Pseudo-Distributions on the Lattice

Joe Karpie
William & Mary / Jefferson Lab

In Collaboration with

Kostas Orginos (W&M / JLab)
Anatoly Radyushkin (Old Dominion U / JLab)
Savvas Zafeiropoulos (Heidelberg U)

Introduction

- Lattice calculations moving from Hadron bulk properties (masses, charges,...) to Hadron distributions (Form Factors, Structure Functions, Distribution functions, …)
- Project Goals
  - Long Term: Study methods of calculating parton distributions from ab initio Lattice QCD
  - Short Term: Understand systematic effects in the simple case of iso-vector quark unpolarized PDF
- Mellin moments and OPE
  - Restricted to low moments by reduced rotational symmetry
- Hadronic Tensor Methods A Chambers et.al (2017) 1703.01153
  - “Light-like” separated Hadronic Tensor K-F Liu et al Phys. Rev. Lett. 72 1790 (1994) , Phys. Rev. D62 (2000) 074501
  - Good lattice cross sections Y.-Q. Ma J.-W. Qiu (2014) 1404.6860 Y.-Q. Ma, J.-W. Qiu (2017) 1709.03018
- Ioffe Time Pseudo Distribution Methods
  - Quasi PDF X. Ji, Phys.Rev.Lett. 110, (2013) J.-W. Chen et.al. (2018) 1803.04393 C Alexandrou et.al. (2018) 1803.02685
  - Pseudo PDF A. Radyushkin Phys.Lett. B767 (2017) K. Orginos, A Radyushkin, JK, S Zafeiropoulos (2017) 1706.05373

What is a pseudo-distribution?

- Standard partonic distributions, particularly collinear distributions, are defined via matrix elements with light like separations
    - Describe probability distribution of quark states
    - Not suitable for lattice calculation
- Pseudo distributions are generalizations of partonic distributions defined via matrix elements with space like separations
    - Do not have probabilistic interpretation
    - Acceptable for lattice calculation
- In the limit that the space like separation goes to 0 the standard distribution is recovered

Parton Distribution Functions

- Cross section factorization $d\sigma_h = f_{h/q} \otimes d\sigma_q$
- Light cone matrix element definition

$$p = (p^+, \frac{m^2}{2p^+}, 0_T)$$

$$f_{h/q}(x,\mu^2) = \int d(\xi^- p^+) e^{\pm ix(\xi^- p^+)}$$

$$\langle h(p)|\bar{\psi}_q(0,\xi^-,0_T)\gamma^+ W((0,\xi^-,0_T);0)\psi_q(0)|h(p)\rangle_{\mu^2}$$

- OPE definition
  - Mellin moments

$$a_n(\mu^2) = \int dx x^{n-1} f(x,\mu^2)$$

  - Local Lorentz Invariant twist 2 matrix element

$$\langle h(p)|\bar{\psi}_q(0,\xi^-,0_T)\gamma^{\{\mu_1} D^{\mu_2} \ldots D^{\mu_n\}}\psi_q(0)|h(p)\rangle_{\mu^2} = a_n(\mu)^2 p^{\{\mu_1} \ldots p^{\mu_n\}}$$

Ioffe Time distribution

$$\nu = p \cdot z$$

B. L. Ioffe, Phys. Lett. 30B, 123 (1969)

- $\mathcal{I}(\nu, \mu^2) = \int_{-1}^{1} e^{i\nu x} f(x, \mu^2)$

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)
I.I. Balitsky and V.M. Braun, Nucl. Phys. B311, 541 (1988)

- Perturbative position space DGLAP evolution

$$\mathcal{I}_v(\nu, \mu_2^2) = \mathcal{I}_v(\nu, \mu_1^2) - \frac{C_F \alpha_s}{2\pi} \log \frac{\mu_2^2}{\mu_1^2} \int_0^1 du \left[\frac{1}{2}\delta(1-u) - (1-u) - 2[\frac{u}{1-u}]_+\right] \mathcal{I}_v(u\nu, \mu_1^2)$$

- CP Even/Odd combinations
    - Even: $q_-(x) = f(x) + f(-x) = q(x) - \bar{q}(x) \equiv q_V(x)$
    - Odd: $q_+(x) = f(x) - f(-x) = q(x) + \bar{q}(x) = q_V(x) + 2\bar{q}(x)$

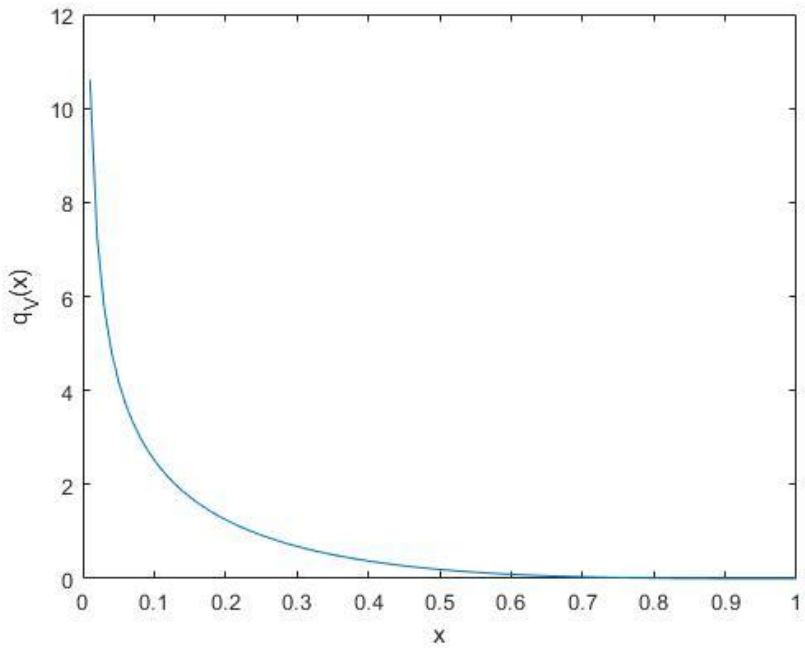
$$\mathfrak{Re}\left[\mathcal{I}(\nu)\right] = \int_0^1 dx \cos(\nu x) q_V(x) \equiv \mathcal{I}_V(\nu)$$

