The MHV Lagrangian and hidden Wilson lines

Piotr Kotko

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P.K., A. Stasto, JHEP 1709 (2017)

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INSTITUTE OF NUCLEAR PHYSICS
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Outline

Anna's talk:

Recursion relations for off-shell MHV currents contain an object $\tilde{J}$ which has a structure exactly like on-shell MHV amplitude but with spinor products continued off-shell.

- $\tilde{J}$ can be constructed from a straight-infinite Wilson line along a polarization vector
- On the other hand $\tilde{J}$ corresponds to the so-called MHV vertices in the Cachazo-Svrcek-Witten (CSW) construction.

What is the connection between the two?

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This talk:

- Lagrangian for the CSW method (the light-front Yang-Mills Lagrangian after certain canonical field transformation) [P. Mansfield (2006)]
- Exact solution to the field transformation
  – constructed from non-light-like Wilson lines, similar to those in $\tilde{J}$
- Consequences

MHV amplitudes

Spinor algebra

Spinor products:

$$\langle ij\rangle = \overline{u}_{-}(k_i)\,u_{+}(k_j) \equiv \epsilon_{\alpha\beta}\lambda_i^{\alpha}\lambda_j^{\beta}, \quad [ij] = \overline{u}_{+}(k_i)\,u_{-}(k_j) \equiv \epsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\beta}$$

where $u_{\pm}(k_i) = \frac{1}{2}(1 \pm \gamma_5)\,u(k_i)$ and $\lambda_i^{\alpha} \equiv u_{+}(k_i)$, $\tilde\lambda_i^{\dot\alpha} \equiv u_{-}(k_i)$.

Momenta k_i are light-like.

Parke-Taylor amplitudes[1]

$$\boxed{\mathcal{M}\left(1^{-}, 2^{-}, 3^{+}, \ldots, n^{+}\right) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}}$$

[1] S.J. Parke, T.R. Taylor, Phys.Rev.Lett. 56, 2459 (1986)

Cachazo-Svrcek-Witten (CSW) Method (1)

General idea

Glue any amplitude from the MHV amplitudes continued off-shell.

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Off-shell continuation of spinors

If k is light-like, we have

$$k_{\alpha\dot{\alpha}} = \lambda_{k\alpha}\tilde{\lambda}_{k\dot{\alpha}} \quad \Longrightarrow \quad \lambda_{k\alpha} = k_{\alpha\dot{\alpha}}\tilde{\lambda}_q^{\dot{\alpha}} / \, [kq]$$

where q is auxiliary light-like momentum.
If k is off-shell we define the off-shell continuation of spinor in the same way:

$$\lambda_{k\alpha}^{(*)} = k_{\alpha\dot{\alpha}}\tilde{\lambda}_q^{\dot{\alpha}}$$

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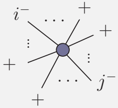
MHV vertices

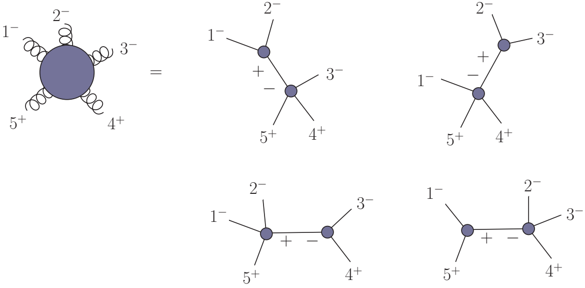
$$\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle} \quad \equiv$$



The spinor products are made from off-shell spinors $\langle ij\rangle = \epsilon_{\alpha\beta}\lambda_i^{(*)\alpha}\lambda_j^{(*)\beta}$.

