Basis light front quantization for the light mesons
with color singlet Nambu-Jona-Lasinio interactions

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The Lagrangian of QCD before gauge fixing:

$$\mathcal{L}_{\mathrm{QCD}} = \overline{\psi}(i\not{D} - m)\psi - \frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a}, \tag{1}$$

is a result of a local $\mathrm{SU}(3)_{\mathrm{c}}$ gauge symmetry:

$$\psi(x) \to \exp\left(-i \sum_{a=1}^{8} T_a \Theta_a(x)\right) \psi(x).$$

Consider quark field with three flavors:

$$\psi = (\mathrm{u},\ \mathrm{d},\ \mathrm{s})^{\mathrm{T}}, \quad m = \mathrm{diag}\,\{m_{\mathrm{u}},\ m_{\mathrm{d}},\ m_{\mathrm{s}}\}. \tag{2}$$

In the chiral limit, there exist global $\mathrm{U}(3)_{\mathrm{L}} \otimes \mathrm{U}(3)_{\mathrm{R}}$ symmetries.

$$P_{\mathrm{R}} = (\mathbf{1} + \gamma_5)/2, \quad P_{\mathrm{L}} = (\mathbf{1} - \gamma_5)/2, \quad \psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\psi.$$

Symmetries of the strong interaction

SYMMETRY	Local gauge	Global chiral	Local chiral
Theory	QCD	NJL	Chiral EFTs
Dof	Quarks and gluons	Quarks	Mesons or baryons
Energy scale	0 to Λ_{GUT}	0 to $1\,\mathrm{GeV}$	Dof dependent

Consider color singlet four-fermion interactions in the three-flavor NJL model. The Lagrangian is given by [Klimt:1989pm]

$$\begin{aligned}\mathcal{L}^{(4)}_{\mathrm{NJL,SU(3)}} &= \overline{\psi}(i\partial\!\!\!/ - m)\psi \\ &+ G_\pi \sum_{i=0}^{8}\left[\left(\overline{\psi}\lambda^i\psi\right)^2 + \left(\overline{\psi}i\gamma_5\lambda^i\psi\right)^2\right] \\ &- G_\rho \sum_{i=0}^{8}\left[\left(\overline{\psi}\gamma_\mu\lambda^i\psi\right)^2 + \left(\overline{\psi}\gamma_\mu\gamma_5\lambda^i\psi\right)^2\right] \\ &- G_{\mathrm{V}}\left(\overline{\psi}\gamma_\mu\psi\right)^2 - G_{\mathrm{A}}\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)^2 .\end{aligned} \tag{3}$$

$\mathrm{SU(3)_V} \otimes \mathrm{SU(3)_A} \otimes \mathrm{U(1)_V} \otimes \mathrm{U(1)_A}$
(isospin) ~~(chiral)~~ (baryonic) ~~(axial)~~

Chiral symmetry is broken by

- nonvanishing quark mass,
- dynamics.

- The $\mathrm{U(1)_A}$ symmetry is broken by field theory effects, which can be accounted for in the NJL model by adding a determinant term:

$$\mathcal{L}_{\mathrm{det}} = G_{\mathrm{D}}\left[\det\overline{\psi}(1+\gamma_5)\psi + \det\overline{\psi}(1-\gamma_5)\psi\right]. \tag{4}$$

Determinants are taken in the flavor space, resulting in six-fermion interactions.

In the two-flavor scenario, the NJL Lagrangian is reduced into

$$\begin{aligned}\mathcal{L}_{\mathrm{NJL,SU(2)}} &= \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{G_\pi}{2}\left[(\overline{\psi}\psi)^2 - \left(\overline{\psi}\gamma_5\vec{\tau}\psi\right)^2\right] \\ &- \frac{G_\rho}{2}\left[(\overline{\psi}\gamma_\mu\vec{\tau}\psi)^2 - \left(\overline{\psi}\gamma_\mu\gamma_5\vec{\tau}\psi\right)^2\right] \\ &- G_{\mathrm{V}}\left(\overline{\psi}\gamma_\mu\psi\right)^2 - G_{\mathrm{A}}\left(\overline{\psi}\gamma_\mu\gamma_5\psi\right)^2, \end{aligned} \tag{5}$$

which is consistent with the three-flavor Lagrangian when determinant terms are added.

