$\frac{|M_n(X,Y)|^2}{|M_n(0,0)|^2} = 1 + 2\alpha\,|z|^2 + 2\beta\,\mathrm{Im}(z^3) + \cdots$$



		$O(p^4)$	elastic	coupled	KLOE	BESIII
charged	a	-1.328	-1.156	-1.142(45)	-1.095(4)	-1.128(15)
charged	b	0.429	0.200	0.172(16)	0.145(6)	0.153(17)
charged	d	0.090	0.095	0.097(13)	0.081(7)	0.085(16)
charged	f	0.017	0.109	0.122(16)	0.141(10)	0.173(28)
charged	g	-0.081	-0.088	-0.089(10)	-0.044(16)	–
neutral					PDG	
neutral	α	+0.0142	-0.0268	-0.0319(34)	-0.0318(15)	
neutral	β	-0.0007	-0.0046	-0.0056	–	

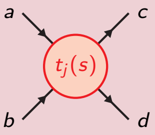
BESIII Collab., Phys. Rev. D**92**,012014 (2015)
KLOE-2 Collab., JHEP **1605**, 019 (2016)

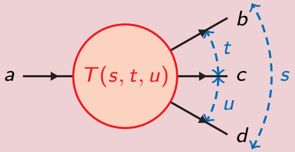
- (Theory) **uncertainty estimation**:
  1. $\eta\pi$ interaction put to zero or to "large"
  2. $10^3 L_3^r = -3.82 \to -2.65$
- General trend: **improve agreement** $[\mathcal{O}(p^4) \to$ elastic $\to$ coupled$]$
- Particularly relevant: $\boldsymbol{\alpha}$.

Khuri-Treiman equations for $\pi\pi$ scattering

JPAC Collab., EPJ,C78,574(2018)

- KT equations for 3-body decays. Crossing: 2-to-2 scattering. Test: $\pi\pi$ scattering.

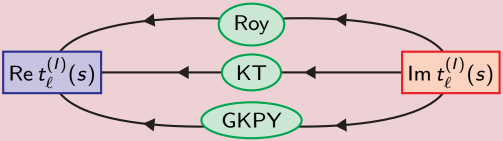




What happens if you apply KT equations to $\pi\pi$ scattering?
JPAC Collab., EPJ,C78,574(2018)

- KT equations for $\pi\pi$ scattering can be written as Roy-like equations:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\text{th}}}^{\infty} dt' \, K_{\ell\ell'}^{II'}(s,t') \, \text{Im}\, t_{\ell'}^{(I')}(t')$$



Results: Analytical comparison with Roy equations

[JPAC Collab., EPJ,C78,574(2018)]

- Roy equations [PL,36B,353(1971)] and KT equations written as:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell', I'} \int_{s_{\rm th}}^{\infty} dt' \, K_{\ell\ell'}^{II'}(s,t') \, \mathrm{Im}\, t_{\ell'}^{(I')}(t')$$

They differ in the expressions for the polynomial ($k_\ell^{(I)}(s)$) and the kernel ($K_{\ell\ell'}^{II'}(s,t')$).

- Restrict KT to
  1. S, P-waves ($t_0^{(0)}$, $t_0^{(2)}$, $t_1^{(1)}$),
  2. one subtraction in each channel: only two subtraction constants.
- Difference between KT and Roy equations amplitudes:

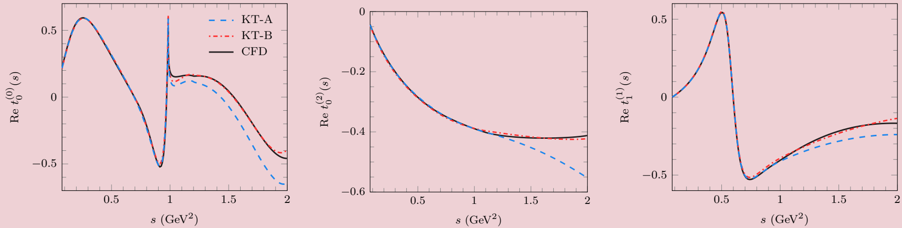
$$(t_{\rm KT})_\ell^{(I)}(s) - (t_{\rm Roy})_\ell^{(I)}(s) = \tilde{k}_\ell^{(I)}(s) - k_\ell^{(I)}(s) + \sum_{\ell', I'} \int_{s_{\rm th}}^{\infty} \mathrm{d}t' \Delta_{\ell\ell'}^{II'}(4m^2, t') \, \mathrm{Im} t_{\ell'}^{(I')}(t')$$

- $\Delta_{\ell\ell'}^{II'}(s,t')$: Difference of kernels is polynomial (logarithmic terms cancel).
- Five conditions that can be fulfilled with the two subtraction constants.

