Lattice Field theory in real time (and finite density)

Paulo Bedaque
University of Maryland

A. Alexandru, G. Basar, S. Lawrence, H. Lamm, G. Ridgway, N. Warrington

Non-perturbative definition/evaluation

$$\langle \mathcal{O}(t)\mathcal{O}(t')\rangle = \frac{\int D\phi \; \mathcal{O}(t)\mathcal{O}(t')e^{iS[\phi]}}{\int D\phi \; e^{iS[\phi]}}$$

Not good for numerical evaluation

Non-perturbative definition/evaluation

$$\langle \mathcal{O}_\tau \mathcal{O}_{\tau'} \rangle = \frac{\int D\phi \; \mathcal{O}_\tau[\phi] \mathcal{O}_{\tau'}[\phi] \; e^{-S_E[\phi]}}{\int D\phi \; e^{-S_E[\phi]}}$$

imaginary time

Non-perturbative definition/evaluation

$$\langle \mathcal{O}_\tau \mathcal{O}_{\tau'} \rangle = \frac{\int D\phi \; \mathcal{O}_\tau[\phi]\mathcal{O}_{\tau'}[\phi] \; e^{-S_E[\phi]}}{\int D\phi \; e^{-S_E[\phi]}}$$

$P[\phi]$

imaginary time

$$\approx \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \mathcal{O}_\tau[\phi]\mathcal{O}_{\tau'}[\phi]$$

Good for numerical evaluation if S_E is real

From euclidean to Minkowski

one point function: $\langle \mathcal{O}_t \rangle = \langle \mathcal{O}_\tau \rangle$

two point function: $\langle \mathcal{O}_\tau \mathcal{O}_{\tau'} \rangle \quad \langle \mathcal{O}_t \mathcal{O}_{t'} \rangle$

related by
analytic continuation
(numerically unstable)

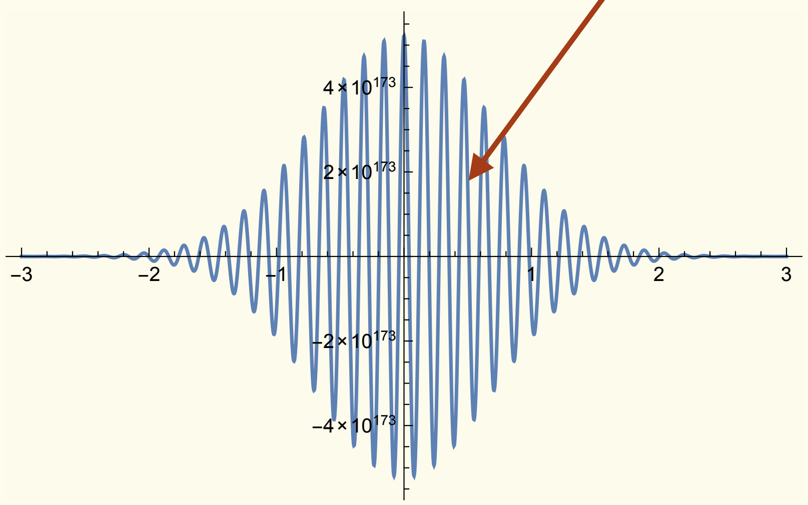
This approach fails if:

- S_E is not real (finite density, …)

- real time observable (transport coefficients, parton distribution functions, …)

Central idea: deform the contour into the complex plane:

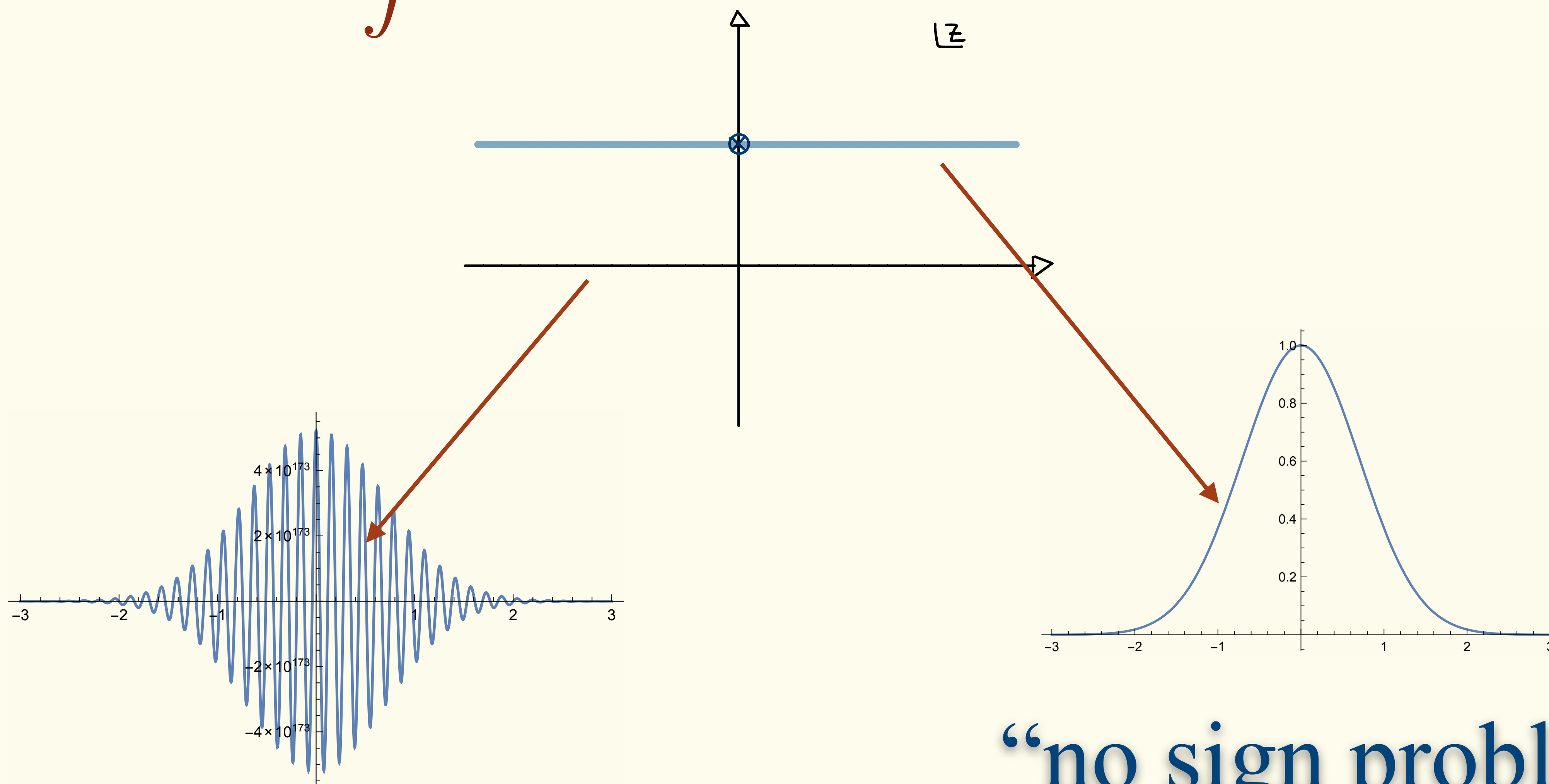
$$\int dx e^{-(z-i20)^2} = \sqrt{\pi}$$



“sign problem”

Central idea: deform the contour into the complex plane:

$$\int dx e^{-(z-i20)^2} = \sqrt{\pi}$$

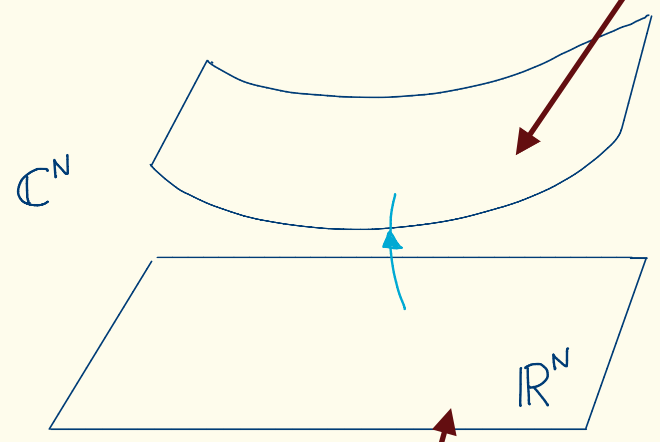


“no sign problem”

Cristoforetti, DiRenzo, Scorzato, ‘12

How to find good deformations ?

new integration manifold



(real) field space

$$\frac{d\phi_i}{dt} = \overline{\frac{\partial S}{\partial \phi_i}} \Rightarrow$$

$$\frac{d\phi_i^R}{dt} = \frac{\partial S^R}{\partial \phi_i^R} = \frac{\partial S^I}{\partial \phi_i^I}$$

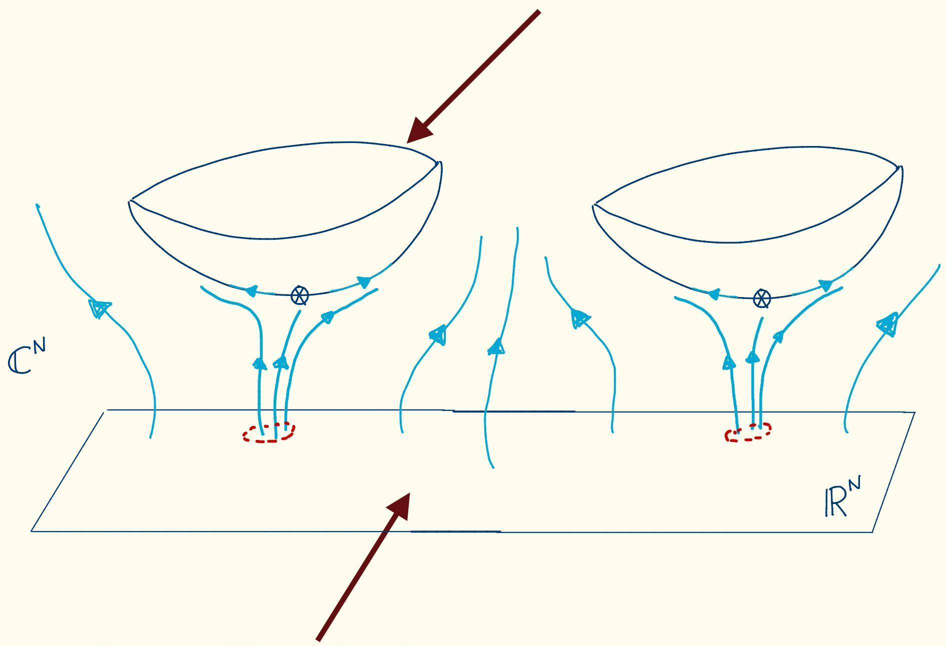
$$\frac{d\phi_i^I}{dt} = \frac{\partial S^R}{\partial \phi_i^I} = -\frac{\partial S^I}{\partial \phi_i^R}$$

gradient flow of S^R, keeps integral well defined

hamiltonian flow of S^I, keeps phase fixed

How to find good deformations ?

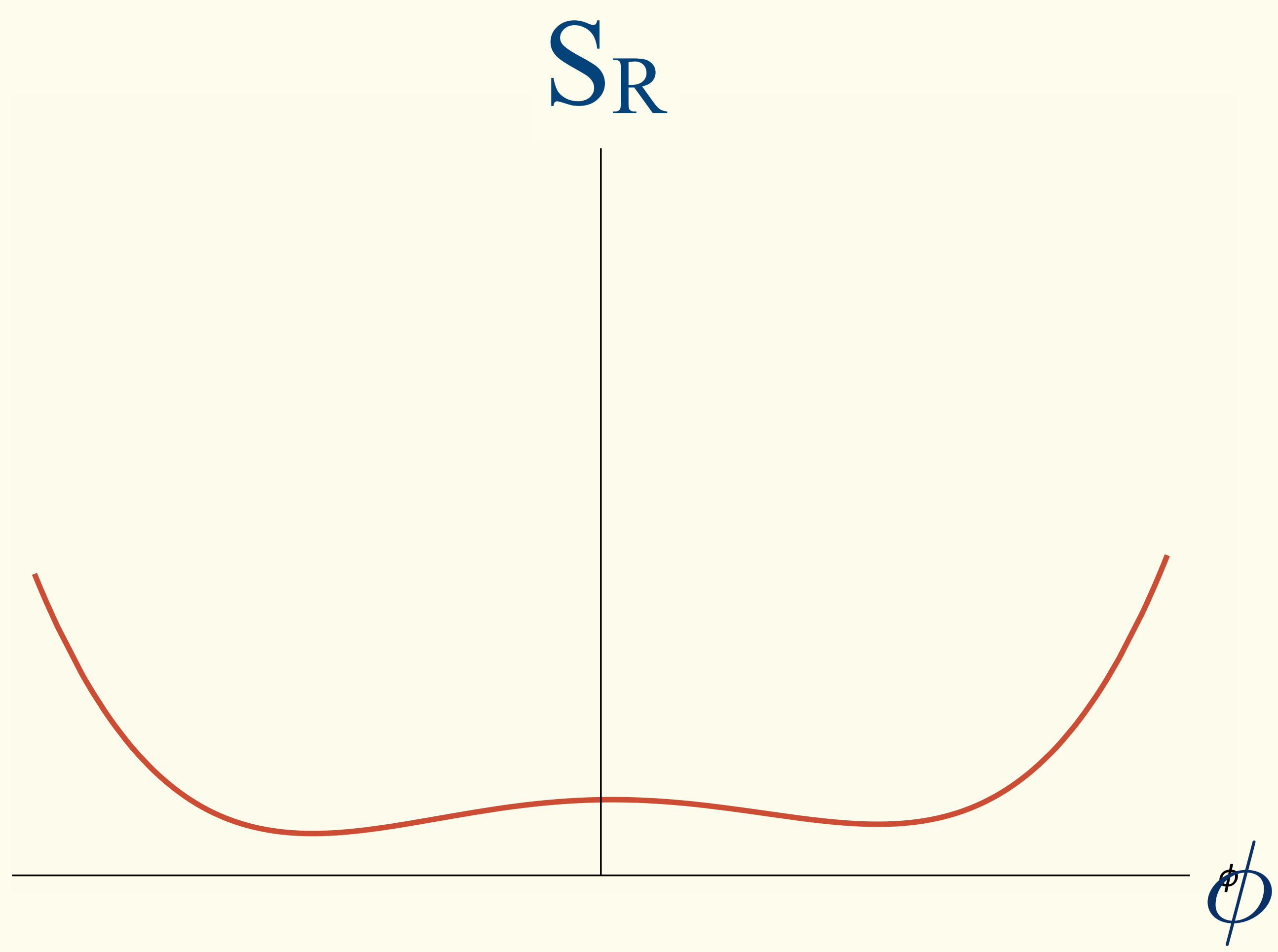
new integration manifold



(real) field space

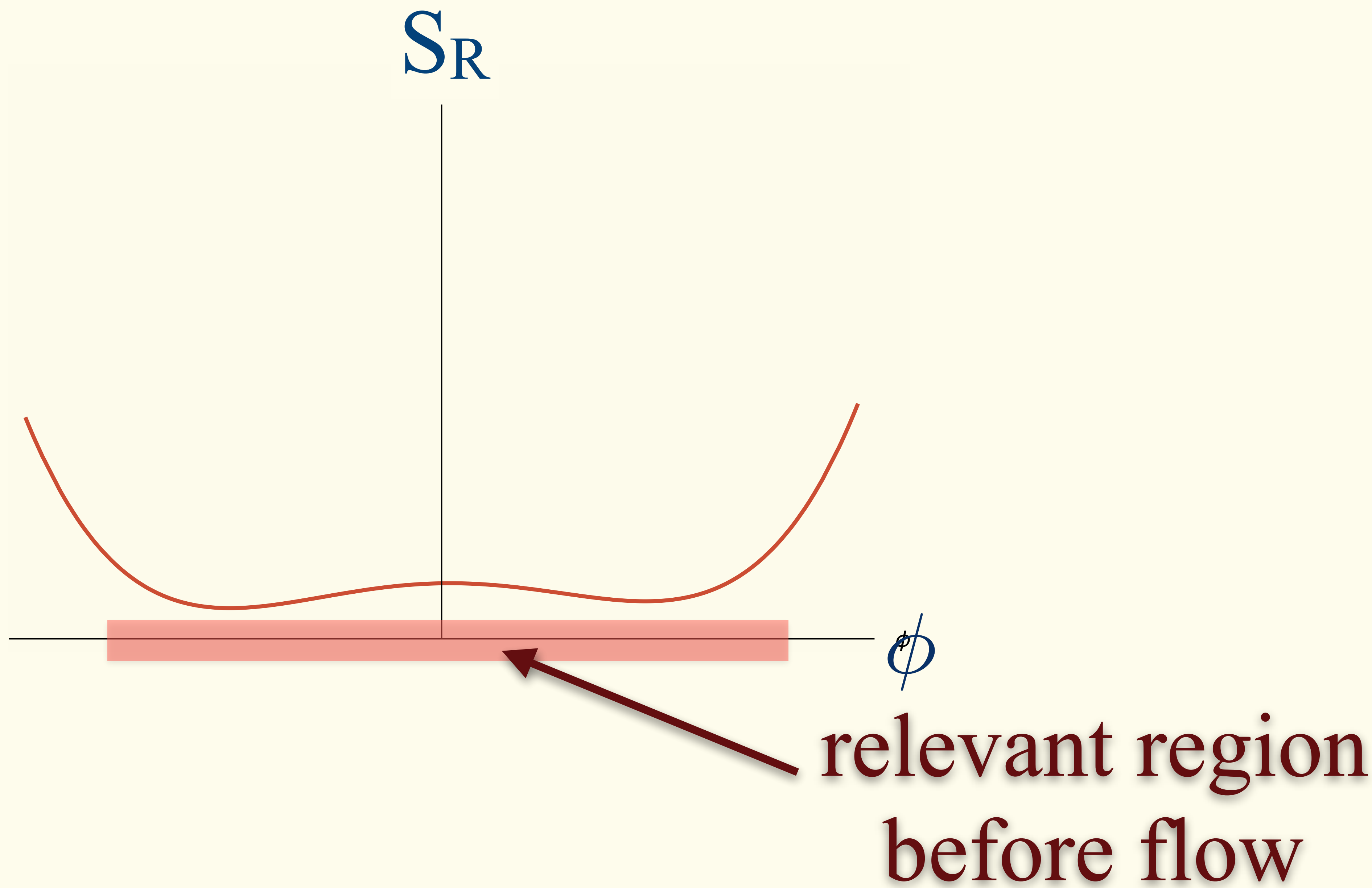
The flow approaches the "thimbles" (multidimensional stationary phase/steepest descent manifolds)

How to find good deformations ?



