All-Opacity Gluon Spectrum for Jet Physics at the EIC

Matthew D. Sievert

Ivan Vitev

Los Alamos NATIONAL LABORATORY

Paper in preparation

QCD Evolution 2018

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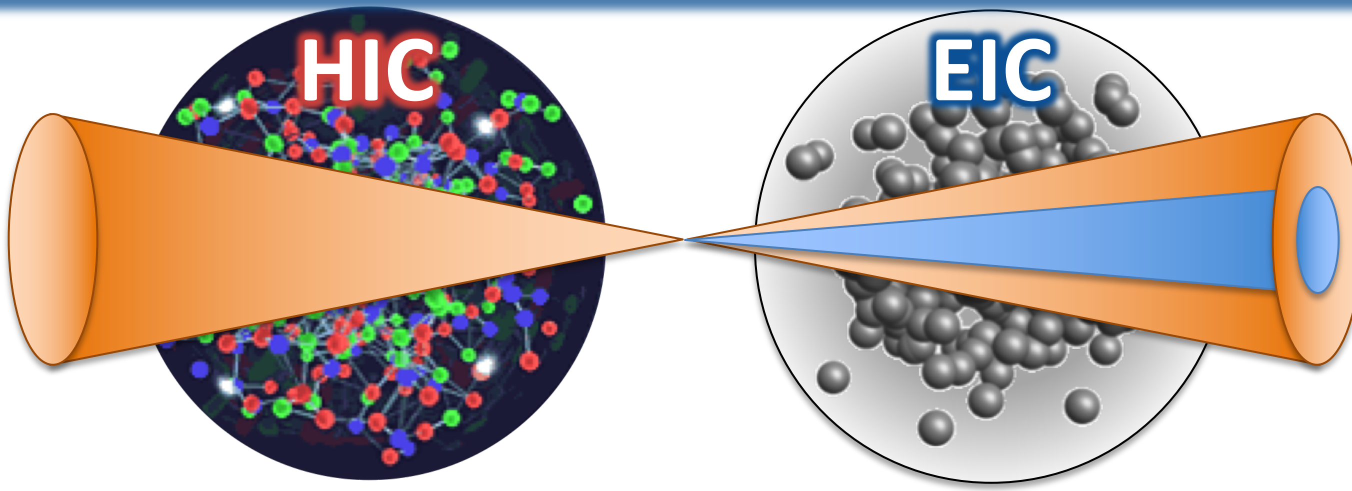
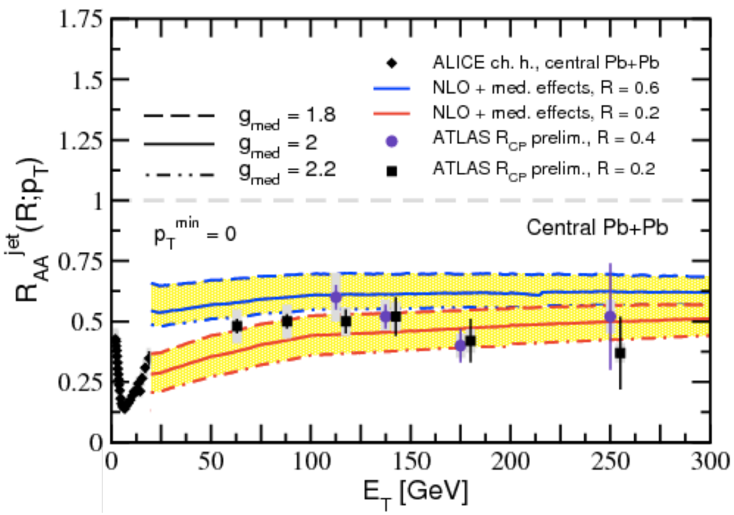
Motivation





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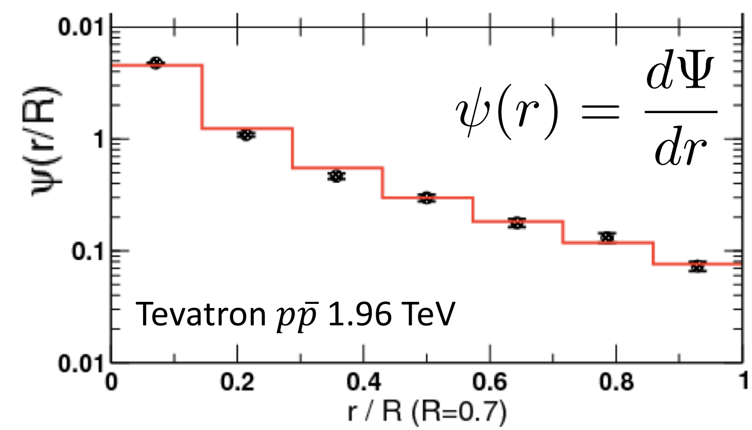

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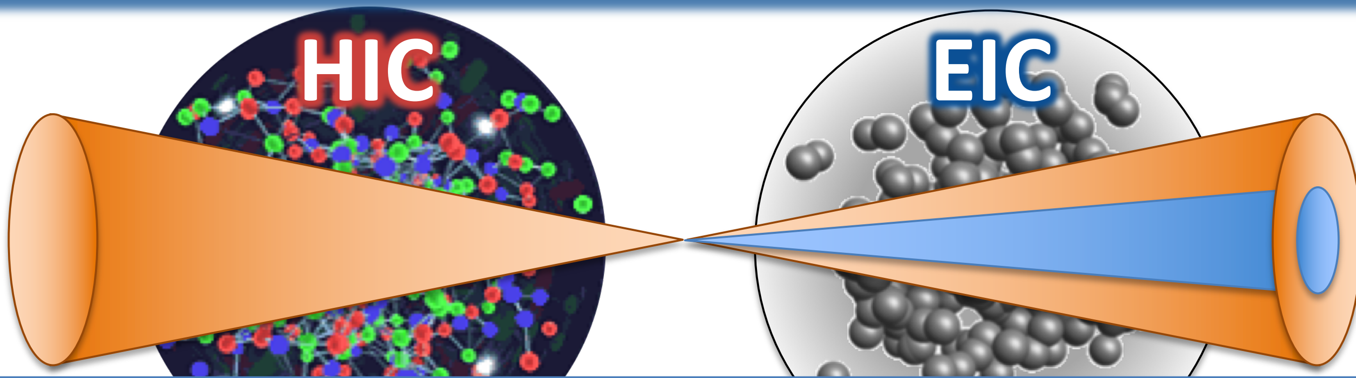
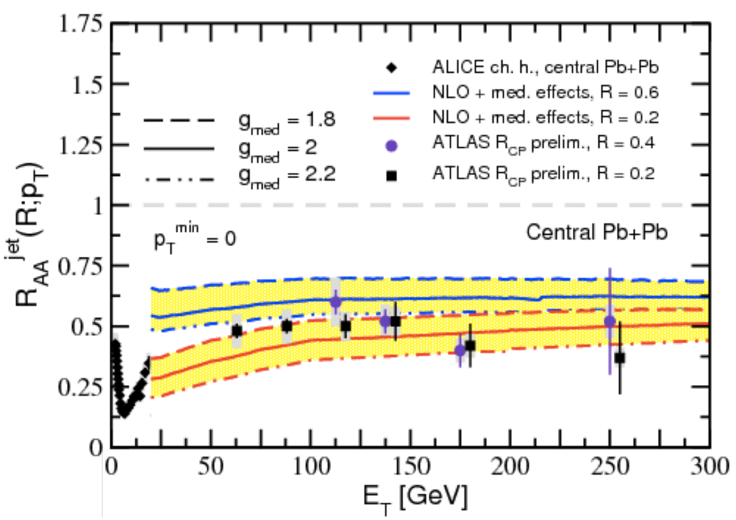


Jets at the EIC and HIC: Complementarity and Universality



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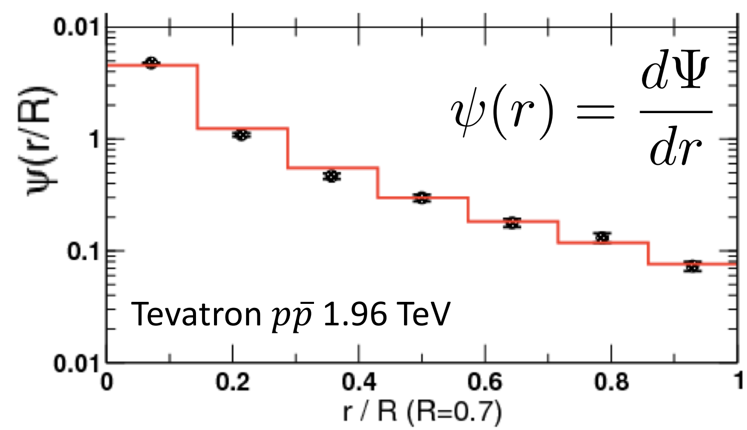

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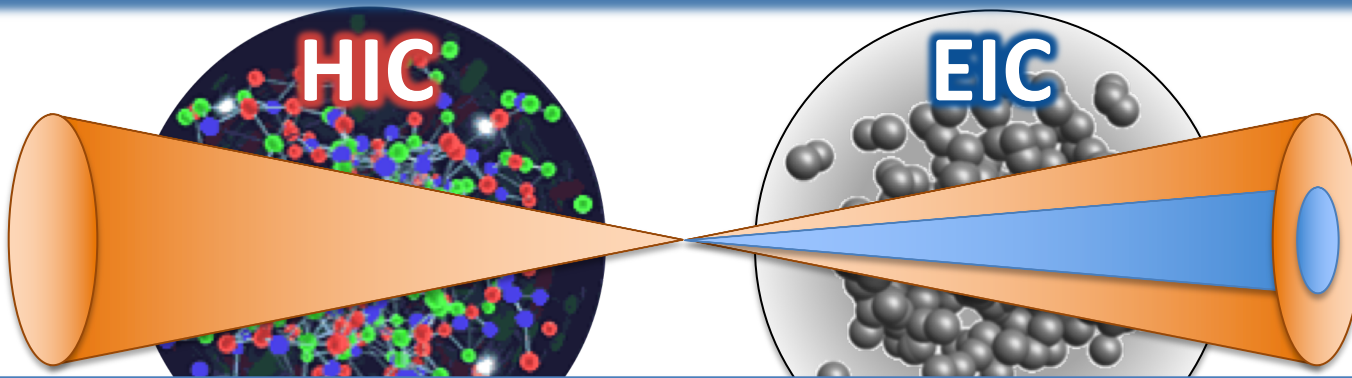