KT equations and Roy equations are equal.

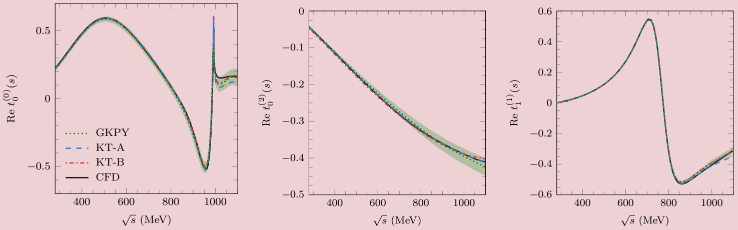
Results: Comparison of amplitudes with GKPY

- Take a succesful parameterization of the amplitude as **input for $\mathrm{Im}t_\ell^{(I)}(s)$**, and compare the **output $\mathrm{Re}t_\ell^{(I)}(s)$** Madrid group, PR,D83,074004(2011)
  - **A:** one subtraction ($\times$ 6), but only 5 free constants. $s_{\max} = 1.0$ GeV^2
  - **B:** two subtractions ($\times$ 6), but only 7 free constants. $s_{\max} = 1.9$ GeV^2



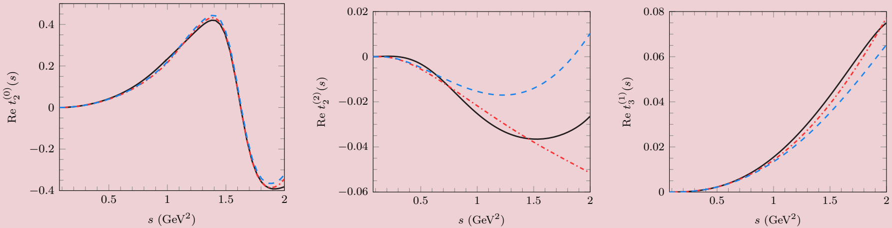
Results: Comparison of amplitudes with GKPY

- Take a succesful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(I)}(s)$**, and compare the **output $\text{Re}t_\ell^{(I)}(s)$** Madrid group, PR,D83,074004(2011)
  - **A:** one subtraction ($\times$ 6), but only 5 free constants. $s_{\text{max}} = 1.0$ GeV^2
  - **B:** two subtractions ($\times$ 6), but only 7 free constants. $s_{\text{max}} = 1.9$ GeV^2



Results: Comparison of amplitudes with GKPY

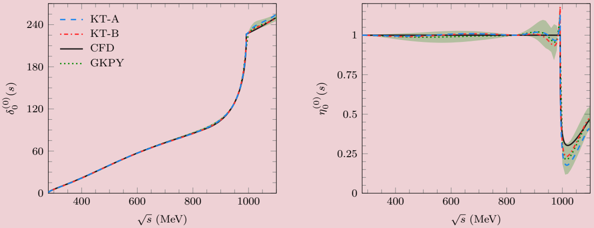
- Take a succesful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(I)}(s)$**, and compare the **output $\text{Re}t_\ell^{(I)}(s)$** Madrid group, PR,D83,074004(2011)
  - **A:** one subtraction ($\times$ 6), but only 5 free constants. $s_{\text{max}} = 1.0$ GeV^2
  - **B:** two subtractions ($\times$ 6), but only 7 free constants. $s_{\text{max}} = 1.9$ GeV^2



13 / 18

Results: Comparison of amplitudes with GKPY

- Take a succesful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(I)}(s)$**, and compare the **output $\text{Re}t_\ell^{(I)}(s)$** Madrid group, PR,D83,074004(2011)
  - **A:** one subtraction ($\times$ 6), but only 5 free constants. $s_{\text{max}} = 1.0$ GeV^2
  - **B:** two subtractions ($\times$ 6), but only 7 free constants. $s_{\text{max}} = 1.9$ GeV^2



Results: Comparison of amplitudes with GKPY (II)

