QCD evolution

May 20-24, 2018
Santa Fe, NM

Extract Hadron's Partonic Structure from Lattice QCD Calculations

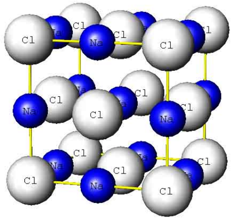
Jianwei Qiu
Theory Center, Jefferson Lab

Based on work done with
T. Ishikawa, Y.-Q. Ma, S. Yoshida, …
and work by many others, …

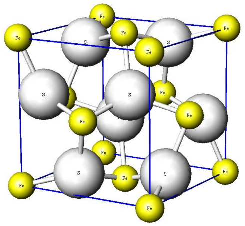
Hadron's partonic structure in QCD

- Structure – "a still picture"

Crystal Structure:

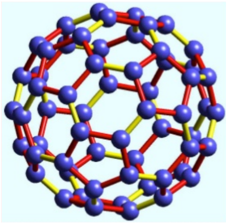


NaCl,
B1 type structure



FeS2,
C2, pyrite type structure

Nano-material:



Fullerene, C60

Motion of nuclei is much slower than the speed of light!

Hadron's partonic structure in QCD

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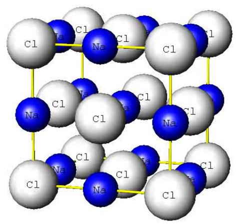

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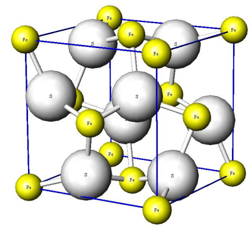

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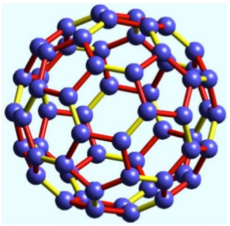

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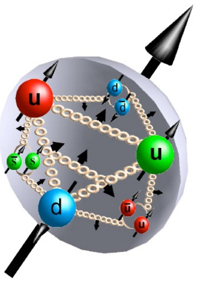

- **No "still picture" for hadron's partonic structure!**

Motion of quarks/gluons is relativistic!

Partonic Structure:

Quantum "probabilities" $\langle P, S | \mathcal{O}(\overline{\psi}, \psi, A^{\mu}) | P, S \rangle$

None of these matrix elements is a direct physical observable in QCD – color confinement!

Hadron's partonic structure in QCD

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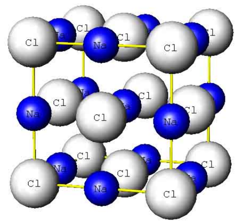

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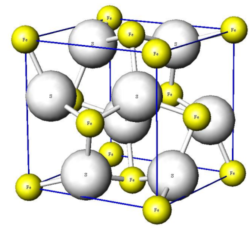

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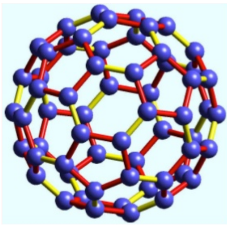

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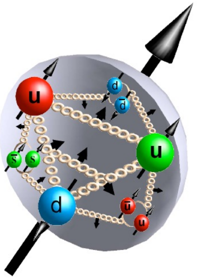

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❑ Accessible hadron's partonic structure?

= Universal matrix elements of quarks and/or gluons

1) can be related to good physical cross sections of hadron(s) *with controllable approximation*,

2) can be calculated in lattice QCD, …

Hadron's partonic structure in QCD

- **Good hadronic cross sections:**

  1) can measured experimentally with precision,
  2) can be factorized into universal matrix elements of quarks and gluons
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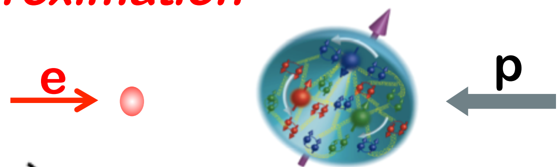
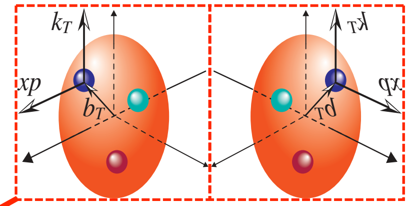
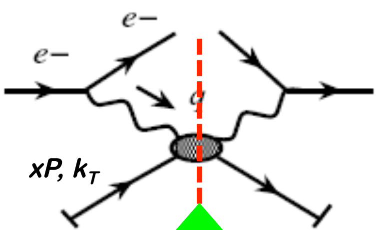




$\sigma_{\text{tot}}^{\text{DIS}}$:



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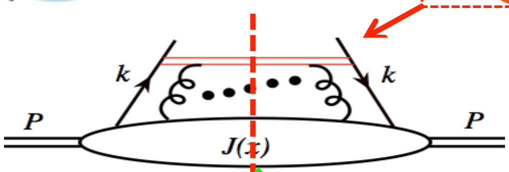

$+\ O\left(\dfrac{1}{QR}\right)$

Factorization

Hard-part
Probe

Parton-distribution
Structure

Power corrections
Approximation

Hadron's partonic structure in QCD

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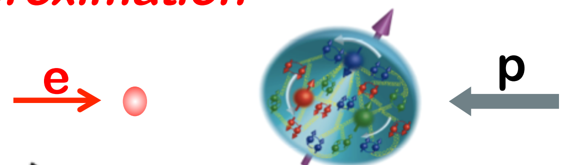

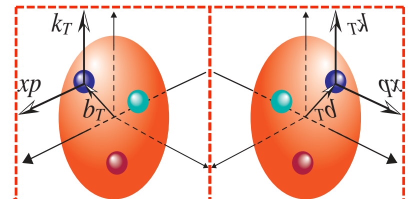

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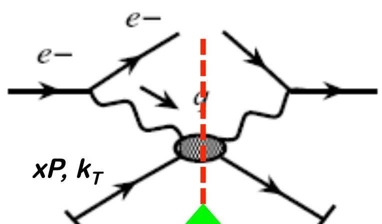

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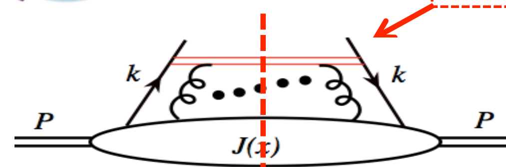

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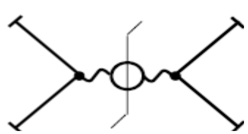