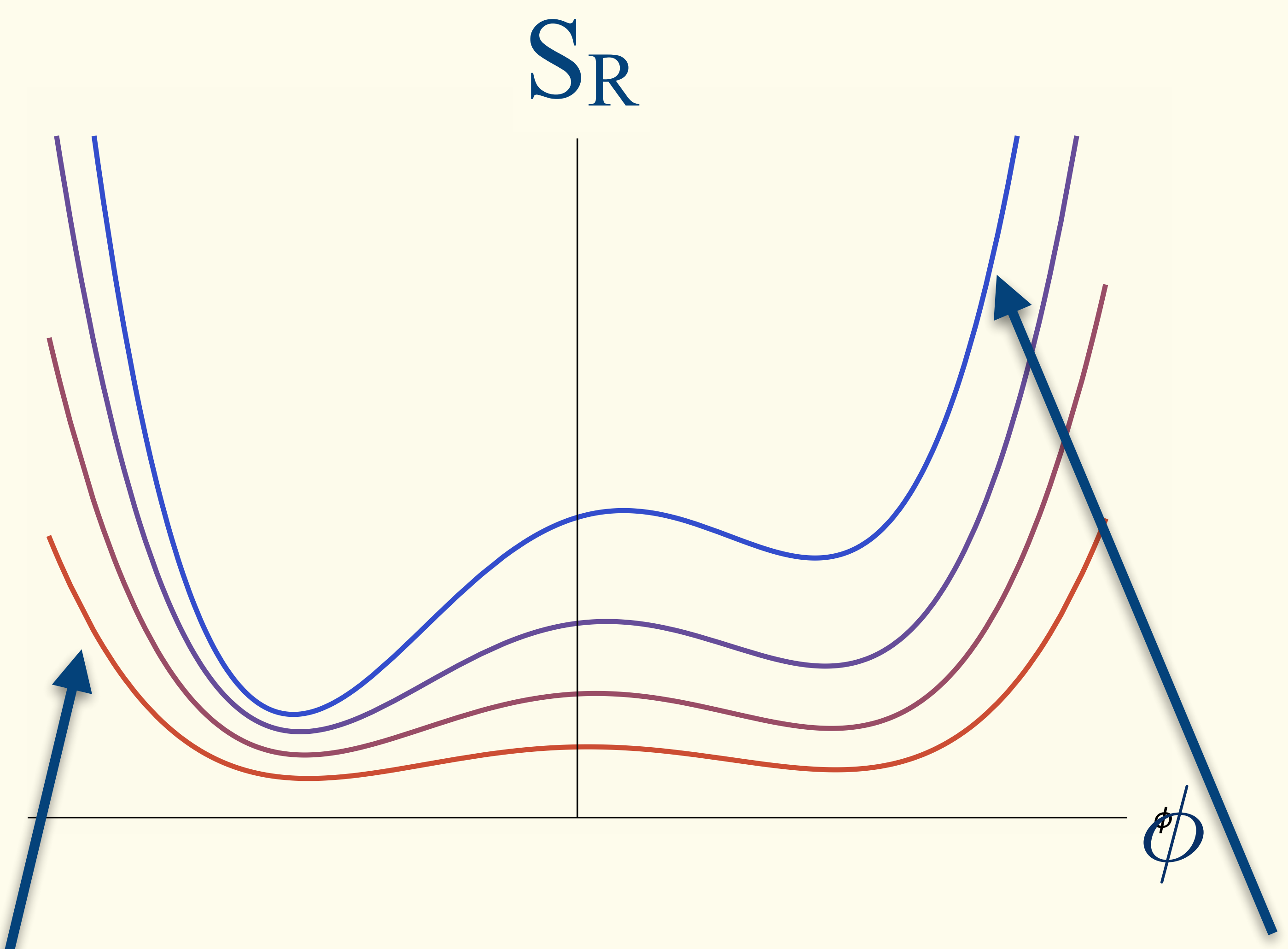
S_R under the flow

How to find good deformations ?



S_R grows under the flow

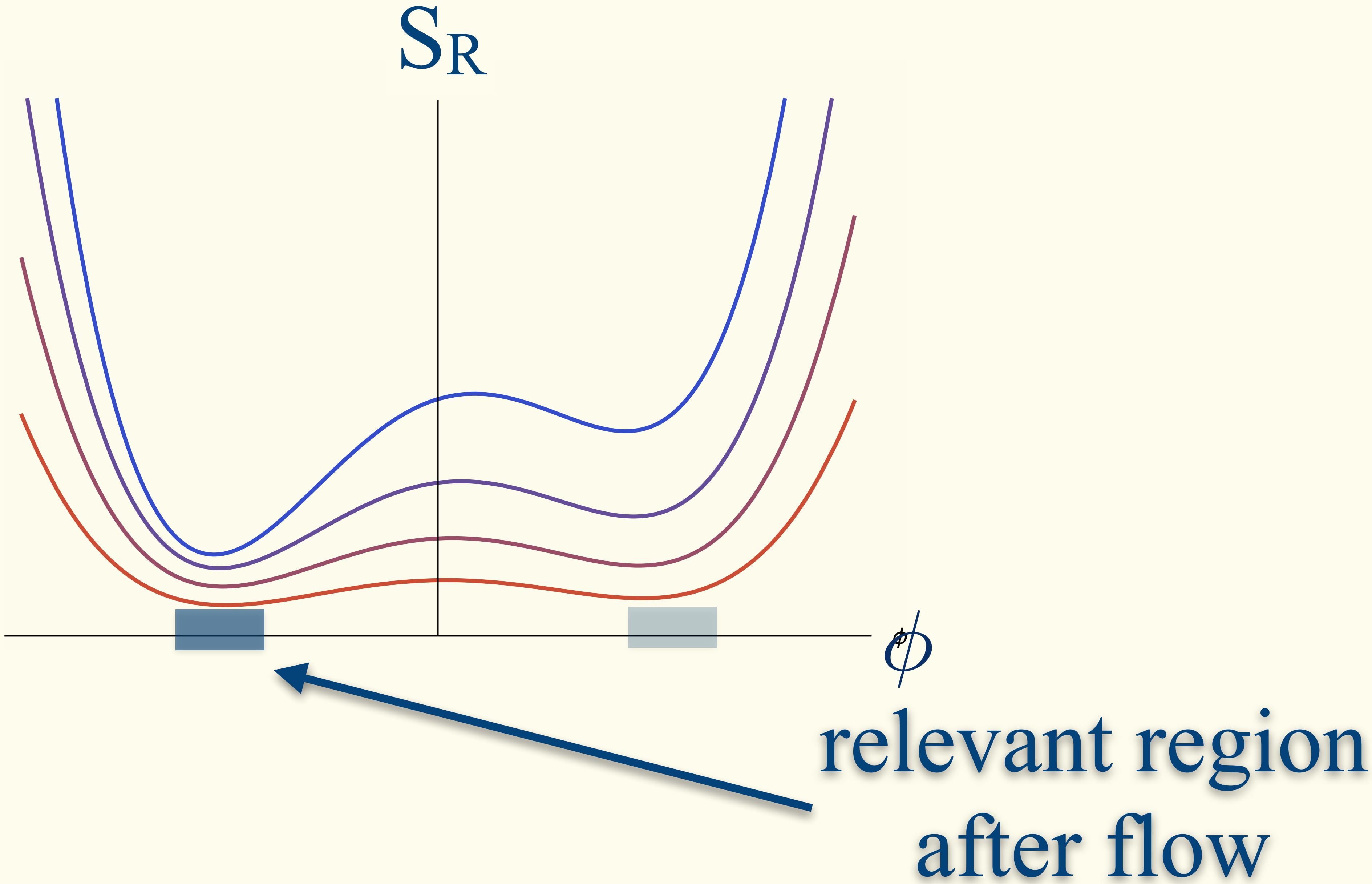
How to find good deformations ?



if this integral exists, so does that

1) homology class preserved by the flow

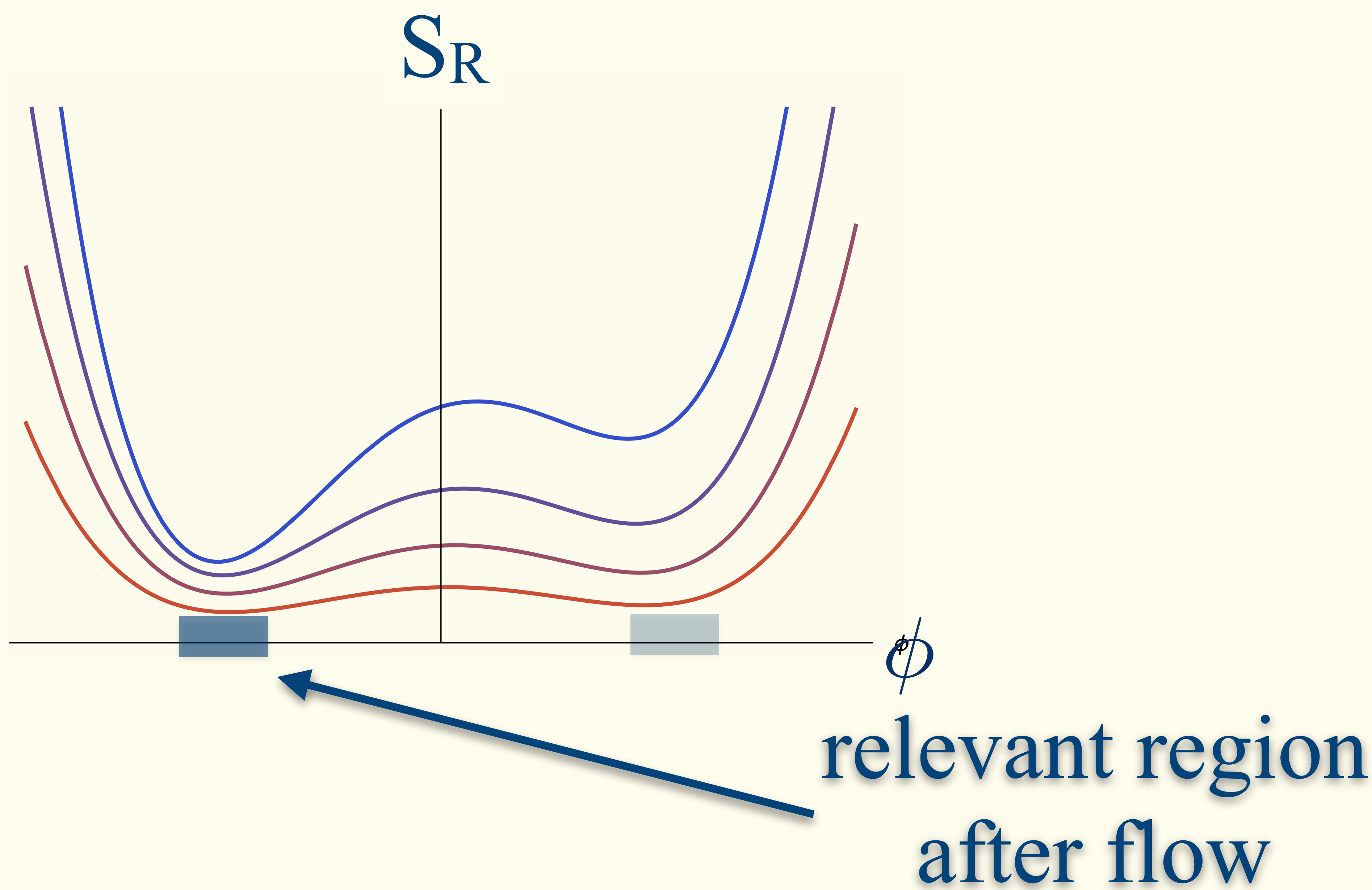
How to find good deformations ?



S_R grows under the flow

S_I stays constant

How to find good deformations ?

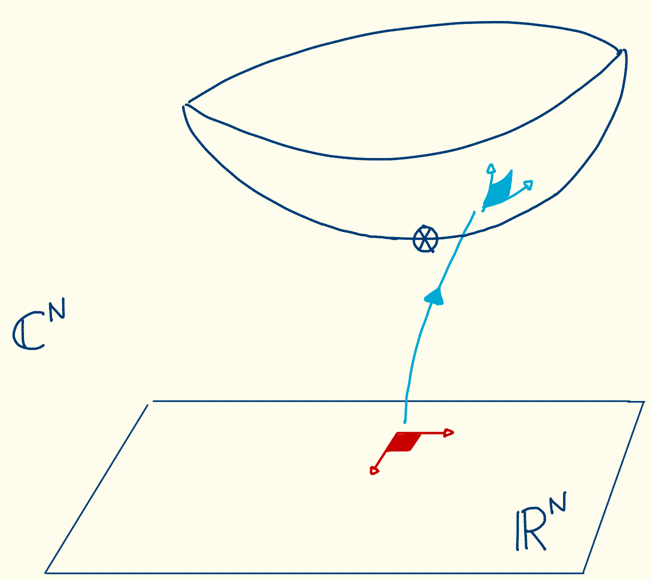


2) sign fluctuations are reduced

The algorithm

$$\langle \mathcal{O} \rangle = \frac{\int d\phi_i \; \mathcal{O} e^{-S_R - iS_I}}{\int d\phi_i \; e^{-S_R - iS_I}} = \frac{\int d\tilde{\phi}_i \; \overbrace{\det\left(\frac{\partial \phi_i}{\partial \tilde{\phi}_i}\right)}^{J} \mathcal{O} e^{-S_R - iS_I}}{\int d\tilde{\phi}_i \; \det\left(\frac{\partial \phi_i}{\partial \tilde{\phi}_i}\right) e^{-S_R - iS_I}}$$

$$= \frac{\int d\tilde{\phi}_i \; \mathcal{O} e^{-iS_I + i\mathrm{Im}J} e^{-\overbrace{(S_R - \mathrm{Re}J)}^{S_{eff}}}}{\int d\tilde{\phi}_i \; e^{-iS_I + i\mathrm{Im}J} e^{-(S_R - \mathrm{Re}J)}} = \frac{\langle \mathcal{O} e^{-iS_I + i\mathrm{Im}J} \rangle_{S_{eff}}}{\langle e^{-iS_I + i\mathrm{Im}J} \rangle_{S_{eff}}}$$



