Constrained Degrees of Freedom of Fermion in Light-Front Dynamics

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Outline

1. Fermion constrained degrees of freedom in LFD
2. Interpolating between the IFD and the LFD
3. e^+e^- annihilation into two scalar particles

Fermion constrained degrees of freedom in LFD

The Lagrangian density for Quantum Electrodynamics is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{1}$$

where

$$D_\mu = \partial_\mu + ieA_\mu, \tag{2}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3}$$

Therefore, the equation of motion for the fermion field ψ is

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi = 0. \tag{4}$$

Fermion constrained degrees of freedom in LFD

Or, written explicitly in light-front coordinates, it is

$$\Big[i\left(\gamma^+\partial_+ + \gamma^-\partial_- + \boldsymbol{\gamma}^\perp\cdot\boldsymbol{\partial}_\perp\right) \\ -e\left(\gamma^+A_+ + \gamma^-A_- + \boldsymbol{\gamma}^\perp\cdot\mathbf{A}_\perp\right) - m\Big]\psi = 0. \tag{5}$$

Define [1]

$$\psi = \psi_+ + \psi_- = P_+\psi + P_-\psi, \tag{6}$$

where the projection operators

$$P_+ = \frac{1}{2}\gamma^-\gamma^+ \quad \text{and} \quad P_- = \frac{1}{2}\gamma^+\gamma^- \tag{7}$$

satisfy

$$P_+ + P_- = 1, \quad P_+P_- = P_-P_+ = 0. \tag{8}$$

Due to $(\gamma^+)^2 = (\gamma^-)^2 = 0$, we have

$$\gamma^+P_- = \gamma^-P_+ = P_-\gamma^+ = P_+\gamma^- = 0.$$

[1] J. Kogut and D. Soper, Phys. Rev. D **1**, 2901(1970)

Fermion constrained degrees of freedom in LFD

Then, after some algebra, one can see that the ψ_- component of the fermion field satisfy the following constraint equation without involving the derivative with respect to the light-front time

$$2\left(i\partial_- - eA_-\right)\psi_- = \left[\left(i\boldsymbol{\partial}_\perp - e\mathbf{A}_\perp\right)\boldsymbol{\gamma}^\perp + m\right]\gamma^+\psi_+, \qquad (9)$$

which reduces in the light-front gauge $A_- = A^+ = 0$ to

$$2i\partial_-\psi_- = \left[\left(i\boldsymbol{\partial}_\perp - e\mathbf{A}_\perp\right)\boldsymbol{\gamma}^\perp + m\right]\gamma^+\psi_+. \qquad (10)$$

Thus, the two components of ψ given by ψ_- in LFD become constrained in the sense that the time dependence of ψ_- is provided by the other fields that satisfy the dynamic equation with the light-front time derivative ∂_+ such as $\mathbf{A}_\perp$ and ψ_+. No new time-dynamic information can be provided by the constrained field ψ_-.

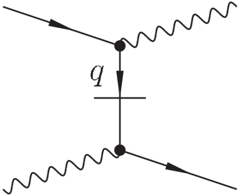
Fermion constrained degrees of freedom in LFD

Further split this constraint field ψ_- into the "free" part $\tilde{\psi}_-$ and the "interaction" part Υ, so that $\psi_- = \tilde{\psi}_- + \Upsilon$:

$$\tilde{\psi}_- = \frac{(i\boldsymbol{\gamma}^\perp \cdot \boldsymbol{\partial}_\perp + m)\gamma^+\psi_+}{2i\partial_-}, \tag{11}$$

$$\Upsilon = \frac{-e\boldsymbol{\gamma}^\perp \cdot \mathbf{A}_\perp \gamma^+ \psi_+}{2i\partial_-}. \tag{12}$$

Then one finds that the $\bar{\Upsilon}(i\gamma^-\partial_-)\Upsilon$ term in the Hamiltonian density gives the fermion instantaneous interaction.