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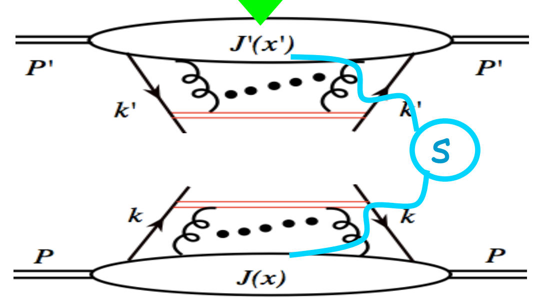

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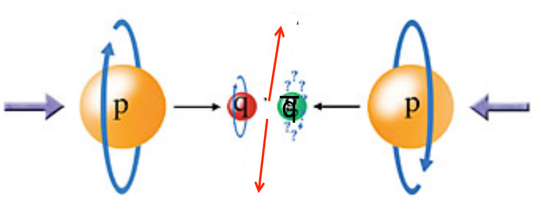

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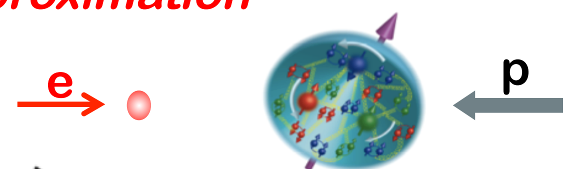
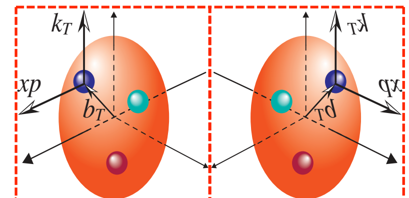
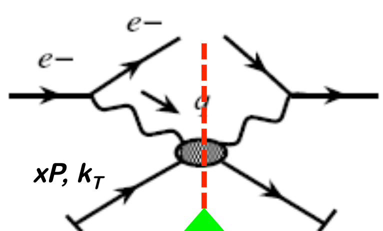




$$\sigma^{\mathrm{DIS}}_{\mathrm{tot}} :$$



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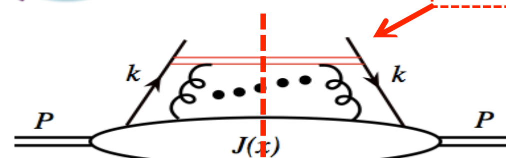

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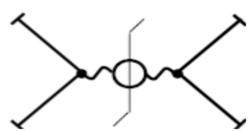

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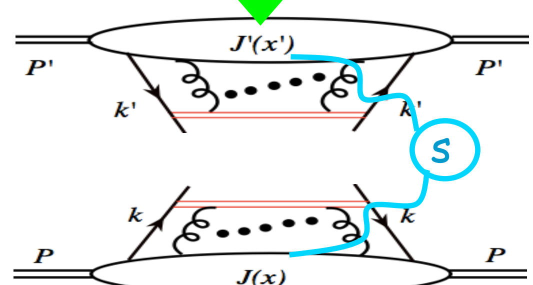

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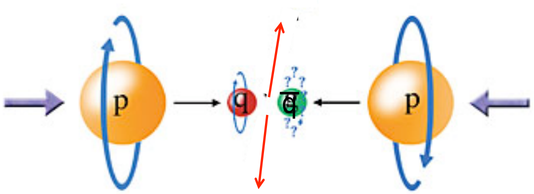

QCD Global analysis:

Need data of "many" good cross sections to be able to extract the x, Q, flavor dependence of the structure, …

Calculate the partonic structure in lattice QCD?

- Answer: Not directly!

Particle nature of quarks/gluons ⇒ Probes with large Q transfer
Large momentum transfer ⇒ Large "+" momentum dominates
Operators on light-cone ⇒ Can't be calculated in lattice QCD

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- **Quasi-PDFs:**

Ji, arXiv:1305.1539

$$\tilde{q}(x,\mu^2,P_z) \equiv \int \frac{d\xi_z}{4\pi} e^{-ixP_z\xi_z} \langle P|\overline{\psi}(\xi_z)\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle$$

Proposed matching:

$$\tilde{q}(x,\mu^2,P_z) = \int_x^1 \frac{dy}{y} Z\left(\frac{x}{y},\frac{\mu}{P_z}\right) q(y,\mu^2) + \mathcal{O}\left(\frac{\Lambda^2}{P_z^2},\frac{M^2}{P_z^2}\right)$$

+ gluon contribution beyond LO

- **Pseudo-PDFs:**

Radyushkin, 2017

$$\mathcal{M}^\alpha(\nu = p\cdot\xi,\xi^2) \equiv \langle p|\overline{\psi}(0)\gamma^\alpha\Phi_v(0,\xi,v\cdot A)\psi(\xi)|p\rangle$$
$$\equiv 2p^\alpha\mathcal{M}_p(\nu,\xi^2) + \xi^\alpha(p^2/\nu)\mathcal{M}_\xi(\nu,\xi^2) \approx 2p^\alpha\mathcal{M}_p(\nu,\xi^2)$$

$$\mathcal{P}(x,\xi^2) \equiv \int \frac{d\nu}{2\pi} e^{ix\nu}\frac{1}{2p^+}\mathcal{M}^+(\nu,\xi^2)$$ with $\xi^2 < 0$

Off-light-cone extension of PDFs: $f(x) = \mathcal{P}(x,\xi^2=0)$ with $\xi^\mu = (0^+,\xi^-,0_\perp)$

- **Other approaches, …**

"OPE without OPE" (Chambers et al. 2017), Hadronic tensor (Liu et al. 1994, …), …

Hadron's partonic structure in Lattice QCD

Ma and Qiu, arXiv:1404.6860
arXiv:1709.03018

- Good "Lattic cross sections":

  = Single hadron matrix element:

  $\sigma_n(\omega, \xi^2, P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle$ with $\omega \equiv P\cdot\xi,\ \xi^2 \neq 0,\ \text{and}\ \xi_0 = 0;$ and

  1) can be calculated in lattice QCD with precision,
     has a well-defined continuum limit (UV+IR safe perturbatively), and
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Collaboration between lattice QCD and perturbative QCD!

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Collaboration between lattice QCD and perturbative QCD!

