

# TMDs at small- $x$ : an Operator Treatment

Yuri Kovchegov

The Ohio State University

work with Dan Pitonyak and Matt Sievert,

arXiv:1706.04236 [nucl-th] and 5 other papers

+ papers in preparation

# Outline

- Goal: understanding the small- $x$  asymptotics of TMDS
  - Helicity: helps evaluate the amount of proton spin coming from small  $x$  partons
  - Transversity: small- $x$  contribution to the proton tensor charge
- Quark Helicity:
  - Quark helicity distribution at small  $x$
  - Small- $x$  evolution equations for quark helicity
  - Small- $x$  asymptotics of quark helicity
- Gluon Helicity:
  - Gluon helicity distribution at small  $x$
  - Small- $x$  evolution equations for gluon helicity
  - Small- $x$  asymptotics of quark helicity TMDs
- Quark Transversity:
  - Quark transversity TMD at small  $x$
  - Small- $x$  evolution equation for quark transversity
  - Small- $x$  asymptotics of quark transversity TMDs

# Main Physical Results

- At large  $N_c$  we get for helicity

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

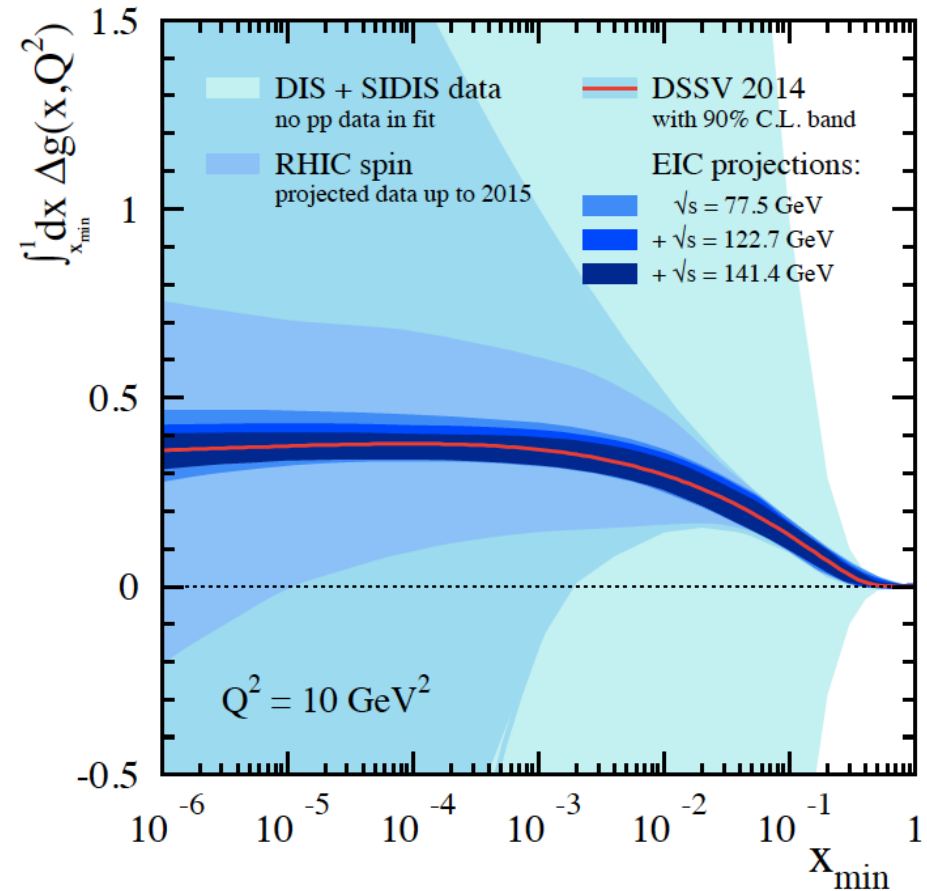
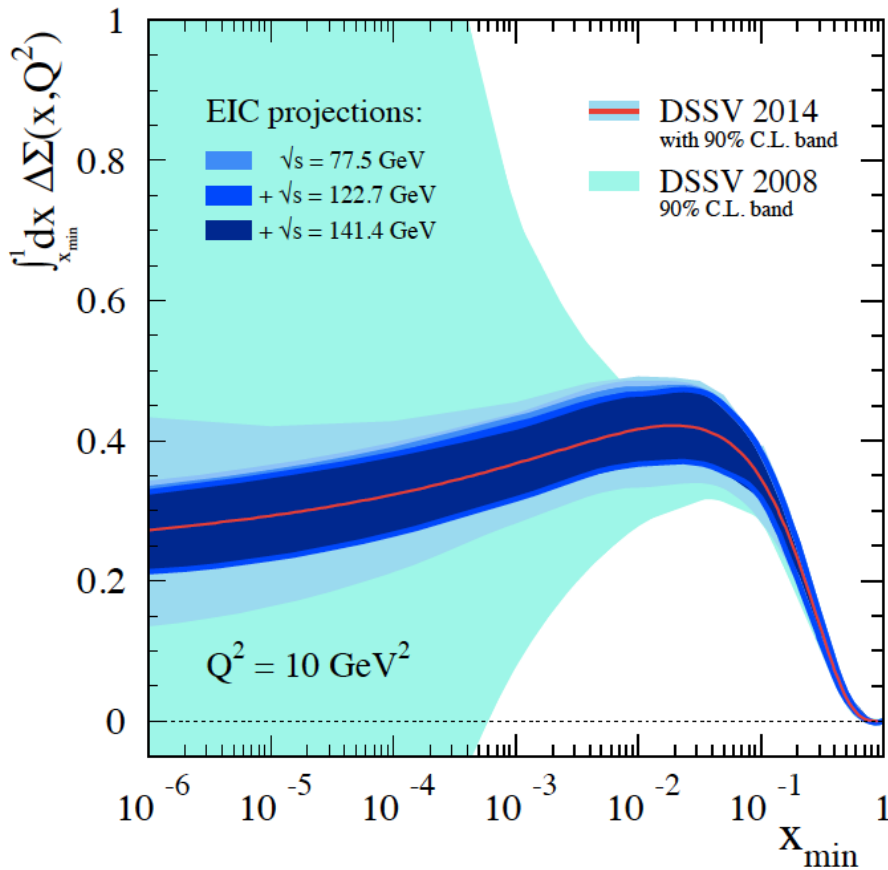
- For transversity TMDs we have (also at large  $N_c$ )

$$h_{1T}^q(x, k_T^2) \sim h_{1T}^{\perp q}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2 \sqrt{\frac{\alpha_s C_F}{\pi}}$$

(Preliminary!)

Our Goal: calculating all PDFs and  
TMDs at Small  $x$

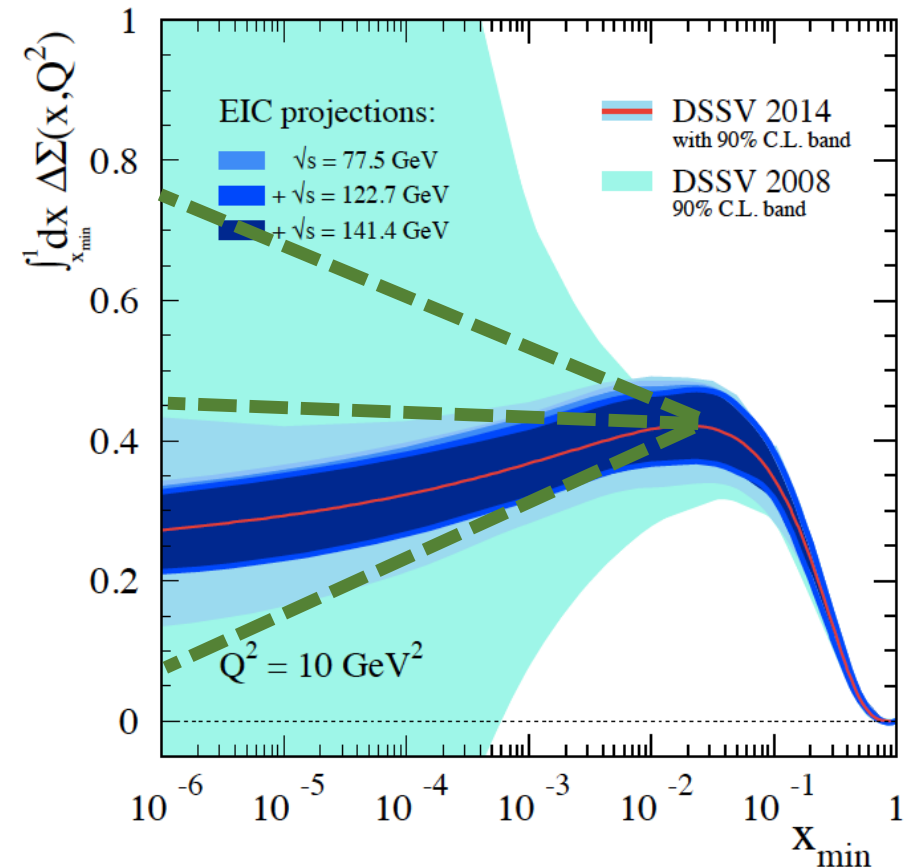
# How much spin is at small x?



- E. Aschenaur et al, [arXiv:1509.06489](https://arxiv.org/abs/1509.06489) [hep-ph]
- Uncertainties are very large at small x!

# Spin at small x

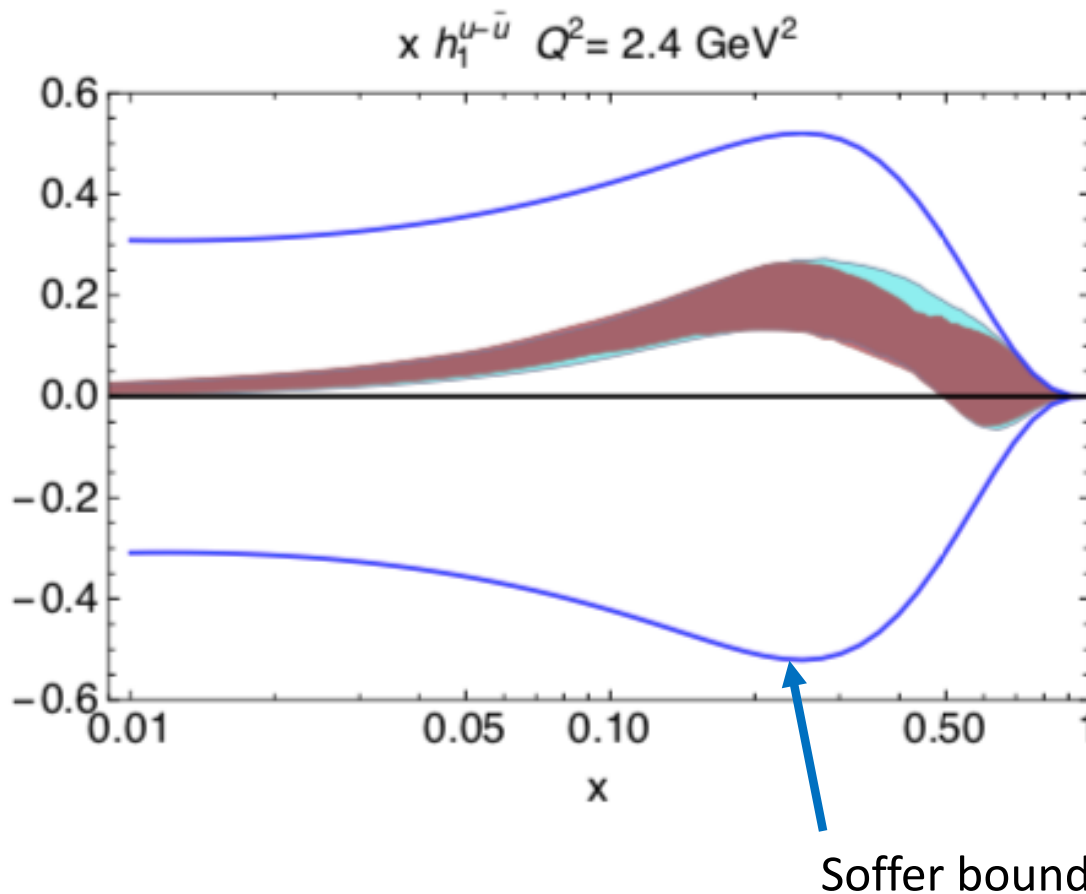
- The goal of this project is to provide theoretical understanding of helicity PDF's at very small x.
- Our work would provide guidance for future hPDF's parametrizations of the existing and new data (e.g., the data to be collected at EIC).
- Alternatively the data can be analyzed using our small-x evolution formalism.