Fermion constrained degrees of freedom in LFD

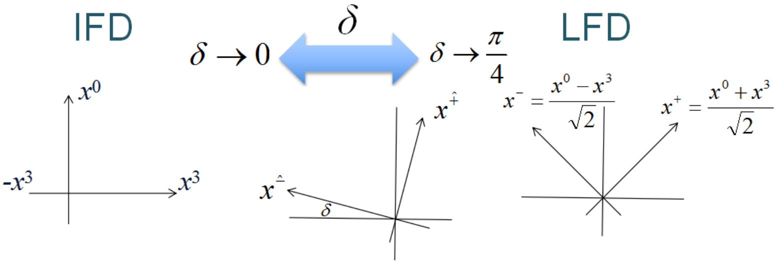
$$
\begin{aligned}
-iV_2 &= -\frac{1}{2}e^2\int d^2\mathbf{x}^\perp dx^- \bar{\tilde{\psi}}(0,\mathbf{x}^\perp,x^-)\gamma^i\tilde{A}_i(0,\mathbf{x}^\perp,x^-) \\
&\times\frac{\gamma^+}{\partial_-}\tilde{A}_j(0,\mathbf{x}^\perp,x^-)\gamma^j\tilde{\psi}(0,\mathbf{x}^\perp,x^-) \\
&= -\frac{1}{4}e^2\int d^2\mathbf{x}^\perp dx^- \bar{\tilde{\psi}}(0,\mathbf{x}^\perp,x^-)\gamma^i\tilde{A}_i(0,\mathbf{x}^\perp,x^-)\gamma^+ \\
&\times\int dx'^-\epsilon(x^- - x'^-)\tilde{A}_j(0,\mathbf{x}^\perp,x'^-)\gamma^j\tilde{\psi}(0,\mathbf{x}^\perp,x'^-), \qquad (13)
\end{aligned}
$$

with

$$
\frac{1}{2}\int dx'^-\epsilon(x^- - x'^-)\, e^{-iq^+(x^- - x'^-)} = \frac{i}{q^+}. \qquad (14)
$$

- This fermion instantaneous interaction is unique to the LF, distinctive from the equal-time forward and backward propagation of the fermion.

Interpolating between the IFD and the LFD

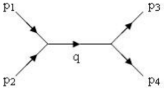


$$\begin{pmatrix} x^{\hat{+}} \\ x^{\hat{1}} \\ x^{\hat{2}} \\ x^{\hat{-}} \end{pmatrix} = \begin{pmatrix} \cos\delta & 0 & 0 & \sin\delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\delta & 0 & 0 & -\cos\delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

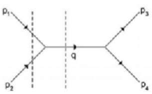
[1] See e.g.
K.Hornbostel, Phys.Rev.D **45**, 3781 (1992)
C.-R. Ji and A.T. Suzuki, Phys. Rev. D **87**, 065015 (2013)

Interpolating between the IFD and the LFD

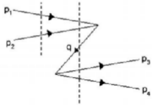
"$e^+e^- \to \mu^+\mu^-$"



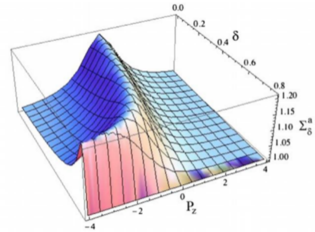
$$= \frac{1}{q^2 - m^2} = \frac{1}{s - m^2}$$

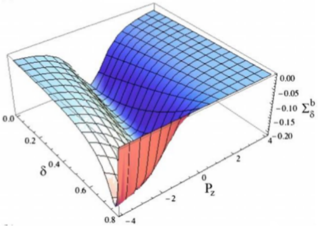


(a)



(b)

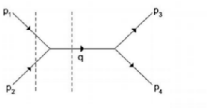




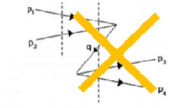
Interpolating between the IFD and the LFD

$$\Sigma^a_\delta + \Sigma^b_\delta = \frac{1}{s - m^2}\,; \quad s = 2\ \mathrm{GeV}^2,\ m = 1\ \mathrm{GeV}.$$

J-shape peak and valley: $P^z = -\sqrt{\frac{s(1-\cos 2\delta)}{2\cos 2\delta}}$.