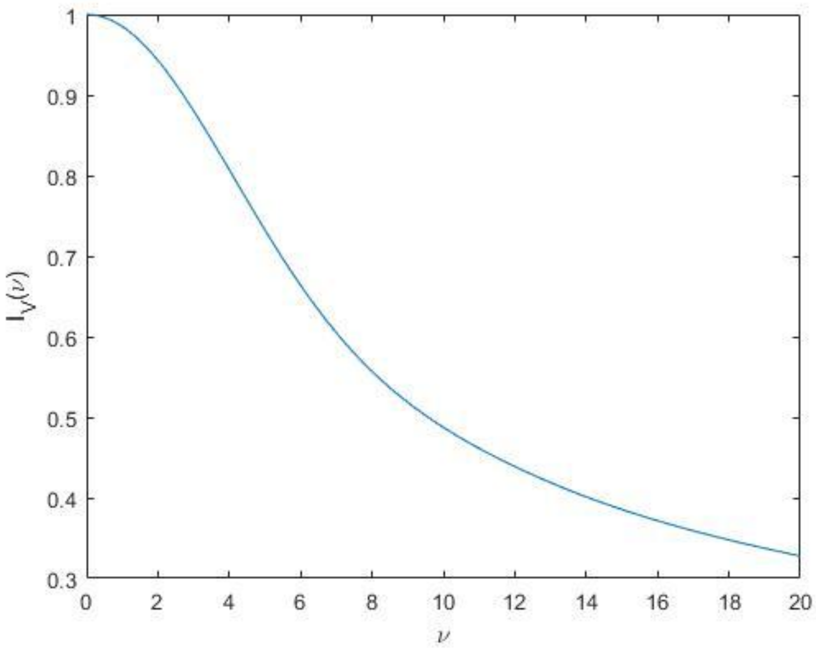
$$\mathfrak{Im}\left[\mathcal{I}(\nu)\right] = \int_0^1 dx \sin(\nu x)(q(x) + \bar{q}(x))$$

A Model Ioffe Time distribution

$$\alpha = -0.5$$
$$\beta = 3$$

$$f_{\alpha\beta}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}$$





Pseudo Ioffe Time Distributions

- A general matrix element of interest

$$M^{\alpha}(z,p) = \langle h(p)|\bar{\psi}_q(z)\gamma^{\alpha}W(z;0)\psi_q(0)|h(p)\rangle$$

- Lorentz decomposition
    - Use of symmetry
    - Choice of p, z, and α can remove higher twist term

$$M^{\alpha}(z,p) = 2p^{\alpha}\mathcal{M}_p(\nu, z^2) + z^{\alpha}\mathcal{M}_z(\nu, z^2)$$

- Relation to ITDF
    - Perturbatively calculable Wilson coefficients for each parton

$$\mathcal{M}(\nu, -z^2) = \sum_i C_i(z^2\mu^2, \alpha_s) \otimes \mathcal{I}_i(\nu, \mu^2) + H.T.$$

A. Radyushkin (2017) 1710.08813
J.-H. Zhang (2018) 1801.03023
T. Izubuchi (2018) 1801.03917

$$M^{\alpha}(z,p) = \langle h(p)|\bar{\psi}_q(z)\gamma^{\alpha}W(z;0)\psi_q(0)|h(p)\rangle$$

$$M^{\alpha}(z,p) = 2p^{\alpha}\mathcal{M}_p(\nu, z^2) + z^{\alpha}\mathcal{M}_z(\nu, z^2)$$

Special Cases

- Light cone PDF

$$p = (p^+, \frac{m^2}{2p^+}, 0_T) \qquad z = (0, z^-, 0_T) \qquad \alpha = +$$

$$\mathcal{M}_p((p^+z^-), 0) = \int_{-1}^{1} dx e^{ix(p^+z^-)} f(x)$$

A. Radyushkin (2017) 1612.05170

- Straight Link “Primordial” TMD

$$p = (p^+, \frac{m^2}{2p^+}, 0_T) \qquad z = (0, z^-, z_T) \qquad \alpha = +$$

$$\mathcal{M}_p((p^+z^-), -z_T^2) = \int_{-1}^{1} dx e^{ix(p^+z^-)} \int d^2k_T e^{ik_T\cdot z_T} F(x, k_T^2)$$

- Pseudo PDF

$$p = (E, 0, 0, p_3) \qquad z = (0, 0, 0, z_3) \qquad \alpha = 0$$

$$\mathcal{M}_p((-z_3 * p_3), -z_3^2) = \int_{-1}^{1} dx e^{ix(-z_3 * p_3)} P(x, -z_3^2)$$

Pseudo PDF vs Quasi PDF

$$0.2GeV \approx 1fm^{-1}$$

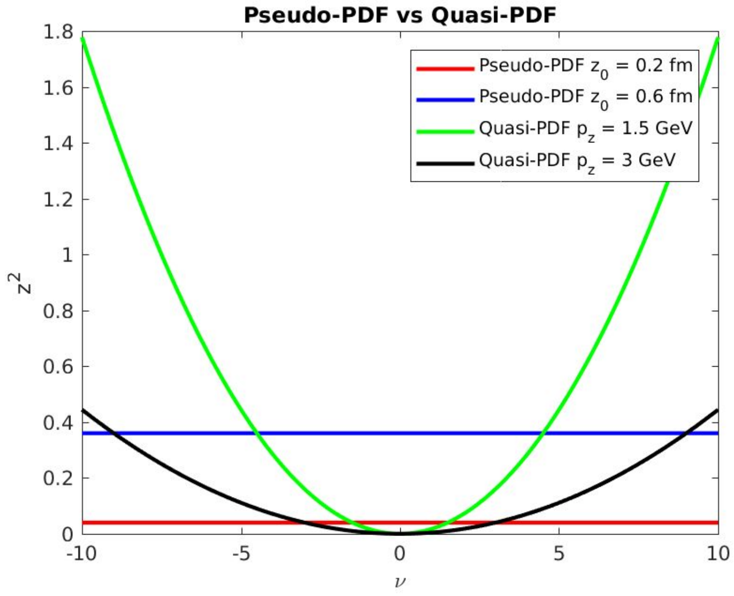
- Both are integrals of pseudo ITDF
  - Pseudo PDF has fixed invariant scale dependence

$$P(x, z_0^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu e^{-i\nu x}\mathcal{M}(\nu, z_0^2)$$