- Threshold parameters (right):

$$\frac{m^{2\ell}}{p^{2\ell}(s)} \mathrm{Re}\, t_\ell^{(I)}(s) = a_\ell^{(I)} + b_\ell^{(I)} \frac{p^2(s)}{m^2} + \cdots \ .$$

- Poles and residues (bottom):

$$t_{II}^{-1}(s) = t_I^{-1}(s) + 2i\sigma(s) \ ,$$

$$t_{II}(s) \simeq \frac{\tilde{g}_p^2}{s - s_p} + \cdots$$

PR,D83,074004('11); PRL,107,072001('11); PL,B749,399('15)

	KT-A	KT-B	GKPY—CFD
$a_0^{(0)}$	0.217	0.213	0.221(9)
$b_0^{(0)}$	0.274	0.275	0.278(7)
$a_0^{(2)}$	−0.044	−0.047	−0.043(8)
$b_0^{(2)}$	−0.078	−0.079	−0.080(9)
$10^3 \cdot a_1^{(1)}$	37.5	37.9	38.5(1.2)
$10^3 \cdot b_1^{(1)}$	5.6	5.7	5.1(3)
$10^4 \cdot a_2^{(0)}$	17.8	17.8	18.8(4)
$10^4 \cdot b_2^{(0)}$	−3.4	−3.4	−4.2(1.0)
$10^4 \cdot a_2^{(2)}$	1.9	1.8	2.8(1.0)
$10^4 \cdot b_2^{(2)}$	−3.2	−3.2	−2.8(8)
$10^5 \cdot a_3^{(1)}$	5.7	5.7	5.1(1.3)
$10^5 \cdot b_3^{(1)}$	−4.0	−4.0	−4.6(2.5)

	KT-A	KT-B	GKPY—CFD
$\sqrt{s_\sigma}$ (MeV)	$(448, 270)$	$(448, 269)$	$(457^{+14}_{-13}, 279^{+11}_{-7})$
$\lvert g_\sigma \rvert$ GeV	3.36	3.37	$3.59^{+0.11}_{-0.13}$
$\sqrt{s_\rho}$ (MeV)	$(762.2, 72.4)$	$(763.4, 73.5)$	$(763.7^{+1.7}_{-1.5}, 73.2^{+1.0}_{-1.1})$
$\lvert g_\rho \rvert$	5.95	6.01	$6.01^{+0.04}_{-0.07}$
$\sqrt{s_{f_0}}$ (MeV)	$(1000, 24)$	$(995, 26)$	$(996 \pm 7, 25^{+10}_{-6})$
$\lvert g_{f_0} \rvert$ (GeV)	2.4	2.3	2.3 ± 0.2
$\sqrt{s_{f_2}}$ (MeV)	$(1275.1, 89.5)$	$(1268.9, 86.4)$	$(1267.3^{+0.8}_{-0.9}, 87 \pm 9)$
$\lvert g_{f_2} \rvert$ (GeV^{-1})	5.6	5.5	5.0 ± 0.3

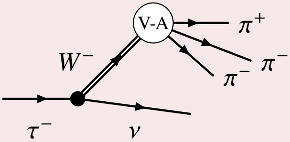
$a_1(1260)$ state

$a_1(1260)$ WIDTH

INSPIRE search

VALUE (MeV)	EVTS	DOCUMENT ID		TECN	COMMENT
250 to 600	**OUR ESTIMATE**				
389 ± 29	**OUR AVERAGE** Error includes scale factor of 1.3.				
430 ±24 ±31		DARGENT	2017	RVUE	$D^0 \to \pi^-\pi^+\pi^-\pi^+$
367 ±9 $^{+28}_{-25}$	420k	ALEKSEEV	2010	COMP	190 $\pi^- \to \pi^-\pi^-\pi^+ Pb'$
• • • We do not use the following data for averages, fits, limits, etc. • • •					
410 ±31 ±30		[1] AUBERT	2007AU	BABR	10.6 $e^+ e^- \to \rho^0\rho^\pm\pi^\mp\gamma$
520 - 680	6360	[2] LINK	2007A	FOCS	$D^0 \to \pi^-\pi^+\pi^-\pi^+$
480 ±20		[3] GOMEZ-DUMM	2004	RVUE	$\tau^+ \to \pi^+\pi^+\pi^-\nu_\tau$
580 ±41	90k	SALVINI	2004	OBLX	$\overline{p}\, p \to 2\,\pi^+ 2\,\pi^-$
460 ±85	205	[4] DRUTSKOY	2002	BELL	$B^{(*)} K^- K^{*0}$
814 ±36 ±13	37k	[5] ASNER	2000	CLE2	10.6 $e^+ e^- \to \tau^+\tau^-$, $\tau^- \to \pi^-\pi^0\pi^0\nu_\tau$