# Transversity and tensor charge

Transversity at small  $x$  is also poorly known, resulting in poor knowledge of the tensor charge:

$$\delta q(Q^2) = \int_0^1 dx h_1(x, Q^2)$$



From the talk by M. Radici at DIS 2018 + see his talk at this meeting

# Quark Helicity at Small $x$ (flavor-singlet case)

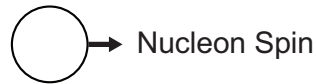
Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]  
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],  
arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph],  
arXiv:1703.05809 [hep-ph] + in preparation



# Observables

- We want to calculate quark helicity PDF and TMD at small x.

## Leading Twist TMDs



		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	$f_1 =$		$h_1^\perp =$ — Boer-Mulders
	L		$g_{1L} =$ — Helicity	$h_{1L}^\perp =$ —
	T	$f_{1T}^\perp =$ — Sivers	$g_{1T}^\perp =$ —	$h_1 =$ — Transversity $h_{1T}^\perp =$ —

# Quark Helicity TMD

- We start with the definition of the quark helicity TMD with a future-pointing Wilson line staple.

$$g_{1L}^q(x, k_T^2) = \frac{1}{(2\pi)^3} \frac{1}{2} \sum_{S_L} S_L \int d^2r dr^- e^{ik \cdot r} \langle p, S_L | \bar{\psi}(0) \mathcal{U}[0, r] \frac{\gamma^+ \gamma^5}{2} \psi(r) | p, S_L \rangle_{r^+=0}$$

- At small-x, in anticipation of the shock-wave formalism, we rewrite the quark helicity TMD as (in  $A^- = 0$  gauge for the + moving proton)

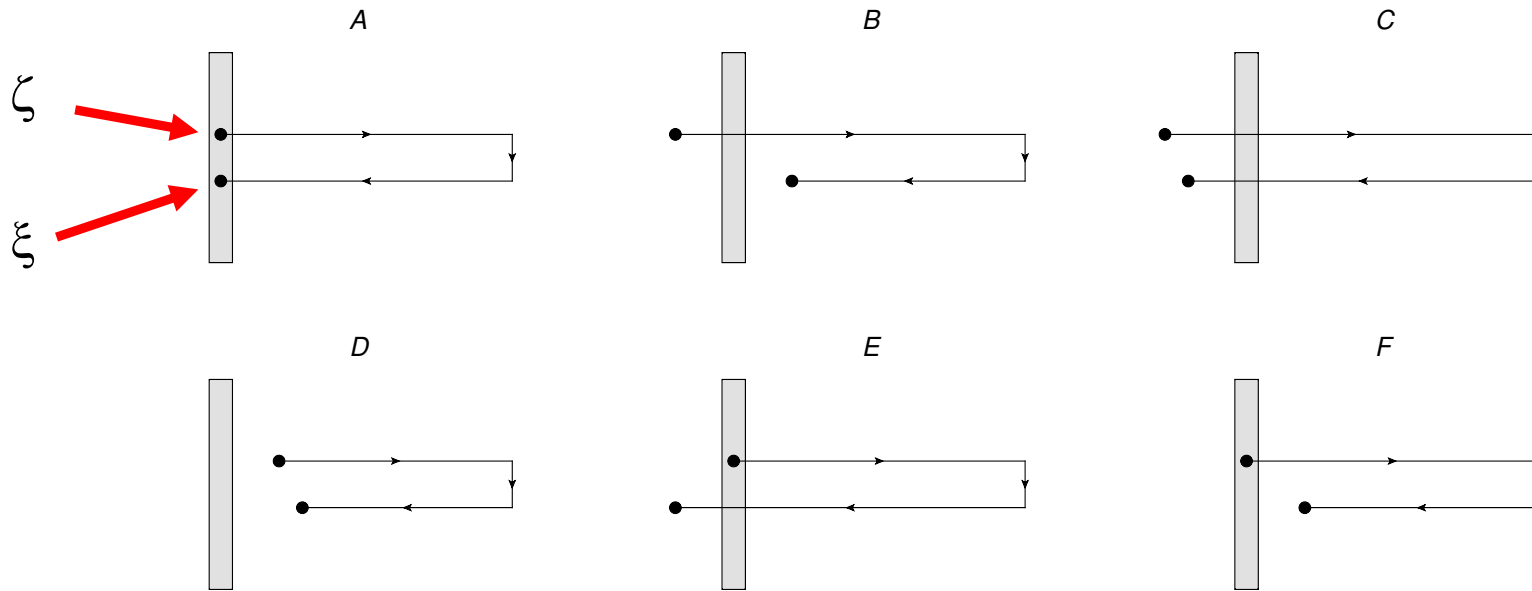
$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta d\zeta^- d^2\xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left( \frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \left\langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-, \infty] V_{\underline{\zeta}}[\infty, \zeta^-] \psi_\beta(\zeta) \right\rangle$$

where the fundamental light-cone Wilson line is

$$V_{\underline{x}}[b^-, a^-] = \text{P exp} \left\{ ig \int_{a^-}^{b^-} dx^- A^+(x^-, \underline{x}) \right\}$$

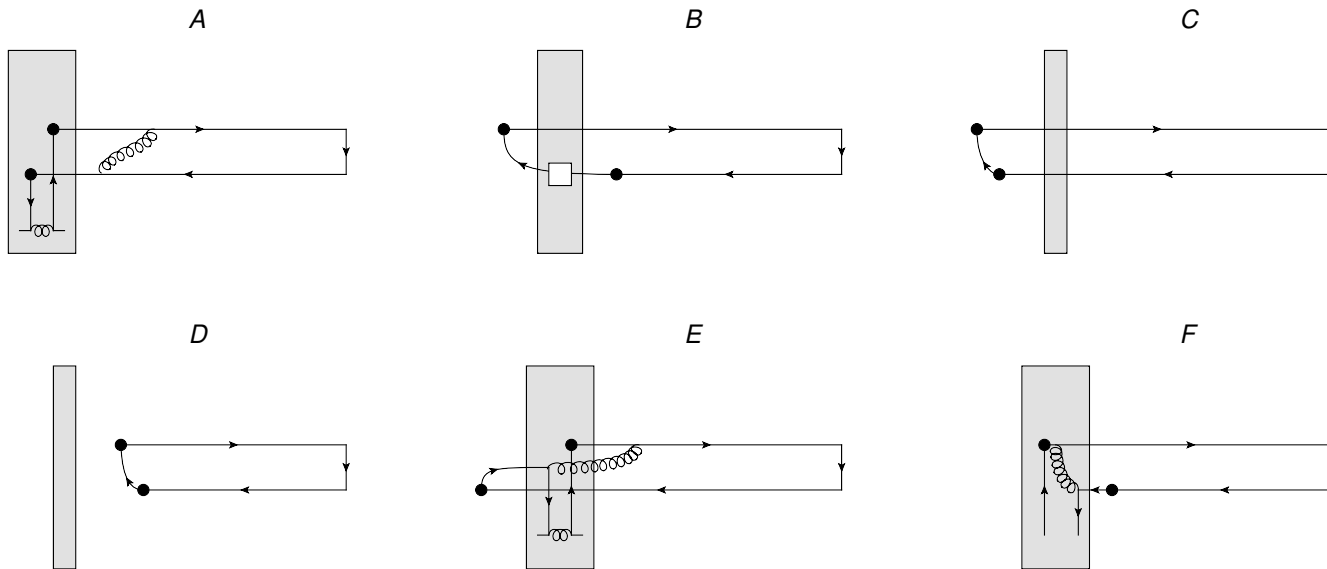
# Quark Helicity TMD at Small x

- At high energy/small-x the proton is a shock wave, and we have the following contributions to the SIDIS quark helicity TMD:



$$g_{1L}^q(x, k_T^2) = \frac{2p^+}{(2\pi)^3} \int d^2\zeta d\zeta^- d^2\xi d\xi^- e^{ik \cdot (\zeta - \xi)} \left( \frac{1}{2} \gamma^+ \gamma^5 \right)_{\alpha\beta} \langle \bar{\psi}_\alpha(\xi) V_{\underline{\xi}}[\xi^-, \infty] V_{\underline{\zeta}}[\infty, \zeta^-] \psi_\beta(\zeta) \rangle$$

