Non-perturbative study of the three-body system using the Bethe-Salpeter approach

Emanuel Ydrefors

Instituto Tecnológico de Aeronáutica (ITA), Brazil

Collaborators: J. H. Alvarenga Nogueira, V. A. Karmanov and T. Frederico

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Introduction

- Understanding the structure of non-perturbative few-body systems, from an fundamental point of view is important for applications in hadron physics, e.g. for studies of the nucleon.
- One important aspect is to obtain a reliable solution directly in Minkowski space, so that dynamical observables such as form factor can be calculated.
- In this talk, the solutions of the Bethe-Salpeter equation for a bound-state system of three bosons, bounded through a (two-body) zero range interaction, using three different approaches are discussed:
  - LF projection, i.e. only retaining the valence component, in Minkowski space.
  - Solution of the BS equation in Euclidean space, through Wick rotation
  - Solution of the BS equation in Minkowski space by direct integration (preliminary results).

Three-body problem with zero-range interaction

- Three-body Bethe-Salpeter equation (Frederico, PLB 282 (1992) 409):

$$v(q,p) = 2\mathrm{i}F(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{(p-q-k)^2 - m^2 + i\epsilon} v(k,p)$$

- Equal-mass case, bare propagators.
- $v(q,p)$ is one of the Faddeev components of the total vertex function.
- $F(M_{12})$: two-body scattering amplitude characterized by scattering length a and $M_{12}^2 = (p-q)^2$.
- 1) $a < 0$: Borromean system, no two-body bound state, 2) $a > 0$: two-body bound state exists.

- LF equation:
  - After the LF projection, i.e. introducing $k_\pm = k_0 \pm k_z$ and integrating over k_-, one obtains the three-body LF equation (Carbonell and Karmanov, PRC 67 (2003) 037001):

$$\Gamma(k_\perp, x) = \frac{F(M_{12})}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1-x-x')} \int_0^\infty \frac{d^2k'_\perp}{M_0^2 - M_3^2} \Gamma(k'_\perp, x')$$

  with $M_0^2 = (k'^2_\perp + m^2)/x' + (k^2_\perp + m^2)/x + ((k'_\perp + k_\perp)^2 + m^2)/(1-x-x')$
- Euclidean BS equation:
  - Through a change of variables $k = k' + \frac{p}{3}$ and $q = q' + \frac{p}{3}$, and a subsequent Wick rotation (Ydrefors et al, PLB 770 (2017) 131):

$$v_{\mathrm{E}}(q'_4, q'_v) = \frac{2F(-M'^2_{12})}{(2\pi)^3} \int_{-\infty}^\infty dk'_4 \int_0^\infty \frac{dk'_v \Pi(q'_4, q'_v, k'_4, k'_v)}{(k'_4 - \frac{\mathrm{i}}{3}M_3)^2 + k'^2_v + m^2} v_{\mathrm{E}}(k'_4, k'_v),$$

  with $M'^2_{12} = (\frac{2}{3}\mathrm{i}M_3 + q'_4)^2 + q'^2_v$. The kernel Π is here given by

$$\Pi(q'_4, q'_v, k'_4, k'_v) = \frac{k'_v}{2q'_v} \log \frac{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v + k'_v)^2 + m^2}{(k'_4 + q'_4 + \frac{i}{3}M_3)^2 + (q'_v - k'_v)^2 + m^2}. \quad (1)$$

- Both the equations can be solved with standard methods, e.g. by using splines.

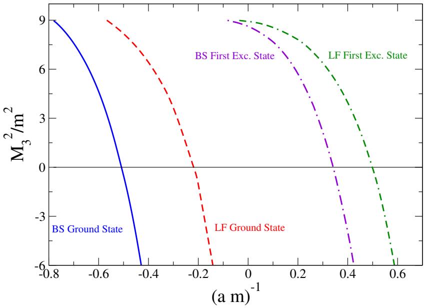
Direct method

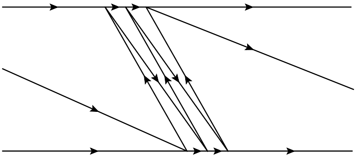
- Direct integration of the BS equation, treating explicitly the singularities.
- The same approach was used by Carbonell and Karmanov (PRD 90 (2014) 056002) to solve the two-body problem (finite-range interaction).
- The equation for the vertex function, $v(q_0, q_v)$ can be written in the "non-singular" form

$$v(q_0,q_v) = \frac{\mathcal{F}(M_{12})}{(2\pi)^4}\int_0^\infty k_v^2 dk_v \Big\{ i\frac{[\Pi(q_0,q_v;\varepsilon_k,k_v)v(\varepsilon_k,k_v)+\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)]}{2\varepsilon_k}$$
$$-2\int_{-\infty}^{0} dk_0 \left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right] \quad (2)$$
$$-2\int_0^\infty dk_0 \left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q;\varepsilon_k,k_v)v(\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right]\Big\},$$