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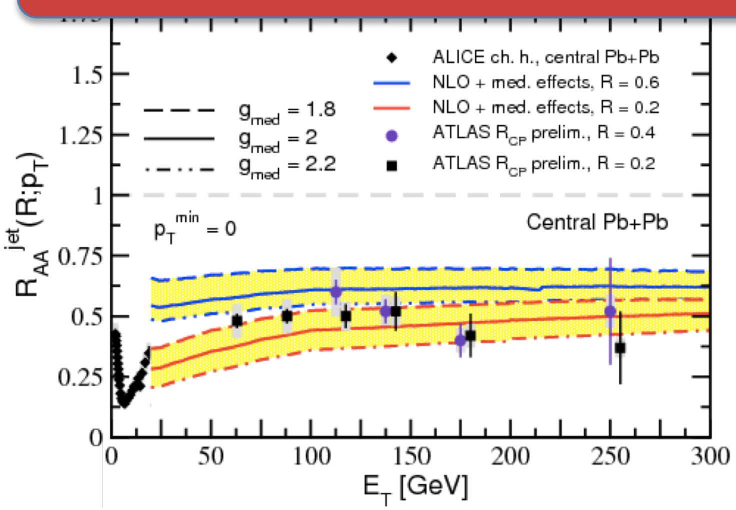
Inclusive Jets vs. Substructure: **Greater Sensitivity** to Medium



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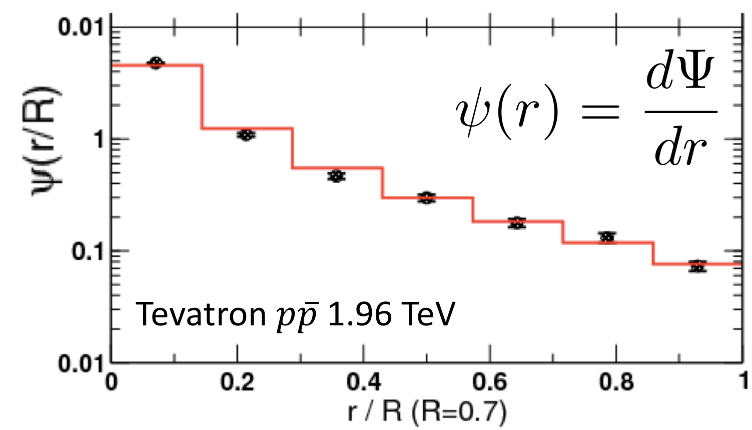

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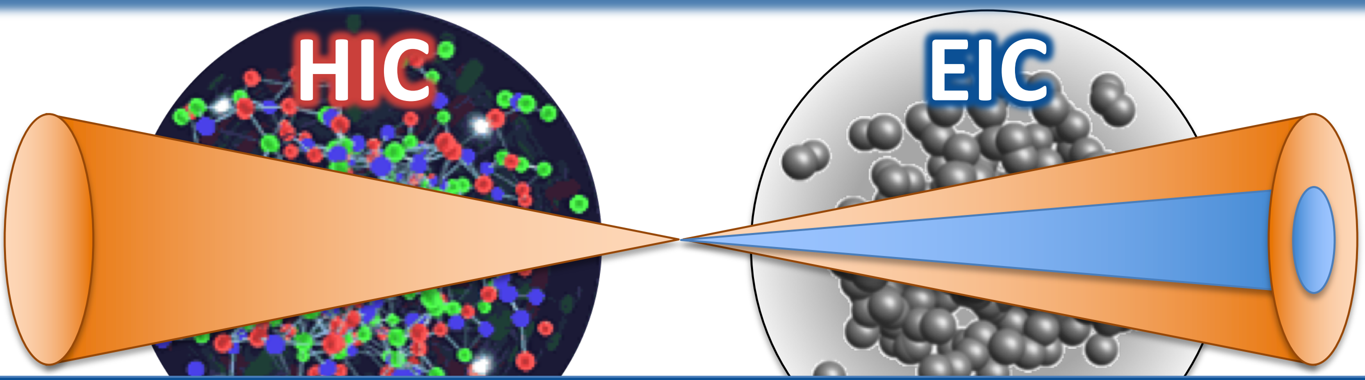

Jets at the EIC and HIC: **Complementarity** and **Universality**

Inclusive Jets vs. Substructure: **Greater Sensitivity** to Medium

This Talk:

- Generalization of the Opacity Expansion to **exact kinematics**
- Exact results for **second order**, easily extended to **higher orders**
- Suitable for **detailed phenomenology** of medium properties

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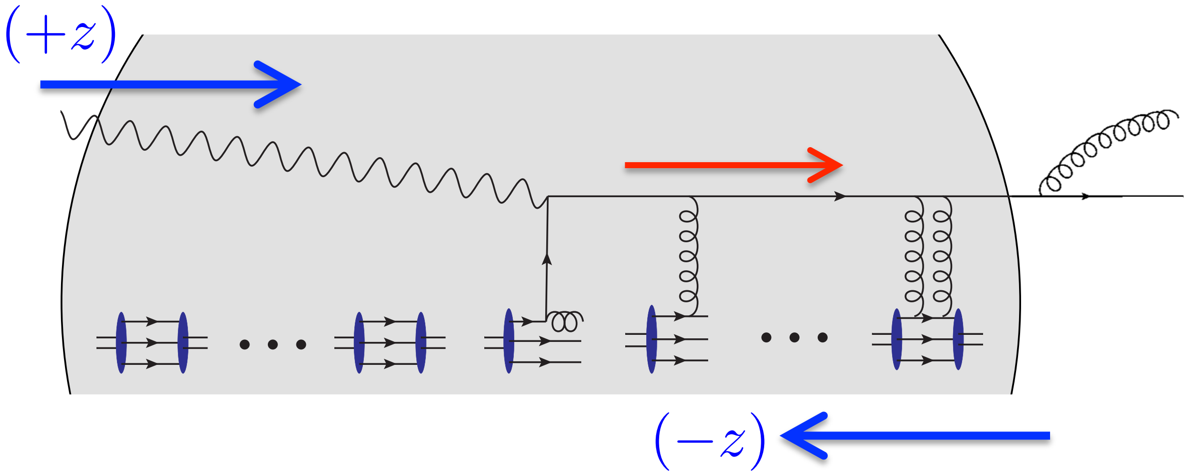
Leading Order Jets at the EIC



Breit Frame

$$p^\mu = \left(p^+ \, , \, \frac{p_T^2}{2p^+} \, , \, \vec{p}_\perp\right)$$

$A^+ = 0$ Light-Cone Gauge

- At LO, **SIDIS produces quark jets** via the handbag diagram

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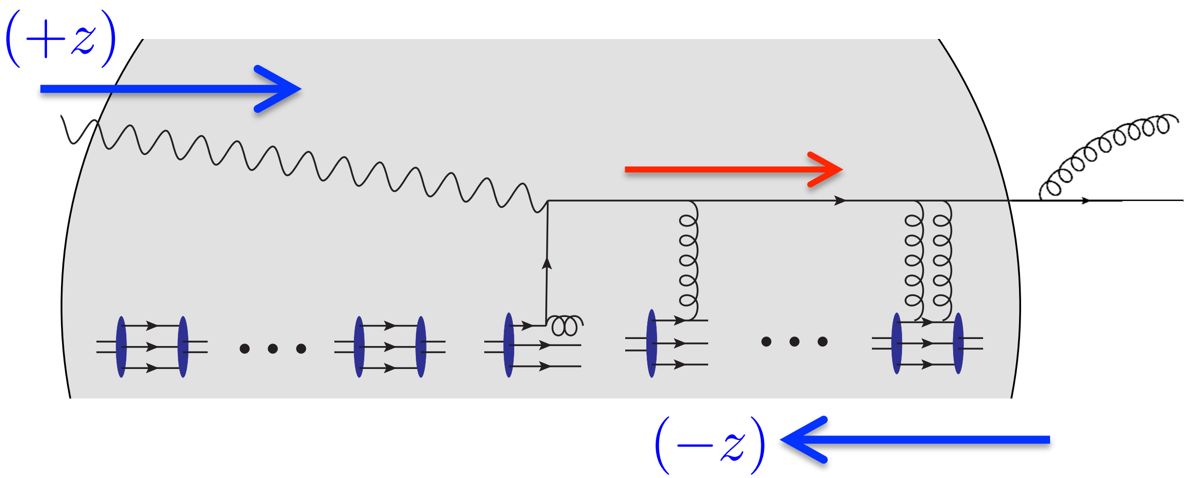

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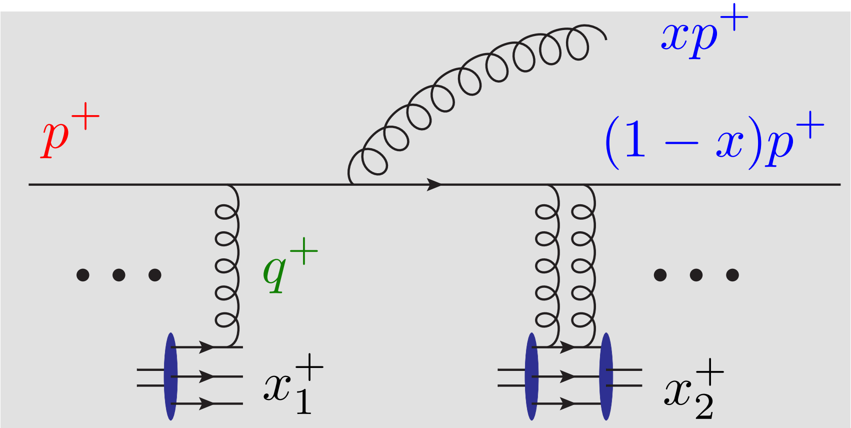
- At LO, **SIDIS produces quark jets** via the handbag diagram
  - At leading twist, **jet substructure (gluon in a quark jet)** is produced by **splitting in the vacuum** (leading log CSS evolution)

$$\frac{d\sigma^{\gamma^* + A \to (jet) + X}}{dx_B \, d^2p \, dQ^2} \propto \int \frac{d^2 r_\perp dr^+}{(2\pi)^3} e^{ip\cdot r} \left\langle \bar{\psi}(0) V_{0_\perp}^\dagger \, \frac{\gamma^+}{2} \, V_{r_\perp} \psi(r) \right\rangle$$

Origins of Jet Modification

- The **modification** of the jet substructure is always **sub-leading twist** and requires **gluon emission inside the medium**.