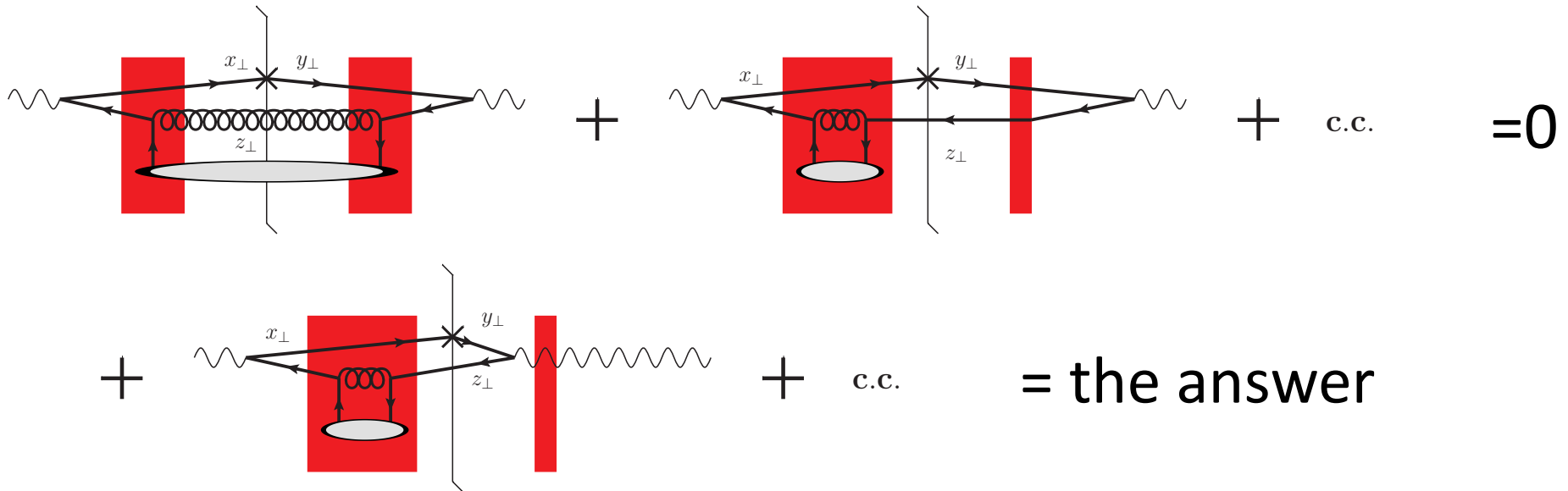
# Quark Helicity TMD at Small $x$



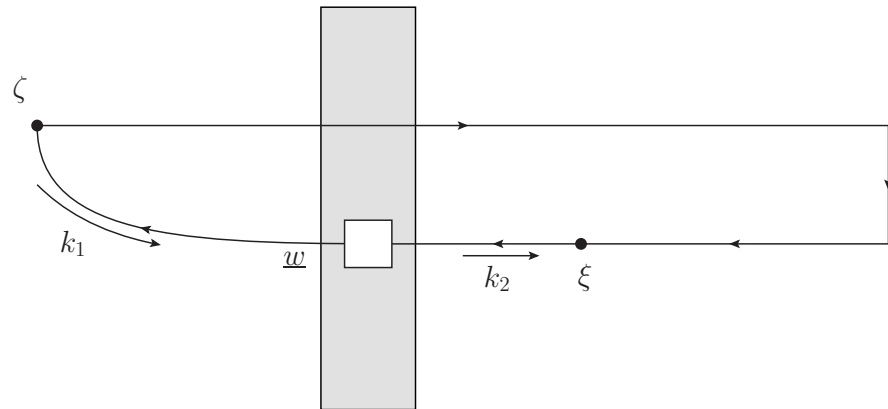
- Clearly diagrams C and D do not transfer spin information from the target, since Wilson lines do not know about polarization.
- Diagram F is energy-suppressed, since the gluon should have no time to be emitted and absorbed inside the shock wave.
- Diagrams of the types A and E++ can be shown to cancel each other.
- We are left with the diagram B.

# Quark Helicity TMD at Small $x$

- Dominance of diagram B can also be obtained by applying crossing symmetry to the SIDIS process (KS '15):



- Compare the last line to the diagram B: reflecting the cc amplitude into the amplitude reduces the above diagram to the one on the right.



# Quark Helicity TMD at Small x

- Evaluating diagram B we arrive at

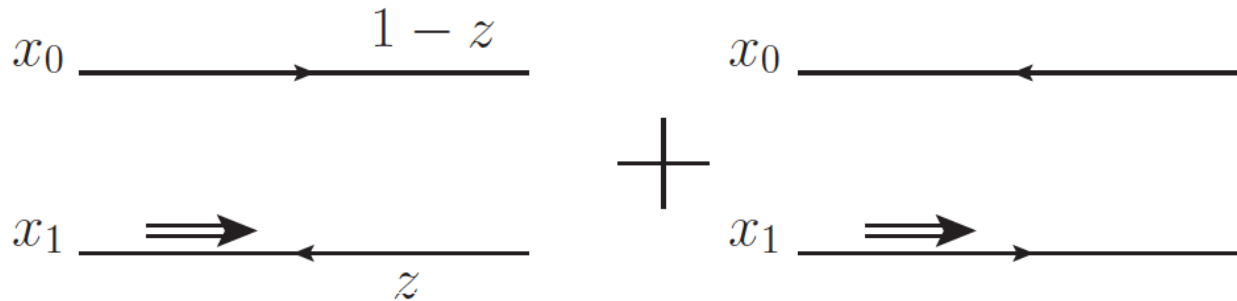
$$g_{1L}^q(x, k_T^2) = \frac{4N_c}{(2\pi)^6} \int d^2\zeta d^2w d^2y e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int_{\Lambda^2/s}^1 \frac{dz}{z} \frac{\underline{\zeta} - \underline{w}}{|\underline{\zeta} - \underline{w}|^2} \cdot \frac{\underline{y} - \underline{w}}{|\underline{y} - \underline{w}|^2} G_{\underline{w}, \underline{\zeta}}(zs)$$

where  $G_{\underline{w}\underline{\zeta}}$  is the polarized dipole amplitude (defined on the next slide).

- Here  $s$  is the cms energy squared,  $\Lambda$  is some IR cutoff, underlining denotes transverse vectors,  $z$  = smallest longitudinal momentum fraction of the dipole momentum out of those carried by the quark and the antiquark
- The same result was previously obtained starting with the SIDIS process (KPS '15) instead of the operator definition of quark helicity TMD: we have thus shown that the two approaches are consistent at small  $x$ .

# Polarized Dipole

- All flavor singlet small-x helicity observables depend on one object, “polarized dipole amplitude”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\left\langle \text{tr} \left[ V_{\underline{0}} V_{\underline{1}}^{pol \dagger} \right] + \text{tr} \left[ V_{\underline{1}}^{pol} V_{\underline{0}}^\dagger \right] \right\rangle\right\rangle(z)$$

unpolarized quark

polarized quark: eikonal propagation,  
non-eikonal spin-dependent interaction

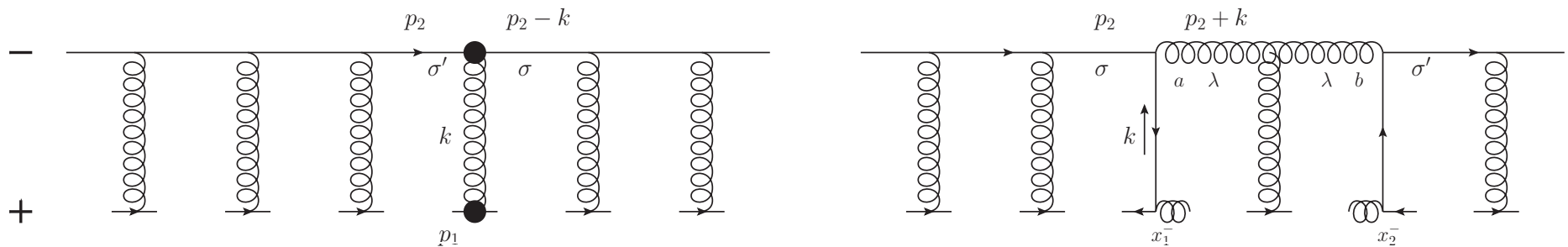
$$V_{\underline{x}} \equiv \mathcal{P} \exp \left[ ig \int_{-\infty}^{\infty} dx^+ A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\left\langle \mathcal{O} \right\rangle\right\rangle(z) \equiv z s \left\langle \mathcal{O} \right\rangle(z)$$

# Polarized fundamental “Wilson line”

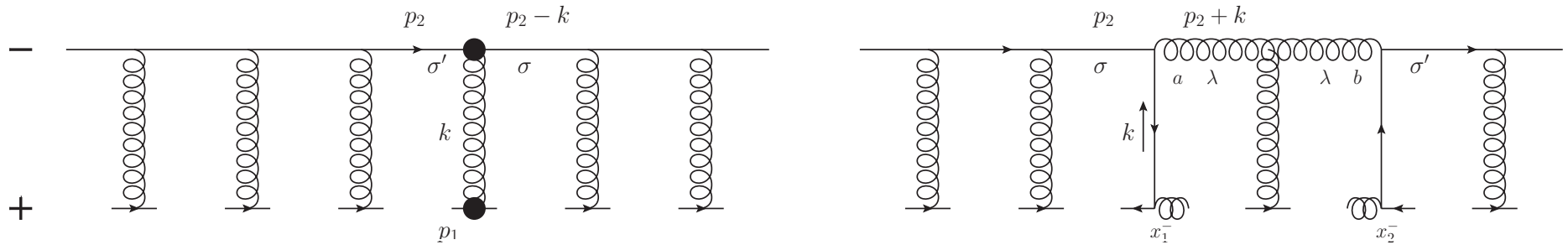
- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized “Wilson line”  $V^{\text{pol}}$ , which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal t-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two t-channel quarks, as shown above.



# Polarized fundamental “Wilson line”



- In the end one arrives at

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int_{-\infty}^{\infty} dx^- V_{\underline{x}}[+\infty, x^-] F^{12}(x^-, \underline{x}) V_{\underline{x}}[x^-, -\infty]$$

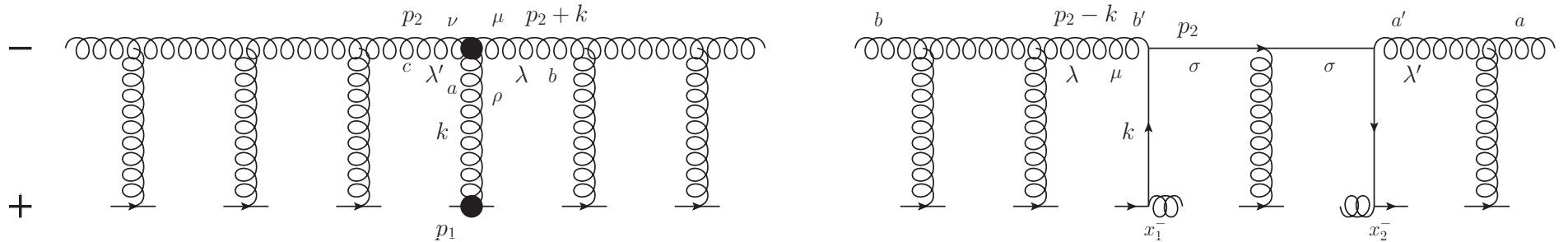
$$- \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[+\infty, x_2^-] t^b \psi_{\beta}(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[ \frac{1}{2} \gamma^+ \gamma^5 \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty].$$

- The first term on the right (the gluon exchange contribution) was known before (KPS '17), the second term (quark exchange) is new.
- We have employed an adjoint light-cone Wilson line

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ ig \int_{a^-}^{b^-} dx^- \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

# Polarized adjoint “Wilson line”

- Quarks mix with gluons. Therefore, we need to construct the adjoint polarized Wilson line --- the leading helicity-dependent part of the gluon scattering amplitude on the longitudinally polarized target.