  - Quasi PDF mixes invariant scales until p_z is effectively large enough

$$Q(x, p_z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu e^{-i\nu x}\mathcal{M}(\nu, \frac{\nu^2}{p_z^2})$$

- Expedite desired limit of $z^2 \longrightarrow 0$
  - Pseudo-PDFs use reduced distributions
  - Quasi-PDFs use LaMET



Pseudo ITD as a Good Lattice Cross Section

- Good Experimental Cross Section - An experiment whose results, Form Factors or asymmetries, is sensitive to a particular PDF.
  - DIS, SIDIS, DY, ….
- Good Lattice Cross Section - A lattice QCD calculable matrix element whose result is sensitive to a particular PDF (Not actually a cross section)
  - Vector-vector currents, Axial-vector currents, any 2 flavor changing currents, Quarks separated by Wilson line, ….
- The matching relation for the pseudo ITD to PDF discusses by Anatoly Radyushkin (see talk from Tuesday) is related to the Wilson coefficients to match good lattice cross sections discussed by JianWei Qiu (see talk from Sunday)

Numerical Lattice Field Theory

- Importance sampling of path integral

$$\langle O(\bar{\psi}, \psi, A_\mu)\rangle = \frac{1}{Z}\int D[\bar{\psi}]D[\psi]D[A_\mu]O(\bar{\psi}, \psi, A_\mu)e^{-S(\bar{\psi},\psi,A_\mu)}$$

$$\approx \frac{1}{N}\sum_i^N F_O(U_\mu^{(i)})$$

- Correlation functions

$$C_2(\vec{p}, T) = \langle O_N(-\vec{p}, T)\bar{O}_N(\vec{p}, 0)\rangle$$

$$C_{op}(O_{op}; \vec{p}, T) = \sum_t \sum_{\vec{x}} \langle O_N(-\vec{p}, T)O_{op}(\vec{x}, t)\bar{O}_N(\vec{p}, 0)\rangle$$

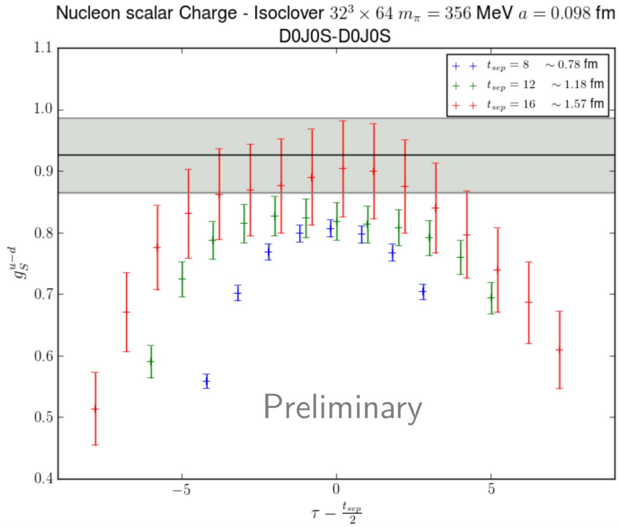
- Feynman-Hellman extraction C. Bouchard et.al Phys. Rev. D 96, no. 1, 014504 (2017)

$$\frac{\langle N(p)|O_{op}|N(p)\rangle}{2E_{N(p)}} = \lim_{T\to\infty}\frac{1}{\tau}(R(T+\tau) - R(T)) \qquad R(T) = \frac{C_{op}(O_{op}; \vec{p}, T)}{C_2(\vec{p}, T)}$$

$$O_q^\alpha(z; T) = \sum_{\vec{x}} \bar{\psi}_q(\vec{x}+\vec{z}, T)\lambda^3\gamma^\alpha W((\vec{x}+\vec{z}, T); (\vec{x}, T))\psi_q(\vec{x}, T)$$

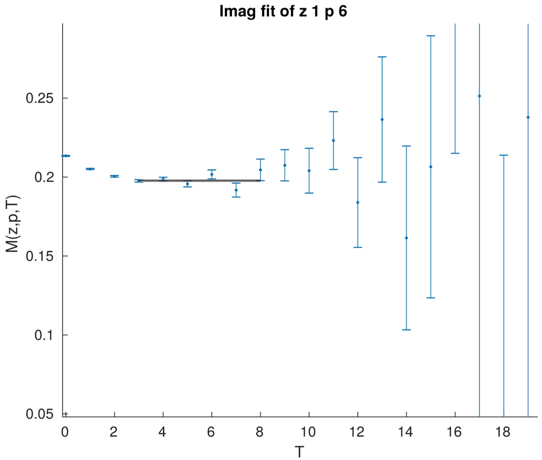
Feynman Hellman matrix element extraction

- 2 parameter extraction
- Questionable Excited state effects



Thanks to Colin Egerer for plot from his LightCone 2018 talk

- 1 parameter extraction
- More clear distinction between excited state, plataeu, noise regions