- Explicitly the two-flavor determinant terms are given by

$$\begin{aligned}&\det\overline{\psi}(1+\gamma_5)\psi + \det\overline{\psi}(1-\gamma_5)\psi \\ &= 2\{\overline{\mathrm{u}}\mathrm{u}\,\overline{\mathrm{d}}\mathrm{d} + \overline{\mathrm{u}}\gamma_5\mathrm{u}\overline{\mathrm{d}}\gamma_5\mathrm{d} - \overline{\mathrm{u}}\mathrm{d}\,\overline{\mathrm{d}}\mathrm{u} - \overline{\mathrm{u}}\gamma_5\mathrm{d}\,\overline{\mathrm{d}}\gamma_5\mathrm{u}\}.\end{aligned} \tag{6}$$

The effective Hamiltonian on the light front

The two-body interactions of the effective Hamiltonian are given by: (light front holography) × (invariant mass ansatz) + (longitudinal confinement) [Li:2015zda,Li:2017mlw] :

$$H_0 = \frac{\vec{k}_\perp^2 + \mathbf{m}^2}{x} + \frac{\vec{k}_\perp^2 + \overline{\mathbf{m}}^2}{1-x} + \kappa^4 x(1-x) r_\perp^2 - \frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2}\partial_x x(1-x)\partial_x. \tag{7}$$

We truncate the light front wavefunction at the valance Fock sector

$$\begin{aligned} &\left|\Psi_{\mathrm{meson}}(P^+, \vec{P}^\perp, j, m_j)\right\rangle \\ &= \sum_{r,s}\int_0^1 \frac{dx}{4\pi x(1-x)} \int \frac{d\vec{\kappa}^\perp}{(2\pi)^2}\, \psi_{rs}(x, \vec{\kappa}^\perp) \\ &\quad \times b_r^\dagger(xP^+, \vec{\kappa}^\perp + x\vec{P}^\perp) d_s^\dagger((1-x)P^+, -\vec{\kappa}^\perp + (1-x)\vec{P}^\perp)|0\rangle, \end{aligned} \tag{8}$$

with $P = k + p$, $x = k^+/P^+$, and $\vec{\kappa}^\perp = \vec{k}^\perp - x\vec{P}^\perp$.

The choice of basis functions

$$\psi_{rs}(x, \vec{\kappa}^{\perp}) = \sum_{nml} \psi(n, m, l, r, s)\phi_{nm}\left(\frac{\vec{\kappa}^{\perp}}{\sqrt{x(1-x)}}\right)\chi_l(x). \quad (9)$$

- The transverse basis function is given by

$$\phi_{nm}\left(\vec{q}^{\perp}; b\right) = \frac{1}{b}\sqrt{\frac{4\pi n!}{(n+|m|)!}}\left(\frac{|\vec{q}^{\perp}|}{b}\right)^{|m|}\exp\left(-\frac{\vec{q}^{\perp 2}}{2b^2}\right) L_n^{|m|}\left(\frac{\vec{q}^{\perp 2}}{b^2}\right) e^{im\varphi}, \quad (10)$$

  with $\tan(\varphi) = q^2/q^1$ and $L_n^{|m|}$ being the associated Laguerre polynomial.