- **Current-current correlators:**

$$\mathcal{O}_{j_1 j_2}(\xi) \equiv \xi^{d_{j_1}+d_{j_2}-2}\, Z_{j_1}^{-1}\, Z_{j_2}^{-1} j_1(\xi)\, j_2(0)$$

with

d_j : Dimension of the current

Z_j : Renormalization constant of the current

Sample currents:

$$j_S(\xi) = \xi^2 Z_S^{-1}[\overline{\psi}_q \psi_q](\xi), \qquad j_V(\xi) = \xi Z_V^{-1}[\overline{\psi}_q \gamma\cdot\xi\psi_q](\xi),$$

$$j_{V'}(\xi) = \xi Z_{V'}^{-1}[\overline{\psi}_q \gamma\cdot\xi\psi_{q'}](\xi), \qquad j_G(\xi) = \xi^3 Z_G^{-1}[-\frac{1}{4}F^c_{\mu\nu}F^c_{\mu\nu}](\xi), \ldots$$

- **Quasi- and pseudo-PDFs:**

$$\mathcal{O}_q(\xi) = Z_q^{-1}(\xi^2)\overline{\psi}_q(\xi)\,\gamma\cdot\xi\Phi(\xi,0)\,\psi_q(0) \qquad \Phi(\xi,0) = \mathcal{P}e^{-ig\int_0^1 \xi\cdot A(\lambda\xi)\,d\lambda}$$

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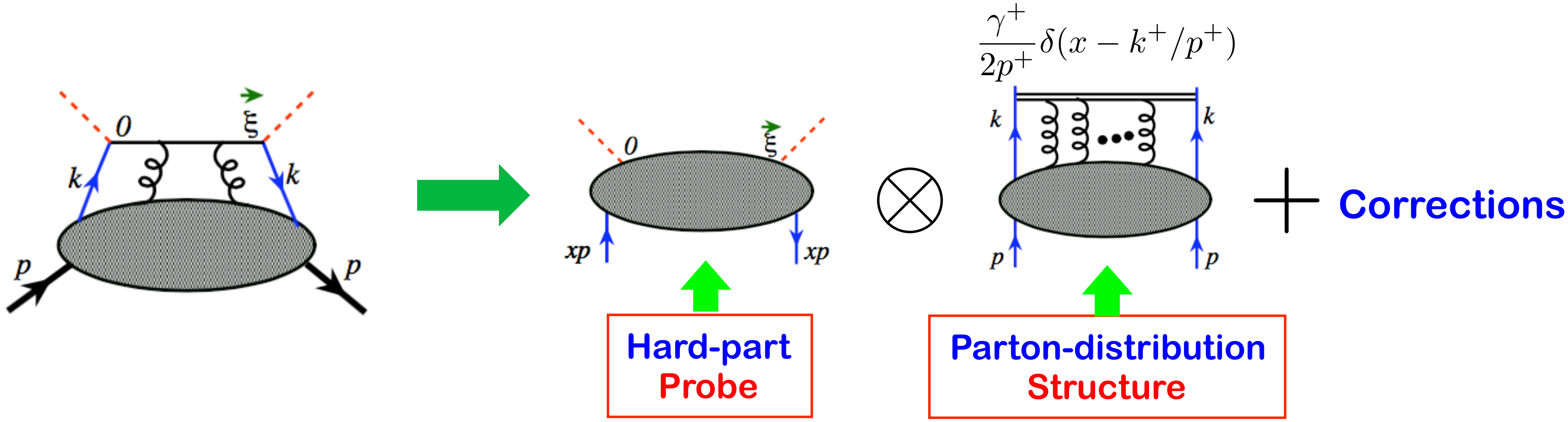
- ❑ **Factorization:**



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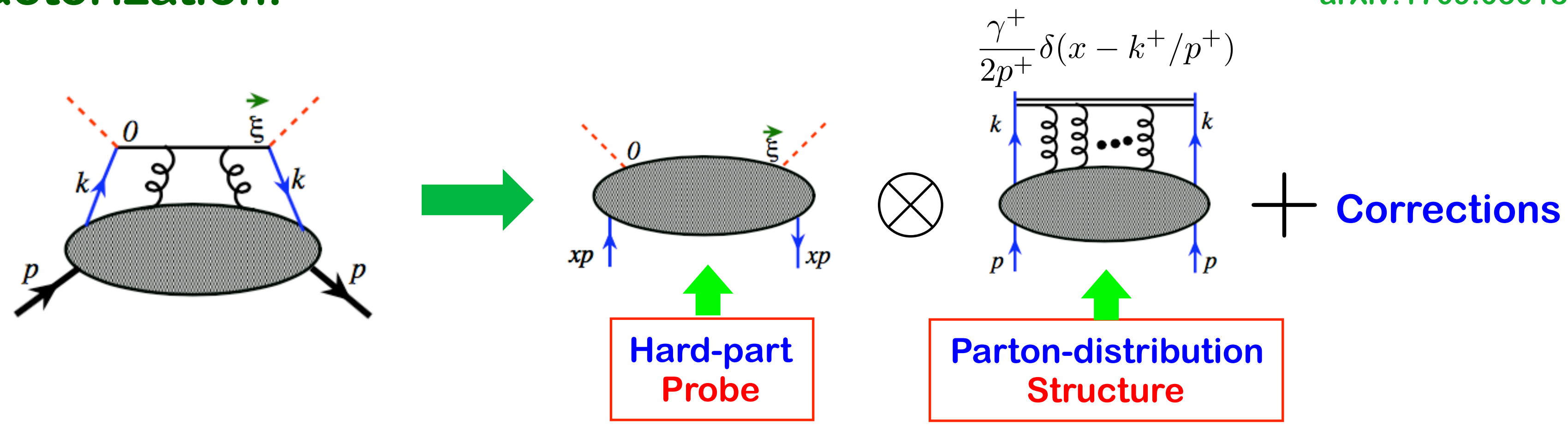

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- **Complementarity and advantages:**
  - Complementary to existing approaches for extracting PDFs,
  - Quasi-PDFs and pseudo-PDFs are special cases,
  - Have tremendous potentials:

Neutron PDFs, … (no free neutron target!)
Meson PDFs, such as pion, …
More direct access to gluons – gluonic current, quark flavor, …

David Richards’
talk on Monday

Renormalization – Summary

- **Current-current correlators – take care by construction:**

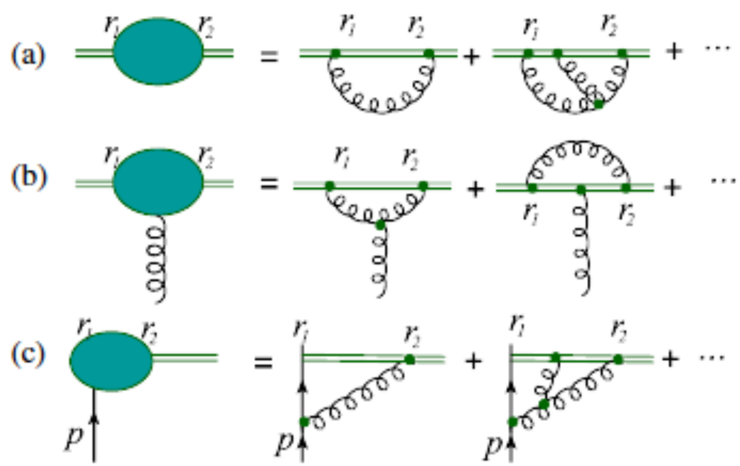
  Construct operators by using renormalizable, or conserved currents