See Axial coupling calculation in Chang et.al. arXiv:1710.06523

Technical Lattice difficulties

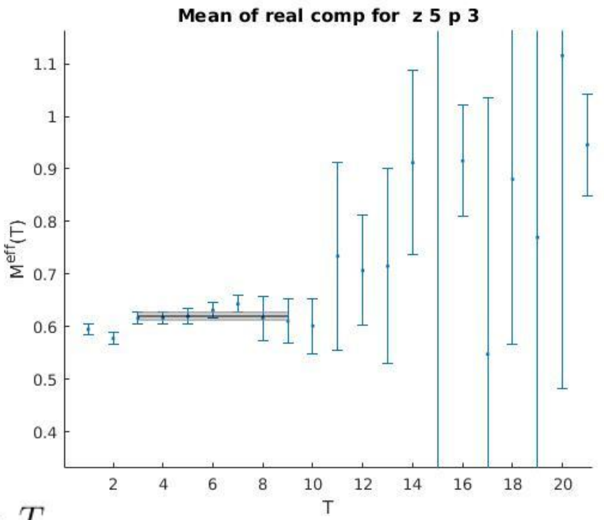
- Excited states contamination
- Reduced Symmetries
- Signal to noise
  - $C_2(p,T) = \langle O_h(p,T) O_h(p,0)^\dagger \rangle \propto e^{-E_h(p)T}$

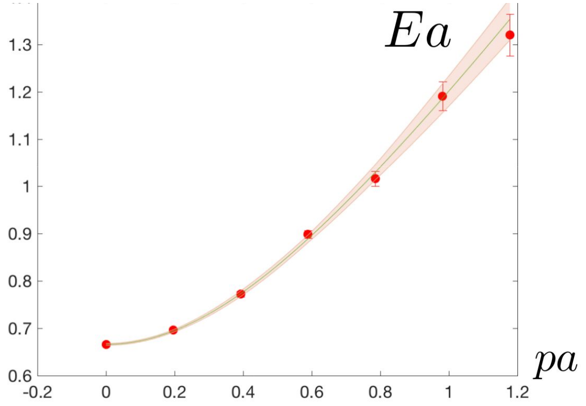
$$var\left[C_2(p,T)\right] = \langle O_h(p,T) O_h(p,T)^\dagger O_h(p,0) O_h(p,0)^\dagger \rangle \propto e^{-n_q m_\pi T}$$

$$\frac{var\left[C_2(p,T)\right]^{\frac{1}{2}}}{C_2(p,T)} \propto e^{(E_h(p) - n_q m_\pi/2)T}$$

  - Momentum smearing Bali et.al. Phys. Rev. D 93, 094515 (2016)
- Use of heavy pions
- Connected and disconnected
- Restriction to low momenta

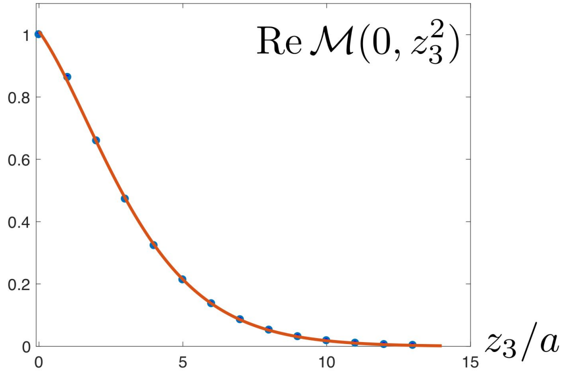
$$apmax = \frac{2\pi}{L}\left(\frac{L}{4}\right) = \frac{\pi}{2} \sim O(1)$$





Renormalization and the Reduced distribution

- Vector current $Z_p^{-1} = M^4(0,p)$
  - Forces matrix elements to give unit charge



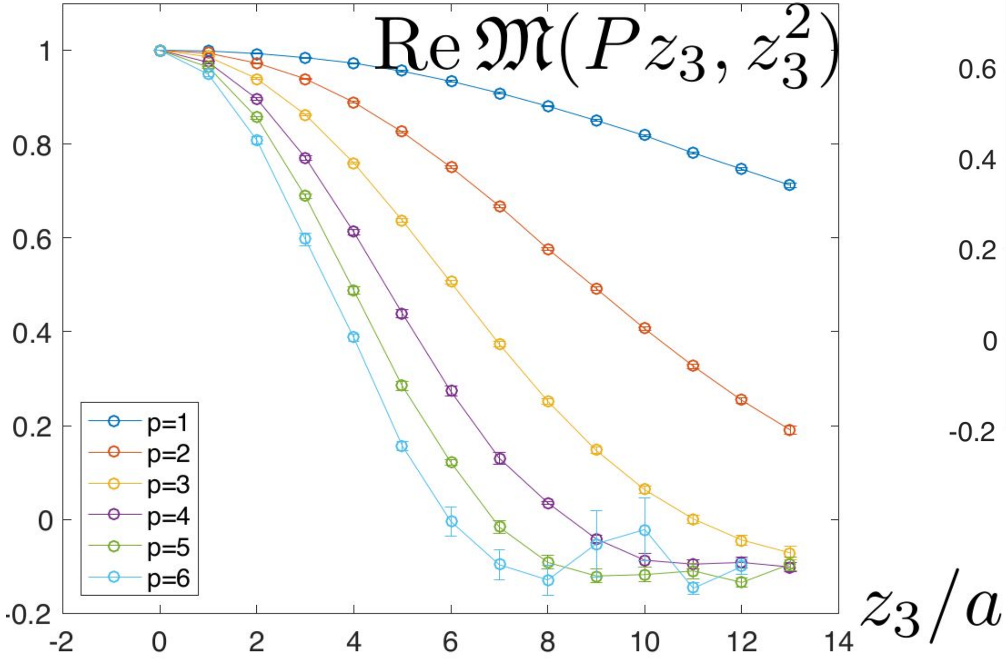
- Reduced distribution $\mathfrak{M}(\nu,z^2) = \dfrac{\mathcal{M}(\nu,z^2)}{\mathcal{M}(0,z^2)}$
  - TMD "Factorization" and suppression of polynomial corrections