Cachazo-Svrcek-Witten (CSW) Method (2)

Example: NMHV amplitude $\mathcal{M}(1^-, 2^-, 3^-, 4^+, 5^+)$



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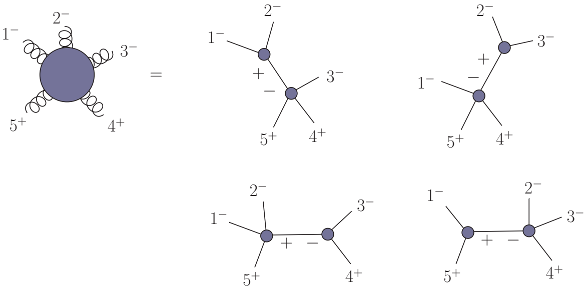

The result:

$$\mathcal{M}\left(1^-, 2^-, 3^-, 4^+, 5^+\right) = \frac{[45]^4}{[12]\,[23]\,[34]\,[45]\,[51]}$$

Yang-Mills action on the light-front (1)

Yang-Mills action

$$S_{\mathrm{Y-M}} = -\frac{1}{4}\int d^4x \, \mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu}$$

where:

$$F^{\mu\nu} = \frac{i}{g'}\left[\mathcal{D}^\mu, \mathcal{D}^\nu\right] \qquad \hat{A}^\mu = A_a^\mu t^a$$

$$\mathcal{D}^\mu = \partial^\mu - ig'\hat{A}^\mu \qquad \left[t^a, t^b\right] = i\sqrt{2} f^{abc} t^c$$

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Light-cone coordinates

Basis vectors:

$$\eta = \frac{1}{\sqrt{2}}\left(1,0,0,-1\right), \quad \tilde{\eta} = \frac{1}{\sqrt{2}}\left(1,0,0,1\right), \quad \varepsilon_\perp^\pm = \frac{1}{\sqrt{2}}\left(0,1,\pm i,0\right)$$

Contravariant coordinates:

$$v^+ = v\cdot\eta, \quad v^- = v\cdot\tilde{\eta}, \quad v^\bullet = v\cdot\varepsilon_\perp^+, \quad v^\star = v\cdot\varepsilon_\perp^-$$

Scalar product: $\quad u\cdot v = u^+w^- + u^-w^+ - u^\bullet w^\star - u^\star w^\bullet$

Three-vectors: $\quad \mathbf{x} \equiv \left(x^-, x^\bullet, x^\star\right), \quad \mathbf{p} \equiv \left(p^+, p^\bullet, p^\star\right)$

Yang-Mills action on the light-front (2)

Yang-Mills action in transverse fields only

- Light cone gauge: $A \cdot \eta = A^{+} = 0$
- Integration of A^{-} fields out of the action

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$$S_{\mathrm{Y-M}}^{(\mathrm{LC})}\left[A^\bullet, A^\star\right] = \int dx^+ \left(\mathcal{L}_{+-}^{(\mathrm{LC})} + \mathcal{L}_{++-}^{(\mathrm{LC})} + \mathcal{L}_{+--}^{(\mathrm{LC})} + \mathcal{L}_{++--}^{(\mathrm{LC})}\right)$$

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$$\mathcal{L}_{+-}^{(\mathrm{LC})}\left[A^\bullet, A^\star\right] = -\int d^3\mathbf{x}\, \mathrm{Tr}\, \hat{A}^\bullet \square \hat{A}^\star$$

$$\mathcal{L}_{++-}^{(\mathrm{LC})}\left[A^\bullet, A^\star\right] = -2ig' \int d^3\mathbf{x}\, \mathrm{Tr}\, \gamma_{\mathbf{x}} \hat{A}^\bullet \left[\partial_- \hat{A}^\star, \hat{A}^\bullet\right]$$

$$\mathcal{L}_{--+}^{(\mathrm{LC})}\left[A^\bullet, A^\star\right] = -2ig' \int d^3\mathbf{x}\, \mathrm{Tr}\, \overline{\gamma}_{\mathbf{x}} \hat{A}^\star \left[\partial_- \hat{A}^\bullet, \hat{A}^\star\right]$$

$$\mathcal{L}_{++--}^{(\mathrm{LC})}\left[A^\bullet, A^\star\right] = -g^2 \int d^3\mathbf{x}\, \mathrm{Tr}\, \left[\partial_- \hat{A}^\bullet, \hat{A}^\star\right] \partial_-^{-2} \left[\partial_- \hat{A}^\star, \hat{A}^\bullet\right]$$

where $\gamma_{\mathbf{x}} = \partial_-^{-1} \partial_\bullet, \quad \overline{\gamma}_{\mathbf{x}} = \partial_-^{-1} \partial_\star.$