(a)



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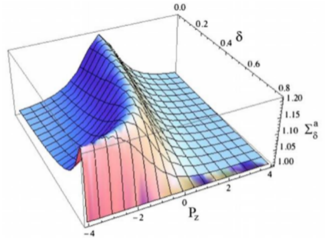
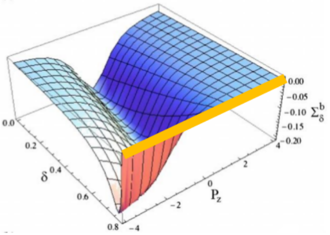
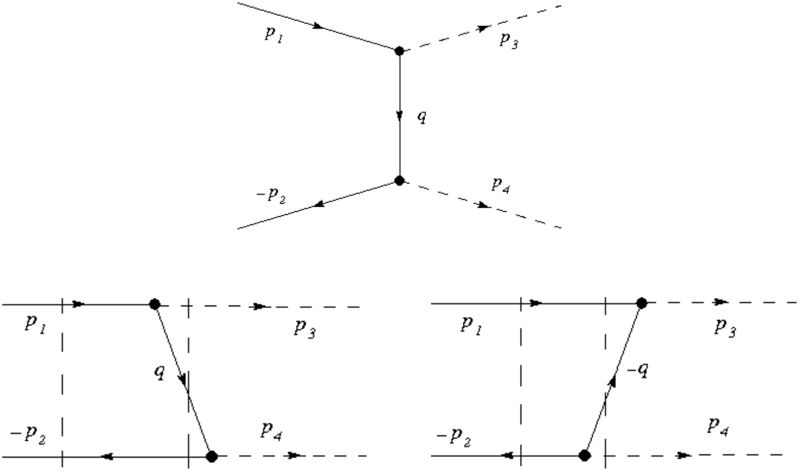




e^+e^- annihilation into two scalar particles



Fermion propagator in interpolation form

$$\Sigma_F = \frac{1}{2Q^{\hat{+}}} \frac{\not{Q}_F + m}{q_{\hat{+}} - Q_{F\hat{+}}}, \quad \Sigma_B = \frac{1}{2Q^{\hat{+}}} \frac{-\not{Q}_B + m}{-q_{\hat{+}} - Q_{B\hat{+}}}, \tag{15}$$

where

$$Q_{F\hat{+}} = \frac{-\mathbb{S}q_{F\hat{-}} + Q^{\hat{+}}}{\mathbb{C}}, \tag{16}$$

$$Q_{B\hat{+}} = \frac{-\mathbb{S}q_{B\hat{-}} + Q^{\hat{+}}}{\mathbb{C}}, \tag{17}$$

and

$$Q^{\hat{+}} = \sqrt{q_{\hat{-}}^2 + \mathbb{C}(\mathbf{q}_\perp^2 + m^2)}, \tag{18}$$

where the 4-momenta $q_F = q$ and $q_B = -q$ are those of the off-shell fermion and anti-fermion, while Q_F and Q_B are the corresponding on-shell 4-momenta.

Light-front limit of the interpolating time-ordered fermion propagators

In the light-front limit $\delta \to \frac{\pi}{4}$, we find that

$$\Sigma_{F,\delta\to\frac{\pi}{4}} = \frac{\not{q}_{on} + m}{q^2 - m^2}, \qquad \Sigma_{B,\delta\to\frac{\pi}{4}} = \frac{\gamma^+}{2q^+}. \tag{19}$$

- The propagator corresponding to positive energy changes to the light-front on-mass-shell propagator.
- The negative energy (anti particle) propagator changes to the light-front instantaneous propagator.