$$\frac{dJ_{ij}}{dt} = \overline{\frac{\partial^2 S}{\partial z_i \partial z_k} J_{jk}}$$

$$J_{ij}(0) = \mathbb{I}$$

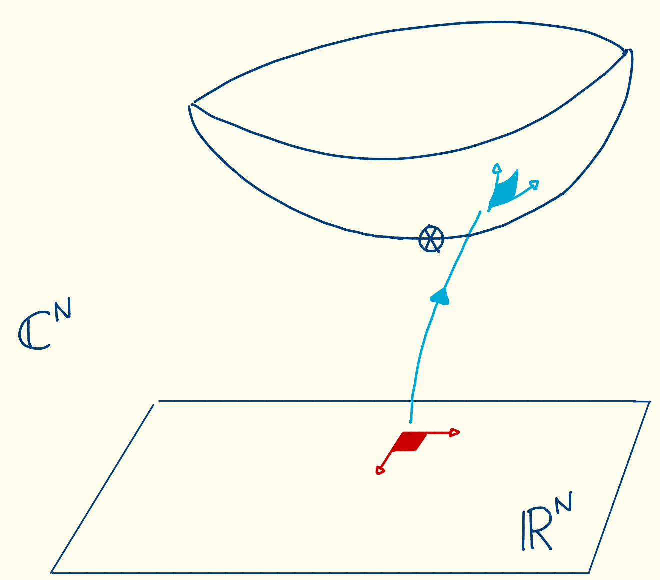
$$J = \det J(T)$$

this is the expensive part

The algorithm

$$\langle \mathcal{O} \rangle = \frac{\int d\phi_i \; \mathcal{O} e^{-S_R - iS_I}}{\int d\phi_i \; e^{-S_R - iS_I}} = \frac{\int d\tilde{\phi}_i \; \overbrace{\det\left(\frac{\partial \phi_i}{\partial \tilde{\phi}_i}\right)}^{J} \mathcal{O} e^{-S_R - iS_I}}{\int d\tilde{\phi}_i \; \det\left(\frac{\partial \phi_i}{\partial \tilde{\phi}_i}\right) e^{-S_R - iS_I}}$$

$$= \frac{\int d\tilde{\phi}_i \; \mathcal{O} e^{-iS_I + i\mathrm{Im}J} e^{-\overbrace{(S_R - \mathrm{Re}J)}^{S_{eff}}}}{\int d\tilde{\phi}_i \; e^{-iS_I + i\mathrm{Im}J} e^{-(S_R - \mathrm{Re}J)}} = \frac{\langle \mathcal{O} e^{-iS_I + i\mathrm{Im}J} \rangle_{S_{eff}}}{\langle e^{-iS_I + i\mathrm{Im}J} \rangle_{S_{eff}}}$$



our algorithm

=

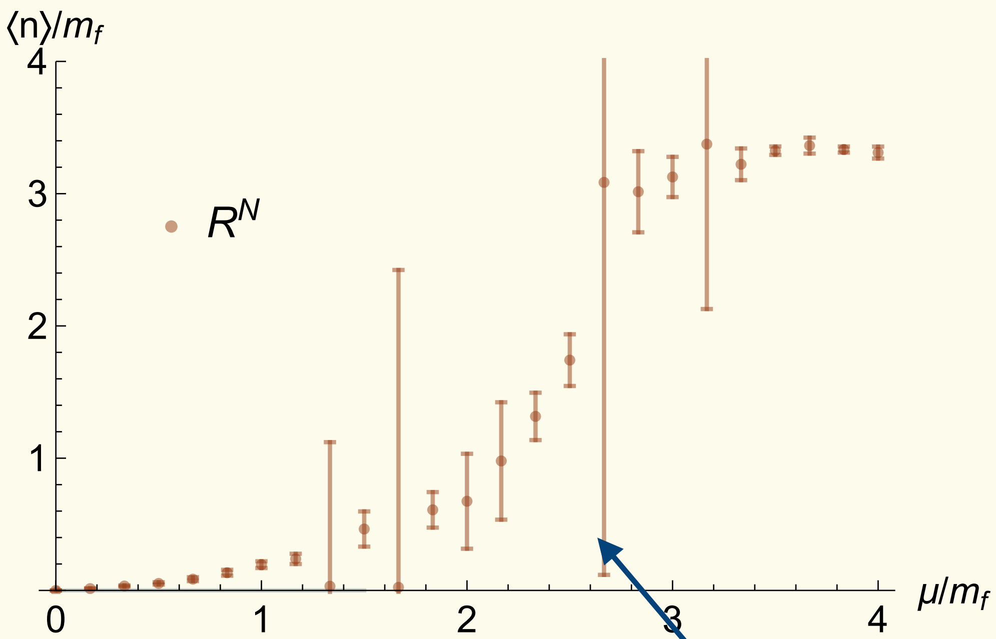
Metropolis in the real space,

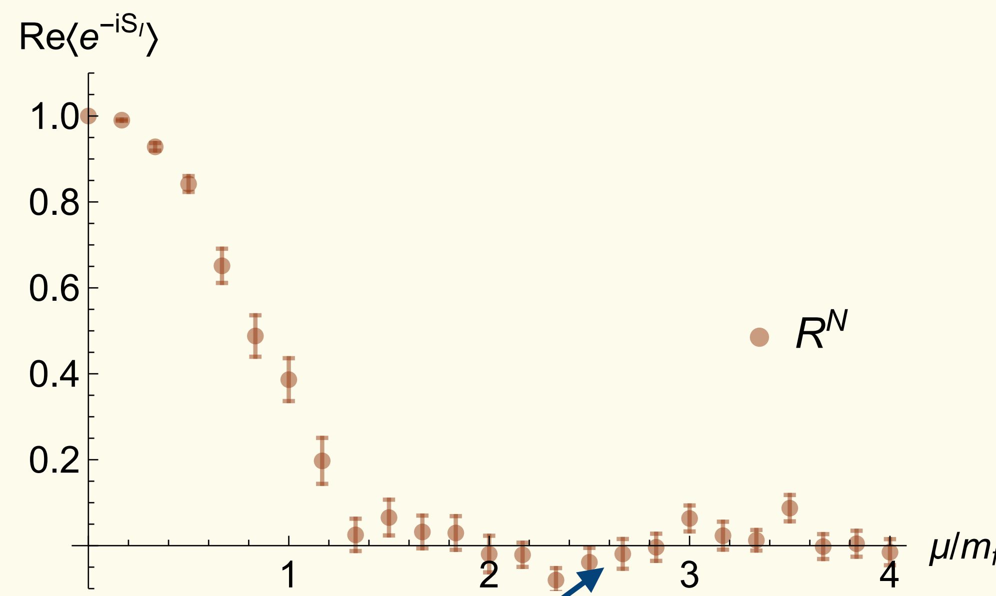
action S_{eff} and

reweighted phase $e^{i\,\mathrm{Im}(\ln J) - i\,\mathrm{Im}(S)}$

Case study: massive Thirring model

Wilson, 10 x10 lattice, N_F=2, am_f=0.3

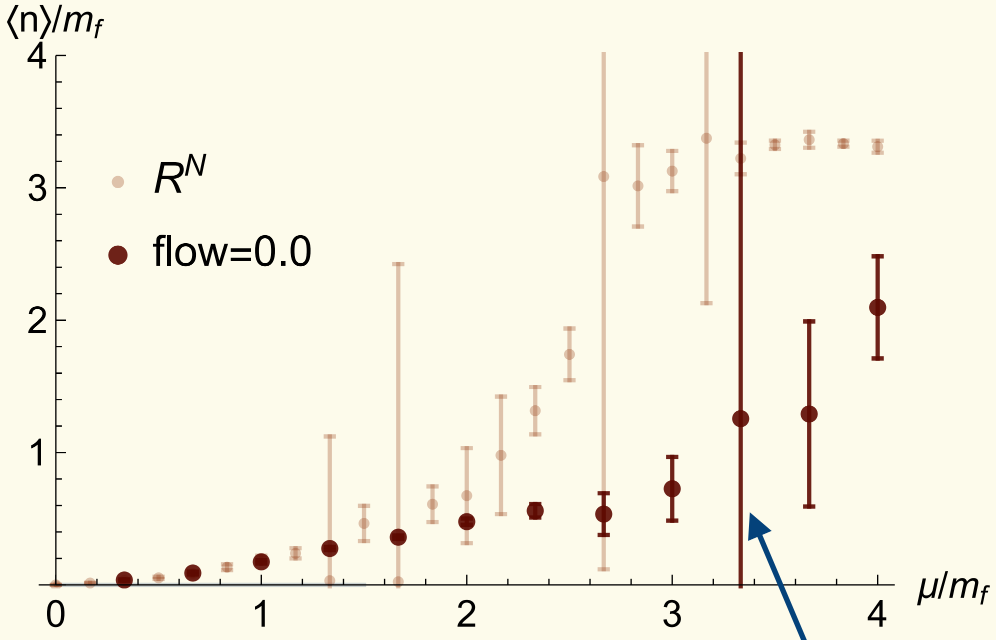


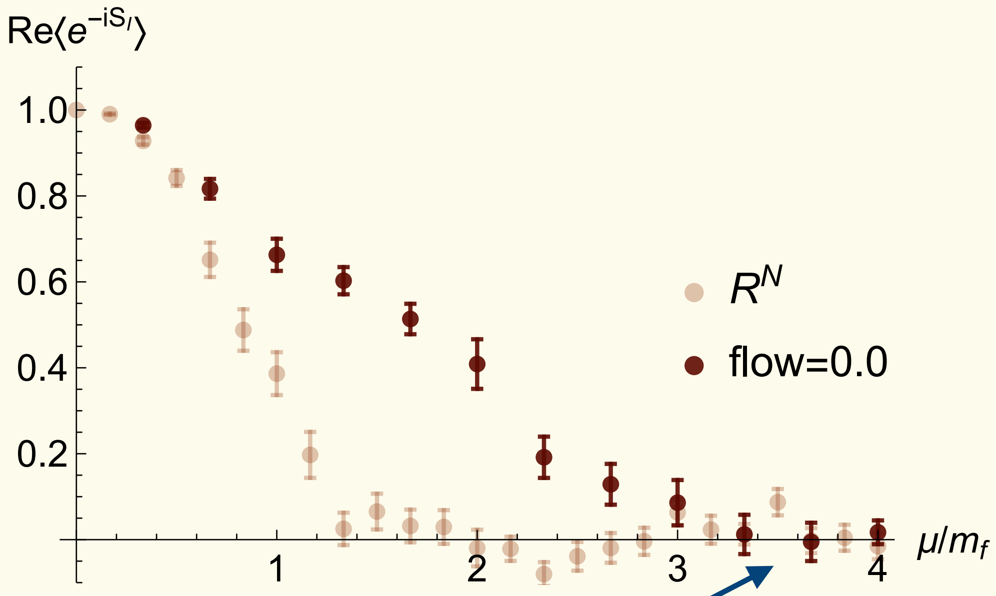


sign problem

Case study: massive Thirring model

Wilson, 10 x10 lattice, $N_F=2$, $am_f=0.3$

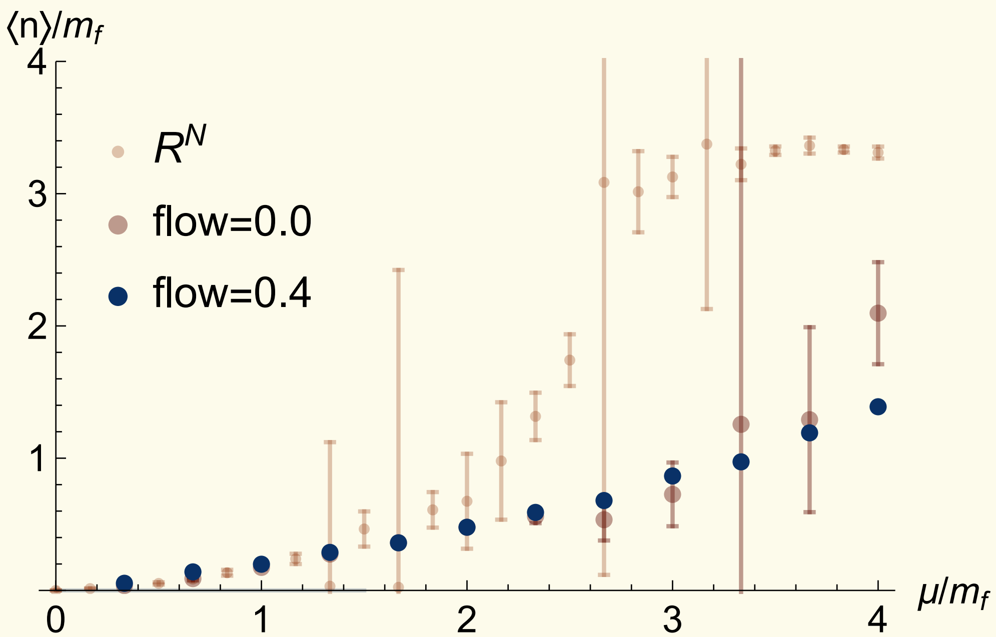


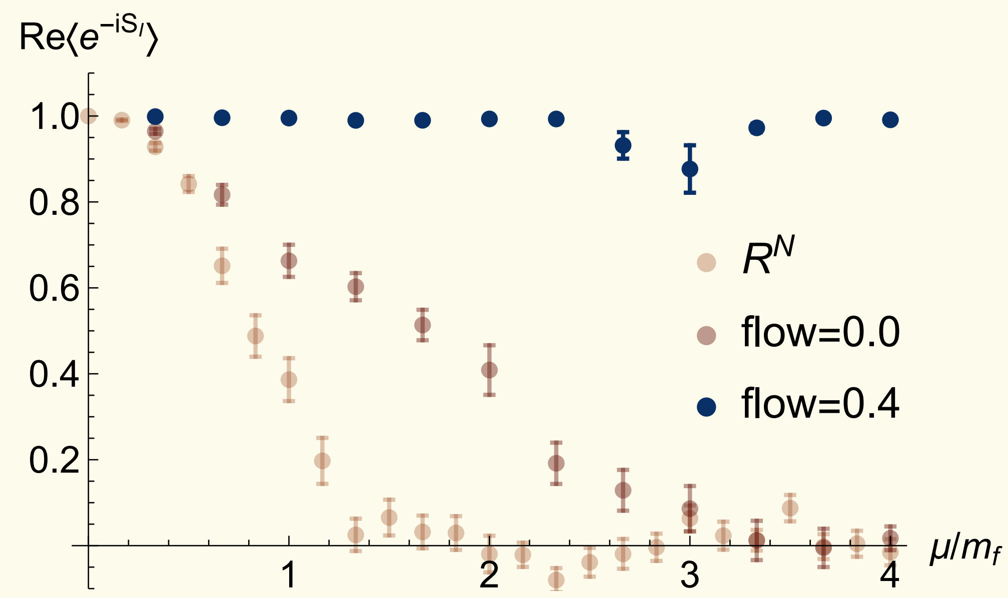


sign problem improved

Case study: massive Thirring model

Wilson, 10 x10 lattice, $N_F=2$, $am_f=0.3$

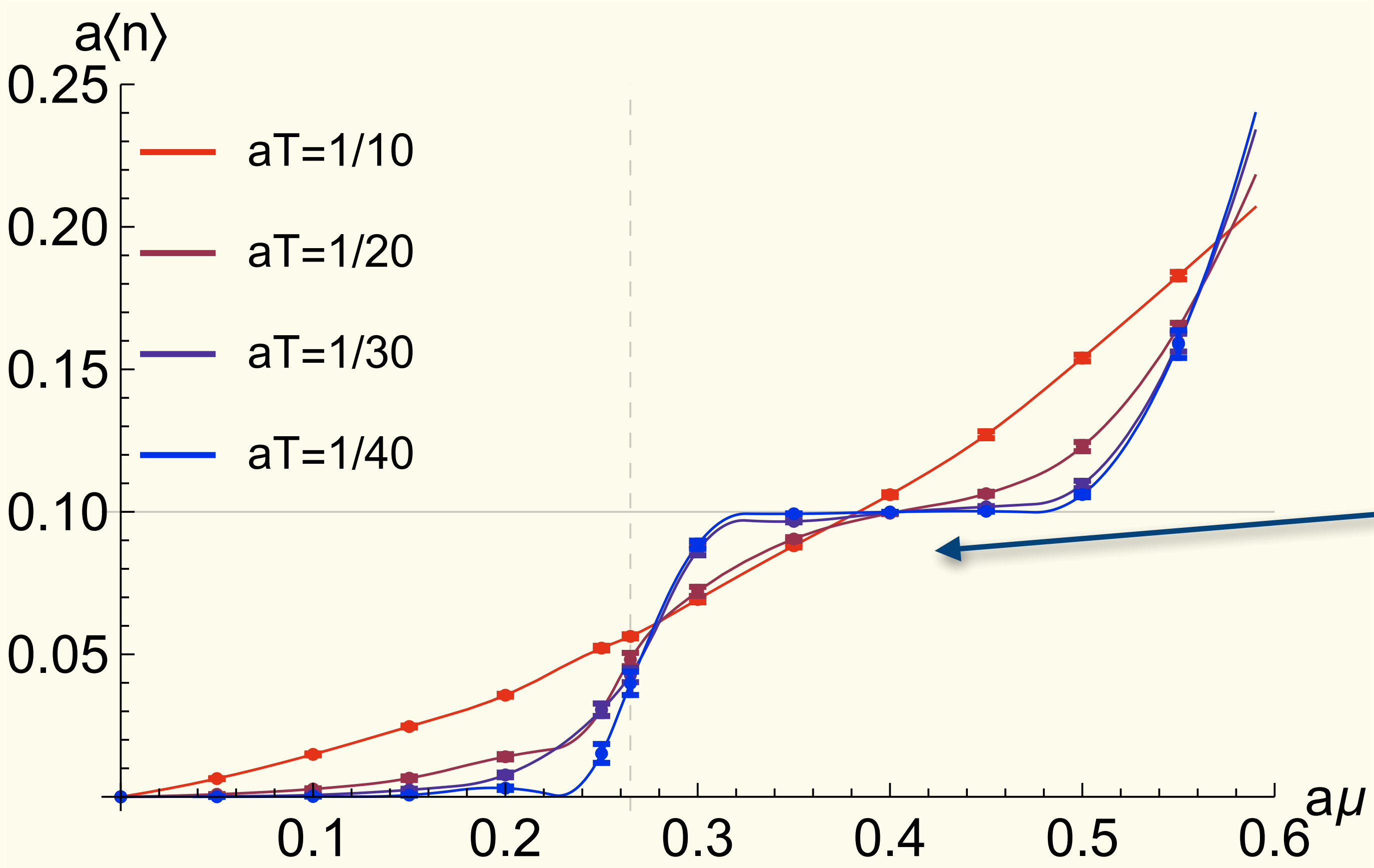




flow done with estimators for the jacobian
(difference reweighted)
no sign problem