The MHV action (1)

Transformation of fields[1]

$$(A^\bullet, A^\star) \to (B^\bullet, B^\star)$$

[1] P. Mansfield, JHEP 03 (2006) 037

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$$\left(A^{\bullet}, A^{\star}\right) \rightarrow \left(B^{\bullet}, B^{\star}\right)$$

1. Transformation is canonical such that $B^{\bullet} = B^{\bullet}\left[A^{\bullet}\right]$

$$\partial_{-} A_a^{\star}\left(\mathbf{x}\right) = \int d^3\mathbf{y}\, \frac{\delta B_c^{\bullet}\left(\mathbf{y}\right)}{\delta A_a^{\bullet}\left(\mathbf{x}\right)} \partial_{-} B_c^{\star}\left(\mathbf{y}\right)$$

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2. The vertex $(+ + -)$ is removed

$$\mathcal{L}_{+-}^{(\mathrm{LC})}[A^{\bullet}, A^{\star}] + \mathcal{L}_{++-}^{(\mathrm{LC})}[A^{\bullet}, A^{\star}] = \mathcal{L}_{+-}^{(\mathrm{LC})}[B^{\bullet}, B^{\star}]$$

$$\boxed{\int d^3\mathbf{y} \, \mathrm{Tr}\left\{ \left[D_{\star}, \gamma_{\mathbf{y}} \hat{A}^{\bullet}(\mathbf{y}) \right] t^c \right\} \frac{\delta B_a^{\bullet}(\mathbf{x})}{\delta A_c^{\bullet}(\mathbf{y})} = \omega_{\mathbf{x}} B_a^{\bullet}(\mathbf{x})}$$

where $\omega_{\mathbf{x}} = \partial_{\bullet}\partial_{\star}\partial_{-}^{-1}$.

[1] P. Mansfield, JHEP 03 (2006) 037

The MHV action (2)

Solution to the transformations in momentum space

$$\tilde{A}^{\bullet}_{a} = \tilde{B}^{\bullet}_{a} + \sum_{n=2}^{\infty} \tilde{\Psi}^{a\{b_1 \dots b_n\}}_{n} \otimes \tilde{B}^{\bullet}_{b_1} \dots \tilde{B}^{\bullet}_{b_n}$$

$$\tilde{A}^{\star}_{a} = \tilde{B}^{\star}_{a} + \sum_{n=2}^{\infty} \tilde{\Omega}^{ab_1\{b_2 \dots b_n\}}_{n} \otimes \tilde{B}^{\star}_{b_1} \tilde{B}^{\bullet}_{b_2} \dots \tilde{B}^{\bullet}_{b_n}$$

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$$\boxed{S_{\mathrm{Y-M}}^{(\mathrm{LC})}\left[\tilde{B}^{\bullet}, \tilde{B}^{\star}\right] = \int dx^{+} \left(\mathcal{L}_{+-}^{(\mathrm{LC})} + \mathcal{L}_{--+}^{(\mathrm{LC})} + \dots + \mathcal{L}_{--+\dots+}^{(\mathrm{LC})} + \dots\right)}$$

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where the MHV vertex is:

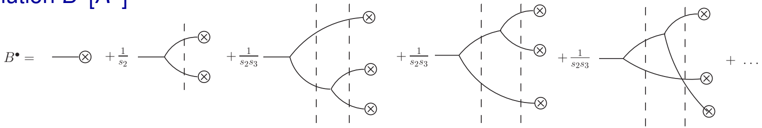
$$\mathcal{L}^{(\mathrm{LC})}_{--+\dots+} = \tilde{\mathcal{V}}^{b_1 \dots b_n}_{--+\dots+} \otimes \tilde{B}^{\star}_{b_1} \tilde{B}^{\star}_{b_2} \tilde{B}^{\bullet}_{b_3} \dots \tilde{B}^{\bullet}_{b_n}$$