Interpolation spinors

The spinors in the interpolation form were studied in [2]

$$u_H^{(+1/2)}(P) = \begin{pmatrix} \sqrt{\frac{P_{\hat{-}}+\mathbb{P}}{2\mathbb{P}}}\sqrt{\frac{P^{\hat{+}}+\mathbb{P}}{\sin\delta+\cos\delta}} \\ P^R\sqrt{\frac{\sin\delta+\cos\delta}{2\mathbb{P}(\mathbb{P}+P_{\hat{-}})}}\sqrt{P^{\hat{+}}+\mathbb{P}} \\ \sqrt{\frac{P_{\hat{-}}+\mathbb{P}}{2\mathbb{P}}}\sqrt{\frac{P^{\hat{+}}-\mathbb{P}}{\cos\delta-\sin\delta}} \\ P^R\sqrt{\frac{\cos\delta-\sin\delta}{2\mathbb{P}(\mathbb{P}+P_{\hat{-}})}}\sqrt{P^{\hat{+}}-\mathbb{P}} \end{pmatrix},$$

$$u_H^{(-1/2)}(P) = \begin{pmatrix} -P^L\sqrt{\frac{\cos\delta-\sin\delta}{2\mathbb{P}(\mathbb{P}+P_{\hat{-}})}}\sqrt{P^{\hat{+}}-\mathbb{P}} \\ \sqrt{\frac{P_{\hat{-}}+\mathbb{P}}{2\mathbb{P}}}\sqrt{\frac{P^{\hat{+}}-\mathbb{P}}{\cos\delta-\sin\delta}} \\ -P^L\sqrt{\frac{\sin\delta+\cos\delta}{2\mathbb{P}(\mathbb{P}+P_{\hat{-}})}}\sqrt{P^{\hat{+}}+\mathbb{P}} \\ \sqrt{\frac{P_{\hat{-}}+\mathbb{P}}{2\mathbb{P}}}\sqrt{\frac{P^{\hat{+}}+\mathbb{P}}{\sin\delta+\cos\delta}} \end{pmatrix},$$

where $P^R = P^1 + iP^2$ and $P^L = P^1 - iP^2$, and the antiparticle spinors are obtained by charge conjugation.

[2] Z. Li, M. An and C.-R. Ji, Phys. Red. D **92**, 105014 (2015)

Center-of-mass kinematics

The kinematics is written as the following. We choose the initial reference frame to be the e^+e^- center of mass frame (CMF), and study the whole landscape of the amplitude change under the boost operation in the z-direction as well as the change of the interpolation angle δ.

$$\begin{aligned}
p_1 &= (E_0, 0, 0, P_e) \\
p_2 &= (E_0, 0, 0, -P_e) \\
p_3 &= (E_0, E_0 \sin\theta, 0, E_0 \cos\theta) \\
p_4 &= (E_0, -E_0 \sin\theta, 0, -E_0 \cos\theta).
\end{aligned}$$

Particle 1 is the incoming electron, particle 2 is the incoming positron, particle 3 and 4 are the two outgoing massless scalar particles.

Interpolating helicity amplitudes of $e^+e^- \rightarrow 2s$

Let's now compute the time-ordered amplitudes for the e^+e^- annihilation into two scalar particles, which are given by

$$\mathcal{M}_a^{\lambda_1,\lambda_2} = \bar{v}_{\lambda_2}(p_2)\cdot\Sigma_a\cdot u_{\lambda_1}(p_1) \tag{20}$$