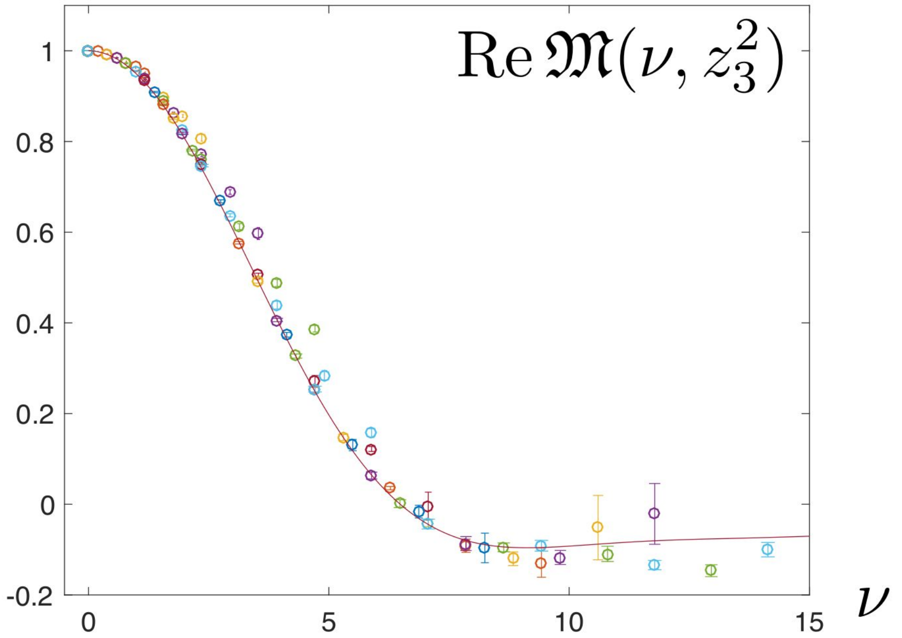
$$F(x,k_T^2) = f(x)g(k_T^2) \quad \mathcal{M}(\nu,z^2) = \mathcal{M}(\nu,0)\mathcal{M}(0,z^2)$$

  - BONUS: UV corrections from Wilson line cancel
  - Effective Reduced element $M^{eff}(T) = (R(T+1) - R(T)) + O(e^{-\Delta T})$

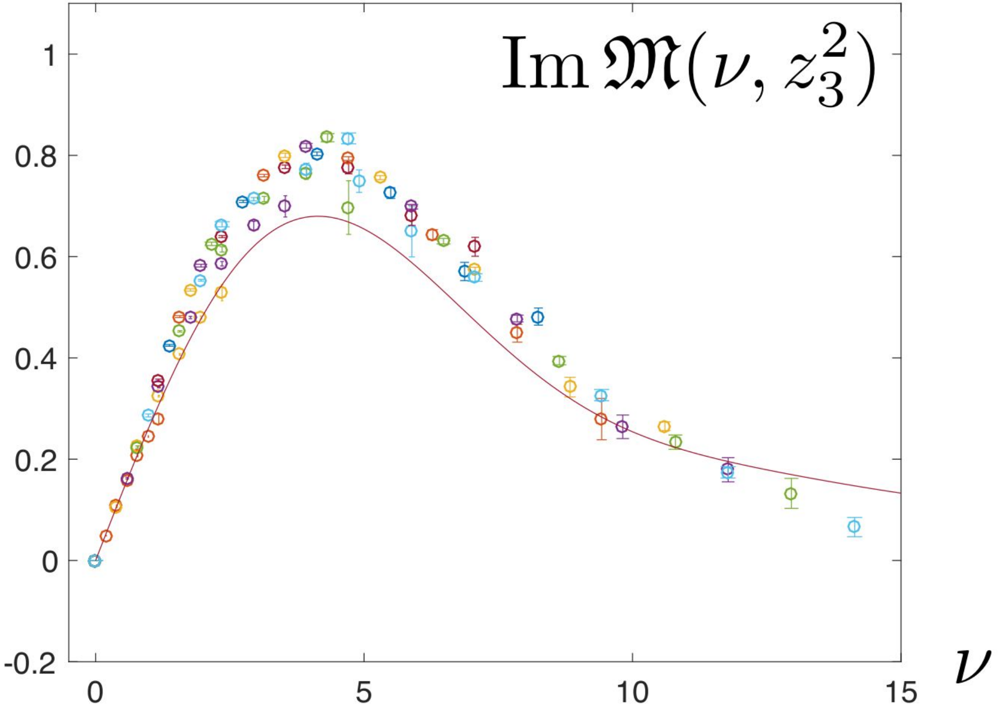
$$\mathfrak{M}^{eff}(\nu,z^2,T) = \left(\frac{M^{eff}(\nu,z^2,T)}{M^{eff}(0,z^2,T)}\right)/\left(\frac{M^{eff}(\nu,0,T)}{M^{eff}(0,0,T)}\right)$$

Quenched Results





Imaginary Component and AntiQuarks



- Imaginary component mixes valence, sea, and antiquark distributions
- Use real component to find valence contribution, the rest is the sea and antiquarks
- Identify anti quark distribution without need of performing inaccurate Fourier transforms and the unreliable low x region
- Qualitatively it gives proper sign for quenched iso-vector quarks

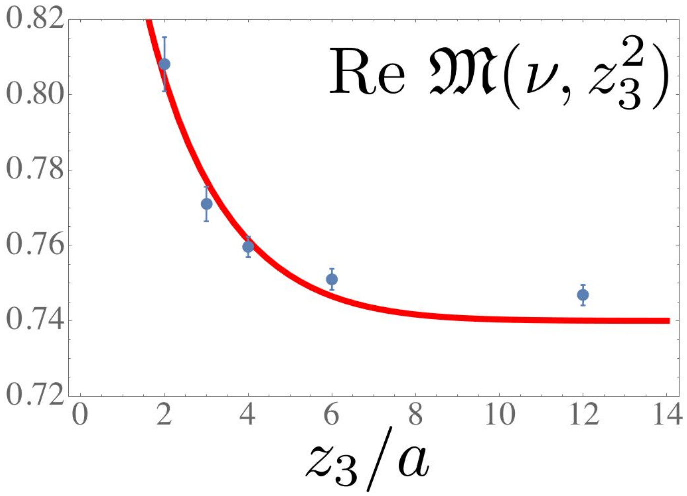
Perturbative Evolution of Lattice data

$$B \otimes \mathfrak{M}(\nu, z^2) = \int_0^1 du B(u) M(u\nu, z^2)$$

$$\mathfrak{M}(\nu, z_0^2) = \mathfrak{M}(\nu, z^2) + \frac{C_F \alpha_s}{2\pi} \log(\frac{z^2}{z_0^2}) B \otimes \mathfrak{M}(\nu, z^2)$$