Case study: massive Thirring model

Staggered, $N_F=2$, $am_f=0.265$

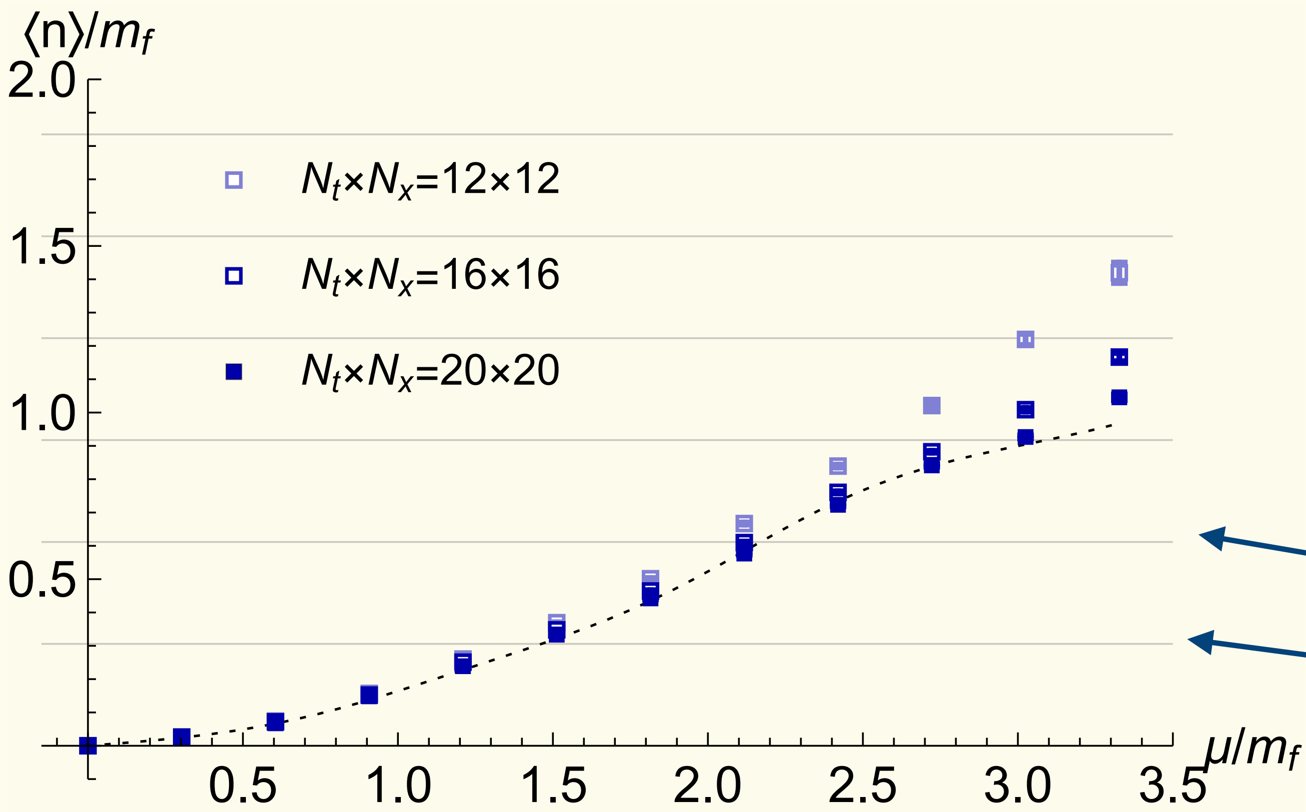


step is missed in a one thimble calculation

cold limit

Case study: massive Thirring model

Staggered, $N_F=2$, $T/m_f=0.302$, $m_fL=3.31$

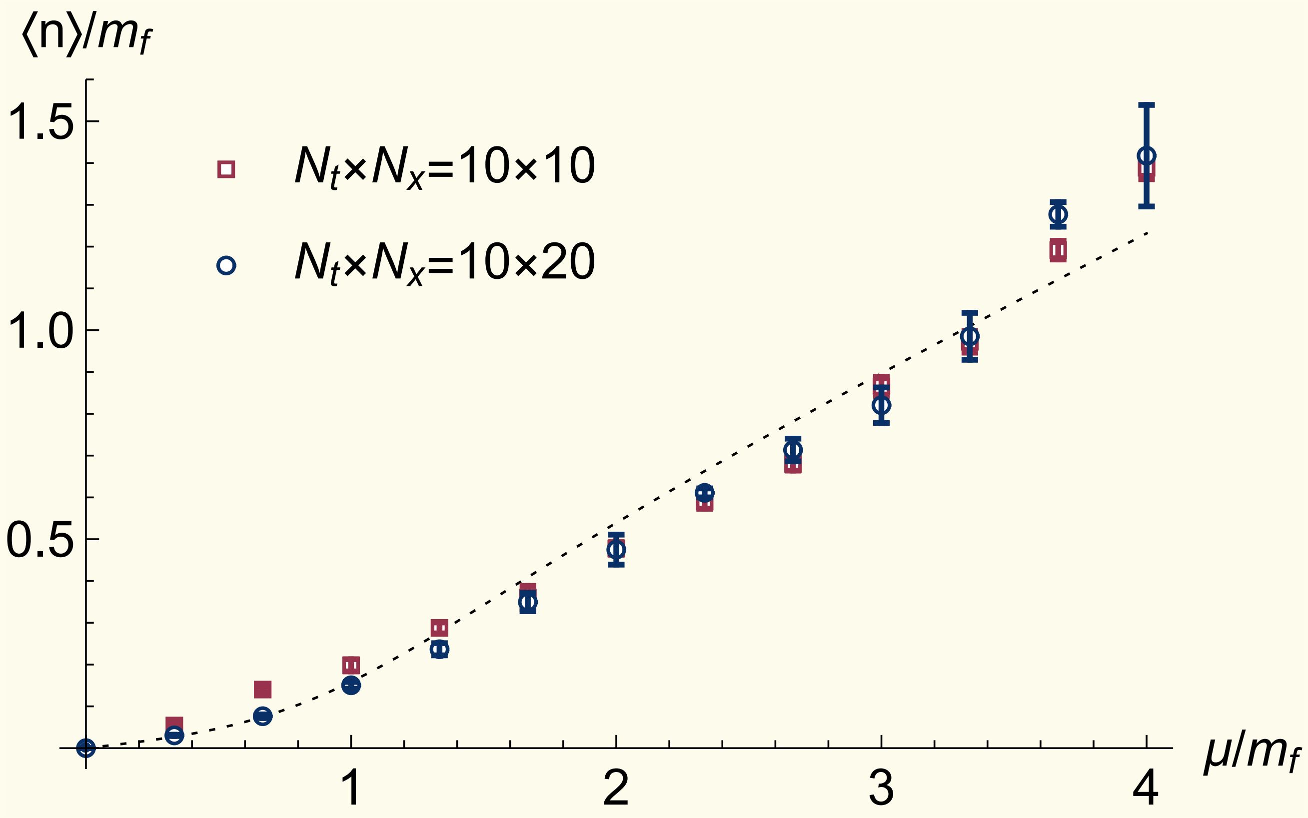


integer particle number

continuum limit

Case study: massive Thirring model

Staggered, N_F=2, T/m_f=0.302

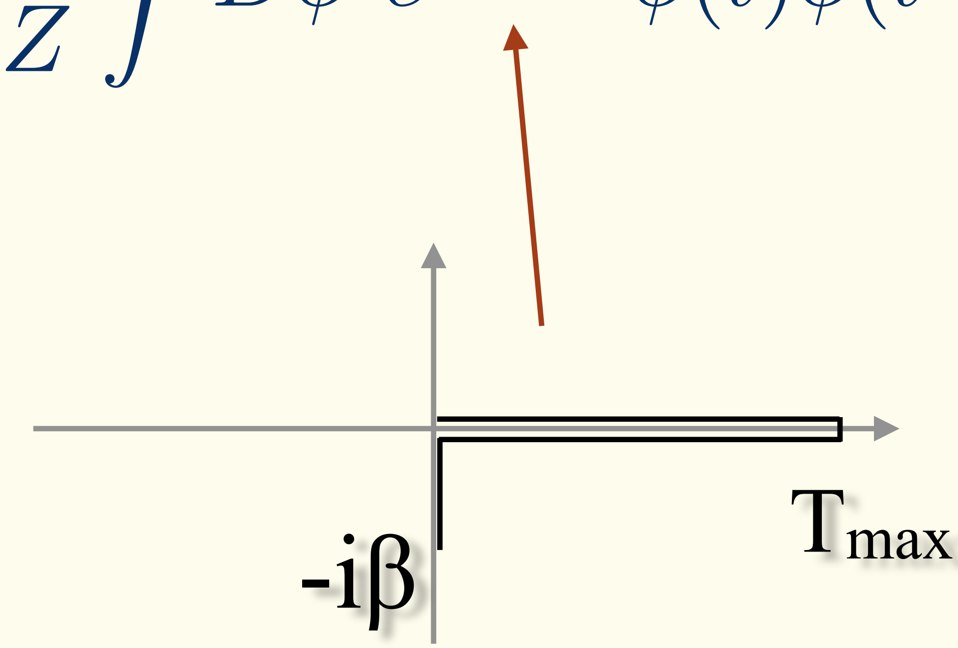


thermodynamic limit

Application: Real Time Dynamics

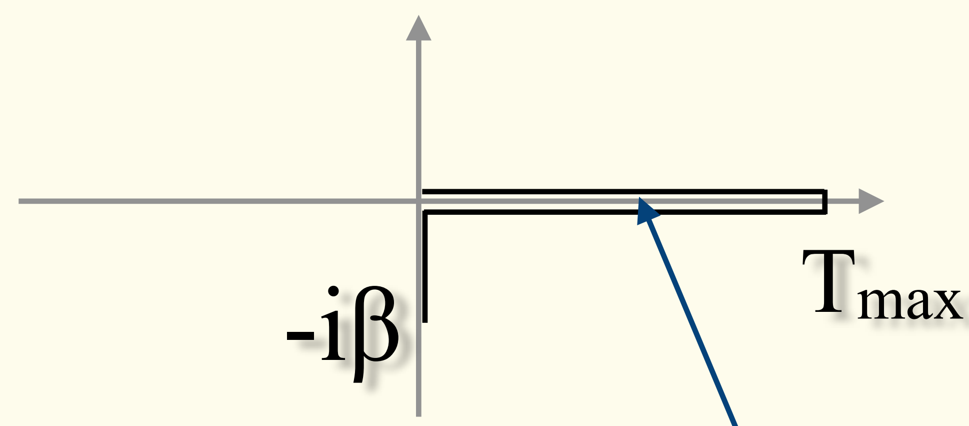
Real time, thermal equilibrium:

$$\langle \phi(t)\phi(t')\rangle_\beta = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}\phi(t)\phi(t')) = \frac{1}{Z}\int D\phi\ e^{iS_c[\phi]}\phi(t)\phi(t')$$



Schwinger-Keldysh
contour
(works also out of equilibrium)

Real Time: The Mother of All Sign Problems

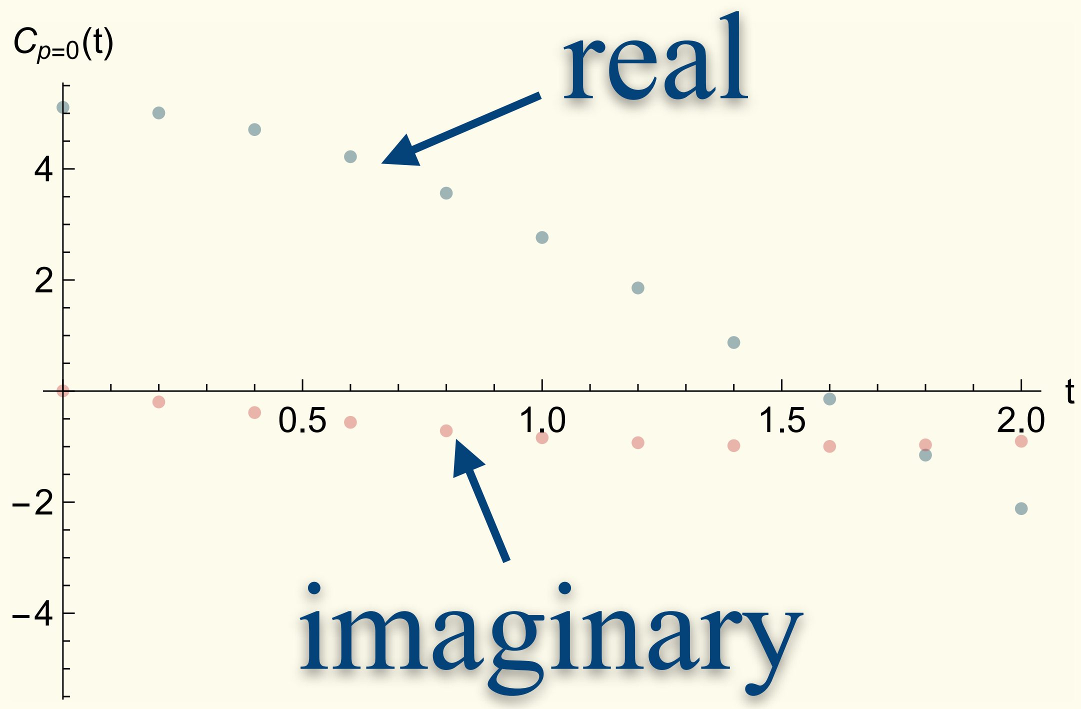


field at a point in the real axis does not contribute to the damping factor in e^{iSc}