- While for the longitudinal basis, we have

$$\chi_l(x; \alpha, \beta) = \sqrt{4\pi(2l+\alpha+\beta+1)}\sqrt{\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}} \times x^{\beta/2}(1-x)^{\alpha/2} P_l^{(\alpha,\beta)}(2x-1), \quad (11)$$

  where $P_l^{(\alpha,\beta)}(z)$ is the Jacobi polynomial.

$$\alpha = \frac{2\overline{\mathbf{m}}(\mathbf{m}+\overline{\mathbf{m}})}{\kappa^2}, \qquad \beta = \frac{2\mathbf{m}(\mathbf{m}+\overline{\mathbf{m}})}{\kappa^2}. \quad (12)$$

The matrix elements for the effective Hamiltonian

$$\left\langle \Psi_{\text{meson}}\left(P'^{+}, \vec{P}'^{\perp}, j', m'_j\right) \middle| H_{\text{eff}} \middle| \Psi_{\text{meson}}\left(P^{+}, \vec{P}^{\perp}, j, m_j\right) \right\rangle$$
$$= 4\pi P^{+}\delta\left(P'^{+} - P^{+}\right)(2\pi)^2\delta\left(\vec{P}'^{\perp} - \vec{P}^{\perp}\right)\sum_{r',s'}\sum_{r,s}$$
$$\times \int_0^1 \frac{dx'}{4\pi x'(1-x')}\int \frac{d\vec{\kappa}'^{\perp}}{(2\pi)^2}\int_0^1 \frac{dx}{4\pi x(1-x)}\int \frac{d\vec{\kappa}^{\perp}}{(2\pi)^2}$$
$$\times \psi^{*}_{r's'}(x', \vec{\kappa}'^{\perp})\, H_{\text{eff}\; r's'rs}(x', \vec{\kappa}'^{\perp}, x, \vec{\kappa}^{\perp})\, \psi_{rs}(x, \vec{\kappa}^{\perp}). \tag{13}$$

When $\kappa = b$, the two-body interaction H_0 is diagonal in the basis representation:

$$\Lambda_0(n, m, l; \mathbf{m}, \overline{\mathbf{m}}, \kappa) = (\mathbf{m}+\overline{\mathbf{m}})^2 + 2\kappa^2(2n+|m|+l+3/2) + \frac{\kappa^4}{(\mathbf{m}+\overline{\mathbf{m}})^2} l(l+1). \tag{14}$$

The matrix elements of the NJL interaction

The flavor decomposition of the direct four-fermion interaction relevant for the valence Fock sector is

$$\int dx^- \int d\vec{x}^{\perp}\, \overline{\psi}_{\mathrm{Q}}(x)\gamma^{?}\psi_{\mathrm{Q}}(x)\, \overline{\psi}_{\mathrm{P}}(x)\gamma^{?}\psi_{\mathrm{P}}(x)$$

$$\to \sum_{s1234} \int d\underline{k}_{1234}\, 4\pi\delta(k_1^+ + k_2^+ - k_3^+ - k_4^+)$$

$$\times (2\pi)^2\delta\left(k_1^\perp + k_2^\perp - k_3^\perp - k_4^\perp\right)$$

$$\times \Big\{ b^\dagger_{\mathrm{Q}1} d^\dagger_{\mathrm{Q}2} d_{\mathrm{P}3} b_{\mathrm{P}4}\, \overline{u}_{\mathrm{Q}1}\gamma^{?} v_{\mathrm{Q}2}\, \overline{v}_{\mathrm{P}3}\gamma^{?} u_{\mathrm{P}4}$$

$$+ b^\dagger_{\mathrm{P}1} d^\dagger_{\mathrm{P}2} d_{\mathrm{Q}3} b_{\mathrm{Q}4}\, \overline{u}_{\mathrm{P}1}\gamma^{?} v_{\mathrm{P}2}\, \overline{v}_{\mathrm{Q}3}\gamma^{?} u_{\mathrm{Q}4}$$

$$- b^\dagger_{\mathrm{Q}1} d^\dagger_{\mathrm{P}2} d_{\mathrm{P}3} b_{\mathrm{Q}4} \overline{u}_{\mathrm{Q}1}\gamma^{?} u_{\mathrm{Q}4}\, \overline{v}_{\mathrm{P}3}\gamma^{?} v_{\mathrm{P}2}$$

$$- b^\dagger_{\mathrm{P}1} d^\dagger_{\mathrm{Q}2} d_{\mathrm{Q}3} b_{\mathrm{P}4} \overline{u}_{\mathrm{P}1}\gamma^{?} u_{\mathrm{P}4}\, \overline{v}_{\mathrm{Q}3}\gamma^{?} v_{\mathrm{Q}2} \Big\}.$$