- **Renormalization of quasi- and pseudo-PDFs:**

  Quasi-quark distributions is multiplicatively renormalizable

$$\tilde{q}_i^R(\xi_z, \mu^2, p_z) = e^{-C_i|\xi_z|} Z_{wi}^{-1} Z_{vi}^{-1} \tilde{q}_i(\xi_z, \mu^2, p_z)$$

Three classes of elementary divergent diagrams:



Ishikawa, Ma, Qiu and Yoshida
arXiv: 1701.03108

Pseudo-quark distributions takes care of the UV renormalization by

$$\mathcal{P}(x, \xi^2) \equiv \int \frac{d\nu}{2\pi} e^{ix\,\nu} \mathcal{M}_{p=p^0}(\nu, \xi^2)/\mathcal{M}_{p=p^0}(0, \xi^2)$$

Different matching

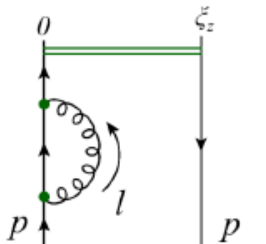
Renormalization – quasi-quark

- **Coordinate-space definition:**

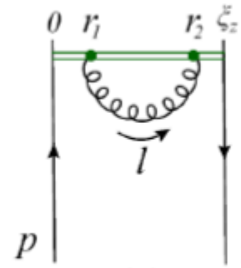
$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z}\langle h(p)|\overline{\psi}_q(\xi_z)\,\frac{\gamma_z}{2}\Phi^{(f)}_{n_z}(\{\xi_z,0\})\,\psi_q(0)|h(p)\rangle$$

- **Why the proof is hard:**
  - Because of z-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences
  - Renormalization of composite operator is needed
- **Broken Lorentz symmetry:**

Both 3D and 4D loop-integration can generate UV divergences



(c)



(a)

$$l^\mu = \bar{l}^\mu + l_z n_z^\mu$$

UV: 4-D integration

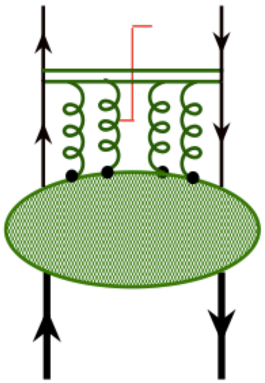
$$\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2(p-l)^2}$$

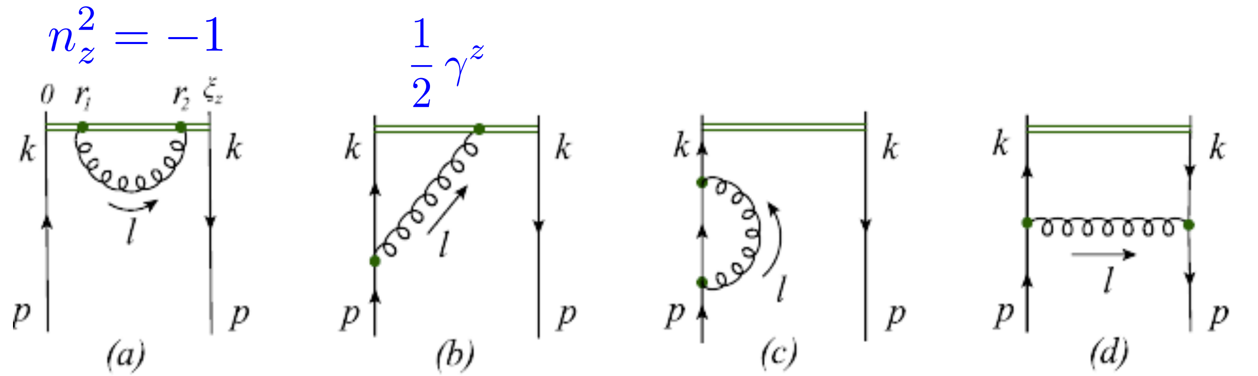
UV: 3-D integration

$$\int \frac{d^3\bar{l}}{l^2} = \int \frac{d^3\bar{l}}{\bar{l}^2 - l_z^2}$$

Renormalization – quasi-quark

- Quasi-quark at one-loop:





- Fig. 1(a):

$$M_{1a} = \frac{e^{ip_z\xi_z}}{p_z}\frac{1}{N_c}\mathrm{Tr}_c[T^aT^a]\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2$$
$$\times\int\frac{d^4l}{(2\pi)^4}e^{-ip_z\xi_z}\,e^{il_z(r_2-r_1)}\left(\frac{-ig_{\mu\nu}}{l^2}\right)$$
$$\times(-ig_sn_z^\mu)(-ig_sn_z^\nu)\mathrm{Tr}\left[\frac{1}{2}\not p\,\frac{1}{2}\gamma_z\right]$$
$$=\frac{\alpha_sC_F}{4i\pi^3}\int_a^{\xi_z-2a}dr_1\int_{r_1+a}^{\xi_z-a}dr_2\int d^4l\frac{e^{il_z(r_2-r_1)}}{l^2}$$

$$M_{1a}\stackrel{\mathrm{div}}{=}-\frac{\alpha_sC_F}{\pi}\frac{|\xi_z|}{a}+\frac{\alpha_sC_F}{\pi}\ln\frac{|\xi_z|}{a}$$

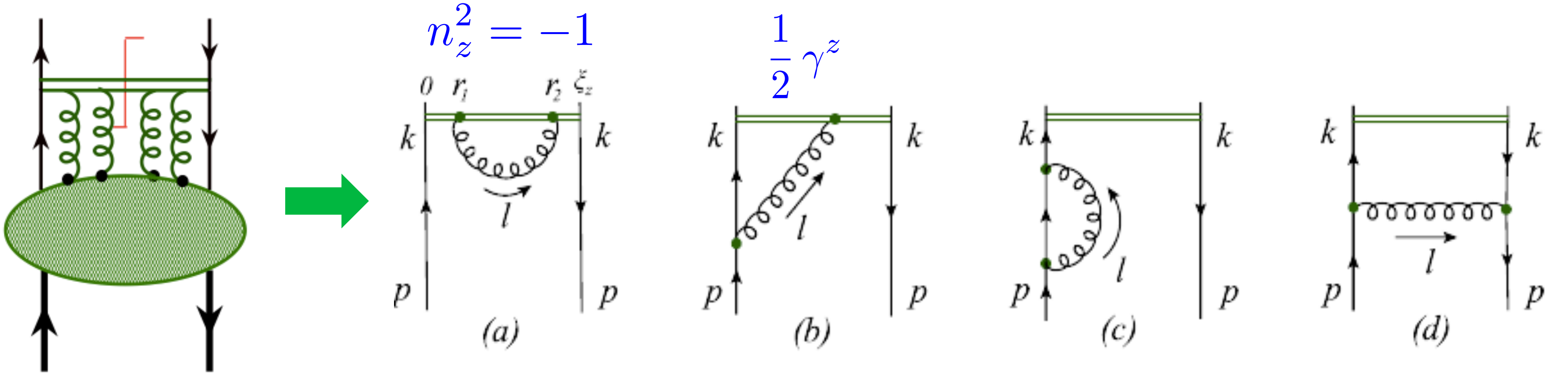
- ✧ Cutoff "a" between fields
- ✧ Conclusion independent of regulator
- ✧ 3D-integration: $d^4l = d^3\bar{l}\,dl_z$