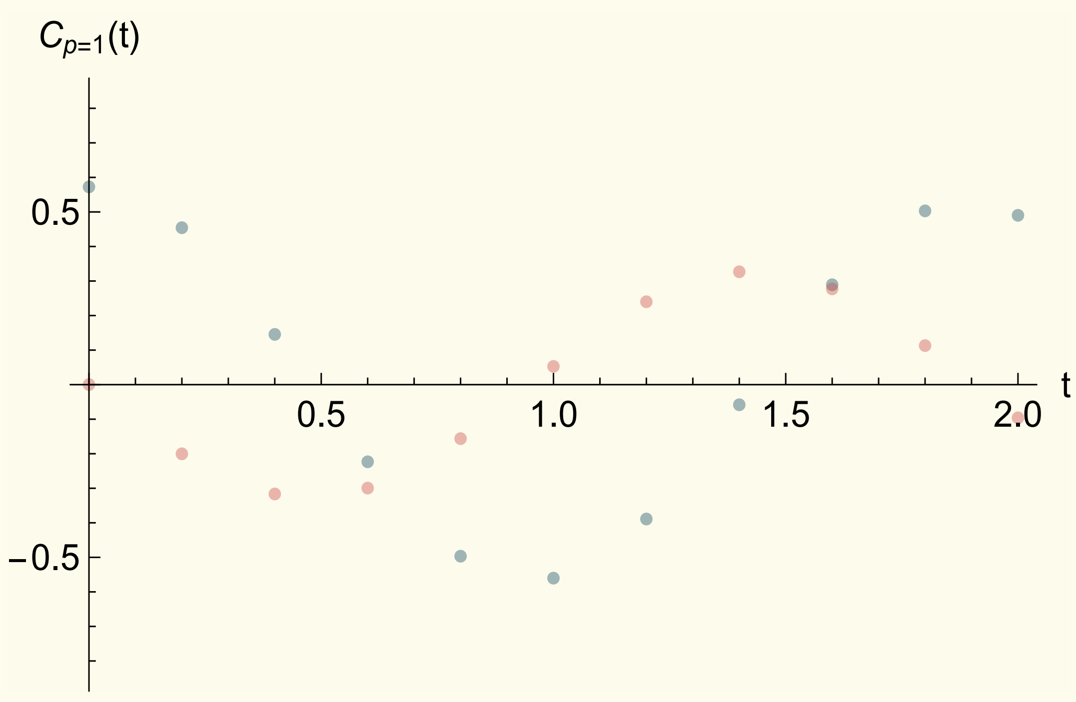
$$\langle e^{i\mathrm{Im}(iS_c)} \rangle = 0$$