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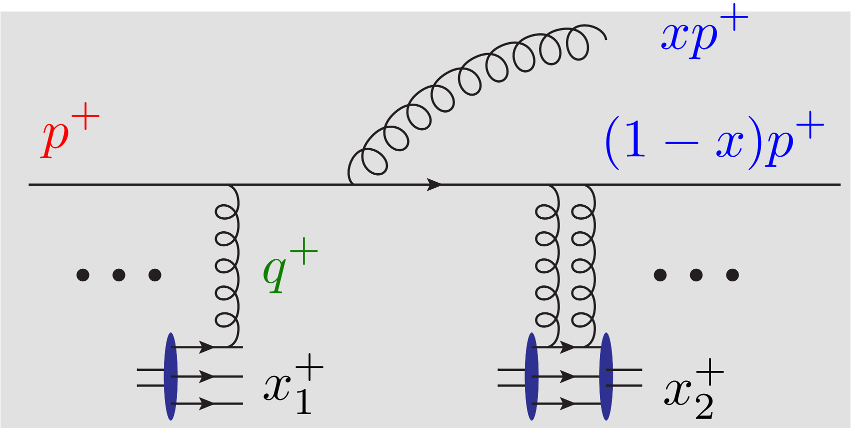

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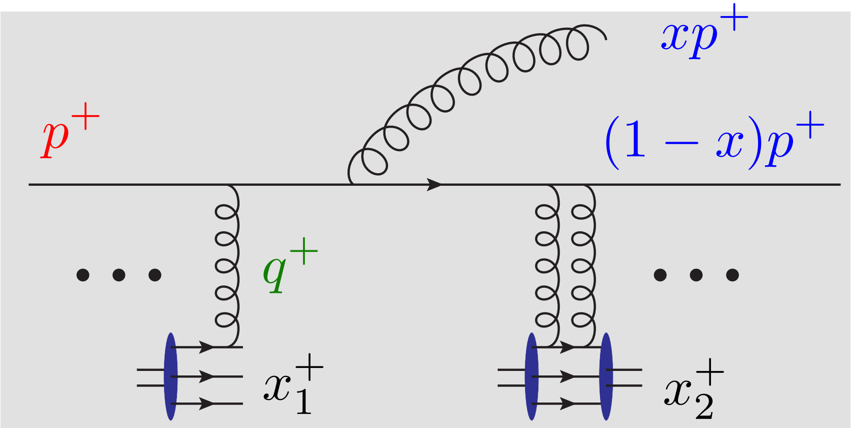

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  - Phases from **bounded gluon emission times**:

$$\int_{x_i^+}^{x_f^+} dt_{LF}\, e^{-i\Delta E^- t_{LF}} \; [-g\,\bar{u}(\gamma\cdot\epsilon^*)u] =$$
$$= \psi(x, \vec{k}_\perp - x\vec{p}_\perp)\left[e^{-i\Delta E^- x_f^+} - e^{-i\Delta E^- x_i^+}\right]$$

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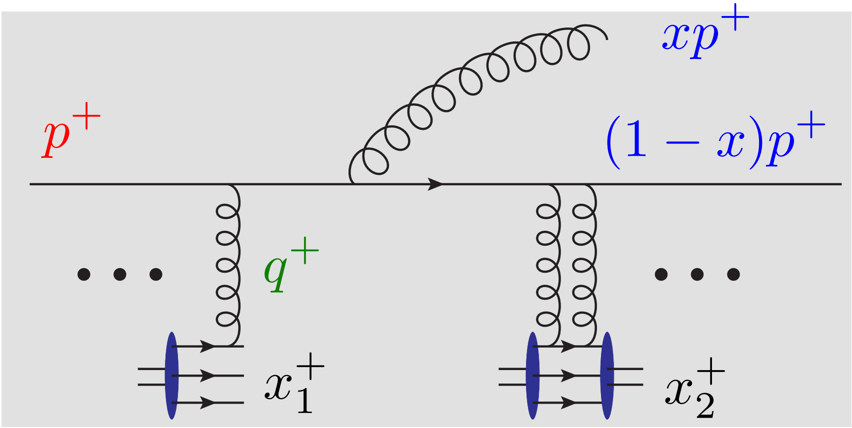

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  - Phases from **changing the virtuality** of the system:

$$e^{-i\left[\Delta E^-(p_f) - \Delta E^-(p_i)\right]x^+}$$

Time-Ordered Perturbation Theory

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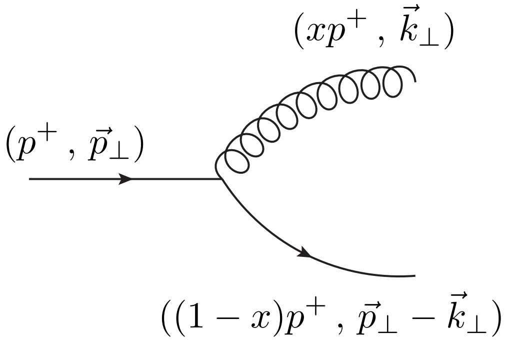
➢ Light-front **wave functions** (i.e., Altarelli-Parisi kernels)



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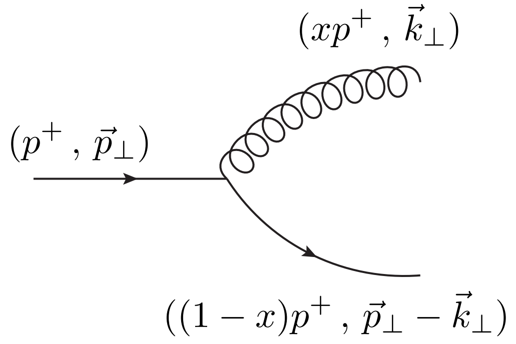

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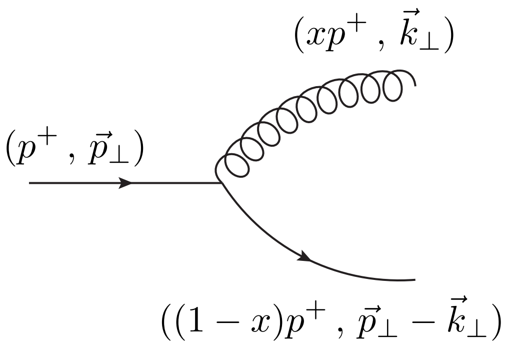

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- Depends only on the **intrinsic transverse momentum**:

$$\vec{\kappa}_\perp = \vec{k}_\perp - x\vec{p}_\perp$$

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$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{ext}^{qG} + \mathcal{L}_{ext}^{3G} + \mathcal{L}_{ext}^{4G}$$

$$\frac{d\sigma^{el}}{d^2q} = \frac{1}{(2\pi)^2}\frac{C_F}{2N_c}[v(q_T^2)]^2$$

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  - Use **Gaussian averaging** of the external fields (leading in α_s for local color neutrality)

$$\left\langle g A_{ext}^{\mu a}(x) \left(g A_{ext}^{\nu b}(y)\right)^* \right\rangle_{med} = g^{\mu +} g^{\nu +} \delta^{ab} \delta(x^+ - y^+) \left[\frac{1}{\lambda_{mfp}^+ C_F} \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}_\perp \cdot(\vec{x}_\perp - \vec{y}_\perp)} \frac{(2\pi)^2}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \right]$$

The Opacity Expansion

- The **longitudinal averaging** over the scattering centers generates factors of the **opacity**:

$$\int_0^{L^+} \frac{dz^+}{\lambda_{mfp}^+} = \frac{L^+}{\lambda_{mfp}^+} = \langle n \rangle$$

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$$\frac{d\sigma^{(jet)+X}}{d^2p\,dy} = \frac{d\sigma^{(jet)+X}}{d^2p\,dy}\bigg|_{vac} + \mathcal{O}\Big(\langle n \rangle\Big) + \mathcal{O}\Big(\langle n \rangle^2\Big) + \cdots$$

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➢ For **fairly small opacities**, the series can be **truncated** at finite opacity

$$\langle n \rangle < \text{few}$$

➢ For **very large opacities**, the series must be **re-summed**.

$$\langle n \rangle \gg 1$$

Sensitivity of the Opacity Expansion

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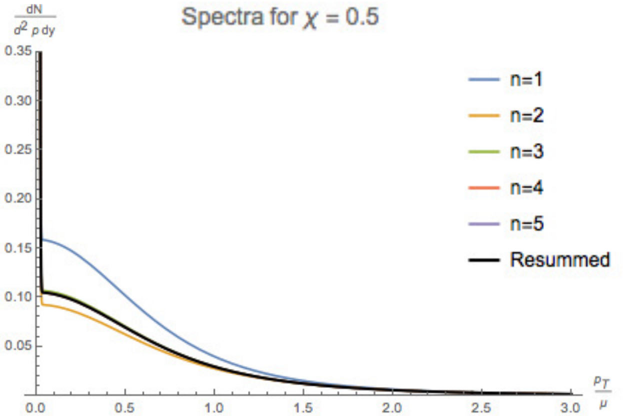