$$\tilde{\mathcal{V}}_{--+\dots+}\left(\mathbf{p}_1, \dots, \mathbf{p}_n\right) = \frac{1}{n!} \left(g'\right)^{n-1} \left(\frac{p_1^{+}}{p_2^{+}}\right)^{2} \frac{\tilde{v}^{*4}_{21}}{\tilde{v}^{*}_{1n} \tilde{v}^{*}_{n(n-1)} \tilde{v}^{*}_{(n-1)(n-2)} \dots \tilde{v}^{*}_{21}}$$

with $\tilde{v}_{(i)(j)} = -p_i^{\star} + p_i^{+}\left(p_j^{\star}/p_j^{+}\right) \sim [ij]$, $\tilde{v}^{*}_{(i)(j)} = -p_i^{\bullet} + p_i^{+}\left(p_j^{\bullet}/p_j^{+}\right) \sim \langle ij \rangle$.

The diagrammatic content of transformations (1)

Solution $B^\bullet[A^\bullet]$



- Vertical dashed lines – energy denominators:

$$D_{1\dots i} = 2\left(E_{\text{initial}} - \sum_{j\,\in\,\text{intermediate}} E_j\right), \quad E_p = \frac{p^\star p^\bullet}{p^+}$$

- Triple gluon vertices – helicity $(-++)$.

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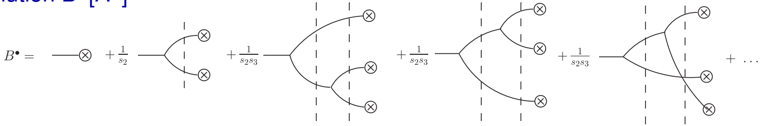

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- Triple gluon vertices – helicity $(- + +)$.

$$\tilde{\Gamma}_n\left(\mathbf{P}; \mathbf{p}_1, \ldots, \mathbf{p}_n\right) = \frac{1}{n!}\left(-g'\right)^{n-1} \frac{1}{\tilde{v}^{*}_{1(1\ldots n)} \tilde{v}^{*}_{(12)(1\ldots n)} \ldots \tilde{v}^{*}_{(1\ldots n-1)(1\ldots n)}} \delta^3\left(\mathbf{p}_{1\ldots n} - \mathbf{P}\right)$$

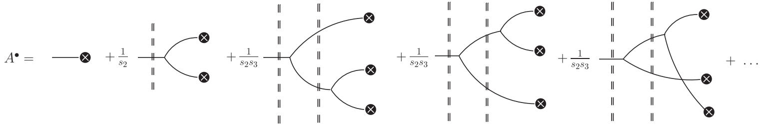
where $p_{1\ldots i} \equiv p_1 + \cdots + p_i$.

$\tilde{\Gamma}_n$ has an interpretation of the gluon wave function[1].

[1] L. Motyka, A. Stasto, Phys.Rev.D 79 (2009) 08016

The diagrammatic content of transformations (2)

Inverse solution $A^{\bullet}[B^{\bullet}]$



- Vertical double-dashed lines – redefined energy denominators:

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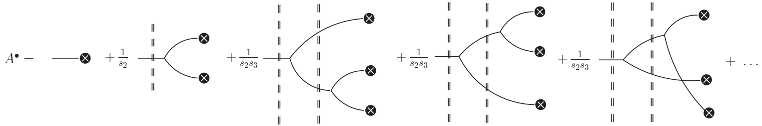

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- Triple gluon vertices – helicity $(-++)$, same as before.

$$\tilde{\Psi}_n\left(\mathbf{P};\mathbf{p}_1,\ldots,\mathbf{p}_n\right) = -\frac{1}{n!}\left(-g'\right)^{n-1}\frac{\tilde{v}^*_{(1\ldots n)1}}{\tilde{v}^*_{1(1\ldots n)}}\frac{1}{\tilde{v}^*_{n(n-1)}\ldots\tilde{v}^*_{32}\,\tilde{v}^*_{21}}\,\delta^3\left(\mathbf{p}_{1\ldots n}-\mathbf{P}\right)$$

$\tilde{\Psi}_n$ has an interpretation of the gluon fragmentation amplitude[1].