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=0.1

weak coupling



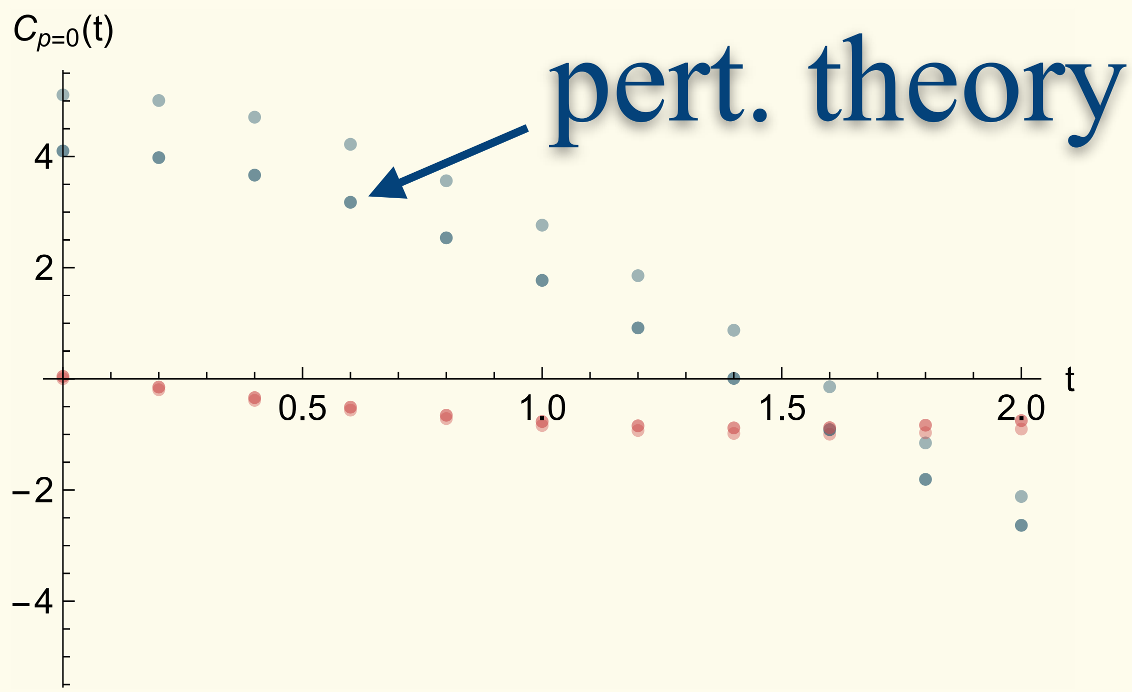
p=0



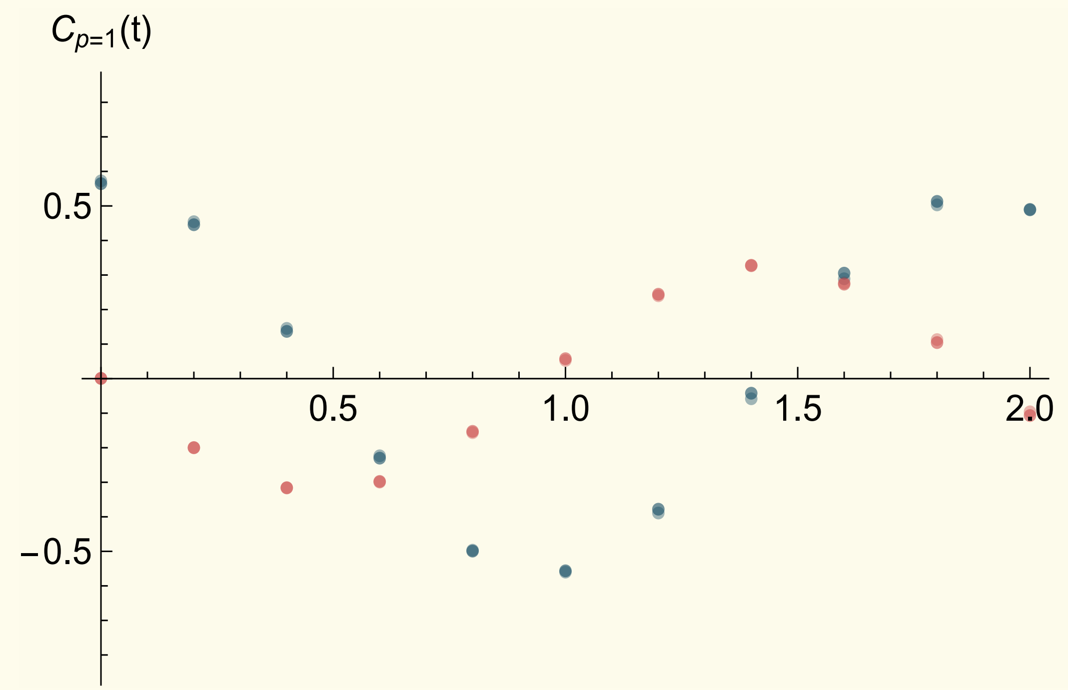
p=2π/L

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=0.1

weak coupling



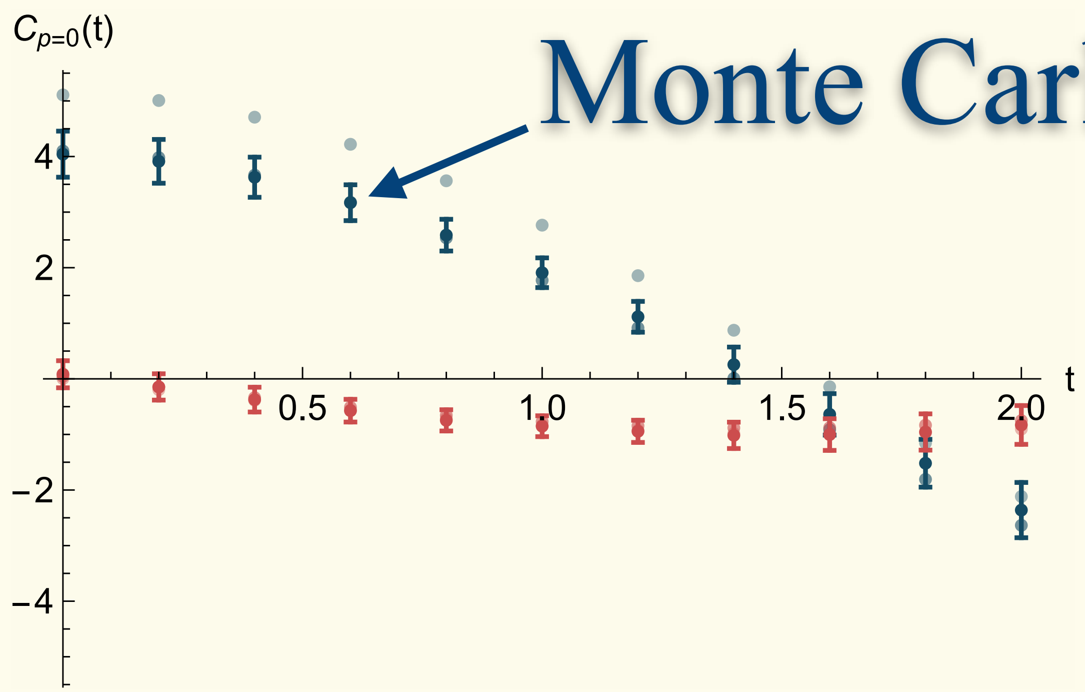
p=0

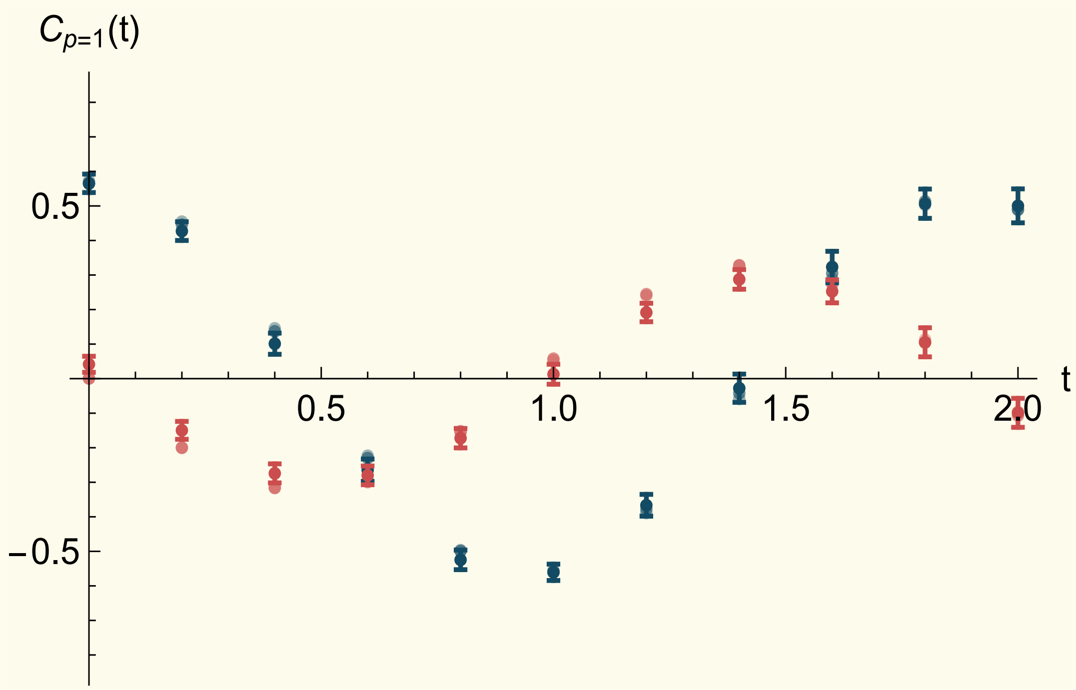


p=2π/L

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=0.1

weak coupling



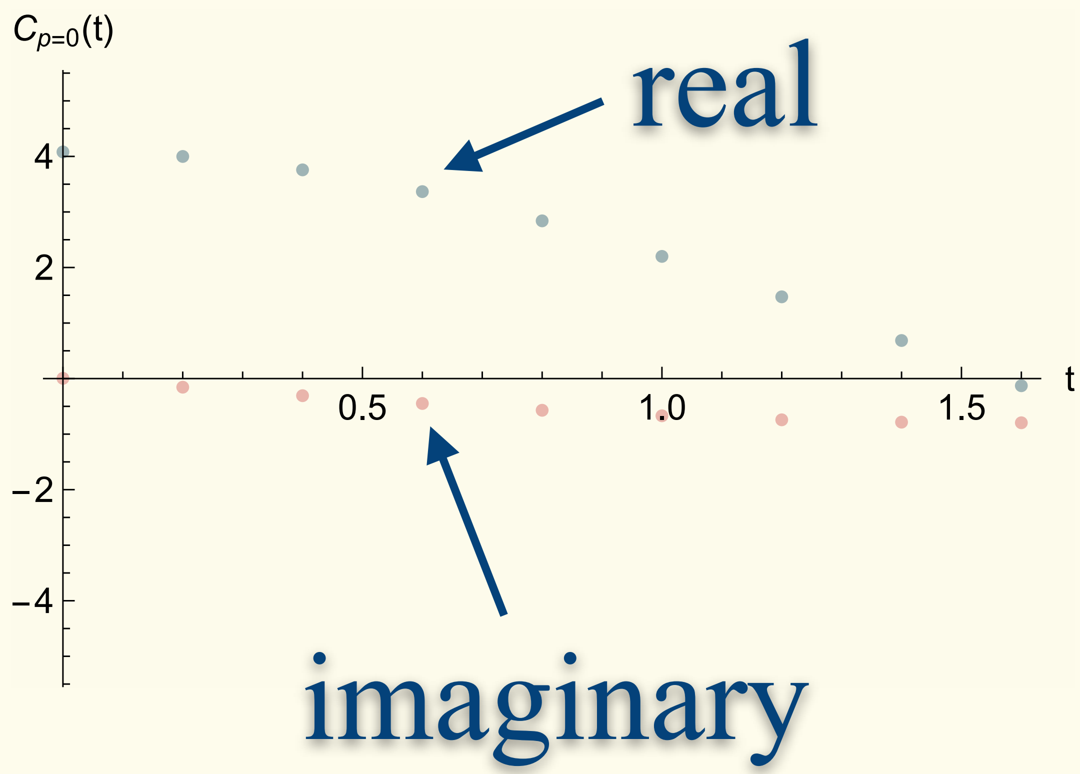


p=0

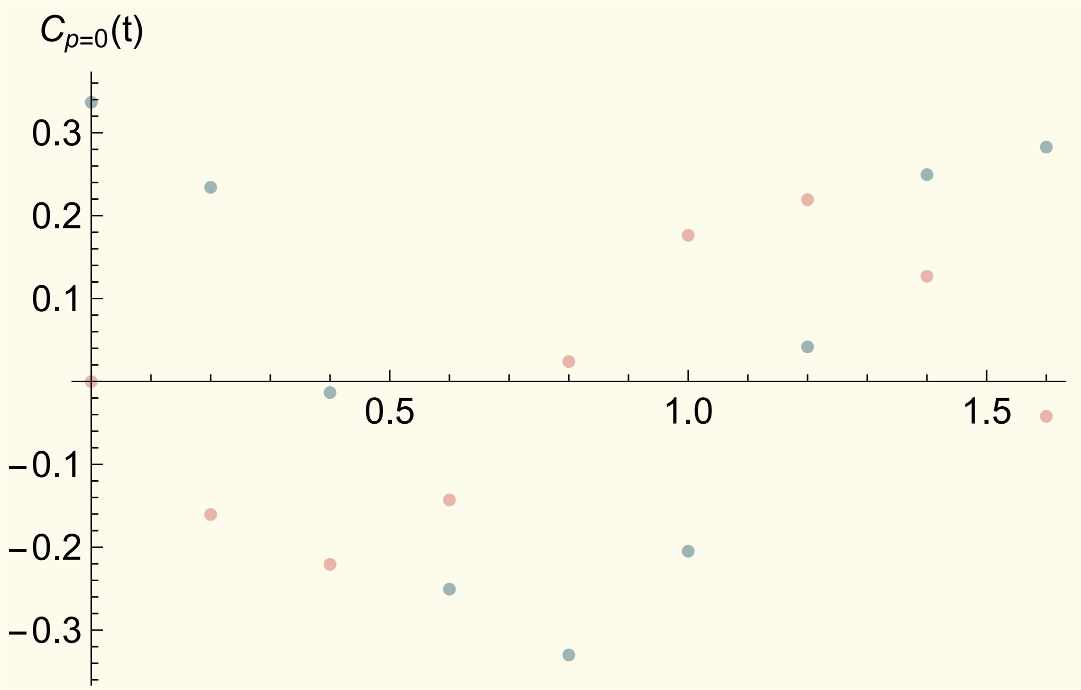
p=2π/L

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=1.0

strong coupling



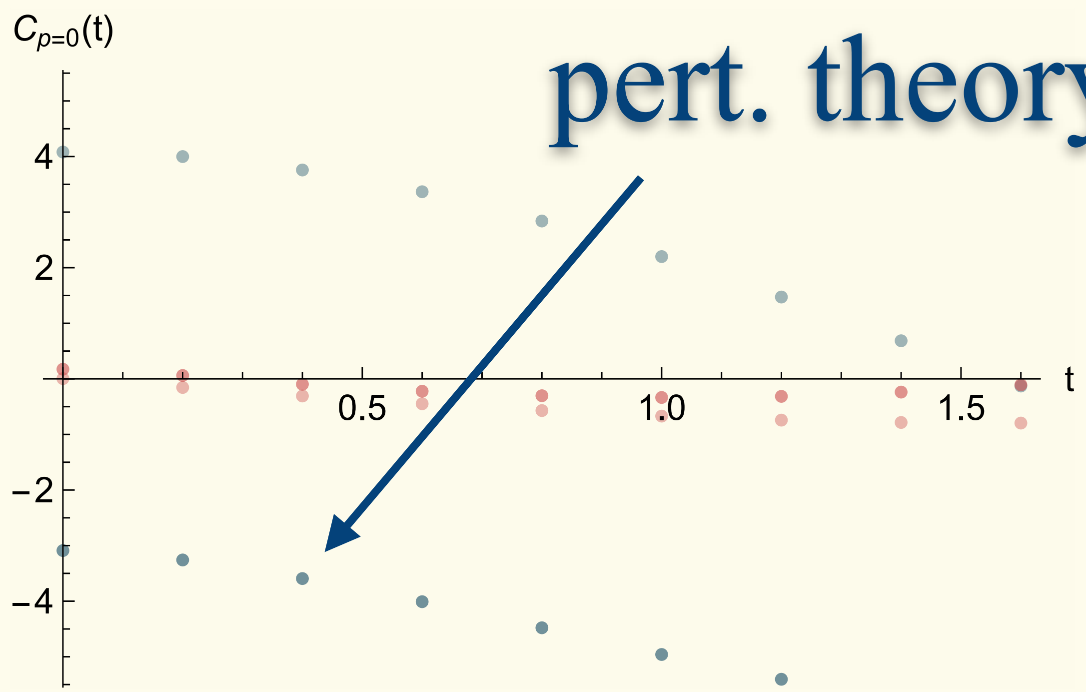
p=0



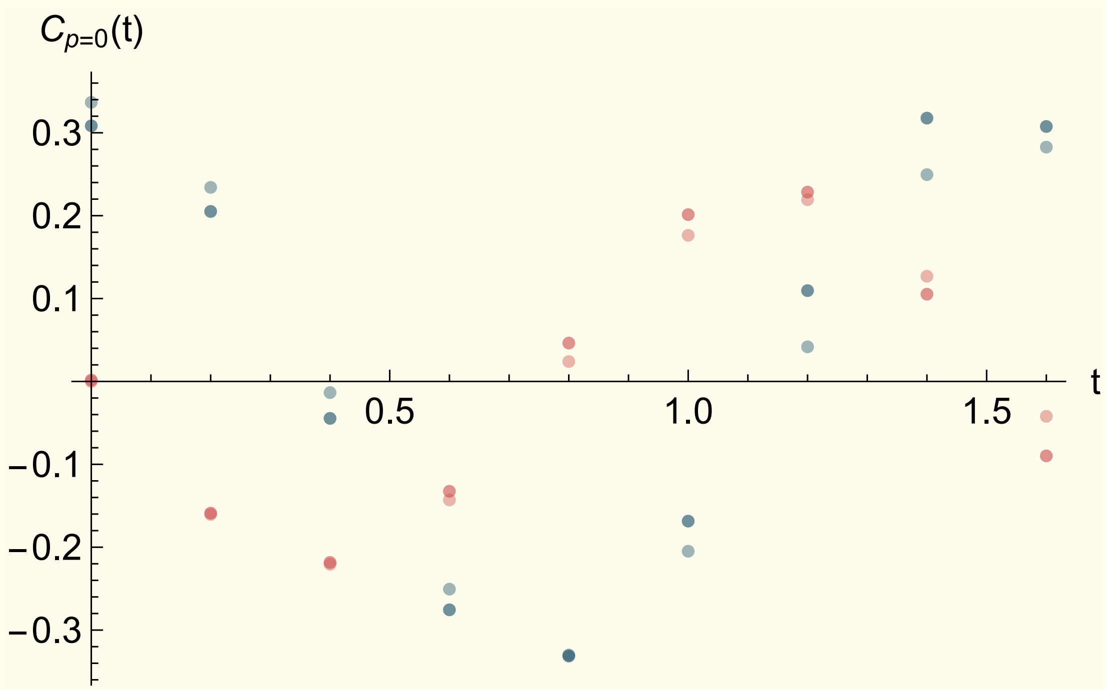
p=2π/L

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=1.0

strong coupling



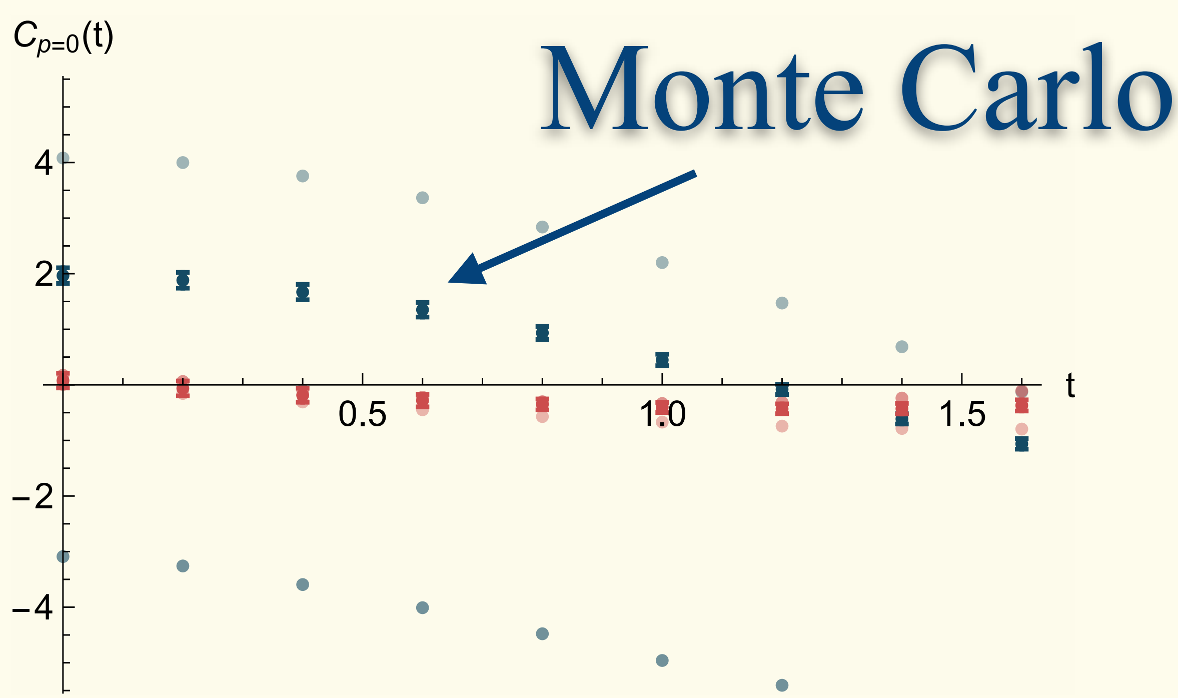
p=0



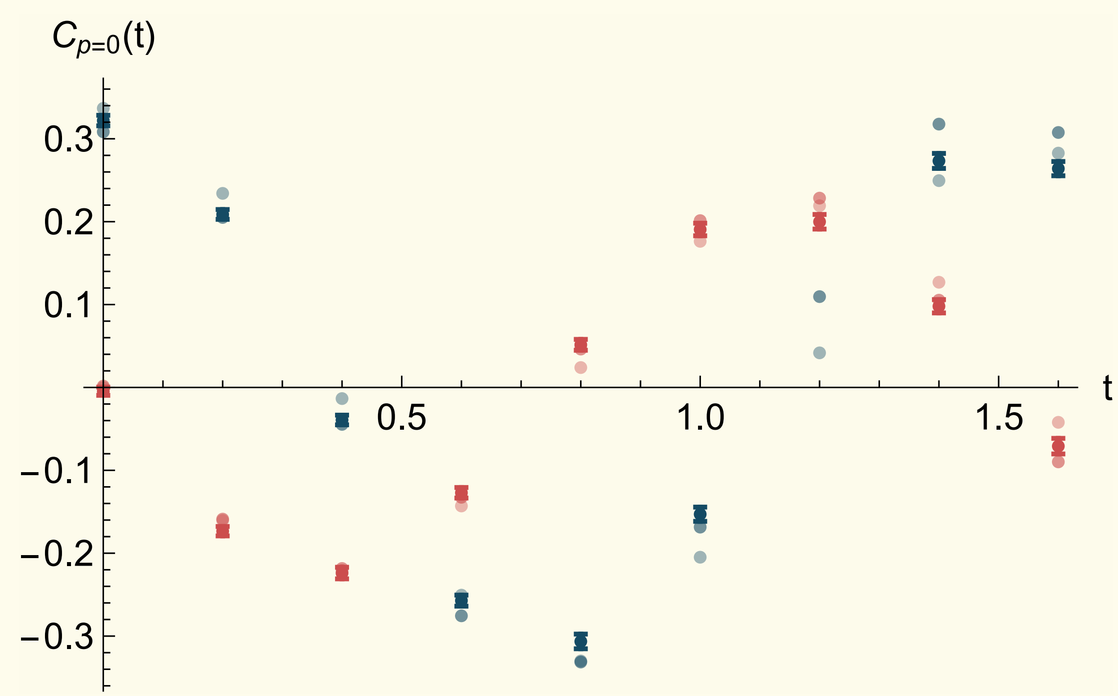
p=2π/L

1+1D φ^4: n_t=10, n_x=10, n_β=2, λ=1.0

strong coupling



p=0

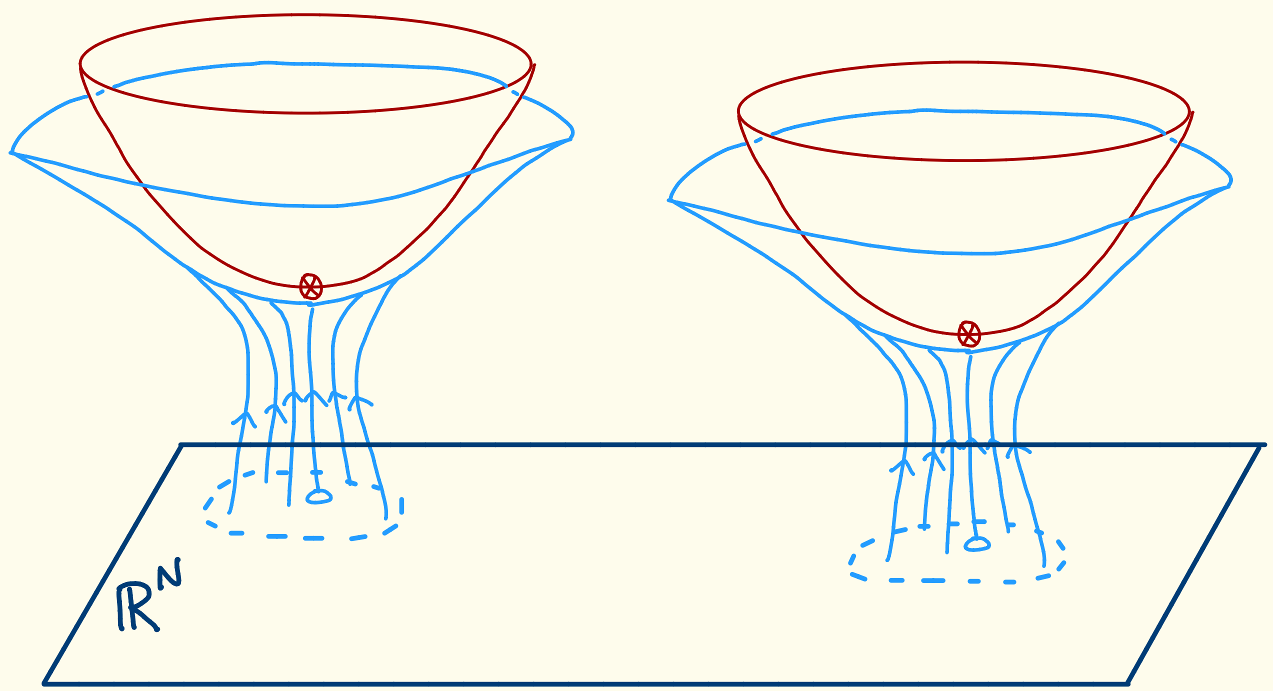


p=2π/L

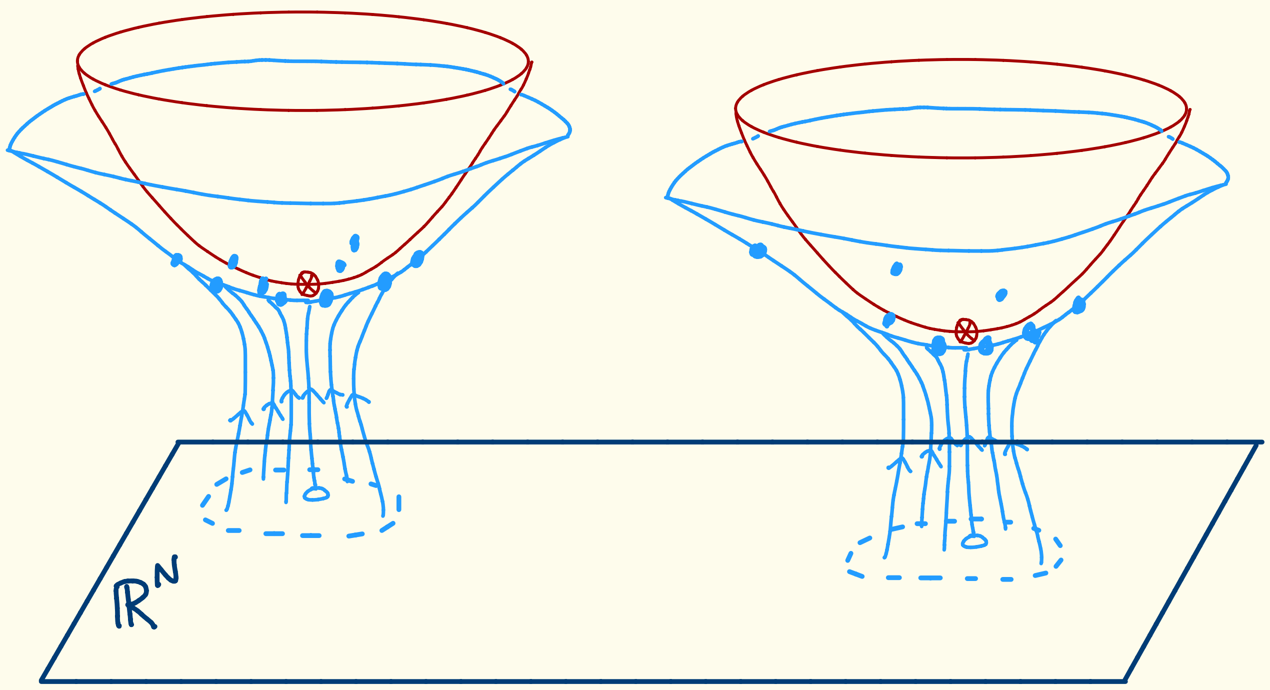
Application: Real Time Dynamics

- Currently limited to small times : $t < 5/T$
- Cost increases sharply with t
- More efficient algorithms/choice of manifolds are necessary
- No reason why other observables (light cone quantities) would be any different.