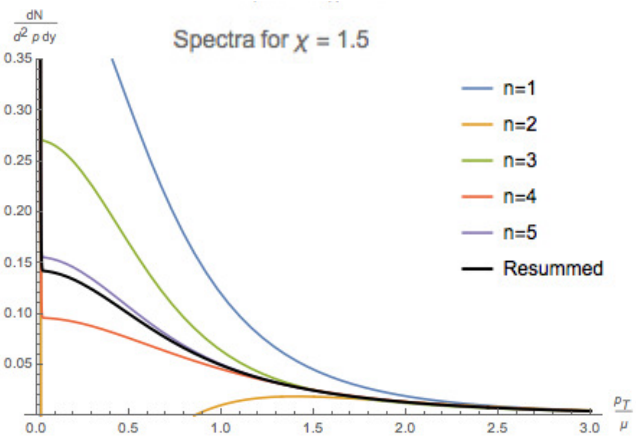

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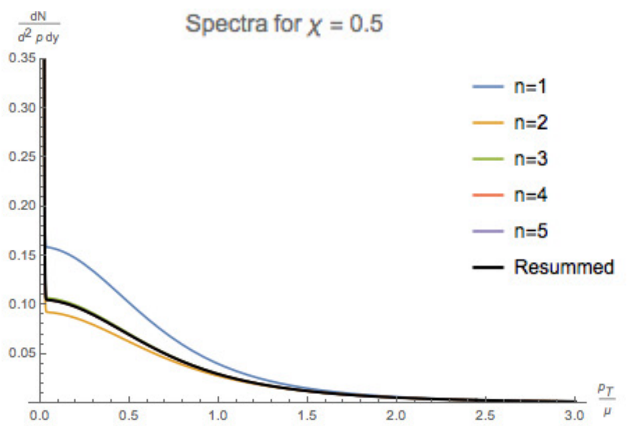
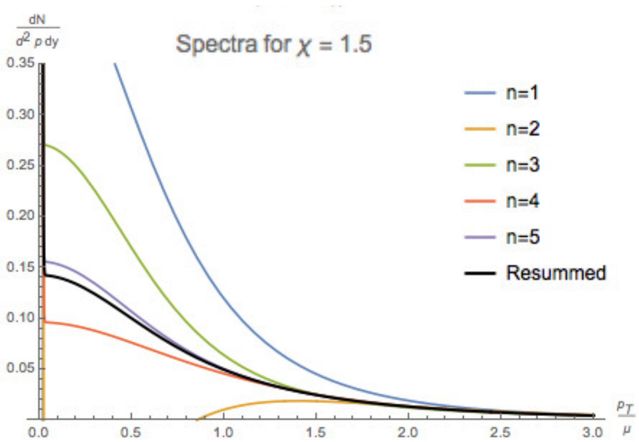
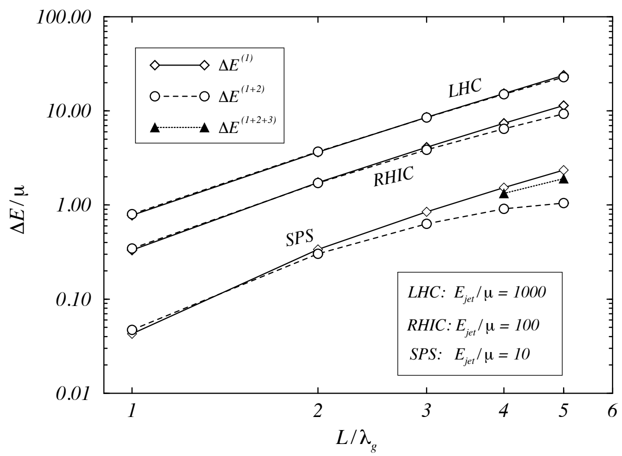
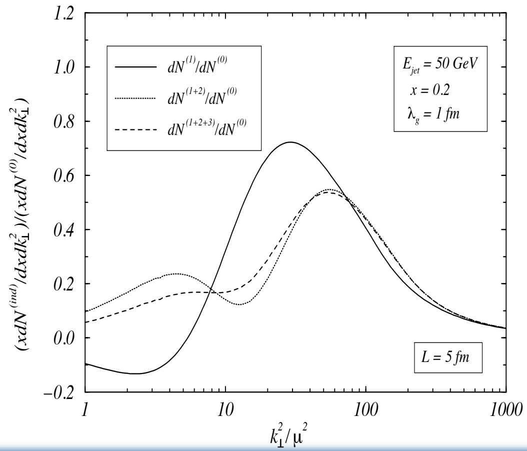




- **Radiative energy loss:**





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  - "Reaction operator" and explicit solution to **any order in opacity**
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  Blaizot, Dominguez, Iancu, Mehtar-Tani [2013] *JHEP 1301 (2013) 143*
  - **Resummation** using **path integrals**: generalized Wilson line
  - Calculations generally require **Gaussian path integrals** to compute: the **"harmonic approximation"** to scattering

The Reaction Operator Beyond Small x

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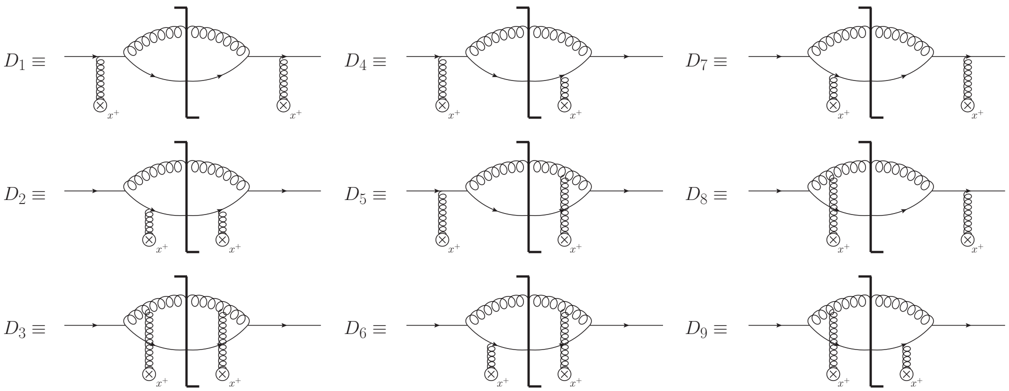
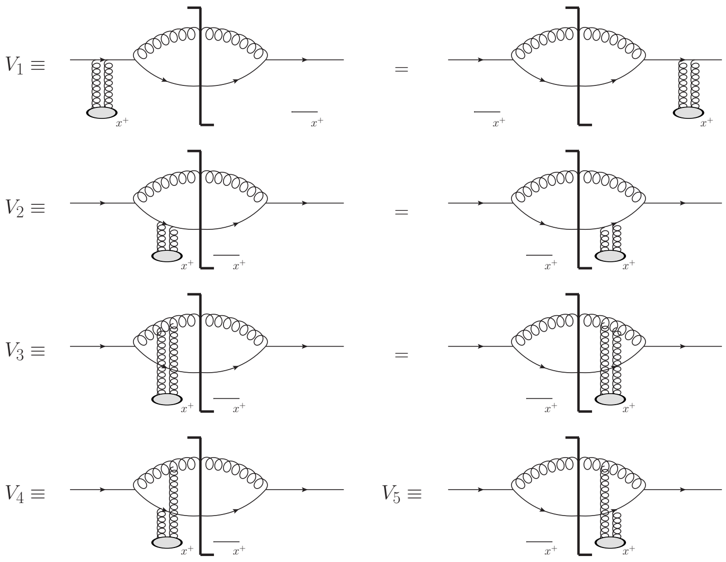


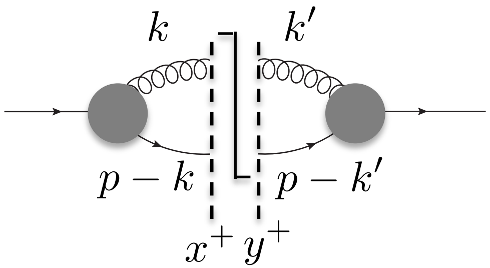


➢ The 17 diagrams appearing at first order serve as a **recursion kernel**

Generalizing the Reaction Operator

- Recursion relations at the level of the **amplitude squared**:
  - Step backward through the last scattering

Trivial multiplicative color factors!



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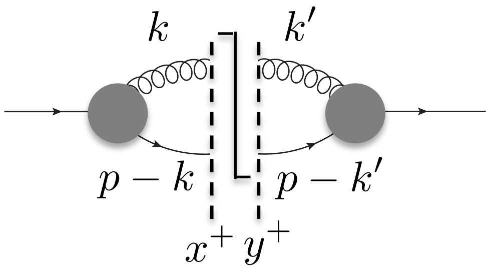
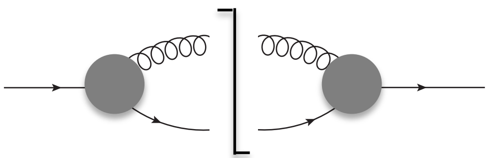


Final state / final state

$$= \int_0^{\min[x^+,y^+]} \frac{dz^+}{\lambda^+_{mfp}} \int \frac{d^2q}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \Bigg[\mathcal{K}_1(k, k', p; x^+, y^+; q, z^+)$$



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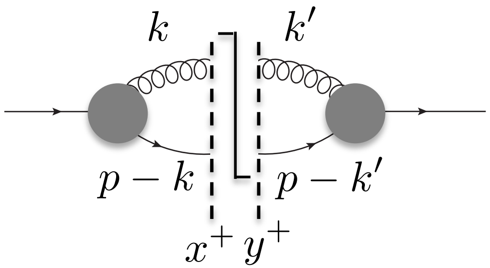

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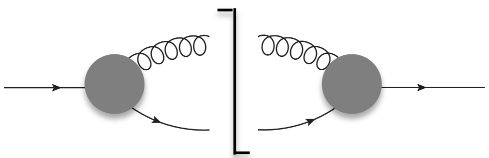
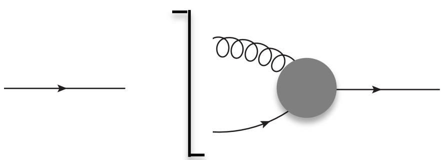


Initial state / final state

$$+\mathcal{K}_2(k, k', p; x^+, y^+; q, z^+)$$



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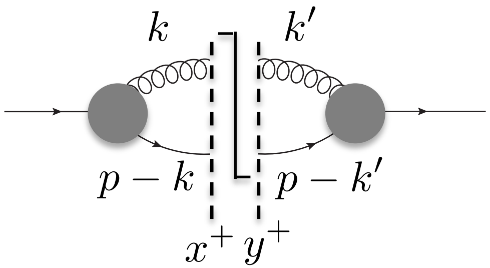