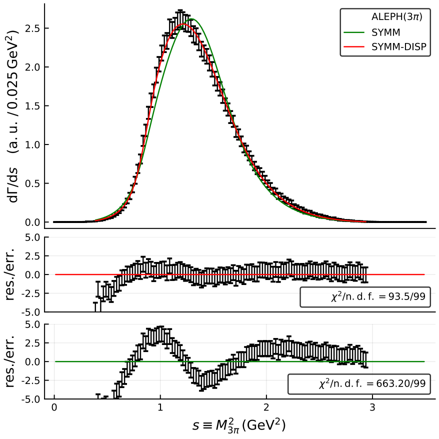
$\tau^- \to \pi^-\pi^+\pi^-\nu$



- V-A: Vector (1^{--}) or Axial (1^{++})
- $I = 1$ due to the charge
- $C_{3\pi} = (-1)^{I+1}$ (neutral partner)
  $\Rightarrow J^{PC} = 1^{++}$

Fit to ALEPH data

[ALEPH, Phys.Rept.421 (2005)][JPAC, PR,D98,096021(2018)]



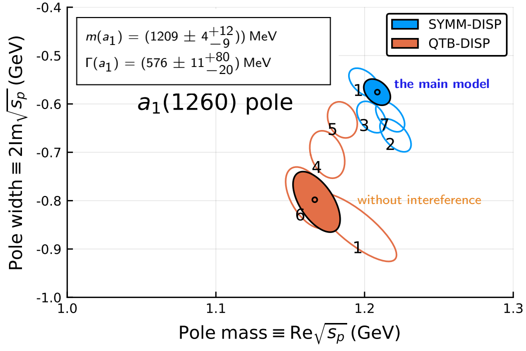
Decay width:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}s} = \frac{\left(s - m_\tau^2\right)^2 \left(s + 2m_\tau^2\right)}{s m_\tau^5} \times \frac{c\rho(s)}{\left| s - m^2 + \frac{i}{2} g^2 C(s) \right|^2}$$

Models:

SYMM/QTB: ρ symmetric or neglect interference.
DISP/NonDISP: $C(s) = \widetilde{\rho}(s)$ or $C(s) = \rho(s)$.

First measurement of the $a_1(1260)$ pole position [JPAC, PR,D98,096021(2018)]

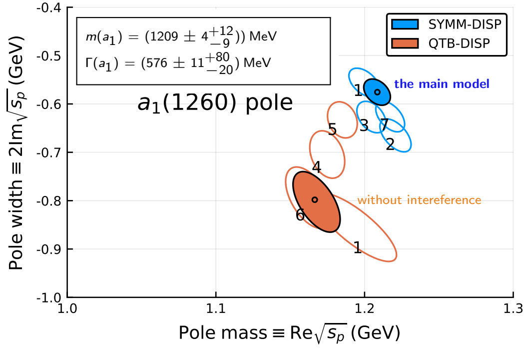


#	Fit studies
1	$s < 2\ \mathrm{GeV}^2$
2	$R' = 3\ \mathrm{GeV}^{-1}$
3	$m'_\rho = m_\rho + 10$ MeV
4	$m'_\rho = m_\rho - 10$ MeV
5	$m'_\rho = m_\rho - 20$ MeV
6	$\Gamma'_\rho = \Gamma_\rho + 5$ MeV
7	$\Gamma'_\rho = \Gamma_\rho - 30$ MeV

Construct KT equations for the decay $a_1(1260) \to 3\pi$

JPAC Collab., *in preparation*

First measurement of the $a_1(1260)$ pole position [JPAC, PR,D98,096021(2018)]



#	Fit studies
1	$s < 2\ \mathrm{GeV}^2$
2	$R' = 3\ \mathrm{GeV}^{-1}$
3	$m'_\rho = m_\rho + 10$ MeV
4	$m'_\rho = m_\rho - 10$ MeV
5	$m'_\rho = m_\rho - 20$ MeV
6	$\Gamma'_\rho = \Gamma_\rho + 5$ MeV
7	$\Gamma'_\rho = \Gamma_\rho - 30$ MeV