  using, e.g, $[k_0^2 - k_v^2 - m^2 + i\epsilon]^{-1} = PV[k_0^2 - \varepsilon_k^2]^{-1} - i\pi/(2\varepsilon_k)[\delta(k_0-\varepsilon_k)+\delta(k_0+\varepsilon_k)]$. Above, where $\varepsilon_k = \sqrt{k_v^2+m^2}$, $k_v = |\vec{k}|$ and the kernel Π only has weak, logaritmic, singularities. For $a < 0$ (considered here) $F(M_{12})$ has no pole.
- The singularities at $k_0 = \pm\varepsilon_k$ were subtracted.
- We have solved the above equation by using a spline expansion for v, i.e. $v(q_0,q_v) = \sum_{ij} C_{ij} S_i(q_0) S_j(q_v)$.

Binding energy versus inverse scattering length (EBS vs LF)





- The (complete) BS equation gives a stronger bound system compared to the LF one for all a.
- For $a < 0$ (i.e. a Borromean system) the solution with the smallest M_3^2, i.e. the formal ground state, is physical.
- However, for $a > 0$, i.e. a two-body bound state exists, the lowest state is unphysical.
- $M_3^2 > -\infty$: No Thomas collapse in the non-relativistic sense, i.e. an effective short-range repulsion.
- The higher-Fock state contributions beyond the valence to the kernel can be interpreted as an effective three-body force of relativistic origin.

Transverse amplitudes

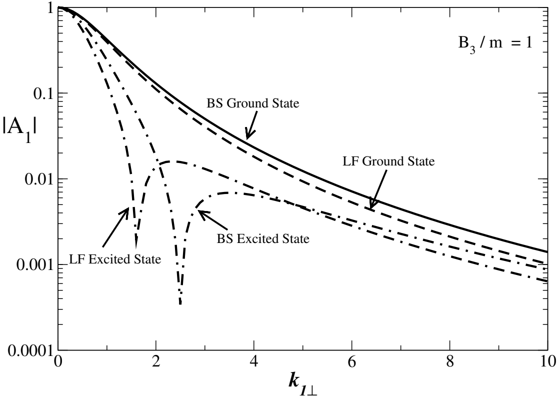
- The LF and (Euclidean) BS vertex functions cannot be directly compared with each other.
- However, we can define the transverse amplitudes

$$A^{\rm LF}(\vec{k}_{1\perp},\vec{k}_{2\perp}) = A_1^{\rm LF} + A_2^{\rm LF} + A_3^{\rm LF} = \frac{-\sqrt{2\pi}}{4} \times \int_0^1 dx_1 \int_0^{1-x_1} \frac{dx_2}{x_1 x_2 (1-x_1-x_2)} \frac{\Gamma(\vec{k}_{1\perp},x_1)+\Gamma(\vec{k}_{2\perp},x_2)+\Gamma(\vec{k}_{3\perp},x_3)}{M_0^2-M_3^2}$$

and

$$A^{\rm EBS}(\vec{k}_{1\perp},\vec{k}_{2\perp}) = A_1^{\rm EBS} + A_2^{\rm EBS} + A_3^{\rm EBS} = -i\int dk_{14}dk_{1z}dk_{24}dk_{2z}[v_E(k_{14},k_{1v})+v_E(k_{24},k_{2v})+v_E(k_{34},k_{3v})]\Pi_1\Pi_2\Pi_3$$

where $\Pi_j^{-1} = (k_{j4}^2 - i\frac{1}{3}M_3)^2 + k_{jz}^2 + k_{j\perp}^2 + m^2$



- In both frameworks the first excited state has one node, and the ground state has no node. This confirms these assignments.
- The extra contributions included in the "full" BS solution has a significant impact on the transverse amplitude, especially for the first excited state.

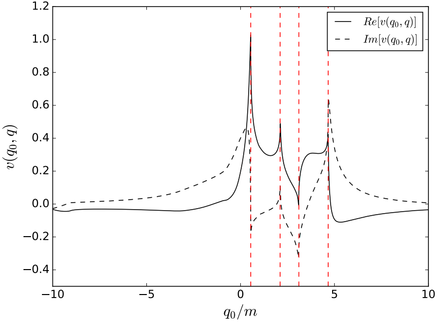
Binding energies: Euclidean solution vs direct method (preliminary results)

- The three-body binding energy (for fixed a) is calculable both in Minkowski and Euclidean spaces.
- In the table are shown for three cases ($a\,m = -1.28, -1.5, -1.705$), the obtained eigenvalue using the B_3 from the Euclidean calculation.