- The calculation is similar to the quark scattering case. It yields

$$\begin{aligned}
 (U_{\underline{x}}^{pol})^{ab} &= \frac{2i g p_1^+}{s} \int_{-\infty}^{+\infty} dx^- (U_{\underline{x}}[+\infty, x^-] \mathcal{F}^{12}(x^+ = 0, x^-, \underline{x}) U_{\underline{x}}[x^-, -\infty])^{ab} \\
 &+ \frac{g^2 p_1^+}{s} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- U_{\underline{x}}^{aa'}[+\infty, x_2^-] \bar{\psi}(x_2^-, \underline{x}) t^{a'} V_{\underline{x}}[x_2^-, x_1^-] \frac{1}{2} \gamma^+ \gamma_5 t^{b'} \psi(x_1^-, \underline{x}) U_{\underline{x}}^{b'b}[x_1^-, -\infty] + \text{c.c.}
 \end{aligned}$$

# Small-x Evolution at large $N_c$

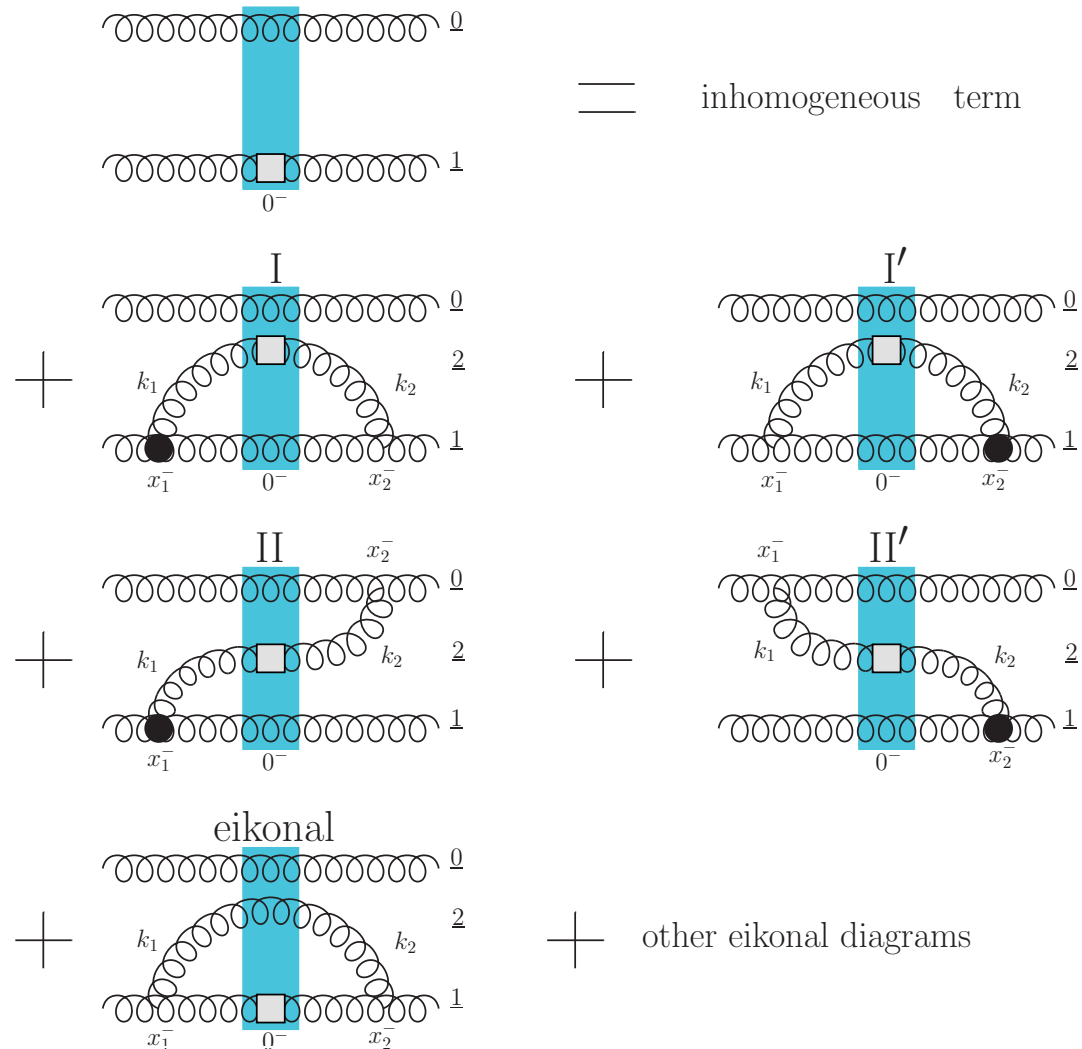
- At large  $N_c$  the evolution is gluon-driven. We will evolve a gluon dipole, remembering that at large  $N_c$  the relation between the adjoint and fundamental longitudinally-polarized gluon dipoles is

$$G_{10}^{adj}(z) = 4 G_{10}(z)$$

(Note that the factor is 4, not 2 like in the unpolarized dipole case.)

# Small-x Evolution at large $N_c$

- We need to sum the following diagrams (box denotes the polarized “Wilson lines”):



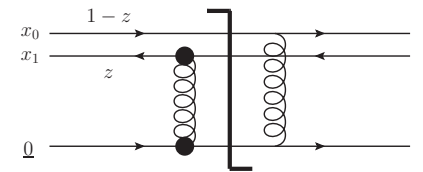
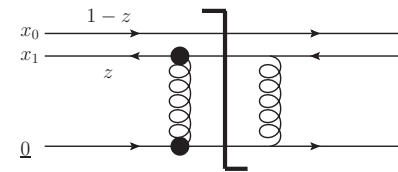
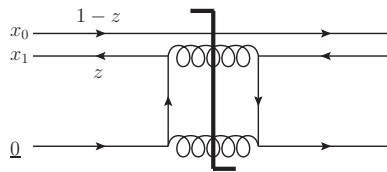
# Large- $N_c$ Evolution

- In the strict DLA limit ( $S=1$ ) and at large  $N_c$  we get (here  $\Gamma$  is an auxiliary function we call the ‘neighbour dipole amplitude’) (KPS ‘15)

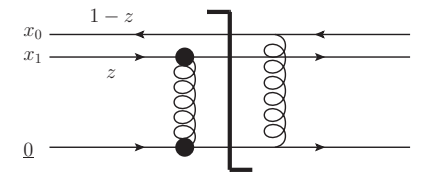
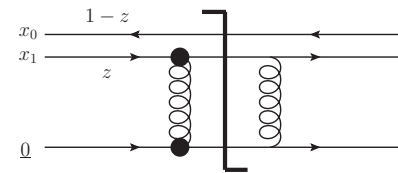
$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} [\Gamma(x_{10}^2, x_{21}^2, z') + 3 G(x_{21}^2, z')]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int_{\frac{1}{z'' s}}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{dx_{32}^2}{x_{32}^2} [\Gamma(x_{10}^2, x_{32}^2, z'') + 3 G(x_{32}^2, z'')]$$

- The initial conditions are given by the Born-level graphs



$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$



$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[ C_F \ln \frac{zs}{\Lambda^2} - 2 \ln(zs x_{10}^2) \right]$$

# Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ln(1/x)$$

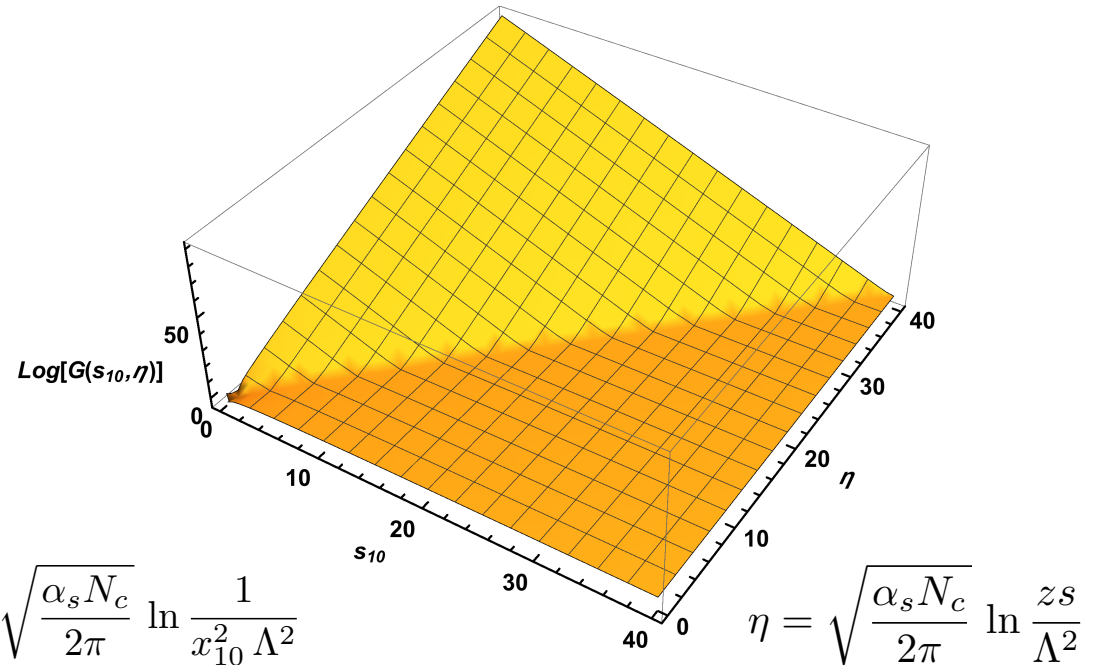
- Helicity evolution resummation parameter is double-logarithmic (DLA):

$$\alpha_s \ln^2 \frac{1}{x}$$

- The second logarithm of  $x$  arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in the UV and IR.
- This was known before: Kirschner and Lipatov '83; Kirschner '84; Bartels, Ermolaev, Ryskin '95, '96; Griffiths and Ross '99; Itakura et al '03; Bartels and Lublinsky '03.

# Quark Helicity at Small x

- These equations can be solved both numerically and analytically. (KPS '16-'17)



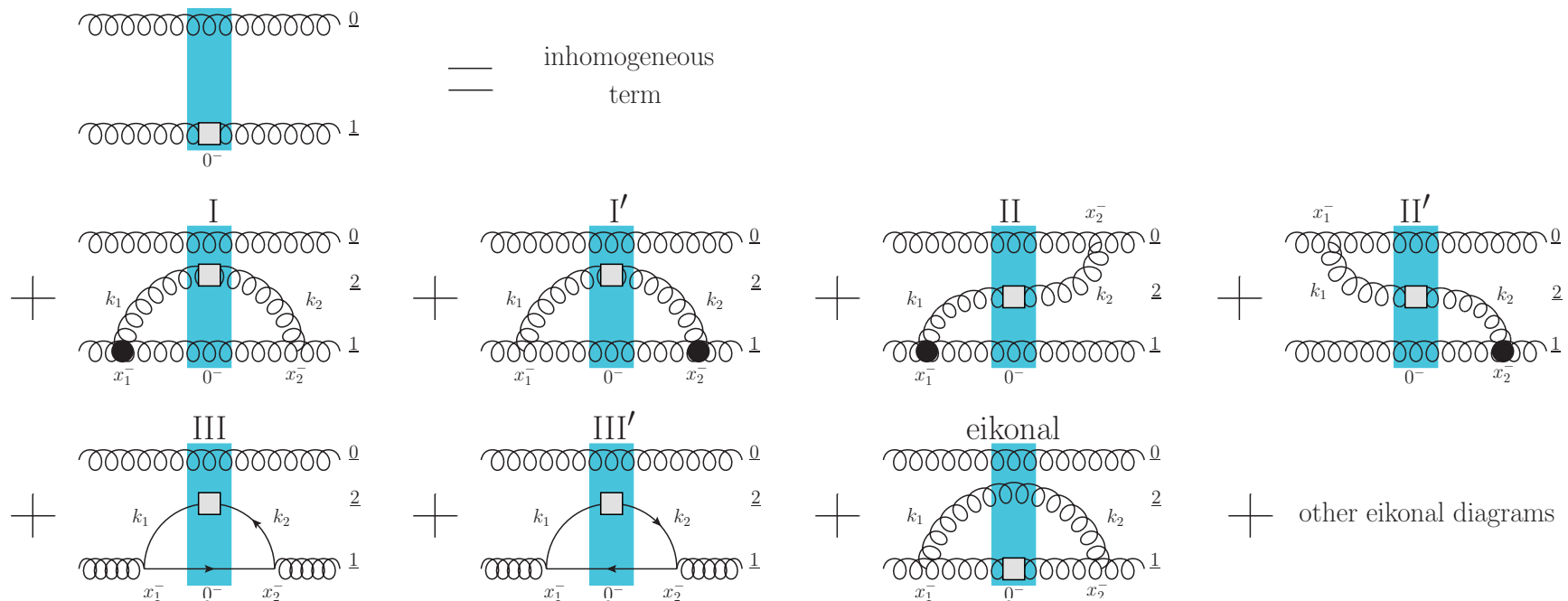
- The small-x asymptotics of quark helicity is (at large  $N_c$ )

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

# Small-x Evolution at large $N_c$ & $N_f$

- At large  $N_c$  &  $N_f$  there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.