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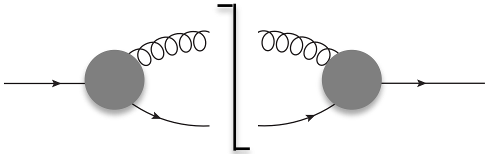
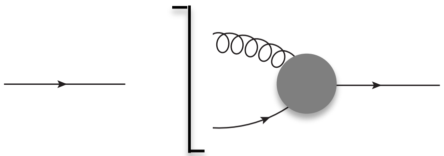


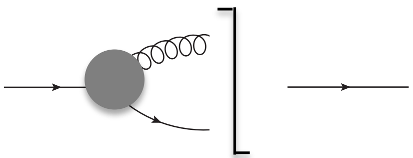
Initial state / final state

$$+\mathcal{K}_2(k, k', p; x^+, y^+; q, z^+)$$



Final state / Initial state

$$+\mathcal{K}_3(k, k', p; x^+, y^+; q, z^+)$$

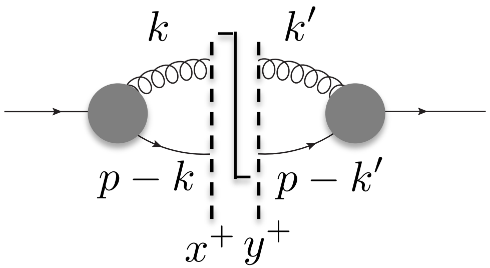


$$\Bigg]$$

Generalizing the Reaction Operator

- Recursion relations at the level of the **amplitude squared**:
  - Step backward through the last scattering

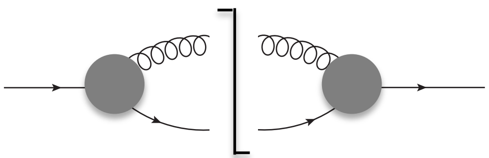
Trivial multiplicative color factors!



$$= \int_0^{\min[x^+,y^+]} \frac{dz^+}{\lambda^+_{mfp}} \int \frac{d^2q}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \Big[$$

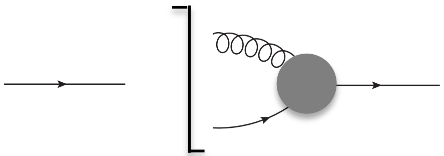
Final state / final state

$$\mathcal{K}_1(k, k', p; x^+, y^+; q, z^+)$$



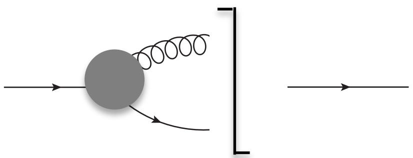
Initial state / final state

$$+\mathcal{K}_2(k, k', p; x^+, y^+; q, z^+)$$



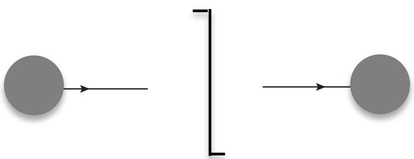
Final state / Initial state

$$+\mathcal{K}_3(k, k', p; x^+, y^+; q, z^+)$$



Initial state / Initial state

$$+\mathcal{K}_4(k, k', p; x^+, y^+; q, z^+)$$



$$\Big]$$

An Example in the Final/Final Sector

$$f^{(N)}_{F/F}(k, p, x^+ | k', p, y^+) = \int_0^{\min[x^+, y^+]} \frac{dz^+}{\lambda^+} \int \frac{d^2q}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \Bigg\{$$

$$+ (15 \text{ more}) \Bigg\}$$

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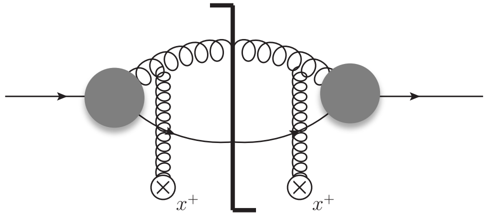
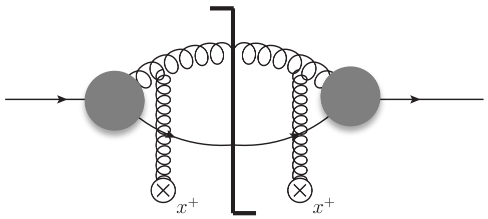


$$\left\{ \frac{N_c}{C_F} \left[e^{i[\Delta E^-(k-xp-(1-x)q)-\Delta E^-(k-xp)]z^+} \times e^{-i[\Delta E^-(k'-xp-(1-x)q)-\Delta E^-(k'-xp)]z^+} \right] \times f^{(N-1)}_{F/F}(k-q,p-q,z^+|k'-q,p-q,z^+) \right.$$

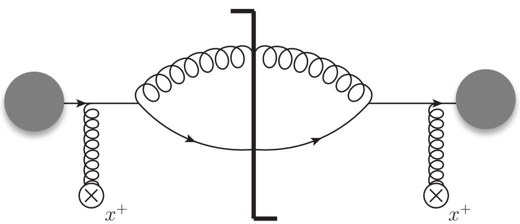
$$+(15\text{ more})\Bigg\}$$

An Example in the Final/Final Sector

$$f^{(N)}_{F/F}(k,p,x^+|k',p,y^+) = \int\limits_0^{\min[x^+,y^+]} \frac{dz^+}{\lambda^+} \int \frac{d^2q}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \Bigg\{$$



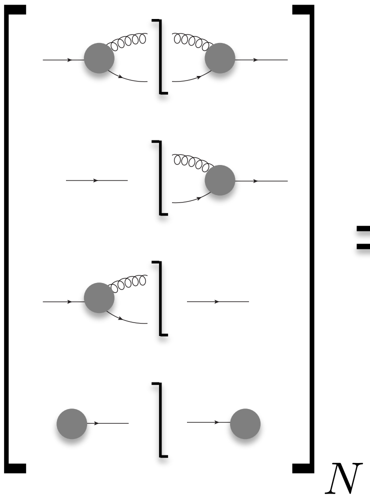
$$\left\{ \frac{N_c}{C_F} \left[e^{i[\Delta E^-(k-xp-(1-x)q)-\Delta E^-(k-xp)]z^+} \right.\right.$$
$$\left. \times\, e^{-i[\Delta E^-(k'-xp-(1-x)q)-\Delta E^-(k'-xp)]z^+} \right]$$
$$\left. \times f^{(N-1)}_{F/F}(k-q,p-q,z^+|k'-q,p-q,z^+) \right.$$



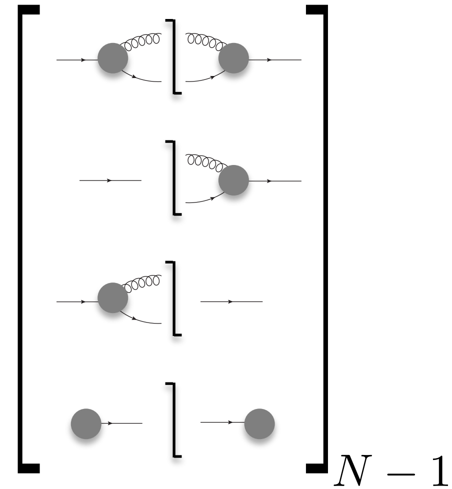
$$\left\{ +\psi(k-xp)\left[e^{-i\Delta E^-(k-xp)x^+} - e^{-i\Delta E^-(k-xp)z^+}\right] \right.$$
$$\times \left[e^{i\Delta E^-(k'-xp)y^+} - e^{i\Delta E^-(k'-xp)z^+}\right]\psi^*(k'-xp)$$
$$\left. \times f^{(N-1)}_{I/I}(p-q,z^+) \right.$$

$$+(15\text{ more})\Bigg\}$$

The Form of the Reaction Operator

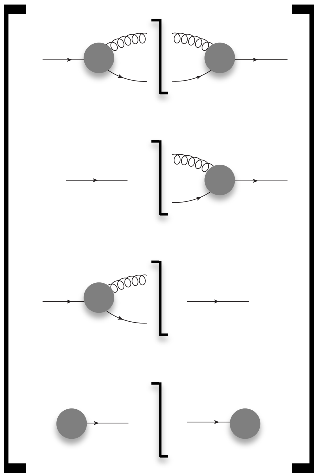


$$
= \overbrace{\begin{bmatrix} \mathcal{K}_1 & \psi \otimes \mathcal{K}_2 & \mathcal{K}_3 \otimes \psi^* & \psi \otimes \mathcal{K}_4 \otimes \psi^* \\ 0 & \mathcal{K}_5 & 0 & \mathcal{K}_6 \otimes \psi^* \\ 0 & 0 & \mathcal{K}_7 & \psi \otimes \mathcal{K}_8 \\ 0 & 0 & 0 & \mathcal{K}_9 \end{bmatrix}}^{RR^\dagger + V1^\dagger + V^\dagger 1}
$$

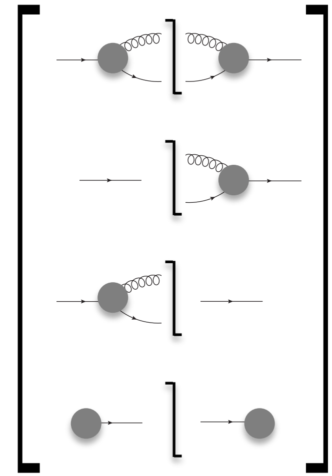


- **Causal** structure: **triangular** matrix
  - ➢ Suggests a particular strategy for **solving analytically**

The Form of the Reaction Operator



$$\left[\;\cdot\;\right]_N = \underbrace{\begin{bmatrix} \mathcal{K}_1 & \psi \otimes \mathcal{K}_2 & \mathcal{K}_3 \otimes \psi^* & \psi \otimes \mathcal{K}_4 \otimes \psi^* \\ 0 & \mathcal{K}_5 & 0 & \mathcal{K}_6 \otimes \psi^* \\ 0 & 0 & \mathcal{K}_7 & \psi \otimes \mathcal{K}_8 \\ 0 & 0 & 0 & \mathcal{K}_9 \end{bmatrix}}_{RR^\dagger + V1^\dagger + V^\dagger 1} \left[\;\cdot\;\right]_{N-1}$$