$a\,m$	B_3/m	λ
-1.28	0.006	$0.999 - 0.0544i$
-1.5	0.395	$1.000 + 0.0023i$
-1.705	1.001	$0.997 + 0.106i$

- Results good for the case $a\,m = -1.5$, but the error in the imaginary is getting quite large for more strongly bound system.
- One reason for the non-zero imaginary part could be the use of finite cuts, i.e. $k_{max}/m = 6.0$ and $k_{0max}/m = 13.0$ (first two cases) and $k_{0max}/m = 15.0$ (third case).
- Euclidean solution obtained without cuts, i.e. using a mapping.

Three-body vertex function in Minkowski space



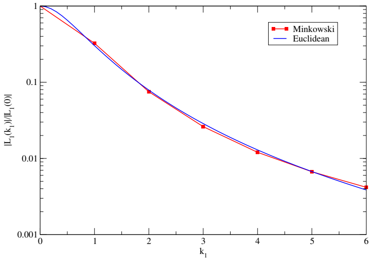
- The figure shows the real and imaginary parts of $v(q_0, q_v)$ at fixed $q_v/m = 0.5$, for the case $B_3/m = 0.395$.
- It is seen that there are four peaks (either singularities or branch cuts). It turns out that they have the positions $q_0 = M_3 \pm \sqrt{q_v^2 + 4m^2}$ and $q_0 = M_3 \pm q_v$, shown by red dashed lines. These are thus moving peaks depending on q_v.
- The non-smooth behavior of v makes the solution of this problem numerically very challenging.

Transverse amplitude in Minkowski space

$$L_1(\vec{k}_{1\perp},\vec{k}_{2\perp}) = \int_{-\infty}^{\infty} dk_{1z} \left\{ \frac{i\pi \left[v(\tilde{k}_{10},k_{1v})\chi(\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) + v(-\tilde{k}_{10},k_{1v})\chi(-\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) \right]}{2\tilde{k}_{10}} \right.$$
$$- \int_0^{\infty} dk_{10} \frac{v(k_{10},k_{1v})\chi(k_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) - v(\tilde{k}_{10},k_{1v})\chi(\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp})}{k_{10}^2 - \tilde{k}_{10}^2}$$
$$\left. - \int_0^{\infty} dk_{10} \frac{v(-k_{10},k_{1v})\chi(-k_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp}) - v(-\tilde{k}_{10},k_{1v})\chi(-\tilde{k}_{10},k_{1z},\vec{k}_{1\perp};\vec{k}_{2\perp})}{k_{10}^2 - \tilde{k}_{10}^2} \right\}, \quad (3)$$

with $\tilde{k}_{10} = \sqrt{k_{1z}^2 + \vec{k}_{1\perp}^2 + m^2}$ and χ is a known function having only weak, square root, singularities.

Results for the transverse amplitudes



- The figure compares (as an example) the modulus of the transverse amplitudes for the case $B_3/m = 0.395$.
- The agreement between the two approaches is good.
- Even though the Minkowski space amplitude, $v(q_0, q_v)$, has a non-smooth behavior, a smooth transverse amplitude is obtained.

Alternative: Nakanishi integral representation

- One alternative in order to avoid the numerical difficulties with the direct method, could be to use the Nakanishi integral representation. This has been used succesfully in the two body-case where the BS amplitude is written in the form:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} \frac{g(\gamma', z'; \kappa^2)}{(\gamma' + \kappa^2 - k^2 - (p \cdot k)z' - i\varepsilon)^3} \quad \kappa^2 = m^2 - M^2/4 \tag{4}$$

- Similarly, in the three-body case, Nakanishi integral representations could be used for $v(q,p)$ and $F(M_{12})$, and thus produce a non-singular integral equation. This is planned for the near future.

Conclusions

- We have in this work studied a system of three bosons interacting through a zero-range potential using three approaches. Namely, 1) using the valence LF equation in Minkowski space, 2) Solving the 4-dimensional Euclidean BS equation, 3) Solving the 4-dimensional BS equation by direct integration in Minkowski space.
- The contributions beyond the valence have large impact both on binding energies and transverse amplitudes. These contributions can be interpreted as an effective three-body force of relativistic origin.
- The direct method is of great interest since it can give a BS amplitude defined in Minkowski space, needed to compute dynamical observables.
- This is work is in progress. However, we have shown that the binding energy (at least for modest B_3) is in fair agreement with the Euclidean. The transverse amplitudes are also in fair agreement.
- Unfortunately, the method is numerically very challenging due the treatment of the many singularities.
- One way to solve this could be to use a Nakanishi integral representation (similarly to the two-body case) for the BS amplitude, and it will be done in the near future.