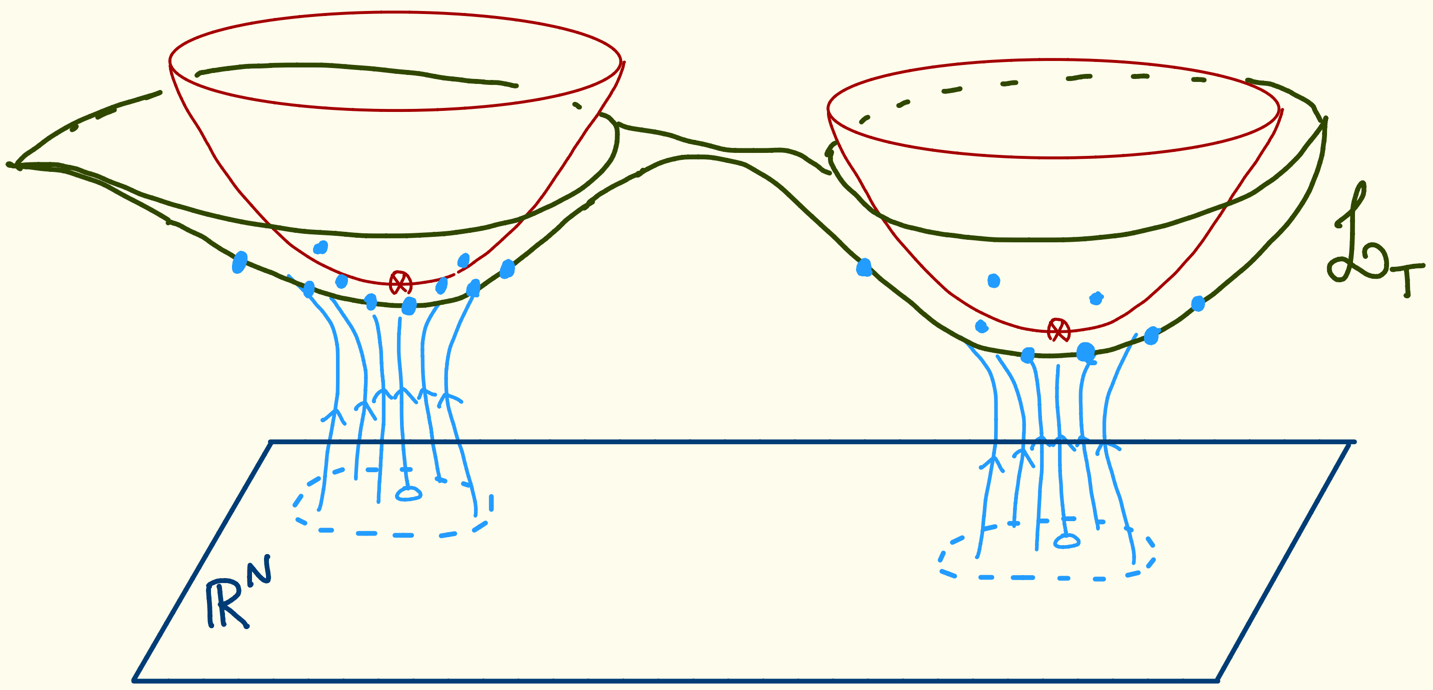
Bypassing the flow: machine learning



Bypassing the flow: machine learning



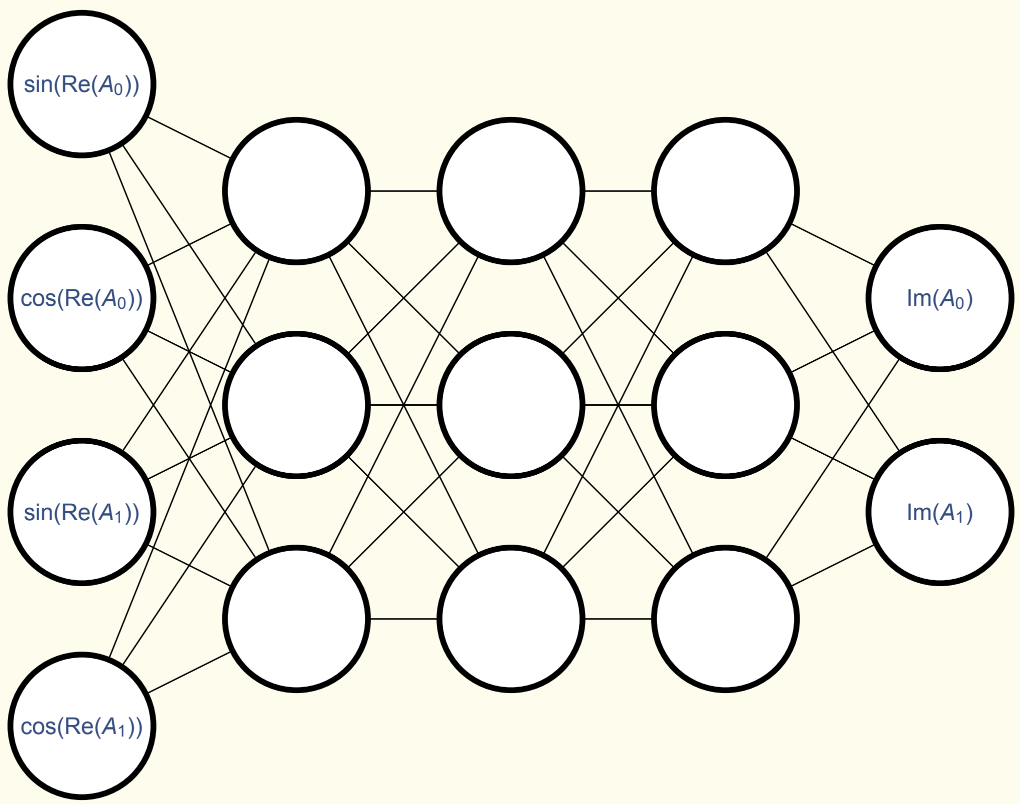
Bypassing the flow: machine learning



$\mathcal{L}_T$=(rough?) interpolation of points of $\mathcal{M}_\mathrm{T}$

Bypassing the flow: machine learning

feed-forward neural net
(supervised training)

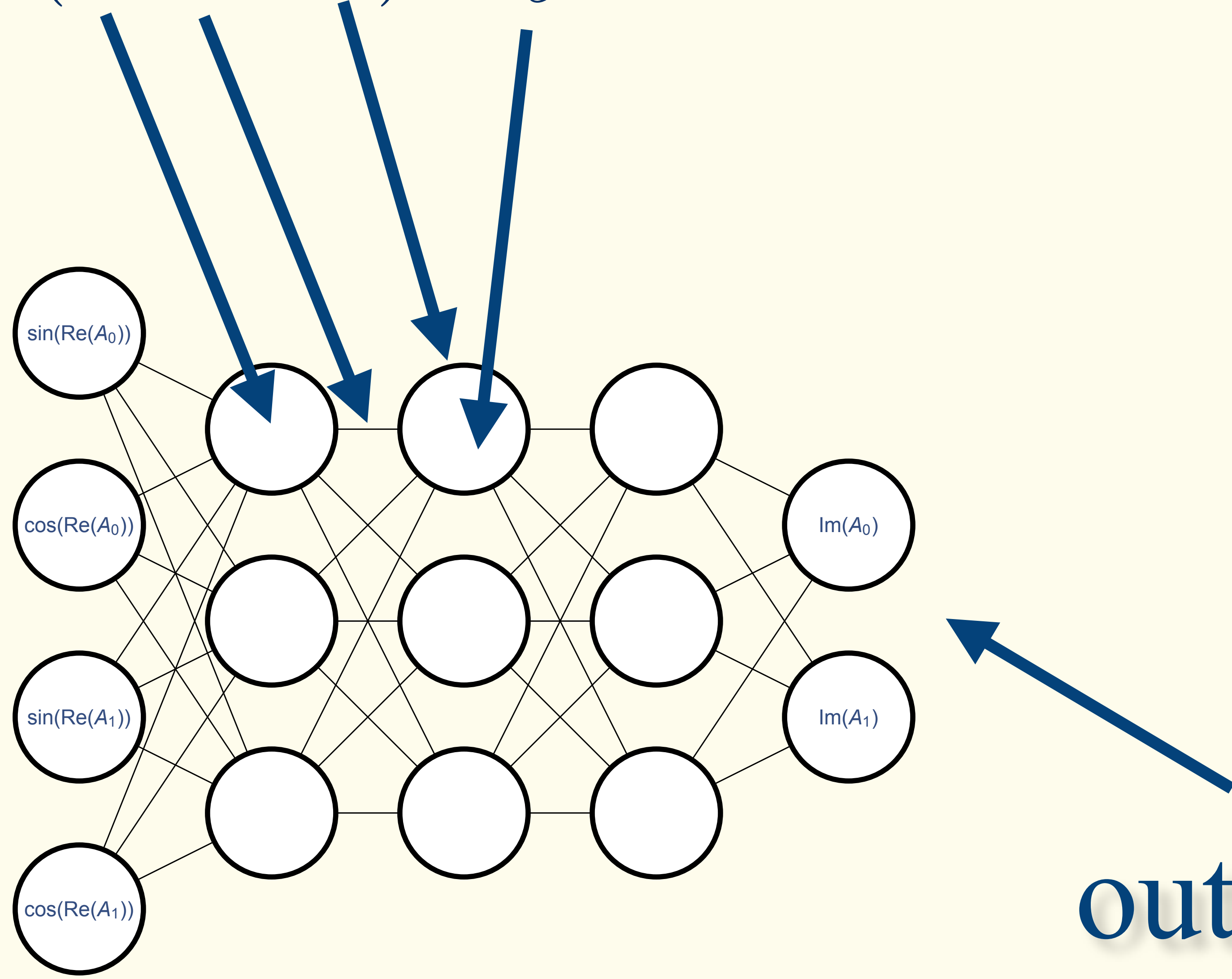


inputs:
Re $A_\mu(x)$

outputs:
Im $A_\mu(0)$

Bypassing the flow: machine learning

$$\sigma(x_i w_i + b) = y$$

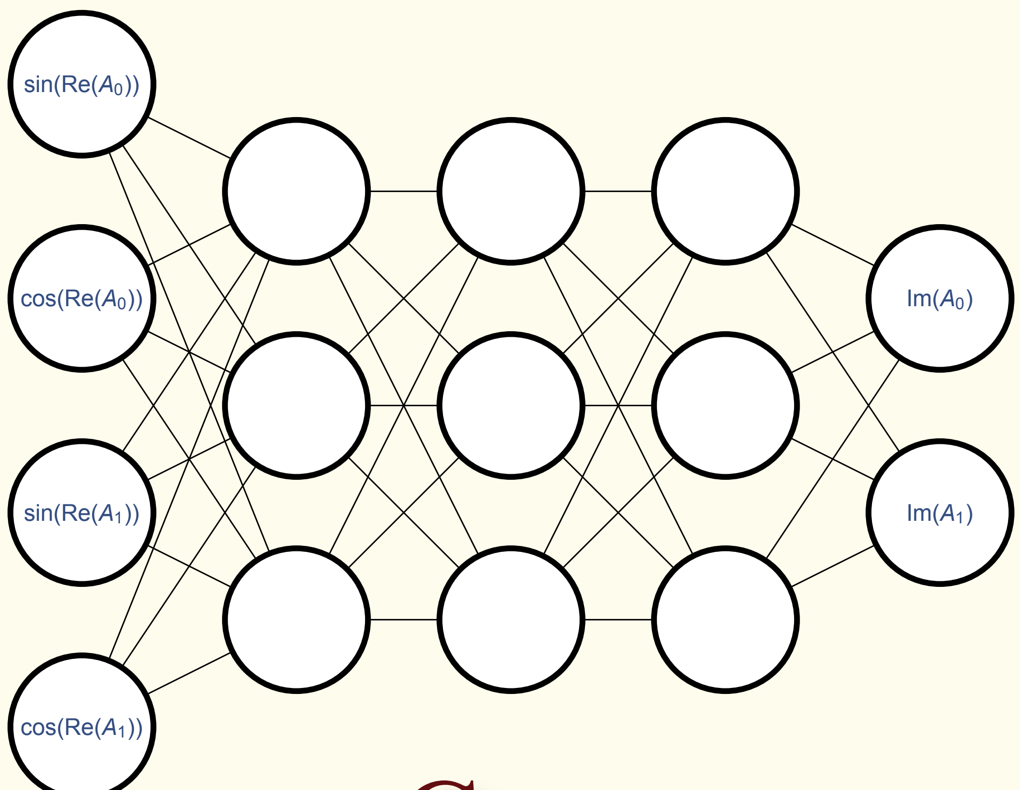


inputs:
Re $A_\mu(x)$

outputs:
Im $A_\mu(0)$

Bypassing the flow: machine learning

feed-forward neural net (supervised training)

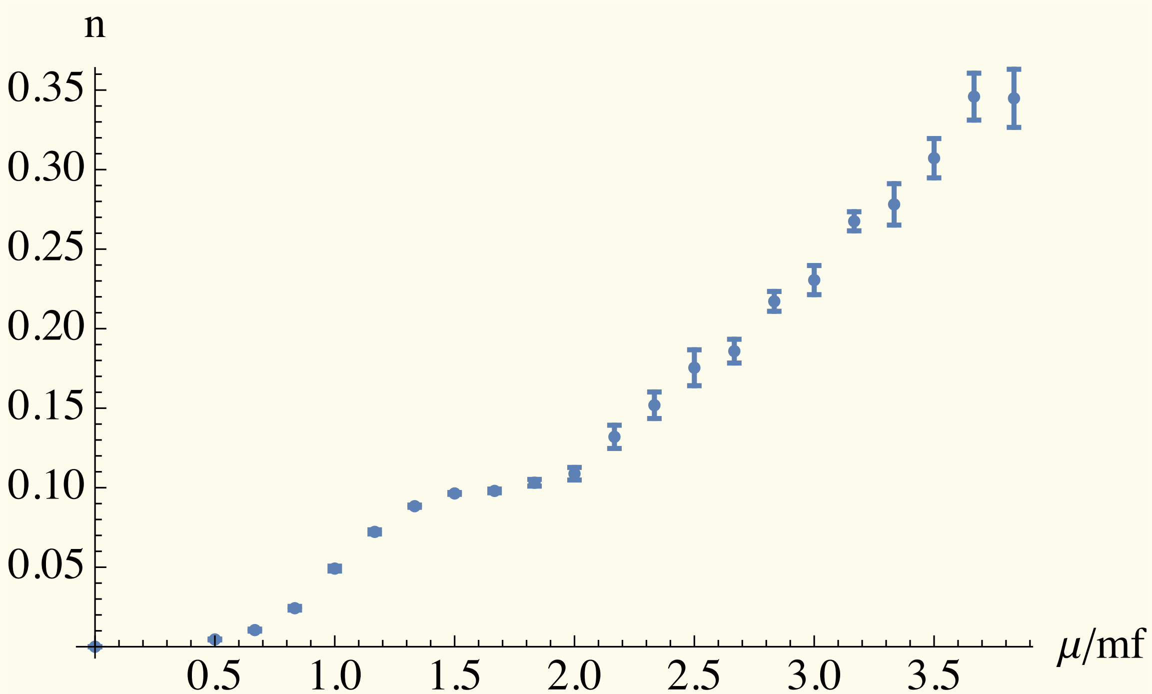


Cost:

flowed configurations >> training >> sampling

Bypassing the flow: machine learning

Wilson, 20 x10 lattice, N_F=2, am_f=0.3



- It finds the shift to tangent plane in the 1st minute; then it finds a better "renormalized tangent plane". Then a curved manifold.
- It doesn't get terribly better fast