$$\int\frac{d^3\bar{l}}{l^2} = \int\frac{d^3\bar{l}}{\bar{l}^2-l_z^2}$$
$$=\int d^3\bar{l}\left(\frac{1}{\bar{l}^2}+\frac{l_z^2}{(\bar{l}^2-l_z^2)\bar{l}^2}\right)$$
$$\int dl_z e^{il_z(r_2-r_1)} = 2\pi\delta(r_2-r_1)$$

1st term vanishes for $r_1 \neq r_2$

Renormalization – quasi-quark

- Quasi-quark at one-loop:



- Complete one-loop contribution:

$$M^{(1)} \stackrel{\text{div}}{=} M_{1a} + 2 \times M_{1b} + 2 \times \frac{1}{2} M_{1c} + M_{1d}$$
$$= \frac{\alpha_s C_F}{\pi}\left(-\frac{|\xi_z|}{a} + 2\ln\frac{|\xi_z|}{a} - \frac{1}{4\epsilon}\right).$$

✧ At one-loop, all 3D integrations are finite
✧ Divergence only come from the region when all momentum components go to infinity

➔ Localized UV divergence in all directions!

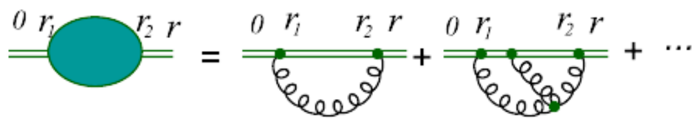
Very different from the UV behavior of normal PDFs: (1, λ^2, λ), $\lambda \to \infty$

Renormalization – quasi-quark

Ishikawa, Ma, Qiu, Yoshida (2017)

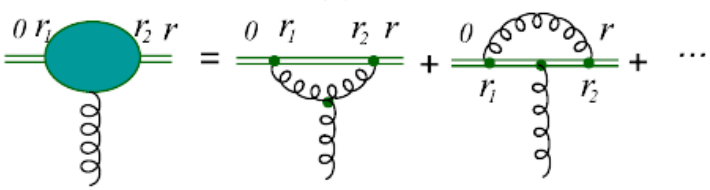
❑ Power counting and divergent sub-diagrams:

(a) - 1/a, ln(1/a):



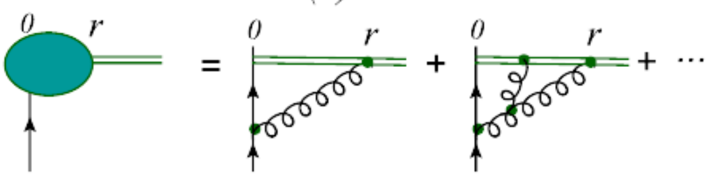
(a)

(b) - ln(1/a):



(b)

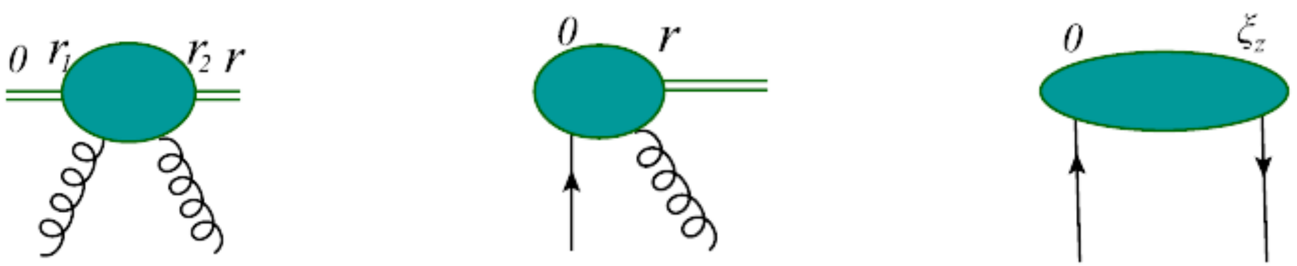
(c) - ln(1/a):



(c)

Happen only when all loop momenta go to infinity – localized!

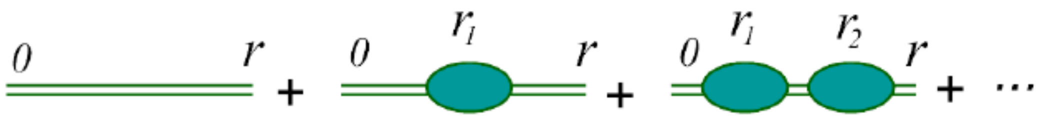
❑ Example of convergent sub-diagrams:



Renormalization – quasi-quark

Ishikawa, Ma, Qiu, Yoshida (2017)

❑ Power divergence:



$$1 + c\int_0^r dr_1 + c^2 \int_0^r dr_1 \int_{r_1}^r dr_2 + \cdots$$

$$= \mathcal{P} e^{\,c\int_0^r dr'} = e^{\,c\,r}\,,$$

- It is allowed to introduce an overall factor $e^{-c|\xi_z|}$ to remove all power UV divergences

❑ Interpretation:

- Mass renormalization of test particle

Dotsenko, Vergeles, NPB (1980)

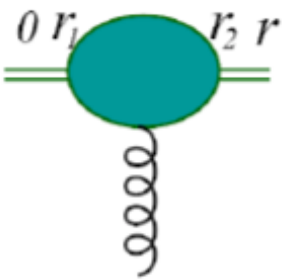
❑ Log divergence in from gauge link:

- Besides power divergence, there are also logarithmic UV divergences
- It is known that these divergences can be removed by a "wave function" renormalization of the test particle, Z_{wq}^{-1}.

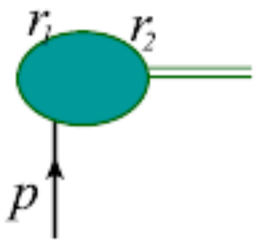
Renormalization – quasi-quark

Ishikawa, Ma, Qiu, Yoshida (2017)

- Log divergence from gluon-gauge link vertex:



- Logarithmic UV: can be absorbed by the coupling constant renormalization of QCD.