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

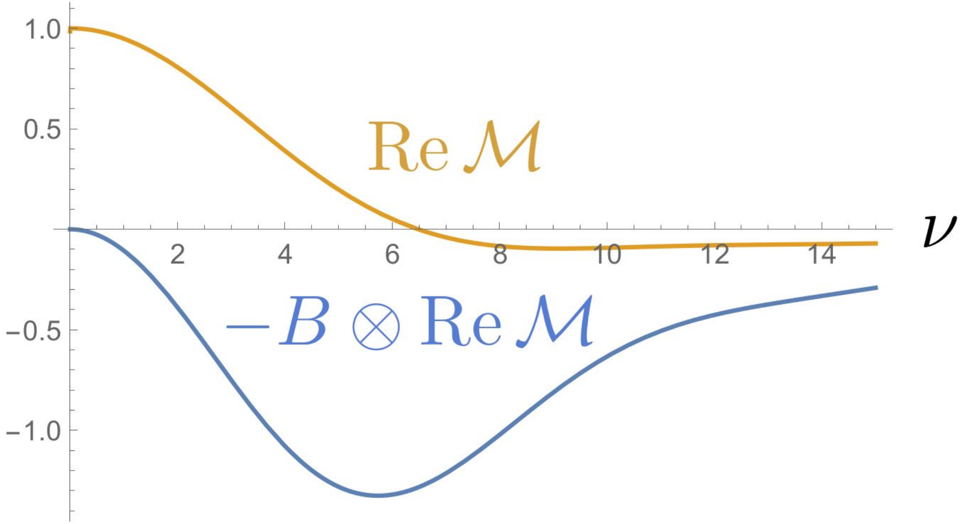
- Improvement of Almost Universal curve
- Freezing of evolution
  - Large separation matrix elements seem z_3 independent