- **Causal** structure: **triangular** matrix
  - ➢ Suggests a particular strategy for **solving analytically**
- Final-state distribution:

$$xp^+ \frac{dN}{d^2k dx\, d^2p dp^+}\bigg|_{\mathcal{O}\left(\langle n \rangle^N\right)} = \frac{C_F}{2(2\pi)^3(1-x)} f_{F/F}^{(N)}(k,p,L^+|k,p,L^+)$$

Validation: N=1 at Finite x

$$
\begin{aligned}
x\,p^+ \frac{dN}{d^2k\,dx\,dp^+\,d^2p}\bigg|_{N=1} &= \frac{C_F}{2(2\pi)^3(1-x)} \int_{0^+}^{L^+} \frac{dz_1^+}{\lambda^+} \int \frac{d^2q}{(2\pi)^2} \left(\frac{(2\pi)^2}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q} \right) \\
&\times \Bigg\{ \left(p^+ \frac{dN_0}{d^2(p-q)\,dp^+} \right) \Bigg[\left|\psi(\underline{k}-x\underline{p})\right|^2 + 2\left(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}) z_1^+\right] \right) \left|\psi(\underline{k}-x\underline{p}+x\underline{q})\right|^2 \\
&\quad + 2\frac{N_c}{C_F}\left(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}-(1-x)\underline{q}) z_1^+\right] \right) \left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q})\right|^2 \\
&\quad + \frac{1}{N_c C_F}\left(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}) z_1^+\right] \right) \psi(\underline{k}-x\underline{p})\psi^*(\underline{k}-x\underline{p}+x\underline{q}) \\
&\quad - \frac{N_c}{C_F}\left(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}-(1-x)\underline{q}) z_1^+\right] \right) \psi(\underline{k}-x\underline{p})\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}) \\
&\quad - \frac{N_c}{C_F}\Bigg(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}) z_1^+\right] - \cos\left[\Delta E^-(\underline{k}-x\underline{p}-(1-x)\underline{q}) z_1^+\right] \\
&\qquad + \cos\left[\left(\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}) - \Delta E^-(\underline{k}-x\underline{p}-(1-x)\underline{q})\right) z_1^+\right] \Bigg) \psi(\underline{k}-x\underline{p}+x\underline{q})\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}) \Bigg] \\
&+ \left(p^+ \frac{dN_0}{d^2p\,dp^+} \right) \Bigg[-\left|\psi(\underline{k}-x\underline{p})\right|^2 - \frac{N_c}{C_F}\left(1 - \cos\left[\Delta E^-(\underline{k}-x\underline{p}) z_1^+\right] \right) \left|\psi(\underline{k}-x\underline{p})\right|^2 \\
&\quad + \frac{N_c}{C_F}\left(\cos\left[\left(\Delta E^-(\underline{k}-x\underline{p}) - \Delta E^-(\underline{k}-x\underline{p}-\underline{q})\right) z_1^+\right] - \cos\left[\Delta E^-(\underline{k}-x\underline{p}) z_1^+\right] \right) \psi(\underline{k}-x\underline{p})\psi^*(\underline{k}-x\underline{p}-\underline{q}) \Bigg] \Bigg\}
\end{aligned}
$$

- Tests **final/final sector** and **correctly reproduces OV [2012]**

Validation: N=2 with Small x, Broad Source

$$xp^+ \frac{dN}{d^2k\,dx\,d^2p\,dp^+}\bigg|_{N=2} = \frac{C_F}{2(2\pi)^3(1-x)} \int\limits_0^{L^+} \frac{dz_2^+}{\lambda^+} \int\limits_0^{z_2^+} \frac{dz_1^+}{\lambda^+} \int \frac{d^2q_1}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q_1} \int \frac{d^2q_2}{\sigma_{el}} \frac{d\sigma^{el}}{d^2q_2}$$

$$\times \frac{2N_c^2}{C_F^2} \Big\{ [\psi(k) - \psi(k-q_1)]\, \psi^*(k-q_1) \left[1 - \cos\left(\Delta E^-(k-q_1) z_1^+\right)\right]$$

$$+ [\psi(k) - \psi(k-q_2)]\, \psi^*(k-q_2) \left[\cos\left(\Delta E^-(k-q_2)(z_2^+ - z_1^+)\right) - \cos\left(\Delta E^-(k-q_2) z_2^+\right)\right]$$

$$- [\psi(k-q_2) - \psi(k-q_1-q_2)]\, \psi^*(k-q_1-q_2) \left[1 - \cos\left(\Delta E^-(k-q_1-q_2) z_1^+\right)\right]$$

$$- [\psi(k) - \psi(k-q_2)]\, \psi^*(k-q_1-q_2) \Big[\cos\left(\Delta E^-(k-q_2)(z_2^+ - z_1^+)\right)$$

$$- \cos\left(\Delta E^-(k-q_1-q_2) z_1^+ + \Delta E^-(k-q_2)(z_2^+ - z_1^+)\right)\Big] \Big\}$$

$$\times \left(p^+ \frac{dN_0}{d^2p\,dp^+} \right)$$

- Tests **all sectors** in the **small-x limit** and **correctly reproduces GLV [2001]**

Exact Results at N = 2: General Features

- There are **4 different shifts** in the initial distribution from **single-** vs. **double-Born scattering**

$$\left(p^+ \frac{dN_0}{d^2p \, dp^+} \right) \qquad \left(p^+ \frac{dN_0}{d^2(p - q_1) \, dp^+} \right)$$

$$\left(p^+ \frac{dN_0}{d^2(p - q_2) \, dp^+} \right) \qquad \left(p^+ \frac{dN_0}{d^2(p - q_1 - q_2) \, dp^+} \right)$$

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$$\left(p^+ \frac{dN_0}{d^2(p-q_2)\, dp^+} \right) \qquad \left(p^+ \frac{dN_0}{d^2(p-q_1-q_2)\, dp^+} \right)$$

- There are **16 different possible arguments** of each wave function

$$\psi(k - xp)$$

$$\psi(k - xp + xq_1) \qquad \psi(k - xp - (1-x)q_1) \qquad \psi(k - xp - q_1)$$

$$\psi(k - xp + xq_2) \qquad \psi(k - xp - (1-x)q_2) \qquad \psi(k - xp - q_2)$$

$$\psi(k - xp + xq_1 + xq_2) \qquad \psi(k - xp + (1-x)q_1 + xq_2) \qquad \psi(k - xp - q_1 + xq_2)$$

$$\psi(k - xp + xq_1 - (1-x)q_2) \qquad \psi(k - xp - (1-x)q_1 - (1-x)q_2) \qquad \psi(k - xp - q_1 - (1-x)q_2)$$

$$\psi(k - xp + xq_1 - q_2) \qquad \psi(k - xp - (1-x)q_1 - q_2) \qquad \psi(k - xp - q_1 - q_2)$$

Exact Results at N = 2: Representative Terms

- **More cosines** with different arguments, and more **differences of terms** within various cosines
  - Consequence of **aggregating impulse phase shifts**

$$\left(p^+ \frac{dN_0}{d^2p\, dp^+}\right) \psi(\underline{k} - x\underline{p} - \underline{q}_1)\, \psi^*(\underline{k} - x\underline{p} - \underline{q}_2) \Bigg[\frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(\underline{k} - x\underline{p})) - \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_1) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p}))$$
$$- \frac{N_c^2}{2C_F^2} \cos(-\delta z_1 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) - \delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p}))$$
$$+ \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_1) - \delta z_1 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) - \delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p})) \Bigg]$$

$$\left(p^+ \frac{dN_0}{d^2(p - q_1)\, dp^+}\right) \psi(\underline{k} - x\underline{p} + x\underline{q}_1)\, \psi^*(\underline{k} - x\underline{p} + x\underline{q}_1 - \underline{q}_2) \Bigg[$$
$$\frac{N_c}{C_F} \cos(-\delta z_1 \Delta E^-(\underline{k} - x\underline{p} + x\underline{q}_1 - q_2) + \delta z_1 \Delta E^-(\underline{k} - x\underline{p} + x\underline{q}_1) - \delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p}))$$
$$- \frac{N_c}{C_F} \cos(-\delta z_1 \Delta E^-(\underline{k} - x\underline{p} + x\underline{q}_1 - q_2) - \delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p}))$$
$$- \frac{N_c}{C_F} \cos(\delta z_1 \Delta E^-(\underline{k} - x\underline{p} + x\underline{q}_1) - \delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p}))$$
$$+ \frac{N_c}{C_F} \cos(-\delta z_2 \Delta E^-(\underline{k} - x\underline{p} - \underline{q}_2) + \delta z_2 \Delta E^-(\underline{k} - x\underline{p})) \Bigg]$$

Conclusions

Bold assertion:

The **reaction operator with exact kinematics** and its explicit realization at any finite order in opacity represent a **complete solution** at L.O to radiative jet energy loss by eikonal external fields.