Construct KT equations for the decay $a_1(1260) \to 3\pi$

JPAC Collab., *in preparation*

Summary

- KT equations are a powerful tool to study **3-body decays**.
- They allow to implement **two-body unitarity** in all the **three channels** (s, t, u).
- For $\eta \to 3\pi$:
  - Not well described by the perturbative chiral amplitudes.
  - We have presented an **extension** of this approach to **coupled channels**. The extension is quite **general**.
  - Effects of $K\bar{K}$ and $\eta\pi$ amplitudes [$f_0(980)$, $a_0(980)$] play some role in the DP parameters, tend to improve.
- For $\pi\pi$ scattering:
  - We have applied KT equations to $\pi\pi$ scattering as benchmark.
  - Restricted to S- and P-waves, KT equations are equal to Roy equations.
  - When other waves are included, good comparison is obtained with GKPY equations.
- In JPAC we have also studied $a_1(1260) \to 3\pi$, where it would be interesting to apply KT equations.

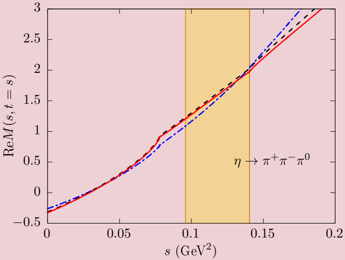
Analysis of three body decays

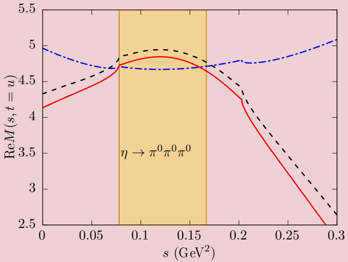
JPAC

Miguel Albaladejo (Jefferson Lab – Theory Center)

April 12, 2019

Results





- Subthreshold region: chiral, elastic, and coupled amplitudes **very close.**
- Adler zero ($s_A \simeq 0.03$ GeV^2):

	NLO	el.	cou.
$s_A/m_{\pi^+}^2 =$	1.42	1.45	1.49

- Substantial influence of coupled channels in the whole region,
- and there is no region in which dispersive and chiral amplitudes agree.
- $T_{\eta\to 3\pi^0} = M_0(s) + M_0(t) + M_0(u) + \ldots$

Quark mass ratio

From the amplitudes $M_I(s)$ one can compute the width up to the unknown factor Q^2:

$$\Gamma = \epsilon_L^2 \int_{4m_\pi^2}^{m_-^2} \mathrm{d}s \int_{t_-(s)}^{t_+(s)} \mathrm{d}t \, |M_0(s) + \cdots|^2$$

$$\epsilon_L = Q^{-2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2} \,, \quad Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

$\Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0)$	
PDG (fit)	1.426(26)
PDG (average)	1.48(5)
CLEO	1.496(43)(35)
chiral $\mathcal{O}(p^4)$	1.425
elastic	1.449
coupled	1.451

Decay	Q elastic	coupled
$\Gamma^{(\mathrm{exp})}_{(\mathrm{neu.})} = 299(11)$ eV	21.9(2)	21.7(2)
$\Gamma^{(\mathrm{exp})}_{(\mathrm{cha.})} = 427(15)$ eV	21.8(2)	21.6(2)

- Effect of inelastic channels $\sim 1\%$ (decreasing)
- Theoretical error on Q:
  - Phase shifts $[s \leqslant 1\ \mathrm{GeV}^2]$: $\sim 1\%$
  - $\mathcal{O}(p^4)$ ampl. $[L_3]$: $\sim 1\%$
  - NNLO ampl.: $\Delta Q_{\mathrm{th.}} = \pm 2.2$

$$Q = 21.6 \pm 0.2 \pm 2.2$$

- Fitted (not matched) polynomial parameters:

Quark mass ratio

From the amplitudes $M_I(s)$ one can compute the width up to the unknown factor Q^2:

$$\Gamma = \epsilon_L^2 \int_{4m_\pi^2}^{m_-^2} \mathrm{d}s \int_{t_-(s)}^{t_+(s)} \mathrm{d}t \, |M_0(s) + \cdots|^2$$

$$\epsilon_L = Q^{-2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2} \,, \quad Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

$\Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0)$	
PDG (fit)	1.426(26)
PDG (average)	1.48(5)
CLEO	1.496(43)(35)
chiral $\mathcal{O}(p^4)$	1.425
elastic	1.449
coupled	1.451