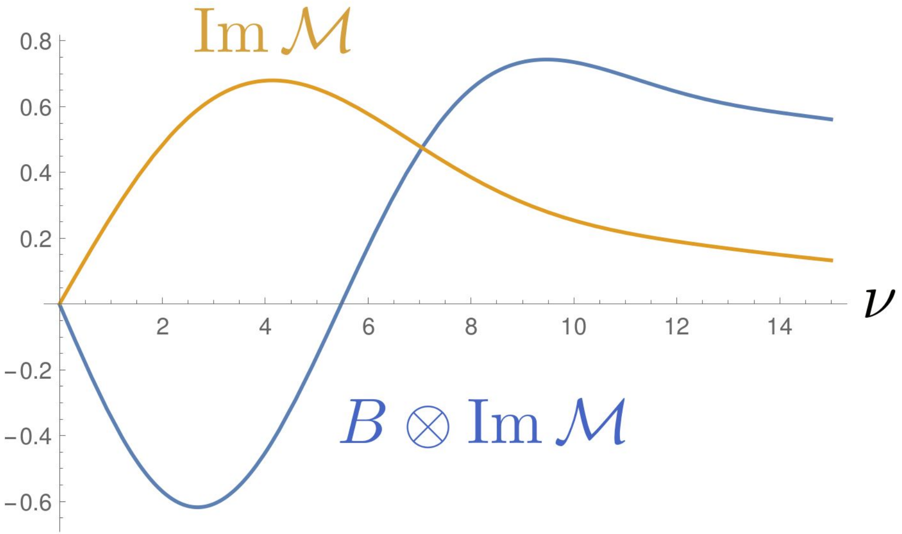


$\nu = 3\pi/4$

Perturbative Evolution of Lattice data

$$B \otimes \mathfrak{M}(\nu, z^2) = \int_0^1 du B(u) M(u\nu, z^2)$$



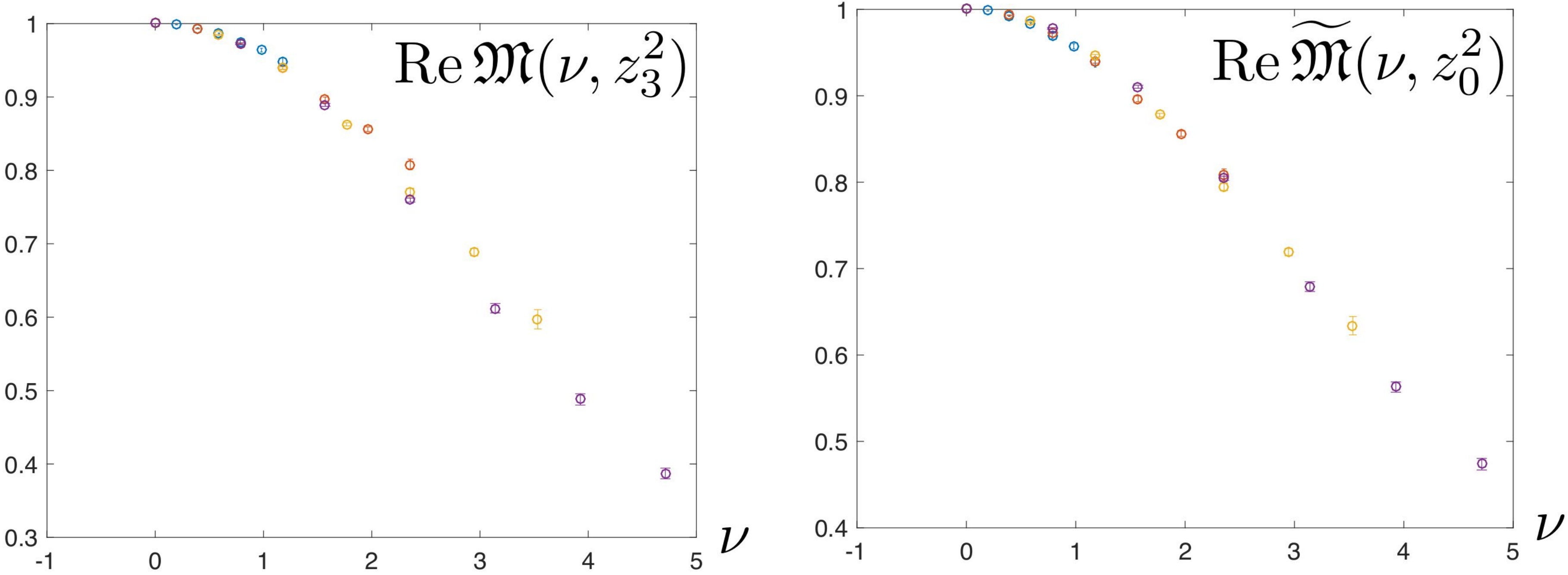


$\alpha = 0.5$

$\beta = 3$

$$f_{\alpha\beta}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}$$

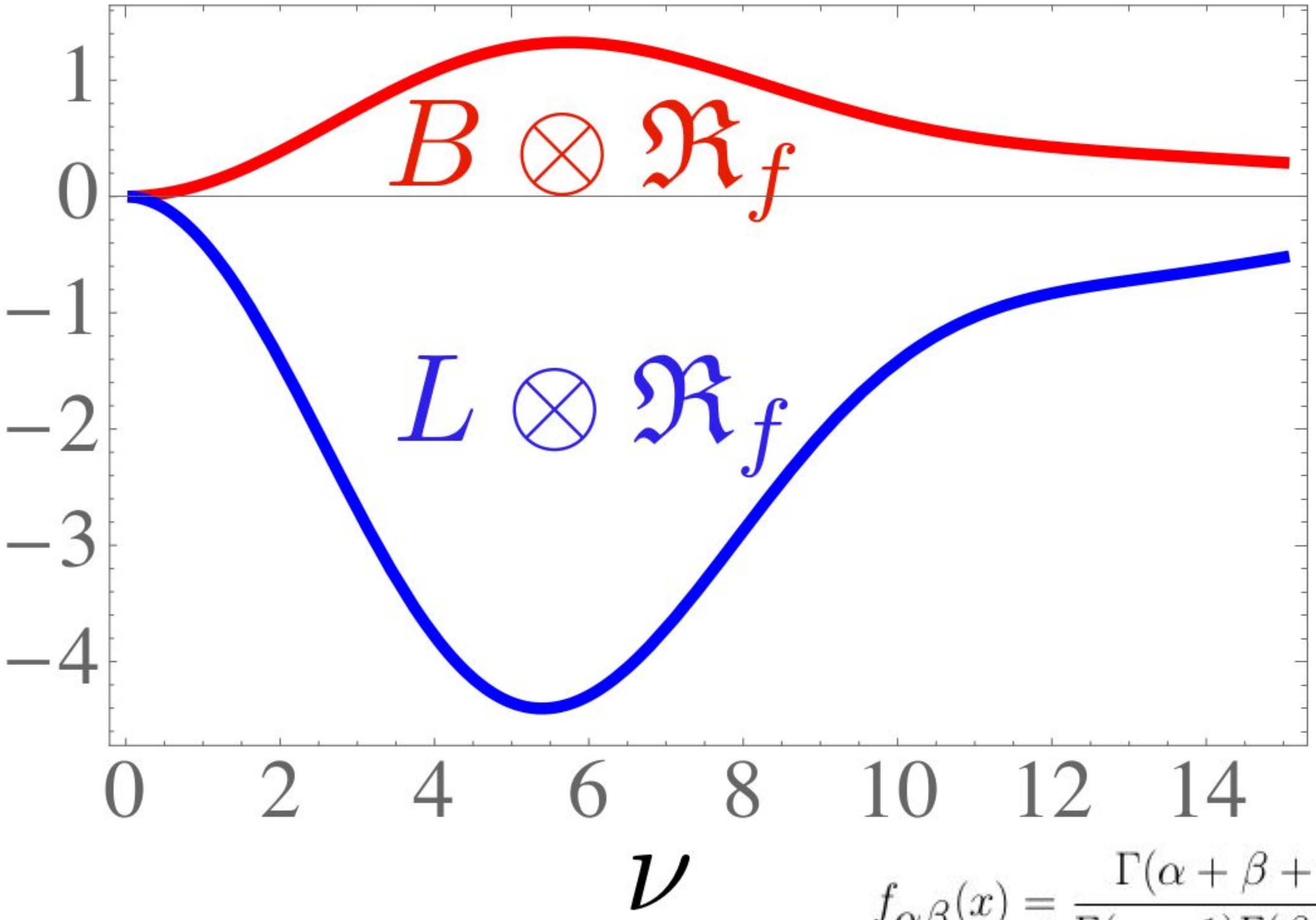
Perturbative Evolution of Quenched Lattice data



Matching Lattice data to Ioffe distribution

- Yet another convolution
- At 1-loop, evolution and matching can be simultaneous
- Allows for a direct relationship between ITDF/PDF and pseudo ITDF
    - No more need for extrapolations in the scale
    - Does require scale to be in regime dominated by pertubative effects
- No real need for pseudo PDFs. Go directly from pseudo ITDF to PDF

$$\begin{aligned}\mathcal{I}(\nu,\mu^2) &= \mathfrak{M}(\nu,z^2) + \frac{C_F\alpha_S}{2\pi}\int_0^1 du\Big(B(u)\left(\log(z^2\mu^2\frac{e^{2\gamma_E}}{4}) + 1\right) \\ &\quad + \left[4\frac{\log(1-u)}{1-u} - 2(1-u)\right]_+\Big)M(u*\nu,z^2) \\ &= \mathfrak{M}(\nu,z^2) + \frac{C_F\alpha_S}{2\pi}\left[(\log(z^2\mu^2\frac{e^{2\gamma_E}}{4}) + 1)B\otimes M(\nu,z^2) + L\otimes M(\nu,z^2)\right]\end{aligned}$$