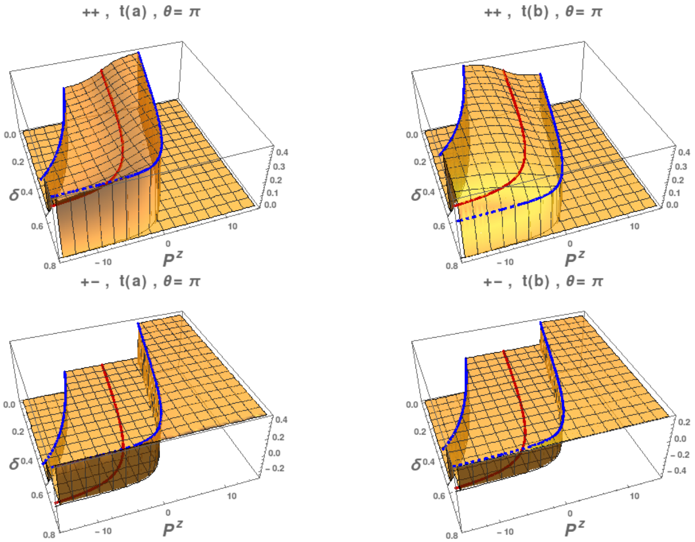
and

$$\mathcal{M}_b^{\lambda_1,\lambda_2} = \bar{v}_{\lambda_2}(p_2)\cdot\Sigma_b\cdot u_{\lambda_1}(p_1), \tag{21}$$

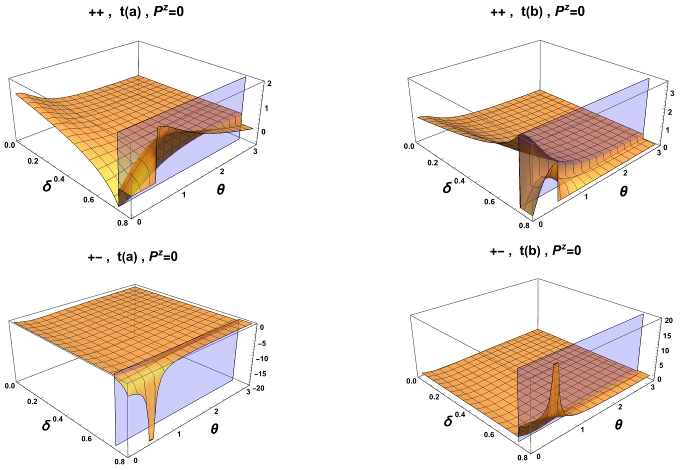
where λ_1 and λ_2 represent the helicities of the initial e^- and e^+ spinors, respectively, and the overall factor such as the coupling constant e, etc., is taken to be 1.
The subscripts a and b are used to represent two different time-ordering, however, depending on the kinematics, either one can correspond to either "forward" or "backward" propagators.
The criterion for the scattering/annihilation angle for the t-channel is found to be

$$\theta_{c,t} = \arccos\left(\frac{P_e}{E_0}\right) \tag{22}$$

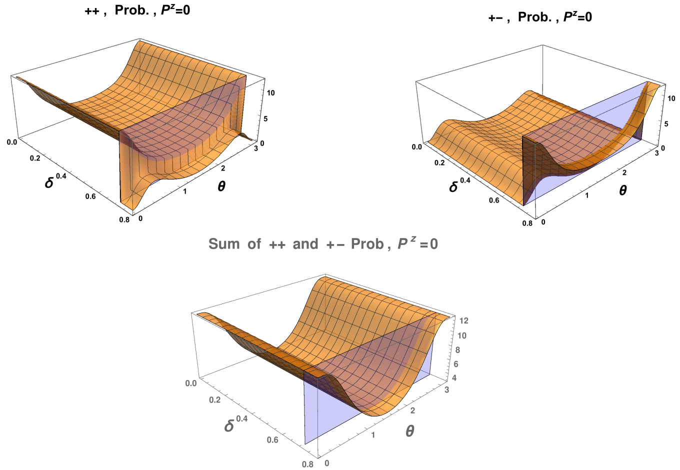
Frame dependence of $e^+e^- \to 2s$ helicity amplitudes



Angular distribution of $e^+e^- \to 2s$ helicity amplitudes



Angular distribution of $e^+e^- \to 2s$ total amplitudes and probabilities in the center-of-mass frame