[1] L. Motyka, A. Stasto, Phys.Rev.D 79 (2009) 08016

The Wilson line solution

The solution $B^\bullet[A^\bullet]$

Introduce a family of 4-vectors lying on a 2-plane:

$$\varepsilon_\alpha^+ = \varepsilon_\perp^+ - \alpha\eta, \implies (-\alpha, -1, 0) \equiv \mathbf{e}_\alpha$$

If $\alpha = p^\bullet/p^+$ it is a polarization vector for a momentum p.

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If $\alpha = p^\bullet/p^+$ it is a polarization vector for a momentum p.

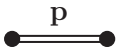
The solution can be expressed through the Wilson line along ε_α^+ integrated over all 'slopes' α (in $A^+ = 0$ gauge):

$$\boxed{B_a^\bullet(\mathbf{x}) = \int_{-\infty}^{\infty} d\alpha\, \mathrm{Tr}\left\{\frac{1}{2\pi i g'} t^a \partial_- \mathbb{P}\exp\left[ig' \int_{-\infty}^{\infty} ds\, \hat{A}^\bullet(\mathbf{x} + s\mathbf{e}_\alpha)\right]\right\}}$$

Diagrammatically (in momentum space):

$ig'\hat{\tilde{B}}^\bullet =$ 

where

 $= \dfrac{i}{\mathbf{e}_\alpha\cdot\mathbf{p}+i\epsilon}$  $= ig'\hat{A}_b^\bullet(\mathbf{p})$

Summary and outlook

Summary

- The Yang-Mills Lagrangian can be canonically transformed to the Lagrangian which contains MHV vertices.
- Such Lagrangian is 'saturated' with fields constructed from Wilson lines extending on a special 2-plane.

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- Such Lagrangian is 'saturated' with fields constructed from Wilson lines extending on a special 2-plane.

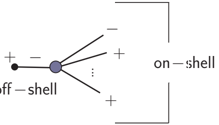
Consequences and open questions

- Some on-shell currents are gauge invariant, in particular the one for $g^{+*} \to g^{-}g^{+} \ldots g^{+}$ (the $\tilde{J}$ current discussed in Anna's talk)
- An inverse functional to the above family of Wilson lines gives the generating functional to the asymptotic solutions of the self-dual Yang-Mills equation (subject related to the integrability of 2D models)
- The MHV Lagrangian vs Lipatov's effective action for high energy QCD

BACKUP

Gauge invariant off-shell currents (1)

Partially reduced Green's function

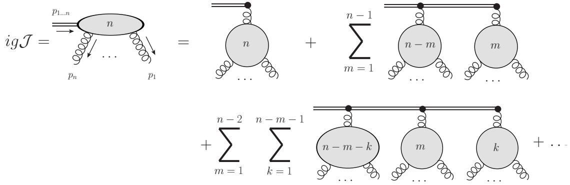


Matrix element of the Wilson line

At tree-level the on-shell fields B can be replaced by A.

$$\mathcal{J}_n\left(p_{1\ldots n}\right) = \int d^4x\, e^{ix\cdot p_{1\ldots n}} \langle 0|\, B_a^{\bullet}\left[A^{\bullet}\right](x)\, |p_1, +; p_2, -; \ldots; p_n, -\rangle$$

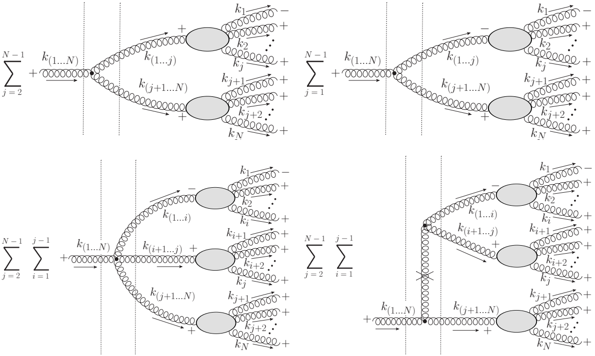
where $|p_i, \pm\rangle$ is on-shell gluon state.