Direct and exchange interactions are related by the Fierz transformations.

$$s \to \frac{1}{4}(s + v + \frac{t}{2} - a - p)$$

$$p \to -\frac{1}{4}(s - v + \frac{t}{2} + a - p)$$

$$v \to \frac{1}{4}(4s - 2v - 2a + 4p)$$

$$a \to -\frac{1}{4}(4s + 2v + 2a + 4p)$$

Example: scalar matrix elements for the basis expansion

$s_1' s_2' s_2 s_1$	$\langle n' m' l' s_1' s_2' \| \overline{u}_{\rm u} u_{\rm u}\, \overline{v}_{\rm d} v_{\rm d} \| n m l s_1 s_2 \rangle$
$++++$	$(-1)^{n'+n+1}(b^2/\pi)\delta_{m',0}\delta_{m,0}\mathbf{m}\overline{\mathbf{m}}\{L'(1/2,1/2)L(-1/2,-1/2) + L'(-1/2,1/2)L(1/2,-1/2) + L'(1/2,-1/2)L(-1/2,1/2) + L'(-1/2,-1/2)L(1/2,1/2)\}$
$\cdots$	$\cdots$

$$
\begin{aligned}
L_l(a,b;\alpha,\beta) &\equiv \int_0^1 \frac{dx}{4\pi}\, x^b (1-x)^a \chi_l(x;\alpha,\beta) \\
&= \sqrt{\frac{2l+\alpha+\beta+1}{4\pi}}\sqrt{\frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}} \\
&\times \sum_{m=0}^{l} \binom{l+\alpha}{m}\binom{l+\beta}{l-m}(-1)^{l-m} B\left(\frac{\beta}{2}+b+m+1, \frac{\alpha}{2}+a+l-m+1\right),
\end{aligned} \tag{15}
$$

where $B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ is the Euler Beta function.

Two-flavor NJL

- Consider the interaction:

$$\mathcal{H}^{\text{eff}}_{\text{NJL}\,\pi} = -\frac{G_{\text{P}\pi}}{2} P^{+} \left[(\overline{\psi}\psi)^2 + (\overline{\psi} i\gamma_5 \vec{\tau} \psi)^2 \right], \tag{16}$$

responsible for the binding of $\pi^{\pm}$.

- Symmetries preserved: $\text{SU}(2)_{\text{V}} \otimes \text{SU}(2)_{\text{A}} \otimes \text{U}(1)_{\text{V}}$.
- The binding of $\rho^{\pm}$ is already taken care of by the confinement interaction in the basis diagonal Hamiltonian.

Three-flavor NJL

- For the binding of $K^{\pm}$:

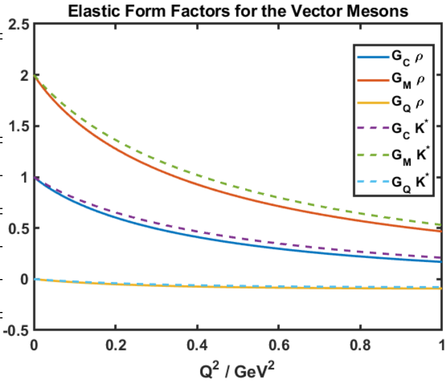
$$\mathcal{H}^{\text{eff}}_{\text{NJL}\,K+} = -G_{\text{P}K} P^{+} [(\overline{\psi}\lambda_a\psi)^2 - (\overline{\psi}\lambda_a\gamma_5\psi)^2]. \tag{17}$$