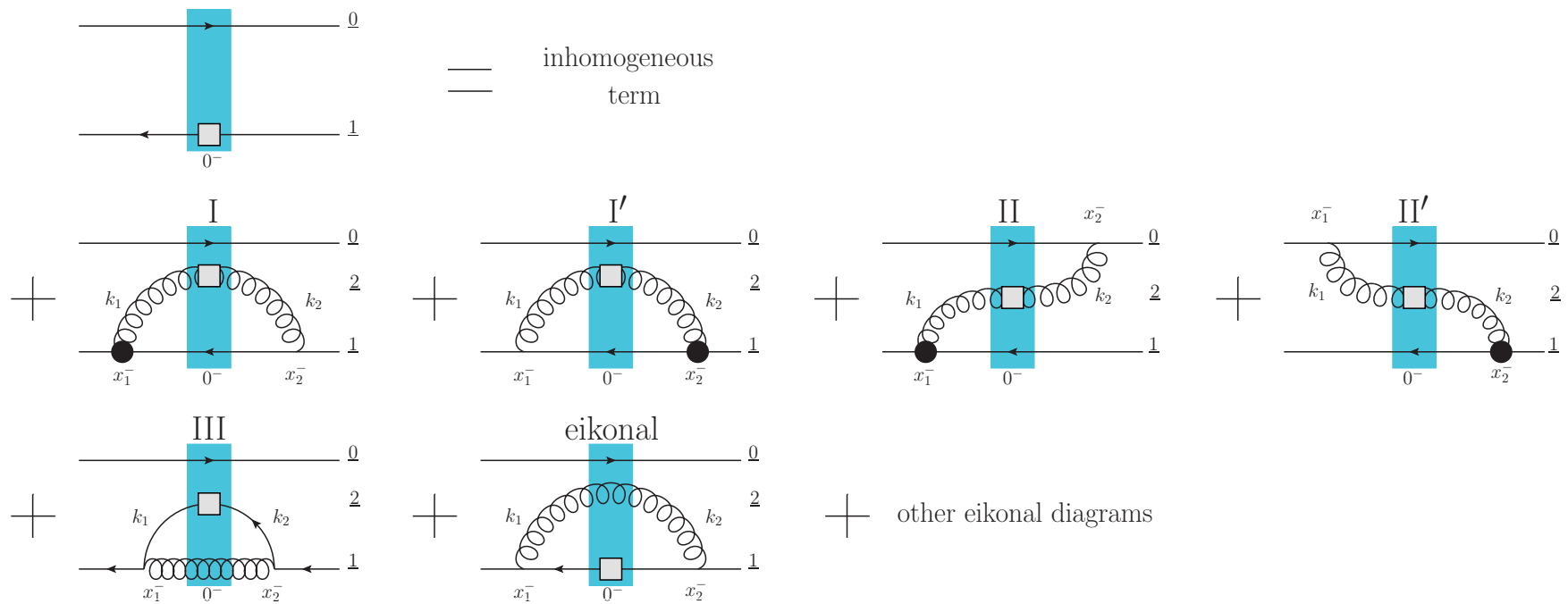
- Here's the adjoint dipole evolution:





# Small-x Evolution at large $N_c$ & $N_f$

- At large  $N_c$  &  $N_f$  there is no simple relation between the adjoint and fundamental polarized dipole amplitudes. We need to construct coupled evolution equations mixing them with each other.
- Here's the fundamental dipole evolution:



# Small-x Evolution at large $N_c$ & $N_f$

- The resulting equations are

$$Q_{10}(zs) = Q_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left\{ \frac{1}{2} \Gamma_{02;21}^{adj}(z') + \frac{1}{2} G_{21}^{adj}(z') + Q_{12}(z') - \bar{\Gamma}_{01;21}(z') \right\}$$

$$+ \frac{\alpha_s N_c}{4\pi} \int_{\Lambda^2/s}^z \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} Q_{21}(z'),$$

$$G_{10}^{adj}(z) = G_{10}^{adj(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^z \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[ \Gamma_{10;21}^{adj}(z') + 3 G_{21}^{adj}(z') \right]$$

$$- \frac{\alpha_s N_f}{2\pi} \int_{\Lambda^2/s}^z \frac{dz'}{z'} \int_{1/(z's)}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} \bar{\Gamma}_{02;21}(z'),$$

These are yet to be solved.

$$\Gamma_{10;21}^{adj}(z') = \Gamma_{10;21}^{adj(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{\max\{\Lambda^2, 1/x_{10}^2\}/s}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[ \Gamma_{10;32}^{adj}(z'') + 3 G_{32}^{adj}(z'') \right]$$

$$- \frac{\alpha_s N_f}{2\pi} \int_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \bar{\Gamma}_{03;32}(z''),$$

$$\bar{\Gamma}_{10;21}(z') = \bar{\Gamma}_{10;21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left\{ \frac{1}{2} \Gamma_{03;32}^{adj}(z'') + \frac{1}{2} G_{32}^{adj}(z'') + Q_{32}(z'') - \bar{\Gamma}_{01;32}(z'') \right\}$$

$$+ \frac{\alpha_s N_c}{4\pi} \int_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int_{1/(z''s)}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} Q_{32}(z'').$$

# Gluon Helicity at Small $x$

Yu.K., D. Pitonyak, M. Sievert, arXiv:1706.04236 [nucl-th]

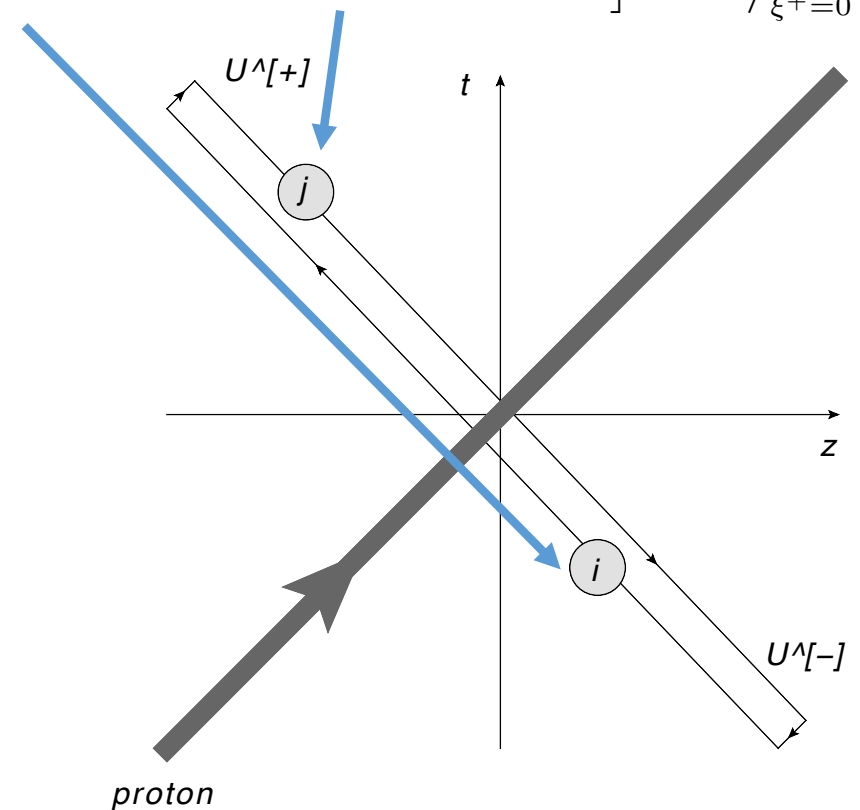
# Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x, k_T^2) = \frac{-2i S_L}{x P^+} \int \frac{d\xi^- d^2\xi}{(2\pi)^3} e^{ixP^+ \xi^- - i\vec{k}\cdot\vec{\xi}} \langle P, S_L | \epsilon_T^{ij} \text{tr} [F^{+i}(0) \mathcal{U}^{[+] \dagger}[0, \xi] F^{+j}(\xi) \mathcal{U}^{[-]}[\xi, 0]] | P, S_L \rangle_{\xi^+=0}$$

- Here  $U^{[+]}$  and  $U^{[-]}$  are future and past Wilson line staples (hence the name 'dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



# Dipole Gluon Helicity TMD

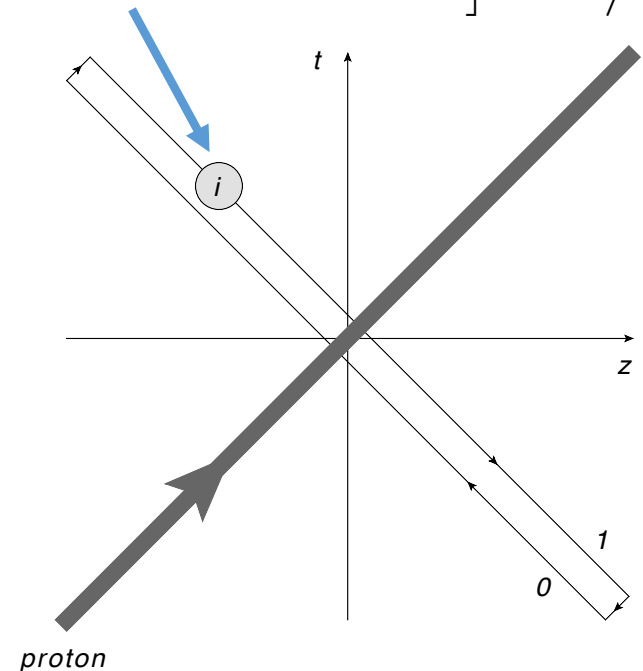
- At small  $x$ , the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G \text{ dip}}(x, k_T^2) = \frac{8i N_c S_L}{g^2 (2\pi)^3} \int d^2 x_{10} e^{i\mathbf{k} \cdot \mathbf{x}_{10}} k_{\perp}^i \epsilon_T^{ij} \left[ \int d^2 b_{10} G_{10}^j(zs = \frac{Q^2}{x}) \right]$$

- Here we obtain a new operator, which is a transverse vector (written here in  $A^-=0$  gauge):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_0[\infty, -\infty] V_1[-\infty, x^-] (-ig) \tilde{A}^i(x^-, \underline{x}) V_1[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- Note that  $k_{\perp}^i \epsilon_T^{ij}$  can be thought of as a transverse curl acting on  $G_{10}^i(z)$  and not just on  $\tilde{A}^i(x^-, \underline{x})$  -- different from the polarized dipole amplitude!