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Caveats:

- Not closed form (yet…)
- Does not resum the opacity series: $\langle n \rangle < \text{few}$
- Assumes eikonal scattering for both partons: $x(1-x) \gg \frac{k_T^2}{s}$
- Neglects other sub-eikonal effects : $\frac{k_T^2}{x(1-x)s} A^{1/3}$
- Does not include additional logarithmic evolution: $\alpha_s \ln A^{1/3} \ll 1$

Outlook

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- **Higher-order terms** are cumbersome but straightforward
- **Matching** onto re-summed results?
- **Solve the triangular matrix structure sequentially**?
  Coordinate space is natural, but the phases are complicated…

Backup Slides

Opacity Estimates in Heavy-Ion Collisions

$$\frac{dN^{ch}}{d\eta} \approx 700$$

RHIC: Au+Au, 200 GeV

$$\frac{dN^{glue}}{dy} \approx 1100$$

Adding ~50% neutral particles, +5 – 10% from Jacobian, assuming approximate parton/hadron duality

$$\rho = \frac{1}{\tau}\frac{dN^{glue}}{dy} \times \frac{1}{A_\perp}$$

Dominant Bjorken expansion

$$\rho = \frac{16}{\pi^2} T^3 \zeta(3)$$

Gas of free gluons in thermal equilibrium

$$\mu = gT$$

Typical coupling constant 1.8 - 2

$$\sigma^{gg} = \frac{9\alpha_s^2\pi}{2\mu^2}$$

LO pQCD cross-section with massive gluon exchange

$$\langle n \rangle = \int_{z_0}^{L} dz\, \rho(z)\, \sigma^{gg}(z)$$

Initial times ~ 0.5 – 1 fm

- Leads to $\langle n \rangle = 3.1$ (most central production) and $\langle n \rangle = 2.4$ (average)

Exact Results at N=2

Full exact results at N=2: Need to be integrated over z_1^+, z_2^+, q_1, q_2

Exact Results at N=2

$$
\begin{aligned}
&\left(p^+ \frac{dN_0}{d^2p\, dp^+}\right)\Bigg\{ \left|\psi(\underline{k}-x\underline{p})\right|^2 \Bigg[\frac{(C_F+N_c)^2}{C_F^2} - \frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p})) + \frac{N_c^2}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad - \frac{N_c(2C_F+N_c)}{2C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(\underline{k}-x\underline{p}))\Bigg] \\
&\quad +\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1)\Bigg[\frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p})) - \frac{N_c^2}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad - \frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}) - \delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1)) \\
&\qquad + \frac{N_c^2}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p}))\Bigg] \\
&\quad +\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_2)\Bigg[-\frac{N_c^2}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) + \frac{N_c(2C_F+N_c)}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad - \frac{N_c(C_F+N_c)}{2C_F^2}\cos((\delta z_1+\delta z_2)(\Delta E^-(\underline{k}-x\underline{p}) - \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2))) \\
&\qquad - \frac{N_c(C_F+N_c)}{2C_F^2}\cos((\delta z_1+\delta z_2)(\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) - \Delta E^-(\underline{k}-x\underline{p}))) \\
&\qquad + \frac{N_c^2}{2C_F^2}\cos(-\delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_1 \Delta E^-(\underline{k}-x\underline{p}) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad + \frac{N_c^2}{2C_F^2}\cos(-\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p}))\Bigg] \\
&\quad (\underline{k}-x\underline{p}-\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_2)\Bigg[\frac{N_c^2}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) - \frac{N_c^2}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad - \frac{N_c^2}{2C_F^2}\cos(-\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad + \frac{N_c^2}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1) - \delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p}))\Bigg] \\
&\quad +\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1-\underline{q}_2)\Bigg[-\frac{N_c^2}{2C_F^2}\cos(-\delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_1 \Delta E^-(\underline{k}-x\underline{p}) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\qquad + \frac{N_c^2}{2C_F^2}\cos(-\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1-\underline{q}_2) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_1 \Delta E^-(\underline{k}-x\underline{p}) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p}))\Bigg]\Bigg\}
\end{aligned}
$$

Exact Results at N=2

$$
\begin{aligned}
\left(p^+ \frac{dN_0}{d^2(p-q_1)\,dp^+}\right) \Bigg\{ \times \left|\psi(\underline{k}-x\underline{p})\right|^2 &\left[-\frac{C_F+N_c}{C_F} + \frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \right] \\
+ \psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_2) &\left[-\frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) + \frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p}) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)) \right] \\
+ \psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1) &\left[\frac{(C_F+N_c)}{C_F^2 N_c}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)) \right. \\
&- \frac{(C_F+N_c)}{C_F^2 N_c} \\
&- \frac{1}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\left. + \frac{1}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \right] \\
+ \psi(\underline{k}-x\underline{p}-\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1) &\left[\frac{1}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \right. \\
&- \frac{1}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1) - \delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&- \frac{1}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&\left. + \frac{1}{2C_F^2}\cos(-\delta z_2 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \right] \\
+ \left|\psi(\underline{k}-x\underline{p}+x\underline{q}_1)\right|^2 &\left[-\frac{2(C_F+N_c)}{C_F} + \frac{2(C_F+N_c)}{C_F}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)) \right] \\
+ \psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1) &\left[-\frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))) \right. \\
&+ \frac{N_c^2}{2C_F^2}\cos(\delta z_1 \Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)) + \delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \\
&+ \frac{N_c(C_F+N_c)}{C_F^2} \\
&\left. - \frac{N_c^2}{2C_F^2}\cos(\delta z_2 \Delta E^-(\underline{k}-x\underline{p})) \right]
\end{aligned}
$$

Exact Results at N=2

$$
\begin{aligned}
&+\psi(\underline{k}-x\underline{p}-\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1)\Bigg[-\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad+\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad+\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad-\frac{N_c^2}{2C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]\\
&+\psi(\underline{k}-x\underline{p}+x\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1)\Bigg[\frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1))\\
&\qquad-\frac{N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)))\\
&\qquad-\frac{N_c(C_F+N_c)}{C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1))\\
&\qquad+\frac{N_c(C_F+N_c)}{C_F^2}\Bigg]\\
&+\left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1)\right|^2\left[-\frac{2N_c(C_F+N_c)}{C_F^2}+\frac{2N_c(C_F+N_c)}{C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-q_1))\right]\\
&+\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-\underline{q}_2)\Bigg[-\frac{1}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-q_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad+\frac{1}{2C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]\\
&+\psi(\underline{k}-x\underline{p}+x\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-\underline{q}_2)\Bigg[\\
&\qquad\frac{N_c}{C_F}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-q_2)+\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad-\frac{N_c}{C_F}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-q_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad-\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\\
&\qquad+\frac{N_c}{C_F}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]
\end{aligned}
$$

Exact Results at N=2

$$+\,\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-\underline{q}_2)\Bigg[$$
$$-\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-\underline{q}_2)+\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$+\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1-\underline{q}_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$+\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$-\frac{N_c^2}{2C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-\underline{q}_2)\Bigg[$$
$$\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)-\underline{q}_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$-\frac{N_c^2}{2C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]$$

$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-\underline{q}_2)\Bigg[$$
$$-\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)-\underline{q}_2)+\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$+\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)-\underline{q}_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$+\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_1)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$
$$-\frac{N_c^2}{2C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]$$

Exact Results at N=2

$$+\,\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-\underline{q}_2)\Bigg[$$

$$\frac{N_c^2}{C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)-\underline{q}_2)+\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$

$$-\frac{N_c^2}{C_F^2}\cos(-\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x)-\underline{q}_2)-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$

$$-\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_1(1-x))-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))$$

$$+\frac{N_c^2}{C_F^2}\cos(-\delta z_2\Delta E^-(\underline{k}-x\underline{p}-\underline{q}_2)+\delta z_2\Delta E^-(\underline{k}-x\underline{p}))\Bigg]\Bigg\} \qquad (29)$$

$$\left(p^+\frac{dN_0}{d^2(p-q_2)\,dp^+}\right)\Bigg\{\;\times\left|\psi(\underline{k}-x\underline{p})\right|^2$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_2)\Bigg[-\frac{1}{2C_F^2}\cos(\delta z_2\Delta E^-(\underline{k}-x\underline{p}+x\underline{q}_2))-\frac{1}{C_FN_c}$$

$$+\frac{(2C_F+N_c)}{2C_F^2N_c}\cos(\delta z_1\Delta E^-(k-xp+xq_2)+\delta z_2\Delta E^-(k-xp+xq_2))\Bigg]$$

$$+\left|\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\right|^2\Bigg[-2-\frac{N_c}{C_F}+\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_2)))-\frac{N_c}{C_F}\cos(\delta z_2\Delta E^-(k-xp+xq_2)))$$

$$+\left(\frac{N_c}{C_F}+2\right)\cos(\delta z_1\Delta E^-(k-xp+xq_2))+\delta z_2\Delta E^-(k-xp+xq_2)))\Bigg]$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1+x\underline{q}_2)\Bigg[\frac{1}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp+xq_2))$$

$$-\frac{1}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1+xq_2)+\delta z_2\Delta E^-(k-px+q_2x))\Bigg]$$

Exact Results at N=2

$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1+x\underline{q}_2)\Big[-\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_2)))+\frac{N_c}{C_F}\cos(\delta z_2\Delta E^-(k-xp+xq_2)))$$
$$+\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_2))-\delta z_1\Delta E^-(k-xp-q_1+xq_2))$$
$$-\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp-q_1+xq_2)+\delta z_2\Delta E^-(k-xp+xq_2))\Big]$$
$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\Big[\frac{N_c}{C_F}+\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-px-q_2(1-x)))$$
$$-\frac{N_c(2C_F+N_c)}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))\Big]$$
$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\Big[\frac{N_c}{C_F}+\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp+xq_2))$$
$$\frac{N_c(C_F+N_c)}{C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp-(1-x)q_2)-(\delta z_1+\delta z_2)\Delta E^-(k-xp+xq_2))$$
$$-\frac{N_c^2}{2C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))$$
$$-\frac{N_c(2C_F+N_c)}{2C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp-(1-x)q_2))$$
$$-\frac{N_c^2}{2C_F^2}\cos(-(\delta z_1+\delta z_2)\Delta E^-(k-xp+xq_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))$$
$$+\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))-\frac{N_c(2C_F+N_c)}{2C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp+xq_2))\Big]$$
$$+\,\psi(\underline{k}-x\underline{p}-\underline{q}_1+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\Big[\frac{N_c^2}{2C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))$$
$$-\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))$$
$$-\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp+xq_2))+\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1+xq_2)+\delta z_2\Delta E^-(k-xp+xq_2)\Big]$$