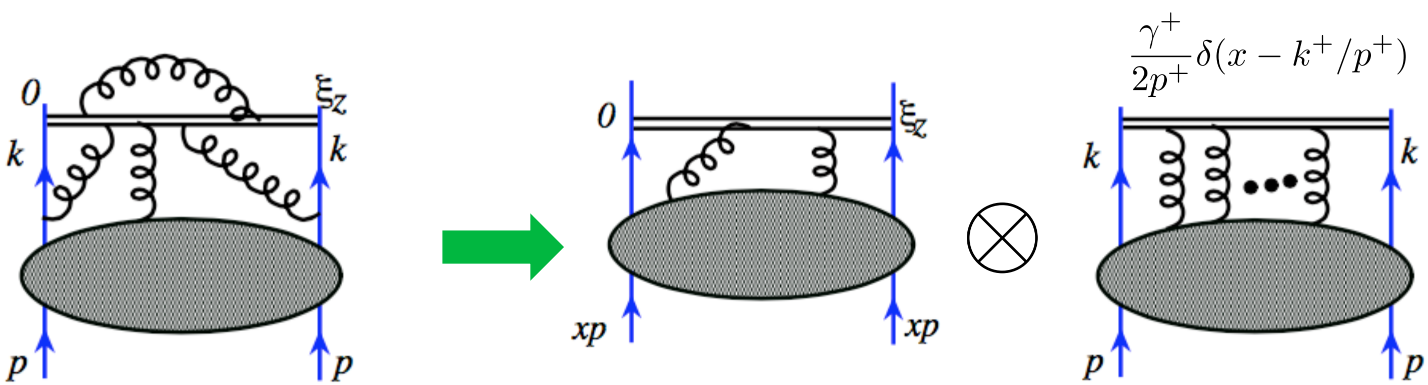
- UV from vertex correction:
  - The most dangerous UV diagram, may mix with other operators
  - Locality of UV divergence: no dependence on $r_2 - r_1$ or p
  - UV divergence is proportional to quark-gaugelink vertex at lowest order, with a constant coefficient
  - A constant counter term is able to remove this UV divergence.

- Renormalization to all orders:
  - Using bookkeeping forests subtraction method, the net effect is to introduce a constant multiplicative renormalizaton factor $Z_{vq}^{-1}$for the quark-gaugelink vertex.

Factorization

- ❑ Does the renormalized lattice cross section and quasi-PDFs share the same CO properties with normal PDFs?

- ❑ Can we extract PDFs from lattice cross section or renormalized quasi-PDFs reliably?



$+$ Corrections

Factorization

Ma and Qiu, arXiv:1404.6860
arXiv:1709.03018

- **Factorized formula for lattice cross section:**

$$\sigma_n(\omega,\xi^2,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x}\, f_a(x,\mu^2) \times K_n^a(x\omega,\xi^2,x^2P^2,\mu^2) + O(\xi^2\Lambda_{\rm QCD}^2)$$

with $f_{\bar{a}}(x,\mu^2) = -f_a(-x,\mu^2)$

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with $f_a(x,\mu^2) = -f_a(-x,\mu^2)$

- **Steps needed to prove:**

Let ξ^2 be small but not vanishing, apply OPE to the operator,

$$\sigma_n(\omega,\xi^2,P^2) = \sum_{J=0}\sum_a W_n^{(J,a)}(\xi^2,\mu^2)\,\xi^{\nu_1}\cdots\xi^{\nu_J} \times \langle P|\mathcal{O}^{(J,a)}_{\nu_1\cdots\nu_J}(\mu^2)|P\rangle$$

Local, symmetric and traceless with spin J

with

$$\langle P|\mathcal{O}^{(J,a)}_{\nu_1\cdots\nu_J}(\mu^2)|P\rangle = 2A^{(J,a)}(\mu^2) \times (P_{\nu_1}\cdots P_{\nu_J} - \text{traces})$$

With reduced matrix element: $A^{(J,a)}(\mu^2) = \langle P|\mathcal{O}^{(J,a)}(\mu^2)|P\rangle$

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with $\Sigma_J(\omega,P^2\xi^2) \equiv \xi^{\nu_1}\cdots\xi^{\nu_J}(P_{\nu_1}\cdots P_{\nu_J} - \text{traces})$

$$= \sum_{i=0}^{[J/2]} C_{J-i}^i(\omega)^{J-2i}\left(-P^2\xi^2/4\right)^i$$

No approximation yet!

Factorization

Ma and Qiu, arXiv:1404.6860
arXiv:1709.03018

- **Approximation – leading power/twist:**

$$A^{(J,a)}(\mu^2) = \frac{1}{S_a}\int_{-1}^{1} dx x^{J-1} f_a(x,\mu^2)$$

With symmetry factor: $S_a = 1, 2$ for $a = q, g$

$$\sigma_n(\omega,\xi^2,P^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2) \times K_n^a(x\omega,\xi^2,x^2P^2,\mu^2) + O(\xi^2\Lambda_{\rm QCD}^2)$$

with $K_n^a = \sum_{J=1} \frac{2}{S_a} W_n^{(J,a)}(\xi^2,\mu^2)\, \Sigma_J(x\omega, x^2P^2\xi^2)$

Note: our proof of factorization is valid only when $|\omega| \ll 1$ **and** $|p^2\xi^2| \ll 1$

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Note: our proof of factorization is valid only when $|\omega| \ll 1$ and $|p^2\xi^2| \ll 1$

- Extrapolate into large ω region:

  - Validity of OPE guarantees that σ_n is an analytic function of ω, so as its Taylor series of ω around ω=0, defined above
  - If we fix ξ to be short-distance, while we increase ω by adjusting p, we can't introduce any new perturbative divergence
  - That is, σ_n remains to be an analytic function of ω unless $\omega = \infty$

Factorization holds for any finite value of ω and $p^2\xi^2$, if ξ is short-distance