$\mathcal{J}$ satisfies the Ward identities.

Gauge invariant off-shell currents (2)

Light-front recurrence relation for off-shell MHV current[1,2,3]



$$J_n^{(--+\cdots+)}\left(p_{1\ldots n}\right) = \mathcal{J}_n\left(p_{1\ldots n}\right) - ig' \sum_{j=2}^{n-1} \mathcal{J}_j\left(p_{1\ldots j}\right) \frac{p^+_{1\ldots n}}{p^+_{j+1\ldots n}\,\tilde{v}^*_{(1\ldots j)(j+1)}} J_{n-j}^{(-+\cdots+)}\left(p_{j+1\ldots n}\right)$$

[1] C. Cruz-Santiago and A. Stasto, Nucl.Phys.B 875 (2013) 368-387
[2] C. Cruz-Santiago, P. Kotko, A. Stasto, Nucl.Phys. B895 (2015) 132-160
[3] P. Kotko, M. Serino, A. Stasto, JHEP 1608 (2016) 026

Inverse transformation in position space

Inverse to the path-ordered exponential

$$\boxed{A^{\bullet}_{a} = \mathrm{Tr}\left\{\frac{i}{g'}\, t^{a} \partial_{-} \mathcal{U}\left[\frac{g'}{2\pi}\hat{B}^{\bullet}\right]\right\}}$$

$$\boxed{\mathcal{U}\left[\hat{\phi}\right] = \sum_{n=1}^{\infty} \int ds_1 d\alpha_1\, \hat{\phi}\left(\mathbf{x}+s_1 \mathbf{e}_{\alpha_1}\right) \prod_{i=2}^{n} \int ds_i d\alpha_i \int_{-\infty}^{0} d\tau_{i-1} \partial_{-} \hat{\phi}\left(\mathbf{x}+\tau_{i-1}\mathbf{e}_{\alpha_{i-1}}+s_i \mathbf{e}_{\alpha_i}\right)}$$

where $\mathbf{e}_{\alpha} = (-\alpha, -1, 0)$ [recall $\mathbf{x} \equiv (x^{-}, x^{\bullet}, x^{\star})$].

The n-th term in the expansion:

$$\mathcal{U}\left[\hat{\phi}\right]^{(n)} = \int ds_1 \ldots ds_n \int d\alpha_1 \ldots d\alpha_n \int_{-\infty}^{0} d\tau_1 \ldots d\tau_{n-1}$$
$$\hat{\phi}\left(\mathbf{x}+s_1\mathbf{e}_{\alpha_1}\right) \partial_{-}\hat{\phi}\left(\mathbf{x}+\tau_1\mathbf{e}_{\alpha_1}+s_2\mathbf{e}_{\alpha_2}\right) \partial_{-}\hat{\phi}\left(\mathbf{x}+\tau_2\mathbf{e}_{\alpha_2}+s_3\mathbf{e}_{\alpha_3}\right) \ldots$$

Geometric interpretation (1)

Define a new object

$$\boxed{\mathfrak{p}_\alpha\left(\tau,\alpha'\right)=\partial_-\int_{-\infty}^{+\infty} ds\,\phi\left(\mathbf{x}+\tau\mathbf{e}_{\alpha'}+s\mathbf{e}_\alpha\right)}$$

- The Wilson line $B^\bullet\left[A^\bullet\right]$ can be expressed in terms of $\mathfrak{p}_\alpha$.
  Set $\phi=\hat{A}^\bullet$. The n-th term in expansion:

$$\int d\alpha\int d\alpha_1\ldots d\alpha_n\int_{-\infty}^{+\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2\ldots\int_{-\infty}^{\tau_{n-1}}d\tau_n\;\underbrace{\hat{\mathfrak{p}}_{\alpha_1}\left(\tau_1,\alpha\right)\hat{\mathfrak{p}}_{\alpha_2}\left(\tau_2,\alpha\right)\ldots\hat{\mathfrak{p}}_{\alpha_n}\left(\tau_n,\alpha\right)}_{\ell_\alpha^{(n)}}$$