Exact Results at N=2

$$
\begin{aligned}
&+\left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\right|^2\Bigg[-\frac{N_c(2C_F+N_c)}{C_F^2}+\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad -\frac{N_c^2}{C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))+\frac{N_c(2C_F+N_c)}{C_F^2}\cos((\delta z_1+\delta z_2)\Delta E^-(k-xp-(1-x)q_2))\Bigg] \\
&+\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1-(1-x)\underline{q}_2)\Bigg[-\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad +\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1-(1-x)\underline{q}_2)\Bigg[\frac{N_c^2}{2C_F^2}\cos(-(\delta z_1+\delta z_2)\Delta E^-(k-xp+xq_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad -\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad -\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1-(1-x)q_2)-(\delta z_1+\delta z_2)\Delta E^-(k-xp+xq_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad +\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))\Bigg] \\
&+\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-\underline{q}_1-(1-x)\underline{q}_2)\Bigg[-\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad +\frac{N_c^2}{C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad +\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2)-\delta z_1\Delta E^-(k-xp-q_1-(1-x)q_2)) \\
&\quad -\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-q_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))\Bigg]\Bigg\}
\end{aligned} \tag{30}
$$

Exact Results at N=2

$$\left(p^+ \frac{dN_0}{d^2(p-q_1-q_2)\,dp^+}\right)\Bigg\{\times \left|\psi(\underline{k}-x\underline{p})\right|^2$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_2)\left[\frac{1}{C_F N_c}-\frac{1}{C_F N_c}\cos(\delta z_2 \Delta E^-(k-xp+xq_2))\right]$$

$$+\left|\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\right|^2\left[2-2\cos(\delta z_2 \Delta E^-(k-xp+xq_2))\right]$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\left[-\frac{N_c}{C_F}+\frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(k-xp-(1-x)q_2))\right]$$

$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\Bigg[-\frac{N_c}{C_F}+\frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(k-xp+xq_2))$$

$$-\frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(k-xp-(1-x)q_2)-\delta z_2 \Delta E^-(k-xp+xq_2))$$

$$+\frac{N_c}{C_F}\cos(\delta z_2 \Delta E^-(k-xp-(1-x)q_2)\Bigg]$$

$$+\left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\right|^2\left[\frac{2N_c}{C_F}-\frac{2N_c}{C_F}\cos(\delta z_2 \Delta E^-(k-xp-(1-x)q_2))\right]$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1+x\underline{q}_2)\Bigg[\frac{1}{2C_F^2 N_c^2}\cos(\delta z_1 \Delta E^-(k-xp+xq_1+xq_2)+\delta z_2 \Delta E^-(k-xp+xq_2))$$

$$-\frac{1}{2C_F^2 N_c^2}\cos(\delta z_2 \Delta E^-(k-xp+xq_2))\Bigg]$$

Exact Results at N=2

$$
\begin{aligned}
&+ \psi(\underline{k} - x\underline{p} + x\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} + x\underline{q}_1 + x\underline{q}_2) \bigg[\frac{1}{C_F N_c} - \frac{1}{C_F N_c} \cos(\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2)) \\
&\qquad + \frac{1}{C_F N_c} \cos(\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2) + \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad - \frac{1}{C_F N_c} \cos(\delta z_2 \Delta E^-(k - xp + xq_2)) \bigg] \\
&+ \psi(\underline{k} - x\underline{p} - (1-x)\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} + x\underline{q}_1 + x\underline{q}_2) \bigg[\frac{1}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad \frac{1}{2C_F^2} \cos(-\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad - \frac{1}{2C_F^2} \cos(-\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad - \frac{1}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \bigg] \\
&+ \left| \psi(\underline{k} - x\underline{p} + x\underline{q}_1 + x\underline{q}_2) \right|^2 \Big[2 - 2\cos(\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2)) \Big] \\
&+ \psi(\underline{k} - x\underline{p})\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 + x\underline{q}_2) \bigg[\frac{1}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad - \frac{1}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 + xq_2) + \delta z_2 \Delta E^-(k - xp + xq_2)) \bigg] \\
&+ \psi(\underline{k} - x\underline{p} + x\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 + x\underline{q}_2) \bigg[-\frac{N_c}{C_F} + \frac{N_c}{C_F} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 + xq_2)) \\
&\qquad - \frac{N_c}{C_F} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 + xq_2) + \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad + \frac{N_c}{C_F} \cos(\delta z_2 \Delta E^-(k - xp + xq_2)) \bigg] \\
&+ \psi(\underline{k} - x\underline{p} - (1-x)\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 + x\underline{q}_2) \bigg[-\frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad - \frac{N_c^2}{2C_F^2} \cos(-\delta z_1 \Delta E^-(k - xp - (1-x)q_1 + xq_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad + \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 + xq_2) + \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\qquad + \frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \bigg]
\end{aligned}
$$

Exact Results at N=2

$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_1+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1+x\underline{q}_2)\bigg[-\frac{N_c}{C_F}+\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2))$$
$$+\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_1+xq_2))$$
$$-\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2)-\delta z_1\Delta E^-(k-xp+xq_1+xq_2))\bigg]$$
$$+\left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1+x\underline{q}_2)\right|^2\left[\frac{2N_c}{C_F}-\frac{2N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2))\right]$$

$$+\,\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\bigg[\frac{1}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))$$
$$-\frac{1}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))\bigg]$$
$$+\,\psi(\underline{k}-x\underline{p}+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\bigg[\frac{1}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))$$
$$\frac{1}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))$$
$$-\frac{1}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))$$
$$-\frac{1}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\bigg]$$
$$+\,\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\bigg[\frac{1}{C_F^2}-\frac{1}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2))$$
$$+\frac{1}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2+xq_1)+\delta z_2\Delta E^-(k-xp-(1-x)q_2))$$
$$-\frac{1}{C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))\bigg]$$

Exact Results at N=2

$$
\begin{aligned}
&+\psi(\underline{k}-x\underline{p}+x\underline{q}_1+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\bigg[\\
&\quad -\frac{N_c}{C_F}\cos(-\delta z_1\Delta E^-(k-xp+xq_1+xq_2)+\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2) \\
&\qquad -\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad +\frac{N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2))+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad +\frac{N_c}{C_F}\cos(-\delta z_1\Delta E^-(k-xp+xq_1+xq_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad -\frac{N_c}{C_F}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\bigg] \\
&+\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\bigg[\\
&\quad \frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2)+\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2) \\
&\qquad -\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad -\frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2+xq_1)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad -\frac{N_c^2}{2C_F^2}\cos(-\delta z_1\Delta E^-(k-xp+xq_2-(1-x)q_1)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2)) \\
&\quad \frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\bigg] \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\psi(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\right|^2\left[\frac{2N_c}{C_F}-\frac{2N_c}{C_F}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2))\right] \\
&+\psi(\underline{k}-x\underline{p})\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-(1-x)\underline{q}_2)\bigg[\\
&\quad \frac{N_c^2}{2C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)) \\
&\quad -\frac{N_c^2}{2C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2))\bigg]
\end{aligned}
$$

Exact Results at N=2

$$
\begin{aligned}
&+ \psi(\underline{k} - x\underline{p} + x\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 - (1-x)\underline{q}_2) \Bigg[\\
&\quad - \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_2 - (1-x)q_1) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\quad \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 - (1-x)q_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2)) \\
&\quad \frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\quad - \frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2)) \Bigg] \\
&+ \psi(\underline{k} - x\underline{p} - (1-x)\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 - (1-x)\underline{q}_2) \Bigg[- \frac{N_c^2}{C_F^2} + \frac{N_c^2}{C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 - (1-x)q_2)) \\
&\quad - \frac{N_c^2}{C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 - (1-x)q_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2)) \\
&\quad + \frac{N_c^2}{C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2)) \Bigg] \\
&+ \psi(\underline{k} - x\underline{p} + x\underline{q}_1 + x\underline{q}_2)\, \psi^*(\underline{k} - x\underline{p} - (1-x)\underline{q}_1 - (1-x)\underline{q}_2) \Bigg[\\
&\quad \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 - (1-x)q_2) - \delta z_1 \Delta E^-(k - xp + xq_2 + xq_1) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta \\
&\quad - \frac{N_c^2}{2C_F^2} \cos(\delta z_1 \Delta E^-(k - xp - (1-x)q_1 - (1-x)q_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\quad - \frac{N_c^2}{2C_F^2} \cos(-\delta z_1 \Delta E^-(k - xp + xq_1 + xq_2) + \delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \\
&\quad \frac{N_c^2}{2C_F^2} \cos(\delta z_2 \Delta E^-(k - xp - (1-x)q_2) - \delta z_2 \Delta E^-(k - xp + xq_2)) \Bigg] \qquad (32)
\end{aligned}
$$

Exact Results at N=2

$$
\begin{aligned}
&+\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1+x\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-(1-x)\underline{q}_2)\Big[\\
&\qquad-\frac{N_c^2}{C_F^2}\cos(-\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2)+\delta z_1\Delta E^-(k-xp-(1-x)q_1-(1-x)q_2)\\
&\qquad\qquad+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\\
&\qquad+\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1-(1-x)q_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\\
&\qquad+\frac{N_c^2}{C_F^2}\cos(-\delta z_1\Delta E^-(k-xp-(1-x)q_1+xq_2)+\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\\
&\qquad-\frac{N_c^2}{C_F^2}\cos(\delta z_2\Delta E^-(k-xp-(1-x)q_2)-\delta z_2\Delta E^-(k-xp+xq_2))\Big]\\
&+\psi(\underline{k}-x\underline{p}+x\underline{q}_1-(1-x)\underline{q}_2)\,\psi^*(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-(1-x)\underline{q}_2)\Big[-\frac{N_c^2}{C_F^2}+\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp+xq_1-(1-x)q_2))\\
&\qquad-\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1-(1-x)q_2)-\delta z_1\Delta E^-(k-xp-(1-x)q_2+xq_1))\\
&\qquad+\frac{N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_2-(1-x)q_1))\Big]\\
&+\left|\psi(\underline{k}-x\underline{p}-(1-x)\underline{q}_1-(1-x)\underline{q}_2)\right|^2\left[\frac{2N_c^2}{C_F^2}-\frac{2N_c^2}{C_F^2}\cos(\delta z_1\Delta E^-(k-xp-(1-x)q_1-(1-x)q_2))\right]\Bigg\}
\end{aligned}
\tag{33}
$$