Coefficient/matching functions

- **Matching coefficients for current-current correlators:**

$$K_n^a = \sum_{J=1} \frac{2}{S_a} W_n^{(J,a)}(\xi^2,\mu^2)\,\Sigma_J(x\omega, x^2P^2\xi^2)$$ ⟹ Need $W_n^{(J,a)}(\xi^2,\mu^2)$

a) Calculate $K_n^a(x\omega,\xi^2,0,\mu)$ - coefficient in CO factorization with $p^2=0$

b) Expand $K_n^a(x\omega,\xi^2,0,\mu)$ in power series of $x\omega$

c) Extract $W_n^{(J,a)}(\xi^2,\mu^2)$ with $\Sigma_J(x\omega,0)=(x\omega)^J$

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- **LO matching:**

$$\Longrightarrow$$

$$k^\mu = xp^\mu$$
$$p^2 = 0$$

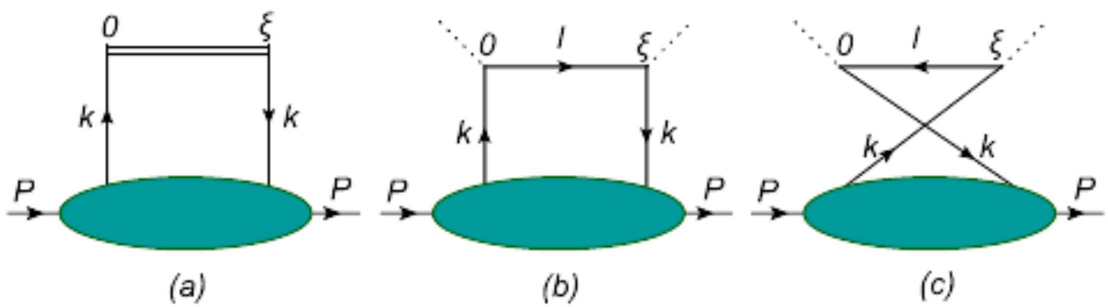

Fig. (a)

$$K_q^{q(0)}(x\omega,\xi^2,0,\mu) = \frac{1}{2}\mathrm{Tr}[\not{k}\not{\xi}]e^{i\xi\cdot k} = 2x\omega e^{ix\omega}$$

$$\Longrightarrow \quad W_q^{(J,q)} = i^{J-1}/(J-1)!$$

Coefficient/matching functions

❑ **Matching coefficients for current-current correlators:**

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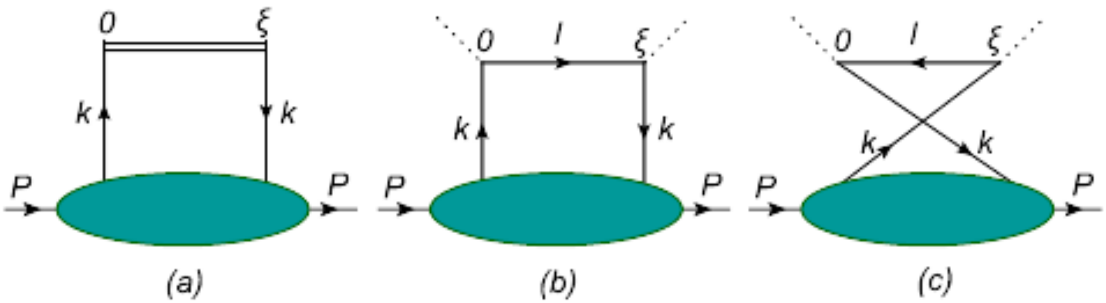

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Fig. (b,c)

$$M_b = \frac{i\xi^4}{2}\int \frac{d^4l}{(2\pi)^4}\frac{\mathrm{Tr}[\not{k}\not{l}]e^{i\xi\cdot(k-l)}}{l^2+i\varepsilon} = \frac{i}{\pi^2}x\omega\, e^{ix\omega} \qquad M_c = M_b^*$$

$$\Rightarrow \quad \widetilde{K}^{q(0)}_{S/V/\widetilde{V}}(x\widetilde{\omega},q^2,0,\mu) = -2i\,\frac{x^2\widetilde{\omega}^2}{1-x^2\widetilde{\omega}^2}$$

Flavor change current
No crossing diagram

Comparison with other approaches

- **Momentum-space version – Fourier transform:**

$$\widetilde{\sigma}_n(\widetilde{\omega}, q^2, P^2) \equiv \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} \sigma_n(P\cdot\xi, \xi^2, P^2)$$

With $\widetilde{\omega} \equiv \frac{2P\cdot q}{-q^2} = \frac{1}{x_B}$, and valid for $\widetilde{\omega}^2 < 1$

$$\widetilde{K}_n^a = \int \frac{d^4\xi}{\xi^4} e^{iq\cdot\xi} K_n^a(xP\cdot\xi, \xi^2, x^2P^2, \mu)$$

Care is needed for the physical region when $\widetilde{\omega}^2 > 1$

Contribution from large $\widetilde{\omega}^2 > 1$ region – poles and cuts

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Contribution from large $\tilde{\omega}^2 > 1$ region – poles and cuts

❑ **Connection to quasi- and pseudo-PDFs:**

With $K_q^{q(0)}$ ➡ $\displaystyle\int \frac{d\omega}{\omega} \frac{e^{-ix\omega}}{4\pi} \sigma_q(\omega, \xi^2, P^2) \approx f_q(x, \mu)$

modulo $O(\alpha_s)$ corrections and higher twist corrections.

With $\xi_0 = 0$, the integral over $\omega = -\vec{\xi}\cdot\vec{P} = -|\vec{\xi}||\vec{P}|\cos\theta$

Quasi-PDFs: $\xi_0 = 0,\ \vec{p} = p_z,\ \vec{\xi} = \xi_z$ **with fixed** p_z

Pseudo-PDFs: $\xi_0 = 0,\ \vec{p} = p_z,\ \vec{\xi} = \xi_z$ **with fixed** ξ_z

One-loop example: quark →quark

Ma and Qiu, arXiv:1404.6860

- **Expand the factorization formula:**

$$\tilde{f}_{i/h}(\tilde{x}, \tilde{\mu}^2, P_z) \approx \sum_j \int_0^1 \frac{dx}{x} \mathcal{C}_{ij}(\frac{\tilde{x}}{x}, \tilde{\mu}^2, P_z)\, f_{j/h}(x, \mu^2)$$

To order α_s :

$$\tilde{f}^{(1)}_{q/q}(\tilde{x}) = f^{(0)}_{q/q}(x) \otimes \mathcal{C}^{(1)}_{q/q}(\tilde{x}/x) + f^{(1)}_{q/q}(x) \otimes \mathcal{C}^{(0)}_{q/q}(\tilde{x}/x)$$

$$\Rightarrow \quad \mathcal{C}^{(1)}_{q/q}(t, \tilde{\mu}^2, \mu^2, P_z) = \tilde{f}^{(1)}_{q/q}(t, \tilde{\mu}^2, P_z) - f^{(1)}_{q/q}(t, \mu^2)$$