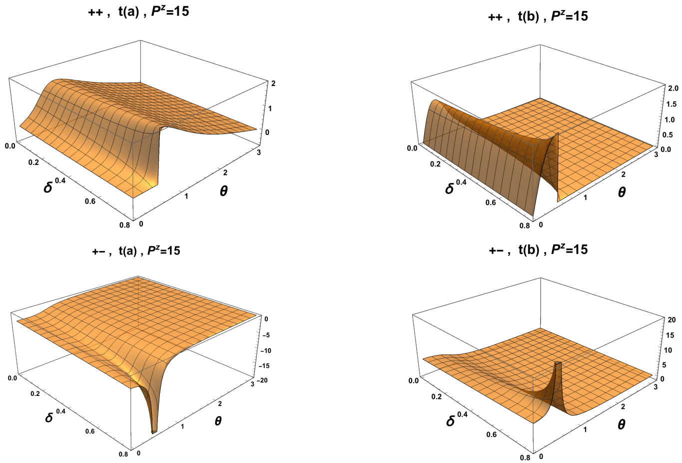


The result agrees with the textbook calculation

$$
\begin{aligned}
|\mathcal{M}|^2_{\text{scalar}} &\equiv \sum_{\lambda_1,\lambda_2} |\mathcal{M}^{\lambda_1,\lambda_2}_{a,t} + \mathcal{M}^{\lambda_1,\lambda_2}_{b,t} + \mathcal{M}^{\lambda_1,\lambda_2}_{a,u} + \mathcal{M}^{\lambda_1,\lambda_2}_{b,u}|^2 \\
&= \frac{2\left(ut + m^2(4s - 5t + 3u) - 15m^4\right)}{(t - m^2)^2} \\
&+ \frac{2\left(tu + m^2(4s - 5u + 3t) - 15m^4\right)}{(u - m^2)^2} \\
&+ \frac{2\left((s+u)u + (s+t)t + 2m^2(3s - t - u) - 30m^4\right)}{(t - m^2)(u - m^2)}, \qquad (23)
\end{aligned}
$$

where $m = m_e$ and the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ are give by $s = 16m_e^2$, $t = (-7 + 4\sqrt{3}\cos\theta)m_e^2$ and $u = -(7 + 4\sqrt{3}\cos\theta)m_e^2$ calculated from the kinematics in CMF with $E_0 = 2m_e$ and $P_e = \sqrt{3}m_e$ for our numerical calculation.

Angular distribution of $e^+e^- \to 2s$ helicity amplitudes in a boosted frame $(P^z = +15m_e)$

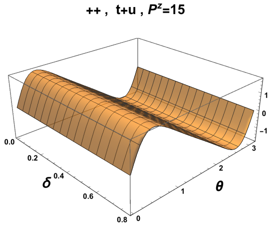


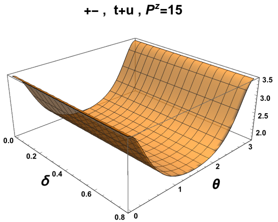
Conclusion

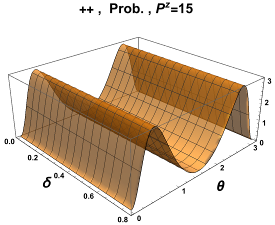
- We find that the fermion constrained degree of freedom is a unique feature only at the exact light-front ($\delta = \frac{\pi}{4}$), and does not exist for any other interpolation angle $(0 \leq \delta < \frac{\pi}{4})$.
- When δ approaches $\frac{\pi}{4}$, the backward fermion propagator changes to the instantaneous fermion propagator of the light-front.
- Infinite Momentum Frame is NOT Light Front Dynamics.

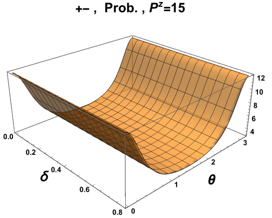
Thank you!