- The expansion in flavor space is given by

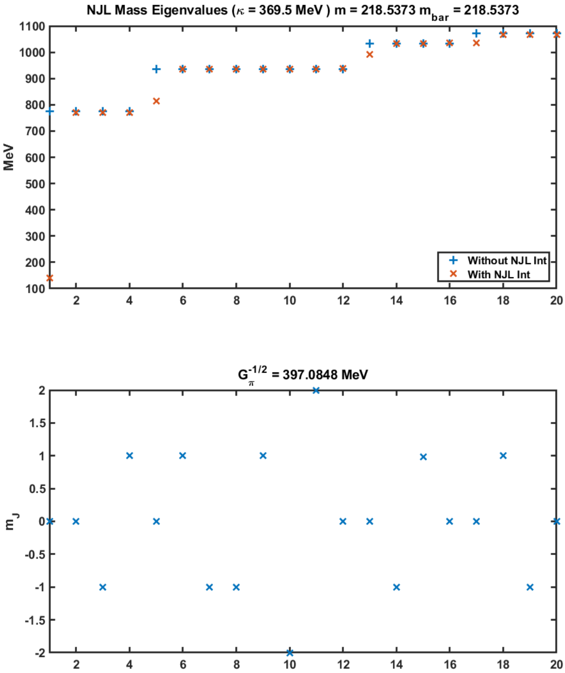
$$\begin{aligned}(\overline{\psi}\,\lambda_a\gamma^{?}\,\psi)^2 &= (\overline{\text{u}}\gamma^{?}\text{u} + \overline{\text{d}}\gamma^{?}\text{d})^2 + (\overline{\text{u}}\gamma^{?}\text{d} + \overline{\text{d}}\gamma^{?}\text{u})^2 \\ &\quad + 2[(\overline{\text{s}}\gamma^{?}\text{s})^2 + 2\,\overline{\text{u}}\gamma^{?}\text{s}\,\overline{\text{s}}\gamma^{?}\text{u} + 2\,\overline{\text{d}}\gamma^{?}\text{s}\,\overline{\text{s}}\gamma^{?}\text{d}].\end{aligned} \tag{18}$$

BLFQ-NJL	model parameters	κ	369.5 MeV	$N_{\max}$	8
m_l	222.2 MeV	$G_{P\pi}$	6.340 GeV^{-2}	$M_{\max}$	2
m_s	398.8 MeV	G_{PK}	7.326 GeV^{-2}	$L_{\max}$	8

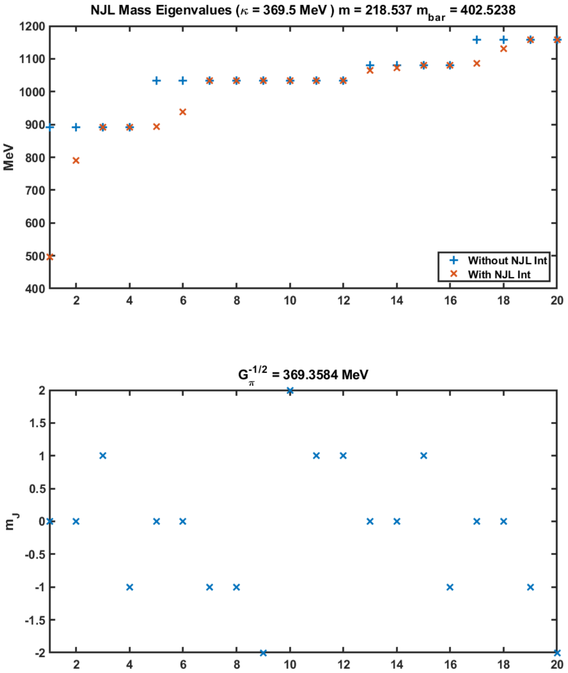
	Mass	Decay constant (MeV)	Charge radius (fm^2)
π^+	139.57	148.08	0.263
ρ^+	775.26	121.02	0.773
K^+	493.68	177.77	0.250
K^{*+}	891.76	106.30	0.608
K_0^{*+}	790.21	49.05	-



Charge radius (fm^2)	π^+	K^+	ρ^+	K^{*+}
BLFQ-NJL	0.263	0.250	0.773	0.608
Hutauruk:2016sug	0.396	0.343		
Carrillo-Serrano:2015uca			0.67	
PDG	0.452(10)	0.256(33)		