# Dipole TMD vs dipole amplitude

- Note that the operator for the dipole gluon helicity TMD

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] (-ig) \tilde{A}^i(x^-, \underline{x}) V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

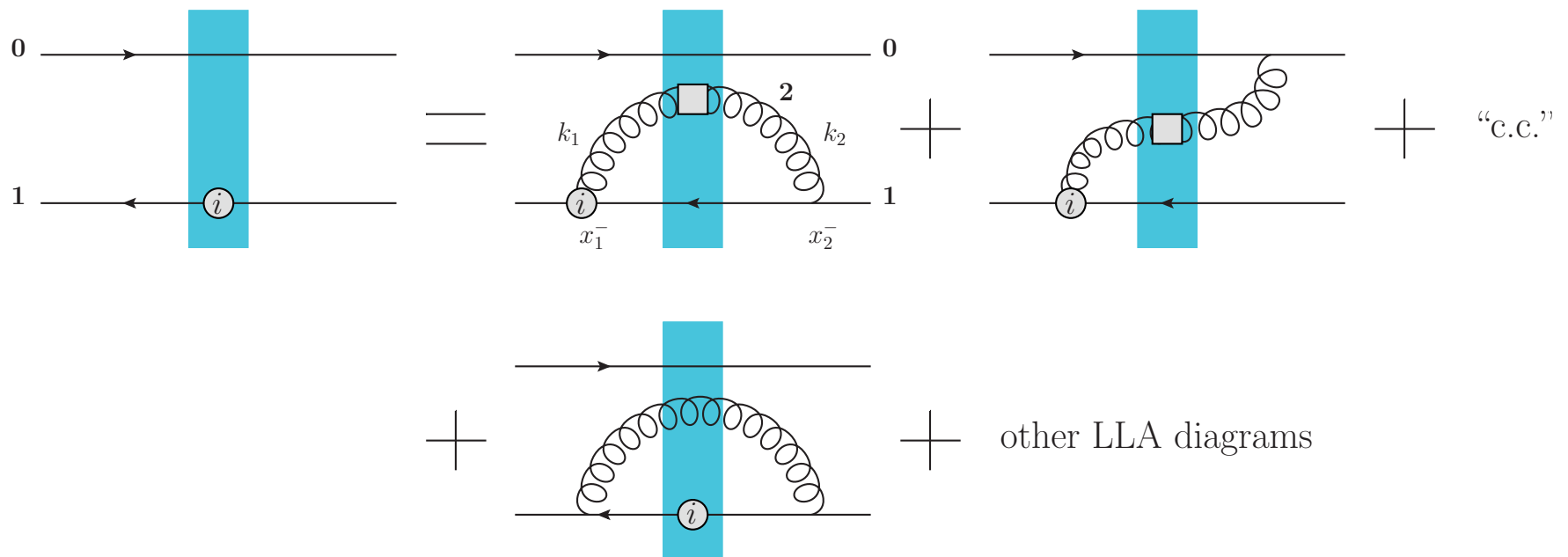
is different from the polarized dipole amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[ V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] (-ig) \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMDs.)
- This is different from the unpolarized gluon TMD case.

# Evolution Equation

- To construct evolution equation for the operator  $G^i$  governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



# Large- $N_c$ Evolution: Equations

- This results in the following evolution equations:

$$\begin{aligned}
 G_{10}^i(zs) &= G_{10}^{i(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{21})_{\perp}^j}{x_{21}^2} \left[ \Gamma_{20,21}^{gen}(z's) + G_{21}(z's) \right] \\
 &\quad - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{20})_{\perp}^j}{x_{20}^2} \left[ \Gamma_{20,21}^{gen}(z's) + \Gamma_{21,20}^{gen}(z's) \right] \\
 &\quad + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[ G_{12}^i(z's) - \Gamma_{10,21}^i(z's) \right] \\
 \\
 \Gamma_{10,21}^i(z's) &= G_{10}^{i(0)}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2x_3 \ln \frac{1}{x_{31}\Lambda} \frac{\epsilon_T^{ij}(x_{31})_{\perp}^j}{x_{31}^2} \left[ \Gamma_{30,31}^{gen}(z''s) + G_{31}(z''s) \right] \\
 &\quad - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2x_3 \ln \frac{1}{x_{31}\Lambda} \frac{\epsilon_T^{ij}(x_{30})_{\perp}^j}{x_{30}^2} \left[ \Gamma_{30,31}^{gen}(z''s) + \Gamma_{31,30}^{gen}(z''s) \right] \\
 &\quad + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int_{\frac{1}{z''s}}^{\min[x_{10}^2, x_{21}^2 \frac{z'}{z''}]} \frac{dx_{31}^2}{x_{31}^2} \left[ G_{13}^i(z''s) - \Gamma_{10,31}^i(z''s) \right].
 \end{aligned}$$



# Large- $N_c$ Evolution: Equations

- Here

$$\Gamma_{20,21}^{gen}(z's) = \theta(x_{20} - x_{21}) \Gamma_{20,21}(z's) + \theta(x_{21} - x_{20}) G_{20}(z's)$$

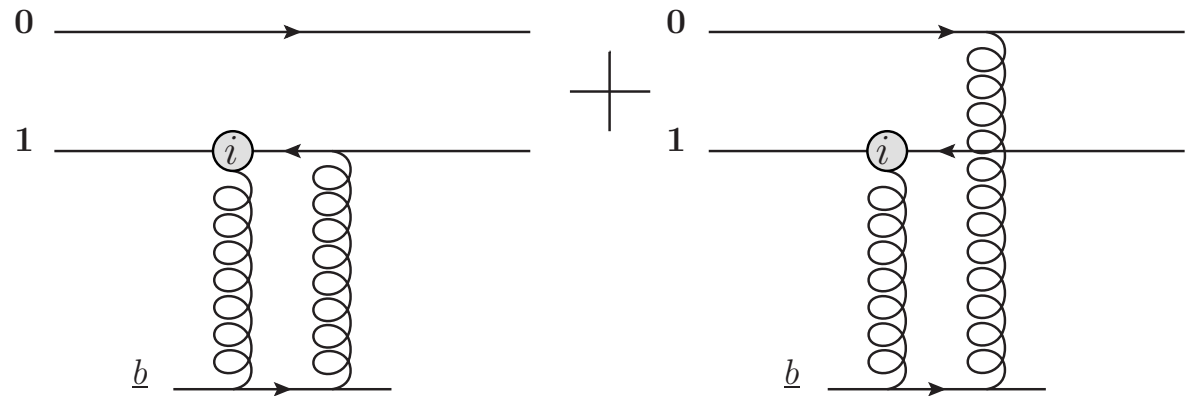
is an object which we know from the quark helicity evolution, as the latter gives us  $G$  and  $\Gamma$ .

- Note that our evolution equations mix the gluon ( $G^i$ ) and quark ( $G$ ) small- $x$  helicity evolution operators:

$$\begin{aligned}
 G_{10}^i(zs) = & G_{10}^{i(0)}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{21})_{\perp}^j}{x_{21}^2} \left[ \Gamma_{20,21}^{gen}(z's) + G_{21}(z's) \right] \\
 & - \frac{\alpha_s N_c}{2\pi^2} \int_{\frac{\Lambda^2}{s}}^z \frac{dz'}{z'} \int d^2x_2 \ln \frac{1}{x_{21}\Lambda} \frac{\epsilon_T^{ij}(x_{20})_{\perp}^j}{x_{20}^2} \left[ \Gamma_{20,21}^{gen}(z's) + \Gamma_{21,20}^{gen}(z's) \right] \\
 & + \frac{\alpha_s N_c}{2\pi} \int_{\frac{1}{x_{10}^2 s}}^z \frac{dz'}{z'} \int_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[ G_{12}^i(z's) - \Gamma_{10,21}^i(z's) \right]
 \end{aligned}$$

# Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2 b_{10} G_{10}^{i(0)}(zs) = \int d^2 b_{10} \Gamma_{10,21}^{i(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c} \pi \epsilon^{ij} x_{10}^j \ln \frac{1}{x_{10} \Lambda}$$

- Note that these initial conditions have no  $\ln s$ , unlike the initial conditions for the quark evolution:

$$\int d^2 b_{10} G_{10}^{(0)}(zs) = \int d^2 b_{10} \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c} \pi \ln(zs x_{10}^2)$$

# Large- $N_c$ Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region yielding the small- $x$  gluon helicity intercept

$$\alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- We obtain the small- $x$  asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

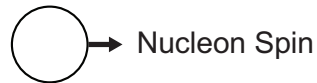
# Quark Transversity at Small $x$

Yu.K., M. Sievert, in preparation (and hence preliminary)

# Observables

- We want to calculate quark transversity TMD at small x:

## Leading Twist TMDs



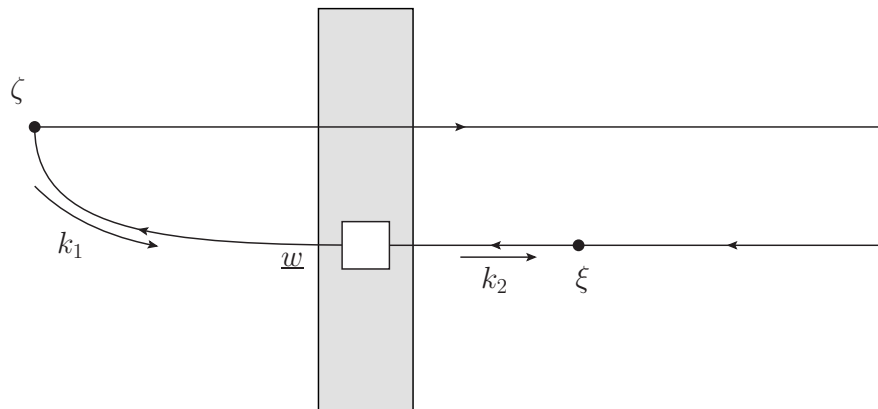
		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	$f_1 =$		$h_1^\perp =$ - Boer-Mulders
	L		$g_{1L} =$ → -  → Helicity	$h_{1L}^\perp =$ → -  →
	T	$f_{1T}^\perp =$ ↑ -  ↓ Sivers	$g_{1T}^\perp =$ ↑ -  ↑	$h_1 =$ ↑ -  ↓ Transversity $h_{1T}^\perp =$ → -  →

# Quark Transversity Operator

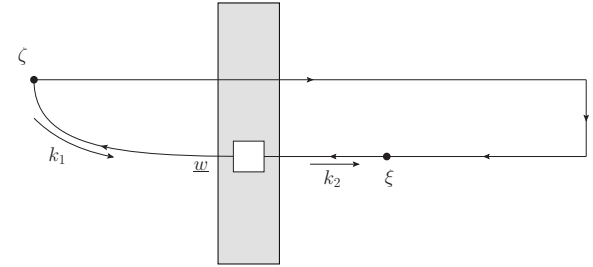
- Analysis of quark transversity is similar to quark helicity: we start with the operator definition (for proton spin in the x-direction)

$$h_{1T}^q(x, k_T^2) + \frac{k_x^2}{M^2} h_{1T}^{\perp q}(x, k_T^2) = \frac{1}{(2\pi)^3} \int d^2r dr^- e^{ik \cdot r} \langle p, S_x = +1 | \bar{\psi}(0) \mathcal{U}[0, r] \frac{\gamma^5 \gamma^+ \gamma^1}{2} \psi(r) | p, S_x = +1 \rangle_{r^+=0}$$

- Diagram analysis again shows that the B-type diagrams dominate:



# Quark Transversity Operator



- Calculating the B-graph contribution we get

$$h_{1T}^q(x, k_T^2) + \frac{k_x^2}{M^2} h_{1T}^{\perp q}(x, k_T^2) = -x \frac{8N_c}{(2\pi)^4} \int d^2x_0 d^2x_1 \int_{\Lambda^2/s}^1 \frac{dz}{z} \int \frac{d^2k_1}{(2\pi)^2} e^{i(\underline{k}_1 + \underline{k}) \cdot \underline{x}_{10}} \frac{1}{\underline{k}_1^2 \underline{k}^2} \left[ \frac{1}{\underline{k}_1^2} + \frac{1}{\underline{k}^2} \right] \\ \times (-2 k_1^x k^x + \underline{k}_1 \cdot \underline{k} - m^2) i T_{10}(zs)$$

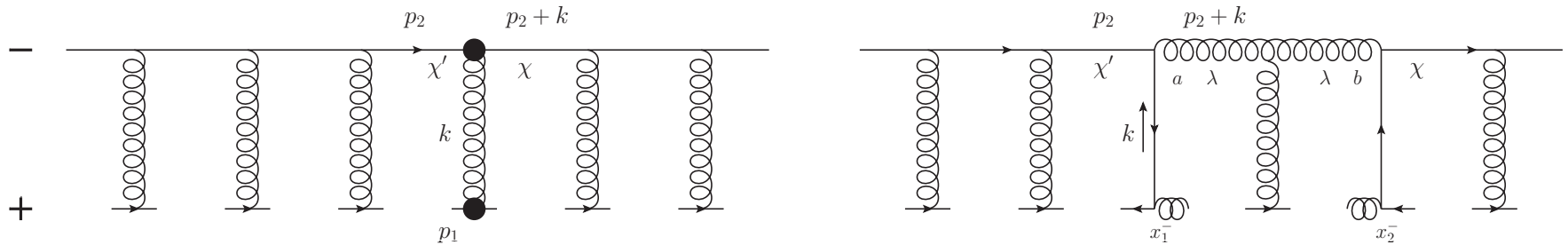
- We have defined a transversely polarized dipole operator

$$T_{10}(zs) \equiv \frac{(zs)^2}{2N_c i} \left\langle \text{tr} \left[ V_{\underline{0}} V_{\underline{1}}^{pol, T \dagger} \right] - \text{tr} \left[ V_{\underline{1}}^{pol, T} V_{\underline{0}}^\dagger \right] \right\rangle_{S_x = +1}$$

- Transverse-spin dependent interaction is suppressed by two (!) powers of energy, hence we rescaled  $T_{10}$  by  $s^2$ .