Decay	Q elastic	Q coupled
$\Gamma^{(\mathrm{exp})}_{(\mathrm{neu.})} = 299(11)$ eV	21.9(2)	21.7(2)
$\Gamma^{(\mathrm{exp})}_{(\mathrm{cha.})} = 427(15)$ eV	21.8(2)	21.6(2)

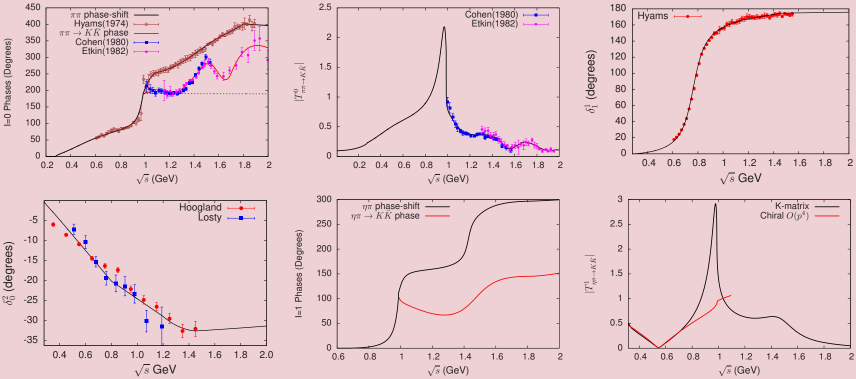
- Effect of inelastic channels $\sim 1\%$ (decreasing)
- Theoretical error on Q:
  - Phase shifts [$s \leqslant 1$ GeV^2]: $\sim 1\%$
  - $\mathcal{O}(p^4)$ ampl. [L_3]: $\sim 1\%$
  - NNLO ampl.: $\Delta Q_{\mathrm{th.}} = \pm 2.2$

$$Q = 21.6 \pm 0.2 \pm 2.2$$

- Fitted (not matched) polynomial parameters:

$$Q_{\mathrm{fit}} = 21.50 \pm 0.67 \pm 0.70$$

Isospin conserving T-matrices



B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rept. **353**, 207 (2001);

R. García-Martín, B. Moussallam, Eur. Phys. J. C**70**, 155 (2010);

B. Moussallam, Eur. Phys. J. C**71**, 1814 (2011);

M. Albaladejo, B. Moussallam, Eur. Phys. J. C**75**, 488 (2015);

Amplitudes $\mathbf{M}_1$ and $\mathbf{M}_2$

An analogous analysis can be done with $M_1(s)$ and $M_2(s)$ amplitudes:

$\mathbf{M}_1(s)$ [P-wave]

$$\mathbf{M}_1(s) = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} (\eta\pi)_{1-} \to (\pi\pi)_{1+} \\ (\eta\pi)_{1-} \to (K\bar{K})_{1+} \end{bmatrix}$$

$$\mathbf{T}_1^1(s) = \begin{bmatrix} (\pi\pi)_1 \to (\pi\pi)_1 & (\pi\pi)_1 \to (K\bar{K})_1 \\ (\pi\pi)_1 \to (K\bar{K})_1 & (K\bar{K})_1 \to (K\bar{K})_1 \end{bmatrix}$$

$$\Delta\mathbf{M}_1(s) = \mathbf{T}_1^{1*}(s)\Sigma^0(s) \times \left[\mathbf{M}_1(s+i\epsilon) + \hat{\mathbf{M}}_1(s)\right]$$

$\mathbf{M}_2(s)$ [S-wave]

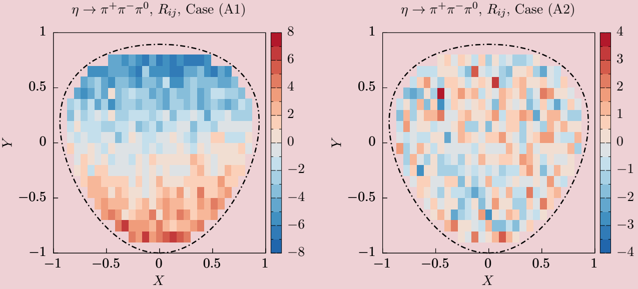
$$\mathbf{M}_2(s) = \begin{bmatrix} M_2 \\ G_{12} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_2 \\ (K\bar{K})_1 \to (\pi\pi)_2 \end{bmatrix}$$

$$t_0^2(s) = t_{(\pi\pi)_2 \to (\pi\pi)_2}$$

$$\text{disc } \mathbf{M}_2(s) = \mathbf{T}^1(s)\Sigma^1(s) \times (\mathbf{M}_2(s-i\epsilon) + \hat{\mathbf{M}}_2(s)) + \sigma_\pi(s)(t_0^2(s))^*(\mathbf{M}_2(s+i\epsilon) + \hat{\mathbf{M}}_2(s))$$