$\alpha = 0.5$
$\beta = 3$

$$f_{\alpha\beta}(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha}(1-x)^{\beta}$$

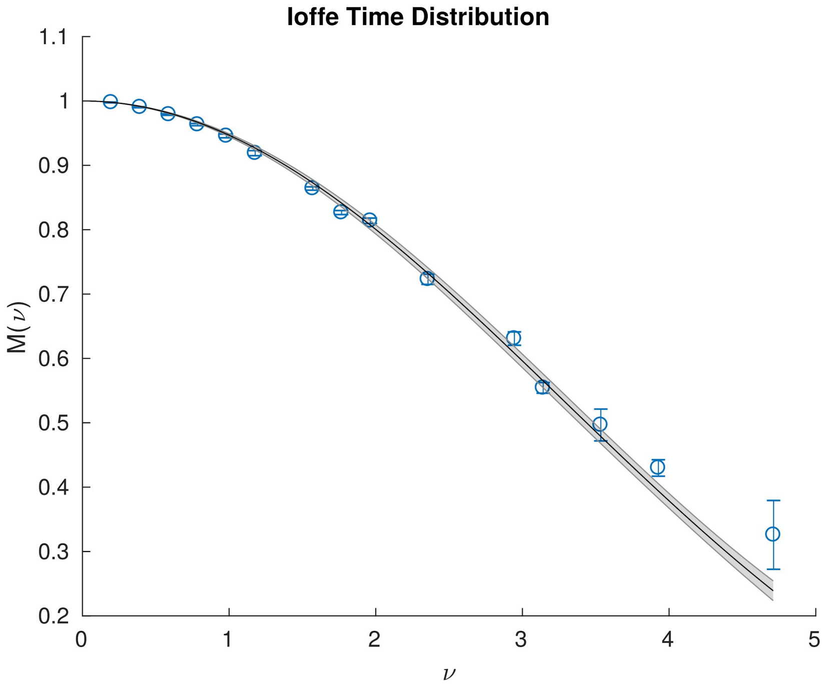
Real component and the Valence Quark distribution

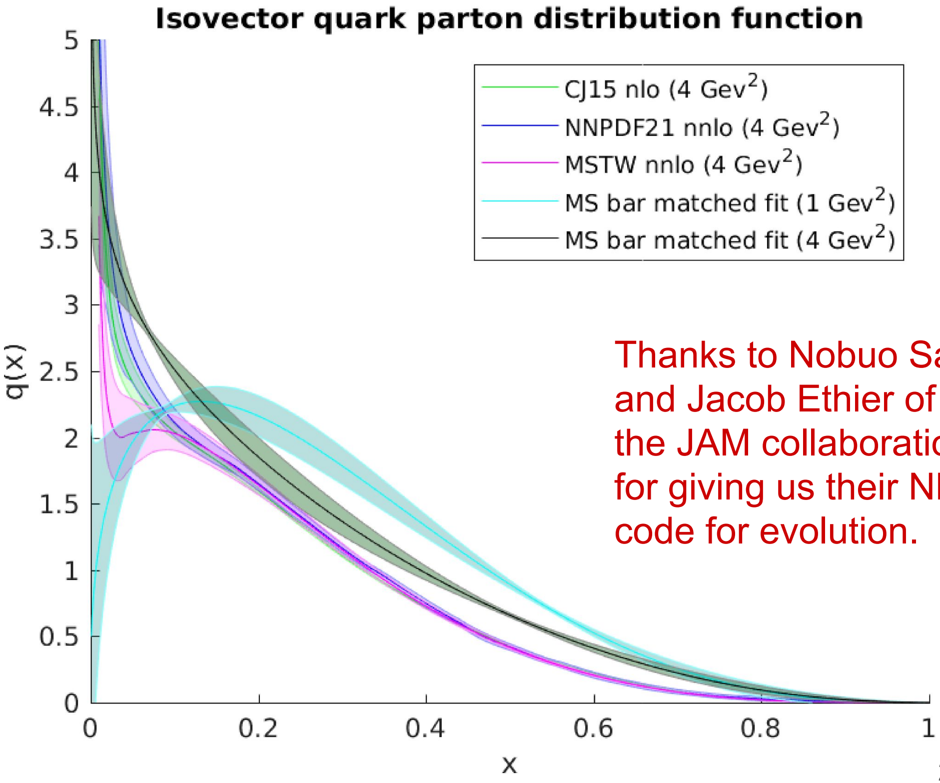
- Want to avoid ill posed inverse Fourier transform
- A general model PDF used by JAM collaboration for fitting

$$f_{abcd}(x) = N_{abcd}\, x^a (1-x)^b (1 + c\sqrt{x} + dx)$$

- Limiting behaviors
  - Regge $a = -0.5$
  - Quark counting $b = 3$
  - Small Corrections $c \sim 0 \quad d \sim 0$

Quenched Pseudo PDF Matched to MS bar Compared to Global fit PDFs





Thanks to Nobuo Sato and Jacob Ethier of the JAM collaboration for giving us their NLO code for evolution.

Summary

- First study of pseudo ITDF analyzed as reduced pseudo PDFs
- Qualitative agreement with PDFs despite few systematics under control
- Treatment of z^2 dependence guided by data allowing for matching to perturbative schemes
- Divergent behavior improves, but not recovered, under proper evolution to 4 GeV^2
- Systematics left to thoroughly study
  - Continuum limit
  - Control of Excited states
  - Finite Volume
  - Physical Pion mass limit
- Once systematics are understood and controlled then any light cone distribution is within reach of the lattice.