Backup: Angular distribution of $e^+e^- \to 2s$ total amplitudes and probabilities in a boosted frame ($P^z = +15m_e$)



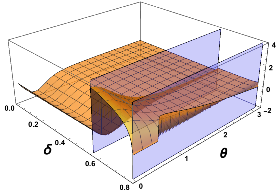




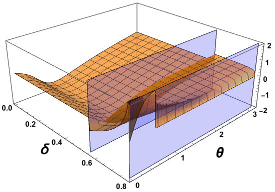


Backup: Angular distribution of $e^+e^- \to 2s$ helicity amplitudes in a boosted frame ($P^z = -15m_e$)

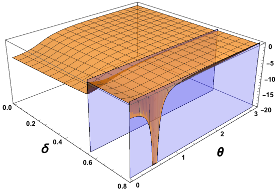
++ , t(a) , P^z=–15



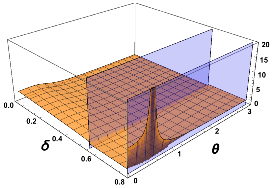
++ , t(b) , P^z=–15



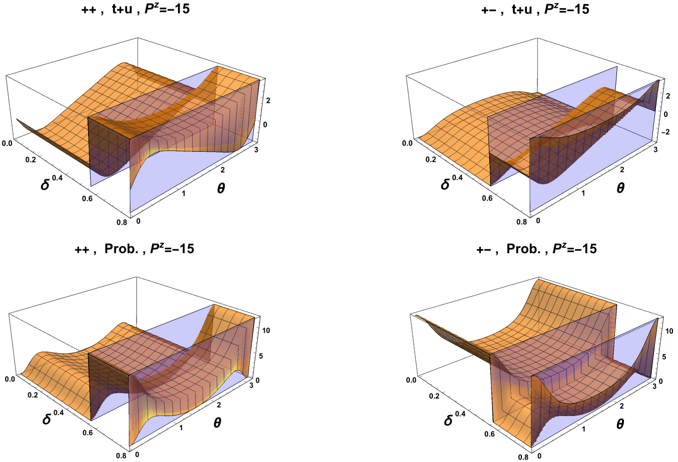
+– , t(a) , P^z=–15



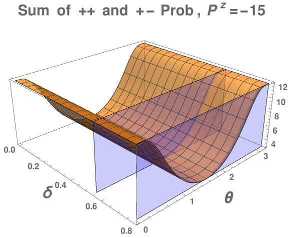
+– , t(b) , P^z=–15

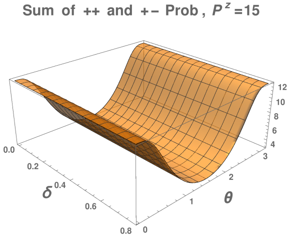


Backup: Angular distribution of $e^+e^- \to 2s$ total amplitudes and probabilities in a boosted frame $(P^z = -15m_e)$



Backup: Angular distribution of $e^+e^- \to 2s$: sum of $++$ and $+-$ probabilities in different frames





Backup: the derivation of the constraint equation for the fermion field

Multiply Eq. (5) from the left by γ^+, and plug in the splitting of ψ:

$$\gamma^+ \left[i \left(\gamma^- \partial_- + \boldsymbol{\gamma}^\perp \cdot \boldsymbol{\partial}_\perp \right) - e \left(\gamma^- A_- + \boldsymbol{\gamma}^\perp \cdot \mathbf{A}_\perp \right) - m \right] (\psi_+ + \psi_-) = 0. \tag{24}$$

Due to $\gamma^- P_+ = 0$, $\gamma^+\gamma^- = 2 - \gamma^-\gamma^+$, $\gamma^+\boldsymbol{\gamma}^\perp = -\boldsymbol{\gamma}^\perp\gamma^+$ and $\gamma^+ P_- = 0$, we have

$$\left(i\gamma^+\boldsymbol{\gamma}^\perp \cdot \boldsymbol{\partial}_\perp - e\gamma^+\boldsymbol{\gamma}^\perp \cdot \mathbf{A}_\perp - \gamma^+ m \right) \psi_+ + 2 \left(i\partial_- - eA_- \right) \psi_- = 0. \tag{25}$$

Passing through γ^+, we get Eq. (9):

$$2 \left(i\partial_- - eA_- \right) \psi_- = \left[\left(i\boldsymbol{\partial}_\perp - e\mathbf{A}_\perp \right) \boldsymbol{\gamma}^\perp + m \right] \gamma^+ \psi_+. \tag{26}$$