- **Consistent approximation:** $\hat{N}_0(s)$, $\hat{G}_{10}(s)$, $\hat{H}_{10}(s)$, $\hat{G}_{12}(s)$: we neglect these LHC functions (would require all the related cross channels amplitudes...).
- Further approximation: For $I = J = 1$, we consider elastic $\pi\pi$.

Fitting



Tour to the complex plane

JPAC, PR D98 096021(2018)

Analytical continuation

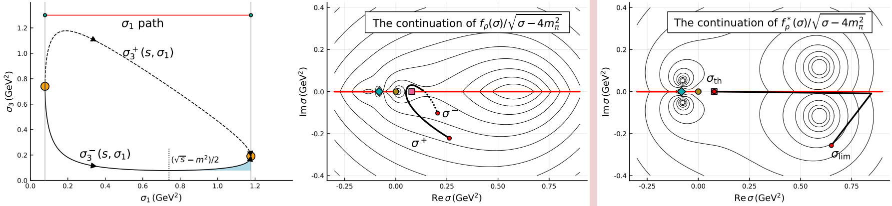
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i\left(\frac{\tilde{\rho}(s)}{2} + \rho(s)\right)\right|.$$

Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors



Tour to the complex plane

JPAC, PR D98 096021(2018)

Analytical continuation

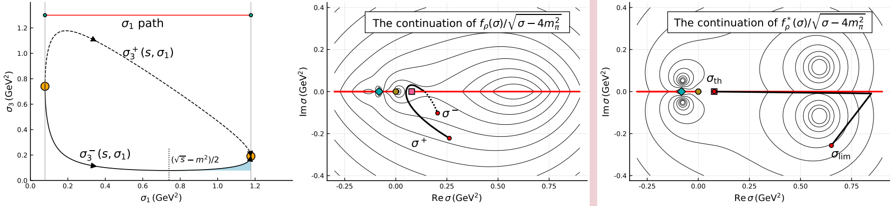
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left(\frac{\tilde{\rho}(s)}{2} + \rho(s) \right) \right|.$$

Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

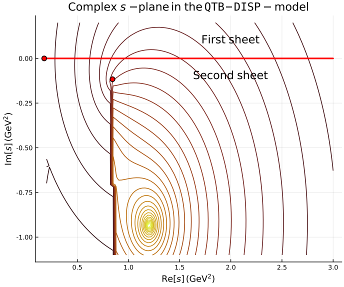
Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors



Tour to the complex plane

JPAC, PR D98 096021(2018)



Analytical continuation

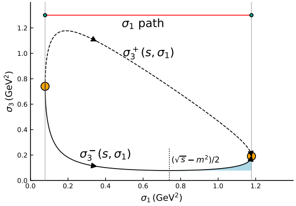
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left(\frac{\tilde{\rho}(s)}{2} + \rho(s) \right) \right|.$$

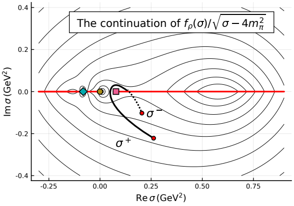
Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

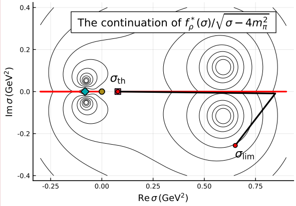
$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors

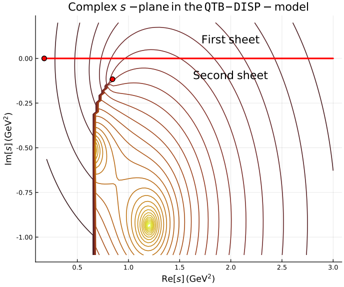






Tour to the complex plane

JPAC, PR D98 096021(2018)