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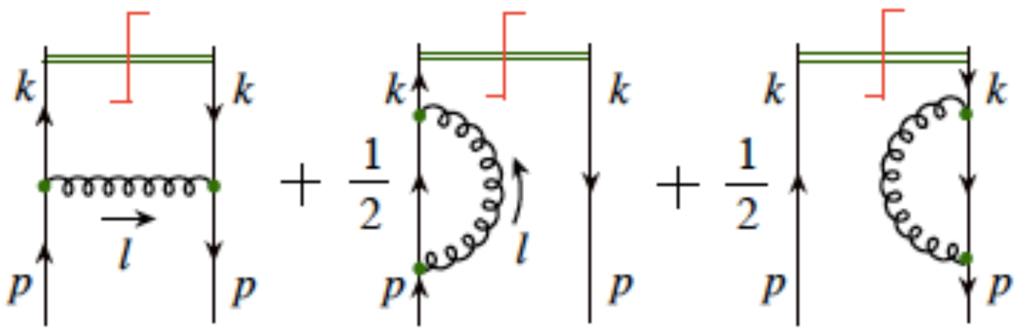
- **Feynman diagrams:**

Same diagrams for both

$\tilde{f}_{q/q}$ and $f_{q/q}$



But, in different gauge: $n_z \cdot A = 0$ for $\tilde{f}_{q/q}$ $\quad n \cdot A = 0$ for $f_{q/q}$

- **Gluon propagator in n_z . A = 0:**

$$\tilde{d}^{\alpha\beta}(l) = -g^{\alpha\beta} + \frac{l^\alpha n_z^\beta + n_z^\alpha l^\beta}{l_z} - \frac{n_z^2\, l^\alpha l^\beta}{l_z^2}$$

with $n_z^2 = -1$

One-loop "quasi-quark" distribution in a quark

Ma and Qiu, arXiv:1404.6860

- **Real + virtual contribution:**

$$\tilde{f}^{(1)}_{q/q}(\tilde{x},\tilde{\mu}^2,P_z) = C_F\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\int_0^{\tilde{\mu}^2}\frac{dl_\perp^2}{l_\perp^{2+2\epsilon}}\int_{-\infty}^{+\infty}\frac{dl_z}{P_z}\left[\delta(1-\tilde{x}-y)-\delta(1-\tilde{x})\right]\left\{\frac{1}{y}\left(1-y+\frac{1-\epsilon}{2}y^2\right)\right.$$
$$\left.\times\left[\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}}\right]+\frac{(1-y)\lambda^2}{2y^2\sqrt{\lambda^2+y^2}}+\frac{\lambda^2}{2y\sqrt{\lambda^2+(1-y)^2}}+\frac{1-\epsilon}{2}\frac{(1-y)\lambda^2}{[\lambda^2+(1-y)^2]^{3/2}}\right\}$$

where $y = l_z/P_z,\ \lambda^2 = l_\perp^2/P_z^2,\ C_F = (N_c^2-1)/(2N_c)$

- **Cancelation of CO divergence:**

$$\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}} = 2\theta(0<y<1)-\left[\mathrm{Sgn}(y)\frac{\sqrt{\lambda^2+y^2}-|y|}{\sqrt{\lambda^2+y^2}}+\mathrm{Sgn}(1-y)\frac{\sqrt{\lambda^2+(1-y)^2}-|1-y|}{\sqrt{\lambda^2+(1-y)^2}}\right]$$

Only the first term is CO divergent for $0 < y < 1$, which is the same as the divergence of the normal quark distribution – necessary!

- **UV renormalization:**

Different treatment for the upper limit of $l_\perp^2$ integration - "scheme"

Here, a UV cutoff is used – other scheme is discussed in the paper

One-loop coefficient functions

Ma and Qiu, arXiv:1404.6860

- MS scheme for $f_{q/q}(x,\mu^2)$

$$\mathcal{C}^{(1)}_{q/q}(t,\tilde{\mu}^2,\mu^2,P_z) = \tilde{f}^{(1)}_{q/q}(t,\tilde{\mu}^2,P_z) - f^{(1)}_{q/q}(t,\mu^2)$$

CO, UV IR finite!

$$\frac{\mathcal{C}^{(1)}_{q/q}(t)}{C_F\frac{\alpha_s}{2\pi}} = \left[\frac{1+t^2}{1-t}\ln\frac{\tilde{\mu}^2}{\mu^2}+1-t\right]_+ + \left[\frac{t\Lambda_{1-t}}{(1-t)^2}+\frac{\Lambda_t}{1-t}+\frac{\mathrm{Sgn}(t)\Lambda_t}{\Lambda_t+|t|}\right.$$
$$\left. -\frac{1+t^2}{1-t}\left[\mathrm{Sgn}(t)\ln\left(1+\frac{\Lambda_t}{2|t|}\right)+\mathrm{Sgn}(1-t)\ln\left(1+\frac{\Lambda_{1-t}}{2|1-t|}\right)\right]\right]_N$$

where $\Lambda_t = \sqrt{\tilde{\mu}^2/P_z^2+t^2}-|t|$, $\mathrm{Sgn}(t)=1$ if $t\geq 0$, and -1 otherwise.

- Generalized "+" description:

$$\int_{-\infty}^{+\infty} dt\left[g(t)\right]_N h(t) = \int_{-\infty}^{+\infty} dt\, g(t)\left[h(t)-h(1)\right]$$

For a testing function

$h(t) \quad t=\tilde{x}/x$

Explicit verification of the CO factorization at one-loop

Note: $\Lambda_t \to \mathcal{O}\left(\frac{\widetilde{\mu}}{P_Z}\right)$ as $P_Z\to\infty$ the linear power UV divergence!

Summary and outlook

- "lattice cross sections" = single hadron matrix elements
  calculable in Lattice QCD, renormalizable + factorizable in QCD

Going beyond the quasi-PDFs

- Extract PDFs by global analysis of data of "Lattice x-sections"

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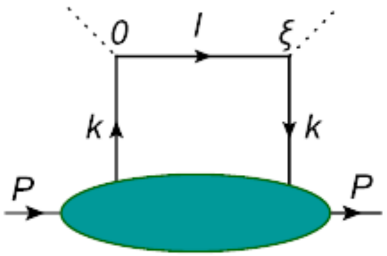
- Conservation of difficulties – an example:



Use heavy-light flavor changing current to suppress noise from the middle propagator:

$$\Rightarrow f_q(x,\mu^2) + f_Q(x,\mu^2) \approx f_q(x,\mu^2) \quad \text{if} \quad m_Q \sim \mu$$

No free-lunch! But, we are trying to find a better way to get our lunch!

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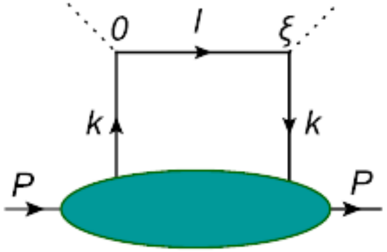

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- Lattice QCD can be used to study hadron structure, but, more works are needed!

Thank you!