# Transversely polarized “Wilson line”

- Next we need to calculate the fundamental transversely polarized “Wilson line” --- the leading transverse-spin dependent part of the quark scattering amplitude on a transversely polarized target.
- The diagrams are similar to the helicity case:

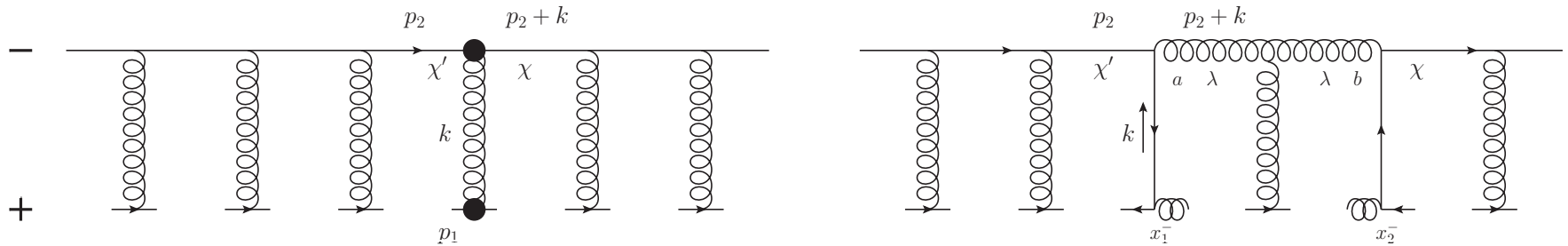


- The result is ( $S^i$  is a unit vector in the transverse spin direction)

$$\begin{aligned}
 V_{\underline{x}}^{pol,T} = & \frac{4g m (p_1^+)^2}{s^2} \int_{-\infty}^{+\infty} dx^- V_{\underline{x}}[+\infty, x^-] S^i [i \epsilon^{ij} F^{-j}(x^-, \underline{x})] V_{\underline{x}}[x^-, -\infty] \\
 & - \frac{g^2 (p_1^+)^2}{2 s^2} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[+\infty, x_2^-] t^b \psi_{\beta}(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[ \left( i \gamma^5 \underline{S} \cdot \underline{\overleftarrow{D}} - \underline{S} \times \underline{\overleftarrow{D}} \right) \gamma^+ \gamma^- \right. \\
 & \left. + (i \gamma^5 \underline{S} \cdot \underline{D} - \underline{S} \times \underline{D}) \gamma^- \gamma^+ \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty]
 \end{aligned}$$



# Transversely polarized “Wilson line”



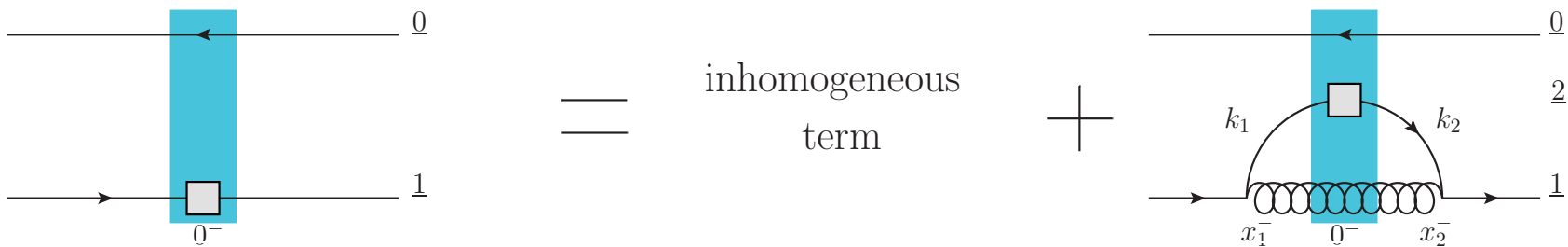
- Note that the interaction with the transverse polarization carrying gluons enters with a factor of quark mass  $m$ , and hence does not give the double logarithmic contribution.

$$\begin{aligned}
 V_{\underline{x}}^{pol,T} &= \frac{4g m (p_1^+)^2}{s^2} \int_{-\infty}^{+\infty} dx^- V_{\underline{x}}[+\infty, x^-] S^i [i \epsilon^{ij} F^{-j}(x^-, \underline{x})] V_{\underline{x}}[x^-, -\infty] \\
 &- \frac{g^2 (p_1^+)^2}{2 s^2} \int_{-\infty}^{\infty} dx_1^- \int_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[+\infty, x_2^-] t^b \psi_{\beta}(x_2^-, \underline{x}) U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[ \left( i \gamma^5 \underline{S} \cdot \overleftarrow{D} - \underline{S} \times \overleftarrow{D} \right) \gamma^+ \gamma^- \right. \\
 &\left. + (i \gamma^5 \underline{S} \cdot \underline{D} - \underline{S} \times \underline{D}) \gamma^- \gamma^+ \right]_{\alpha\beta} \bar{\psi}_{\alpha}(x_1^-, \underline{x}) t^a V_{\underline{x}}[x_1^-, -\infty]
 \end{aligned}$$

- Therefore, at DLA, quarks do not mix with gluons as they evolve to small  $x$ , and we do not need the gluon polarized “Wilson line”.

# Evolution equation for quark transversity

- Constructing the evolution equation for the transverse polarized fundamental dipole amplitude is straightforward (though a little cumbersome). Diagrammatically it is



- We get

$$T_{10}(zs) = T_{10}^{(0)}(zs) + \frac{\alpha_s N_c}{2\pi} \int_{\Lambda^2/s}^1 \frac{dz'}{z'} \int_{1/z's}^{x_{10}^2 z/z'} \frac{dx_{21}^2}{x_{21}^2} T_{10}(z's)$$

# Small-x Asymptotics of Quark Transversity

- Solution of the transversity evolution equation is straightforward.
- The resulting small-x asymptotics is

$$h_{1T}^q(x, k_T^2) \sim h_{1T}^{\perp q}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2\sqrt{\frac{\alpha_s C_F}{\pi}}$$

- Note the suppression by  $x^2$  compared to the unpolarized quark TMDs.
- For  $\alpha_s = 0.3$  we get

$$h_{1T}^q(x, k_T^2) \sim h_{1T}^{\perp q}(x, k_T^2) \sim x^{0.286}$$

- This certainly satisfies the Soffer bound, but is not likely to produce much tensor charge from small x.

$$\delta q(Q^2) = \int_0^1 dx h_1(x, Q^2)$$

# Conclusions

- We now have a well-defined operator prescription for finding the small- $x$  asymptotics of any TMD (either at large- $N_c$  or at large  $N_c$ & $N_f$ ).

- We have

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

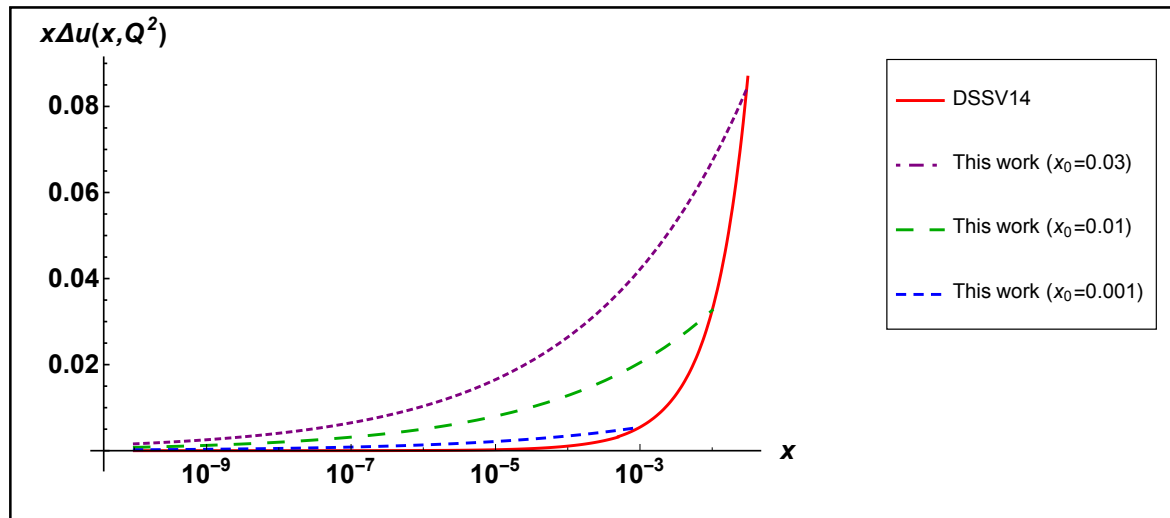
$$h_{1T}^q(x, k_T^2) \sim h_{1T}^{\perp q}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_t^q} \quad \text{with} \quad \alpha_t^q = -1 + 2 \sqrt{\frac{\alpha_s C_F}{\pi}}$$

- Future helicity work may involve including running coupling corrections + solving the large- $N_c$ & $N_f$  equations + OAM at small  $x$  to constrain the spin+OAM coming from small- $x$  quarks and gluons.