Analytical continuation

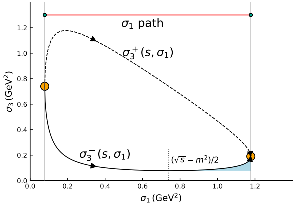
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i \left(\frac{\tilde{\rho}(s)}{2} + \rho(s) \right) \right|.$$

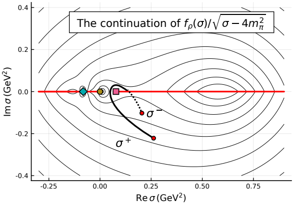
Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

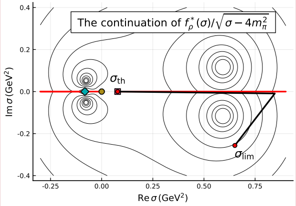
$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors

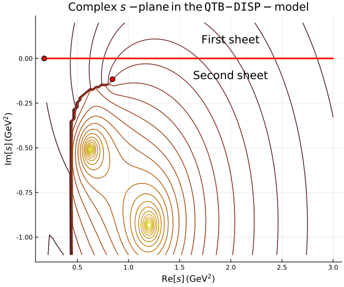






Tour to the complex plane

JPAC, PR D98 096021(2018)



Analytical continuation

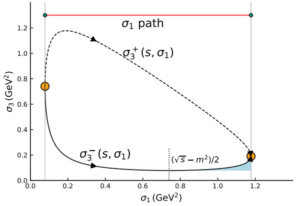
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i\left(\frac{\tilde{\rho}(s)}{2} + \rho(s)\right)\right|.$$

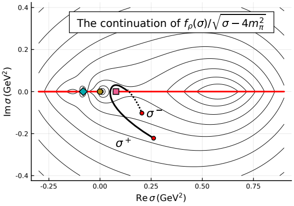
Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

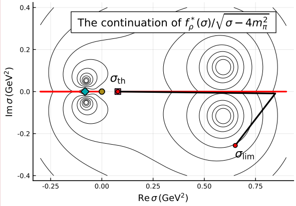
$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors

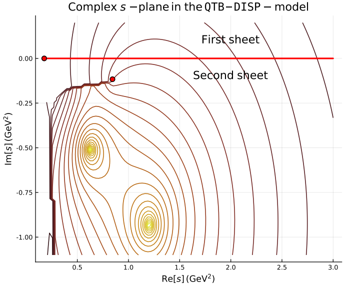






Tour to the complex plane

JPAC, PR D98 096021(2018)



Analytical continuation

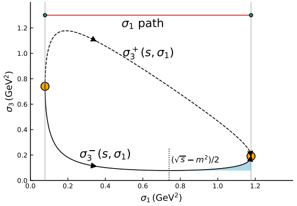
$$|t_{ll}^{-1}(s)| = \left| \frac{m^2 - s}{g^2} - i\left(\frac{\tilde{\rho}(s)}{2} + \rho(s)\right)\right|.$$

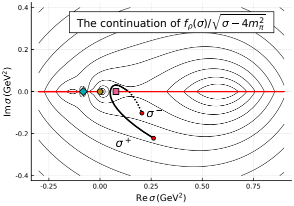
Analytical continuation of $\rho(s)$: integral over the Dalitz plot for the complex energy

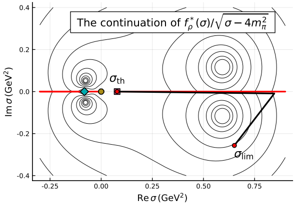
$$\rho(s) = \sum_{\lambda} \int \mathrm{d}\Phi_3 \left| f_\rho(\sigma_1) d_{\lambda 0}(\theta_{23}) - f_\rho(\sigma_3) d_{\lambda 0}(\hat{\theta}_3 + \theta_{12}) \right|^2$$

Analytic contuation of ρ-meson decay amplitude f_ρ

- Breit-Wigner amplitude with the dynamic width
- P-wave Blatt-Weisskopf factors







The spurious pole in the Breit-Wigner model

Energy dependent width, stable particles

$$t(s) = \frac{1}{m^2 - s - im\Gamma(s)}, \quad \Gamma(s) = \Gamma_0 \frac{p(s)}{p(m^2)} \frac{m}{\sqrt{s}}, \quad p(s) = \frac{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{2\sqrt{s}}$$

Example: $m_1 = 140\,\text{MeV},\ m_2 = 770\,\text{MeV},\ m = 1.26\,\text{GeV},\ \Gamma_0 = 0.5\,\text{GeV}$