NJL Mass Eigenvalues (κ = 369.5 MeV) m = 218.5373 m_bar = 218.5373
MeV
Without NJL Int
With NJL Int
G_π^-1/2 = 397.0848 MeV
m_J


NJL Mass Eigenvalues (κ = 369.5 MeV) m = 218.537 m_bar = 402.5238
MeV
Without NJL Int
With NJL Int
G_π^-1/2 = 369.3584 MeV
m_J

Spin matrix elements for the direct scalar interaction

$s_1' s_2' s_2 s_1$	$\overline{u}_{\mathrm{us}1'}(p_1') u_{\mathrm{us}1}(p_1)\, \overline{v}_{\mathrm{ds}2}(p_2) v_{\mathrm{ds}2'}(p_2')$
$++++$	$-\mathbf{m}\overline{\mathbf{m}} \left(\sqrt{\frac{x'}{x}} + \sqrt{\frac{x}{x'}} \right) \left(\sqrt{\frac{1-x'}{1-x}} + \sqrt{\frac{1-x}{1-x'}} \right)$
$+++-$	$\overline{\mathbf{m}} \left(\sqrt{\frac{x'}{1-x'}} q^{\mathrm{L}} - \sqrt{\frac{x}{1-x}} q'^{\mathrm{L}} \right) (2 - x' - x)$
$++-+$	$\mathbf{m}(x'+x) \left(\sqrt{\frac{1-x}{x}} q'^{\mathrm{L}} - \sqrt{\frac{1-x'}{x'}} q^{\mathrm{L}} \right)$
$++--$	$-(x'+x-2x'x) q'^{\mathrm{L}} q^{\mathrm{L}} + \sqrt{x'(1-x')x(1-x)} (q'^{\mathrm{L}2} + q^{\mathrm{L}2})$
$+-++$	$\mathbf{m}(x'+x) \left(\sqrt{\frac{1-x'}{x'}} q^{\mathrm{R}} - \sqrt{\frac{1-x}{x}} q'^{\mathrm{R}} \right)$
$+-+-$	$(1-x')x q'^{\mathrm{L}} q^{\mathrm{R}} + x'(1-x) q'^{\mathrm{R}} q^{\mathrm{L}} - \sqrt{x'(1-x')x(1-x)} (q'^{\mathrm{L}} q'^{\mathrm{R}} + q^{\mathrm{L}} q^{\mathrm{R}})$

$s_1' s_2' s_2 s_1$	$\overline{u}_{\mathrm{us}1'}(p_1') u_{\mathrm{us}1}(p_1)\, \overline{v}_{\mathrm{ds}2}(p_2) v_{\mathrm{ds}2'}(p_2')$
$+--+$	$++++$
$+---$	$+++-$
$-+++$	$-\overline{\mathrm{m}} \left(\sqrt{\frac{x'}{1-x'}} q^{\mathrm{R}} - \sqrt{\frac{x}{1-x}} q'^{\mathrm{R}} \right) (2 - x' - x)$
$-++-$	$++++$
$-+-+$	$x'(1-x) q'^{\mathrm{L}} q^{\mathrm{R}} + x(1-x') q'^{\mathrm{R}} q^{\mathrm{L}} - \sqrt{x'(1-x')x(1-x)} (q'^{\mathrm{L}} q'^{\mathrm{R}} + q^{\mathrm{L}} q^{\mathrm{R}})$
$-+--$	$++-+$
$--++$	$-(x' + x - 2x'x) q'^{\mathrm{R}} q^{\mathrm{R}} + \sqrt{x'(1-x')x(1-x)} (q'^{\mathrm{R}2} + q^{\mathrm{R}2})$
$--+-$	$+-++$
$---+$	$-+++$
$----$	$++++$

- The u-spinor mass is $\mathbf{m}$. The v-spinor mass is $\overline{\mathbf{m}}$.
- The q' and q are the transverse momenta to be integrated.