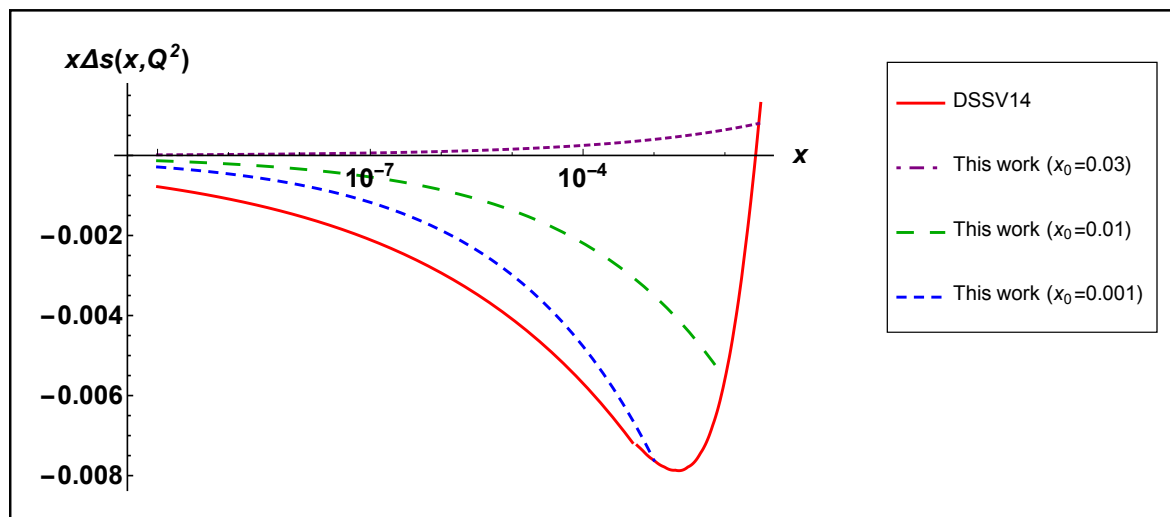
# Backup Slides

# Impact of our $\Delta\Sigma$ on the proton spin

- We have attached a  $\Delta\tilde{\Sigma}(x, Q^2) = N x^{-\alpha_h}$  curve to the existing hPDF's fits at some ad hoc small value of  $x$  labeled  $x_0$  :

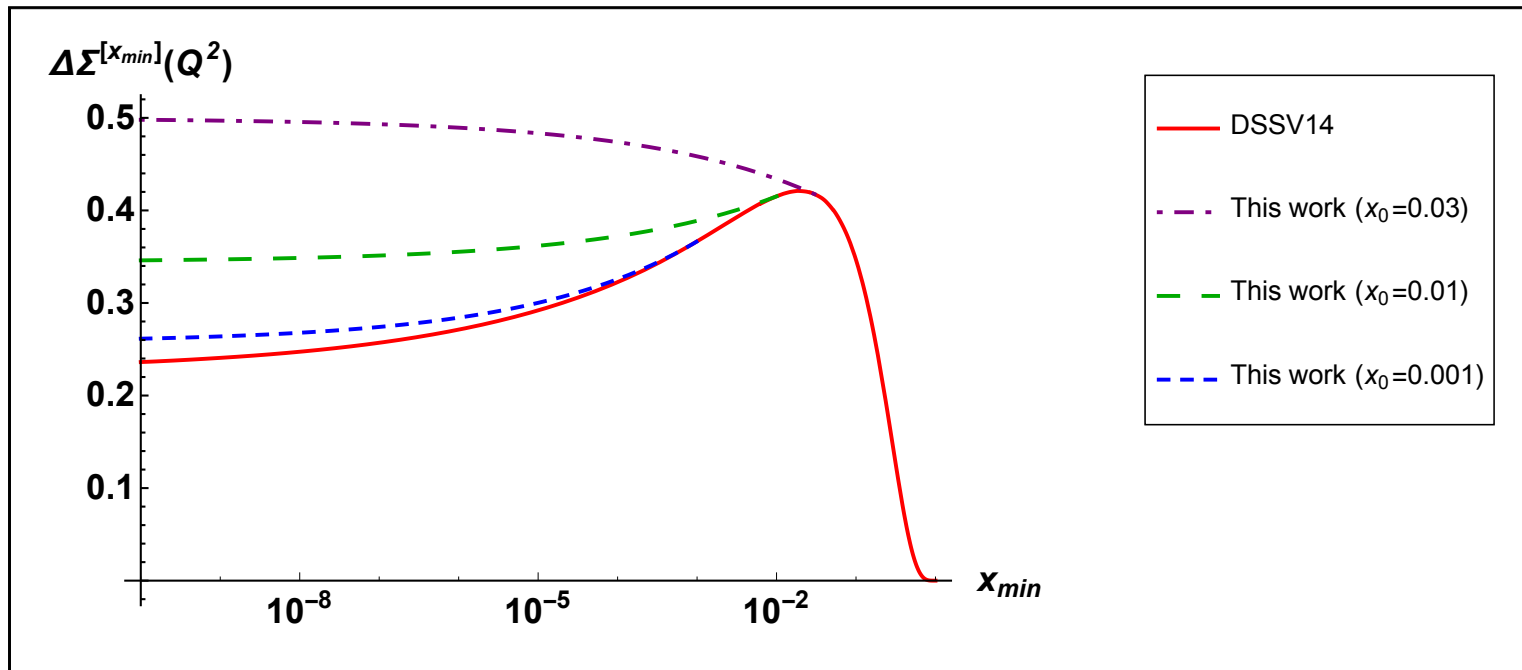


“ballpark”  
phenomenology



# Impact of our $\Delta\Sigma$ on the proton spin

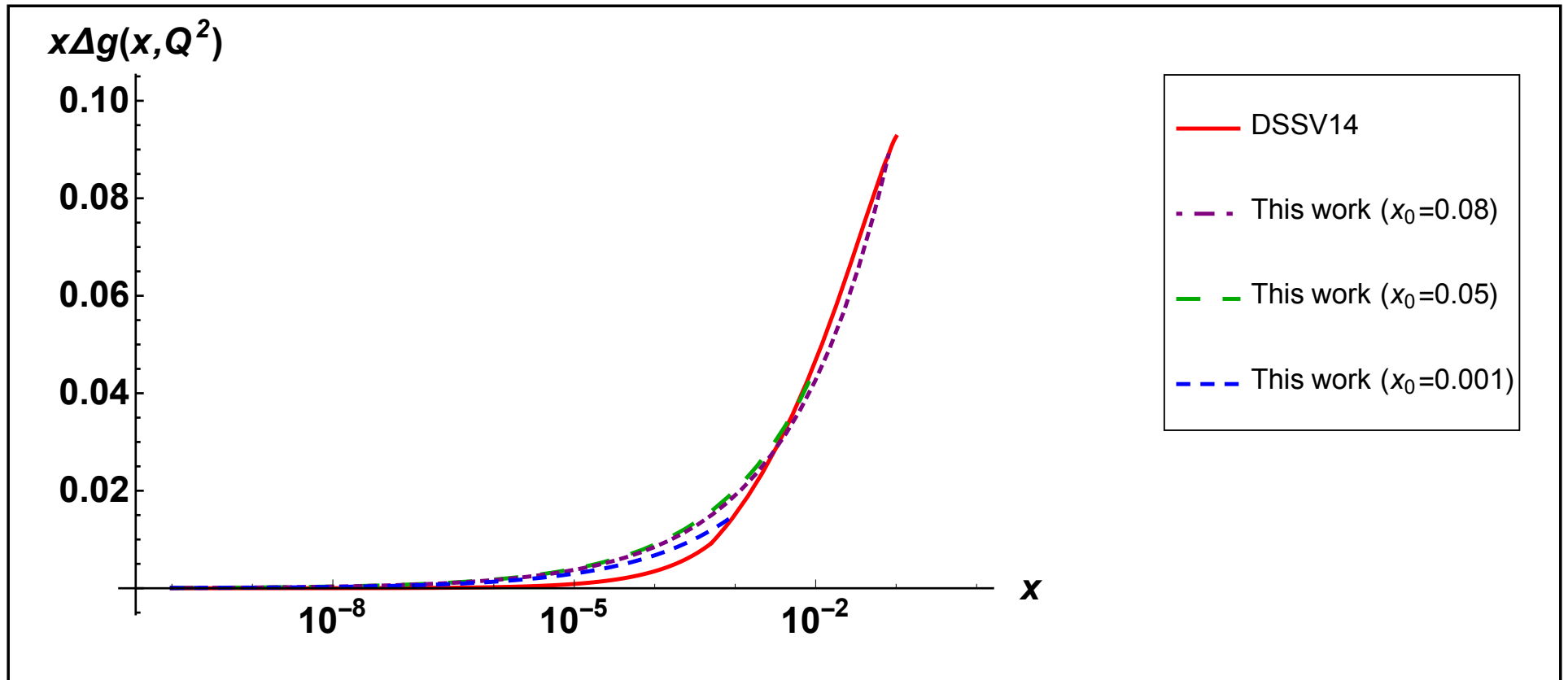
- Defining  $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^1 dx \Delta\Sigma(x, Q^2)$  we plot it for  $x_0=0.03, 0.01, 0.001$ :



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

# Impact of our $\Delta G$ on the proton spin

- We have attached a  $\Delta\tilde{G}(x, Q^2) = N x^{-\alpha_h^G}$  curve to the existing hPDF's fits at some ad hoc small value of  $x$  labeled  $x_0$  :

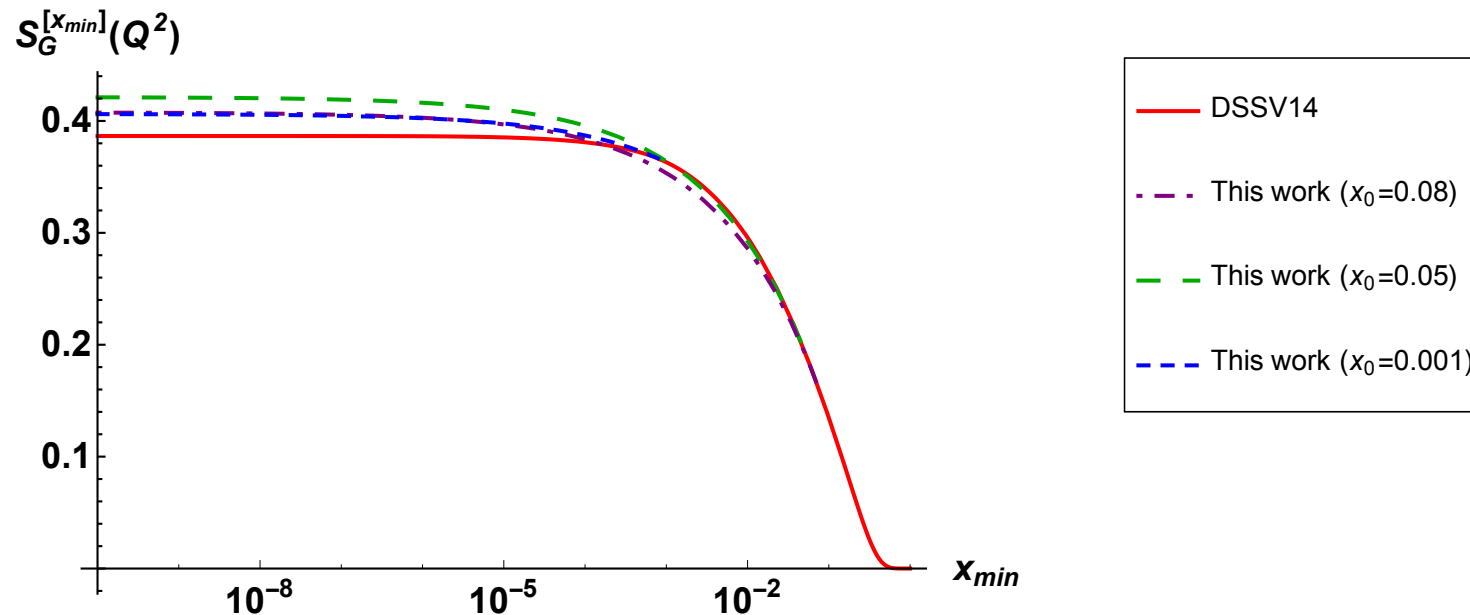


“ballpark”  
phenomenology



# Impact of our $\Delta G$ on the proton spin

- Defining  $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^1 dx \Delta G(x, Q^2)$  we plot it for  $x_0=0.08, 0.05, 0.001$ :



- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.