Bypassing the flow entirely: SOMME

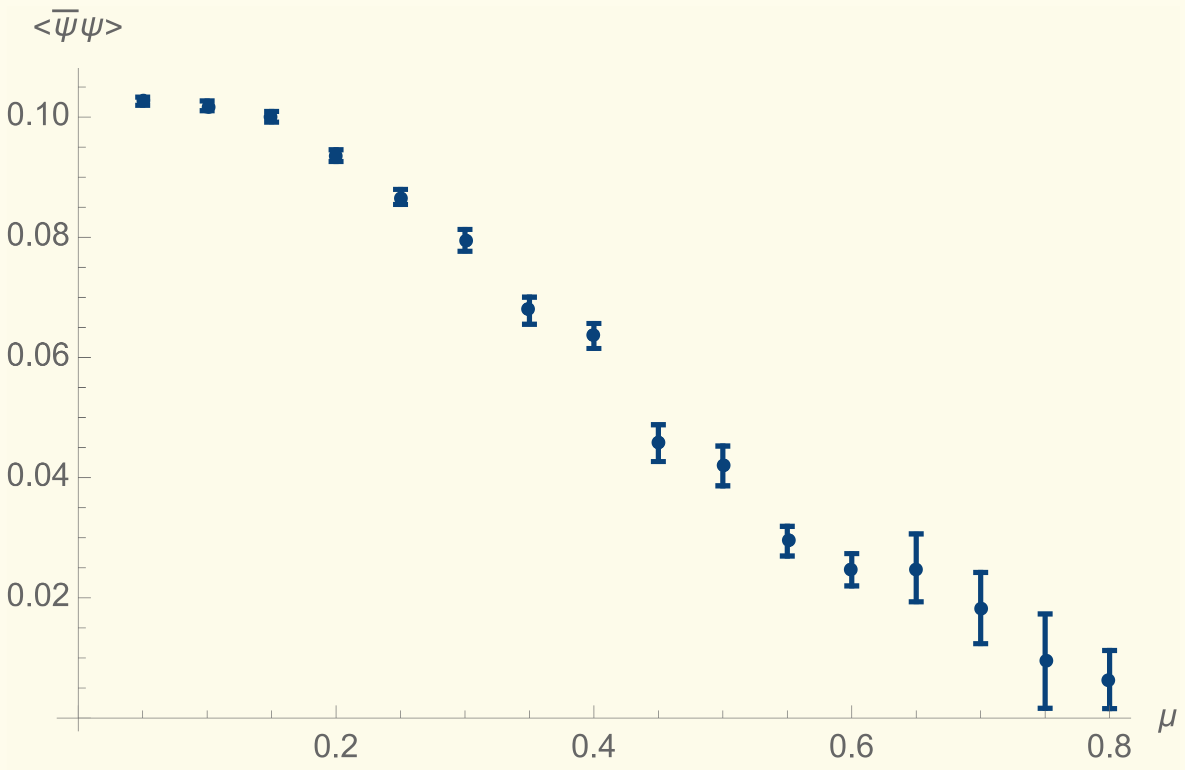
Maximize the average sign:

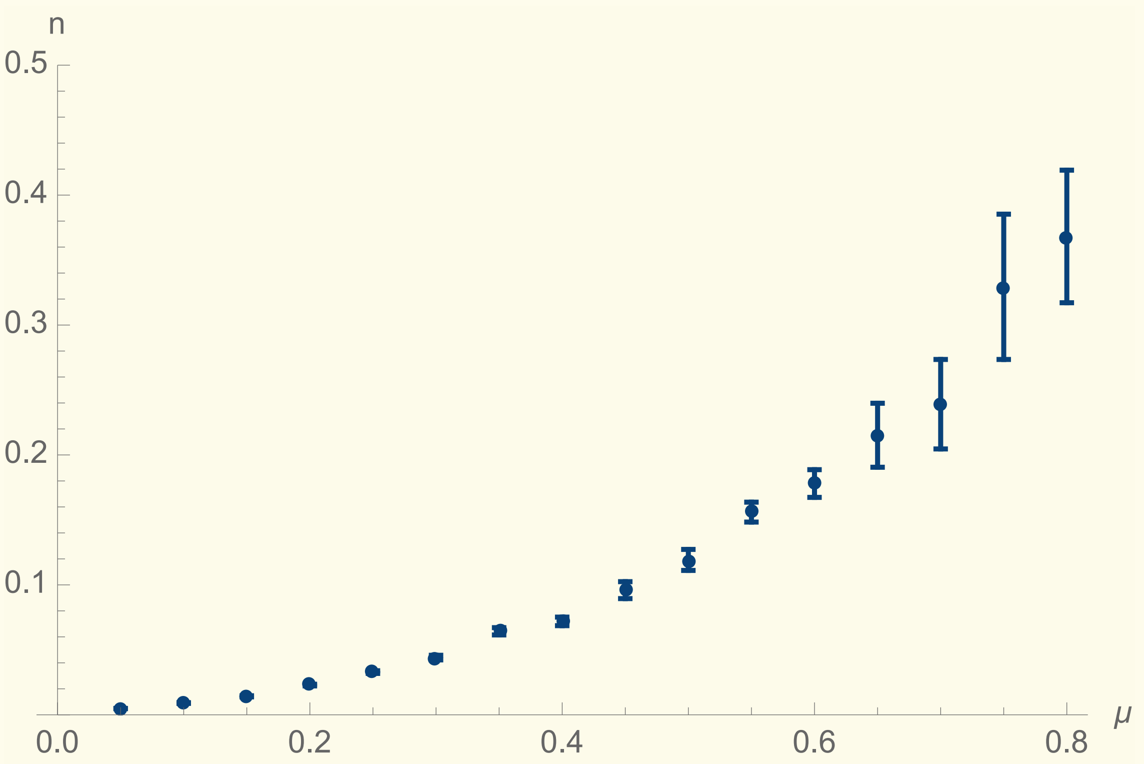
$$\langle e^{i\alpha} \rangle = \frac{\int d\phi \; e^{-iS_I[\phi]} e^{-S_R[\phi]}}{\int d\phi \; e^{-S_R[\phi]}}$$

in the space of manifolds. A short sign-problem free Monte Carlo at every step

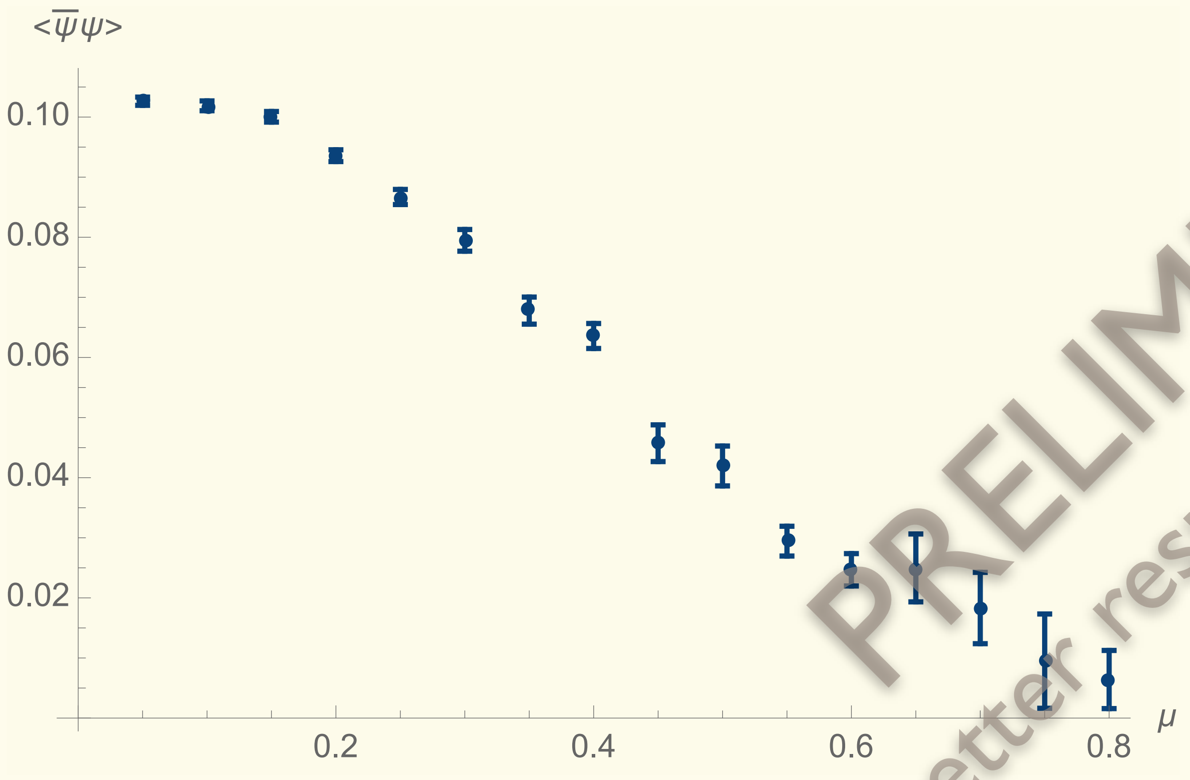
simple ansatz: $\tilde{\phi}(x) = \phi_R(x) + itf(\phi_R(x))$

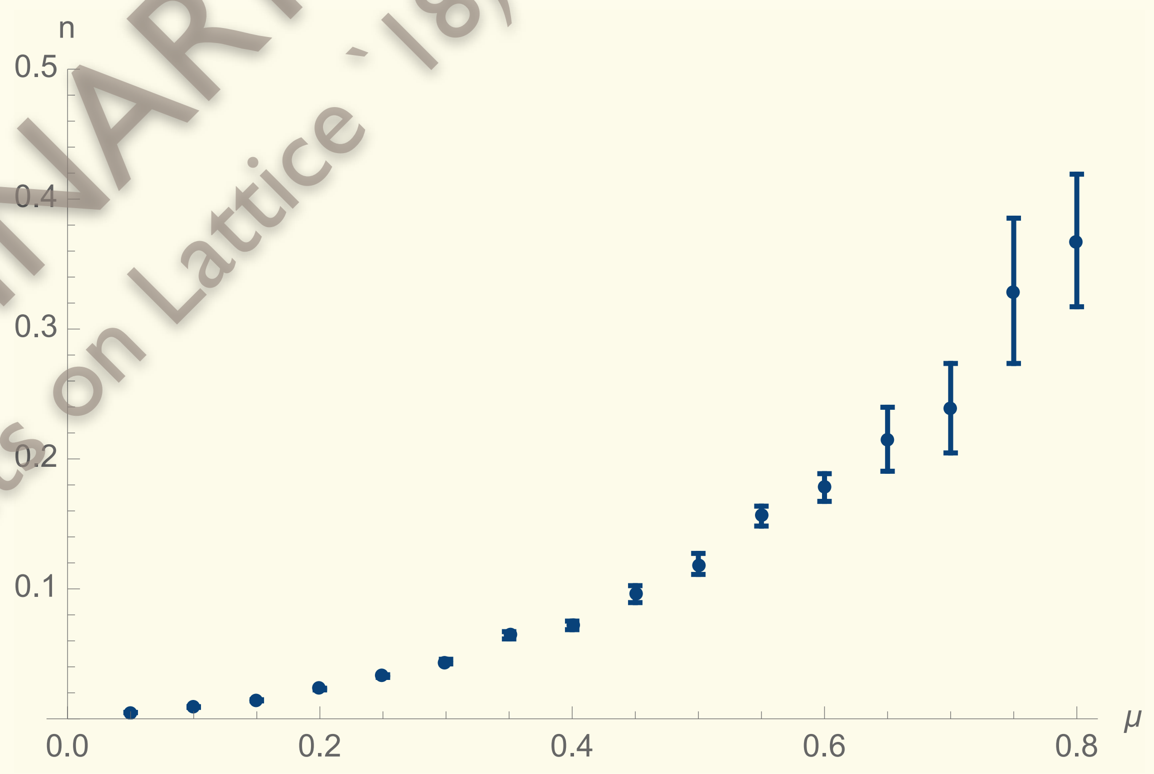
Thirring model in a 6x6x6 lattice:





Thirring model in a 6x6x6 lattice:



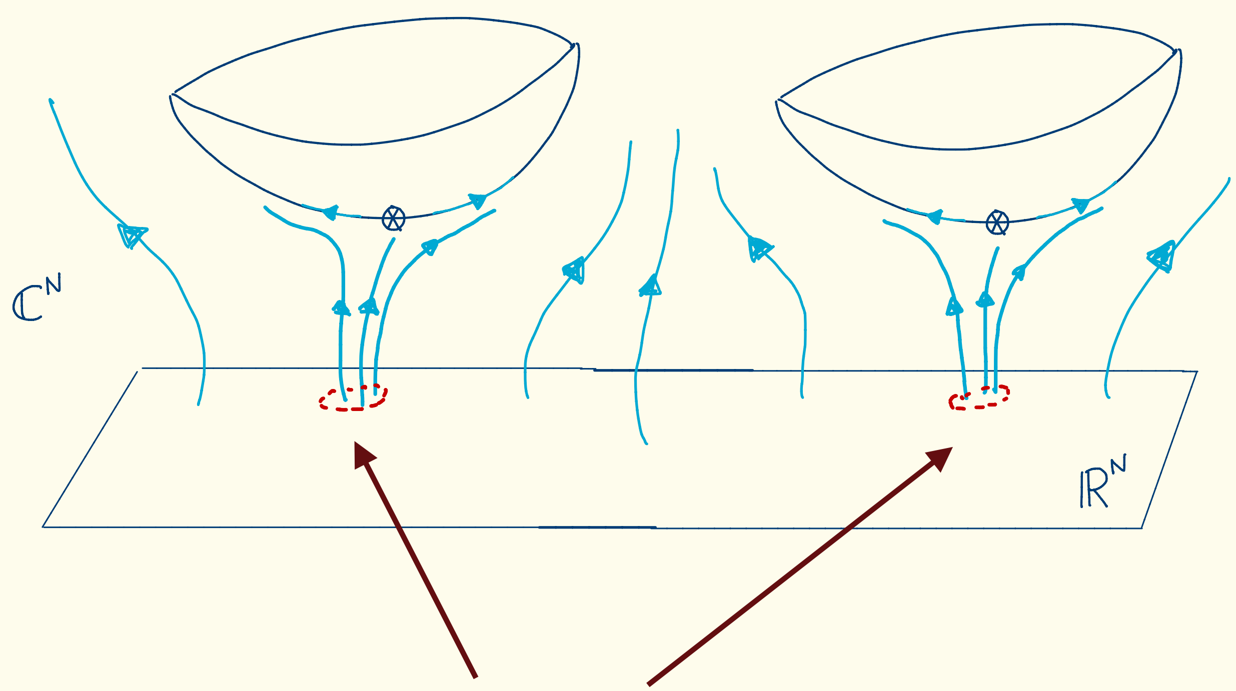


PRELIMINARY
(better results on Lattice `18)

To take home:

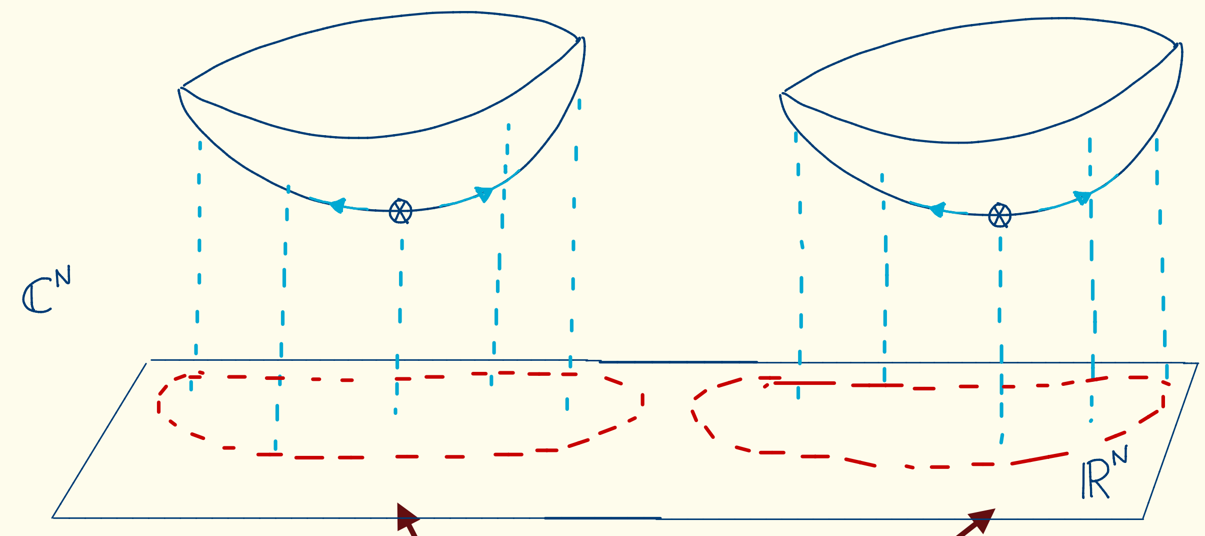
- Deforming the integration on complex space is a good thing
- Thimbles are just one possibility; insight, by humans or machines, will give other the possibilities
- Jacobians are expensive: estimators, “Grady-style” algorithm, ansatze, alternative flows, machine learned manifolds, …
- Progress is being made at a pace of one dimension/year. 1+1 D QED is coming out.

Bypassing the flow: machine learning



isolated modes: trapping

Bypassing the flow: machine learning



no trapping

continuous map between manifolds:

$$\tilde{\phi} = \phi + iNetwork[\phi]$$