- The inverse functional $A^\bullet\left[B^\bullet\right]$
  Set $\phi=\hat{B}^\bullet$. The n-th term in expansion:

$$\int d\alpha\int d\alpha_1\ldots d\alpha_n\int_{-\infty}^{+\infty}d\tau_1\int_{-\infty}^{0}d\tau_2\ldots\int_{-\infty}^{0}d\tau_n\;\underbrace{\hat{\mathfrak{p}}_{\alpha_1}\left(\tau_1,\alpha\right)\hat{\mathfrak{p}}_{\alpha_2}\left(\tau_2,\alpha_1\right)\hat{\mathfrak{p}}_{\alpha_3}\left(\tau_3,\alpha_2\right)\ldots}_{\mathfrak{L}_\alpha^{(n)}}$$

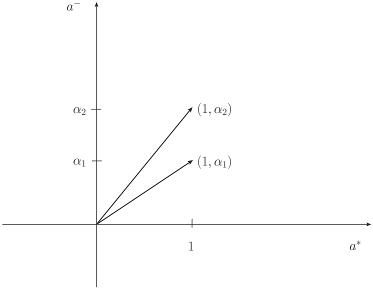
Geometric interpretation (2)

Vector field in the 2D space

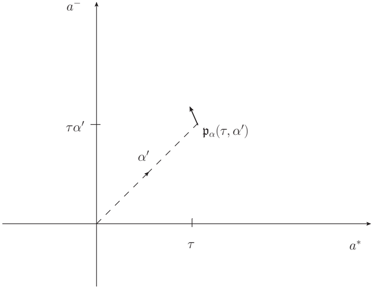
Consider 2D space made of points $(a^\star, a^-)\,,\ a^\star, a^- \in \mathbb{R}$.

In that space

$$\mathbf{e}_\alpha \to (1, \alpha)$$

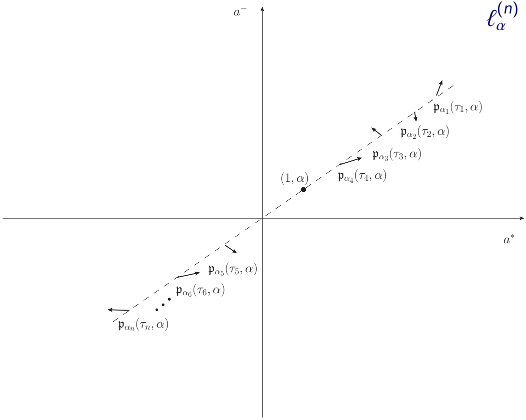


The object $\mathfrak{p}_\alpha\,(\tau, \alpha')$ can be thought of as a vector attached in a point $(\tau, \tau\alpha')$ and having a direction given by α.



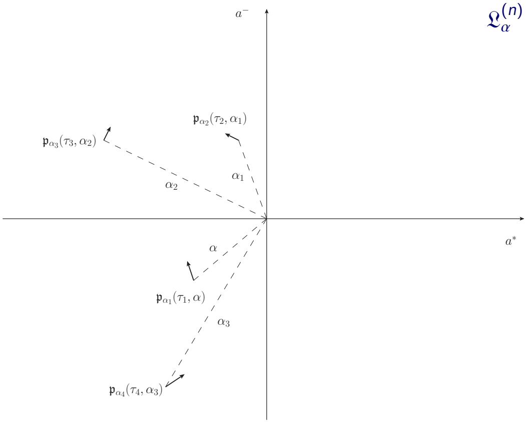
Geometric interpretation (3)

Objects $\ell_\alpha^{(n)}$ and $\mathfrak{L}_\alpha^{(n)}$ in 2D space



Geometric interpretation (3)

Objects $\ell_\alpha^{(n)}$ and $\mathfrak{L}_\alpha^{(n)}$ in 2D space



Geometric interpretation (3)

Objects $\ell_\alpha^{(n)}$ and $\mathfrak{L}_\alpha^{(n)}$ in 2D space