Mesons as relativistic bound states in a Minkowski-space approach

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Motivation

"The patchwork quilt of QCD"

*"There exists a wide range of **non-perturbative methods**, all of which trying to approach **QCD** from one way or another. One should not see these different techniques as competing, but as patchwork trying to cover all aspects of QCD."*

Nele Vandersickel
in "*A study of the Gribov-Zwanziger action: from propagators to glueballs*"
PhD Thesis, (2011)

How can we combine knowledge?

Outline

- **Part I** - Results from CST (Covariant Spectator Theory) applied to the study of heavy and heavy-light mesons;

in collaboration with Alfred Stadler (**University of Évora**), Elmar Biernat and Teresa Peña (**IST**)

arXiv:1408.1834 in PRD 90 (2014) ; arXiv:1608.08065 in PLB 764 (2017); arXiv:1707.09303 in PRD 96 (2017)

Elmar Biernat's talk, Monday

- **Part II** - Comparison of two Minkowski-space approaches to heavy quarkonia: CST and BLFQ (Basis Light-Front Quantization);

in collaboration with Yang Li, James P. Vary, Pieter Maris (**Iowa State University**)

arXiv:1705.06178 in EPJC 77 (2017)

James Vary's talk, Monday

- **Summary and Outlook**

Part I - Results from CST applied to the study of mesons

Main goal of this work: construct a *manifestly covariant* theoretical model for *all mesons* interpreted as quark-antiquark bound states, and derived directly in *Minkowski space-time.*

***Why*?**

➡ Manifest covariance: strongly constrains spin-dependence of interactions,

We were able to "test" this!

➡ To complement, for instance, effective field theories, which usually separate physics at different scales (challenging, huge mass range: pion (~0.14 GeV) up to bottomonium (~10 GeV).

Spectroscopy from Upsilon to D mesons (ground states & higher radial excitations) rms ~ 30 MeV.

➡ Minkowski space-time is advantageous over Euclidean formulations (although a number of singularities have to be handled numerically) because no analytic continuations are needed to calculate, e.g., form factors, even in the time-like region.

Simple model for the pion electromagnetic form factor.

arXiv:1310.7465 in PRD 90 (2014)

A covariant model for mesons

How?

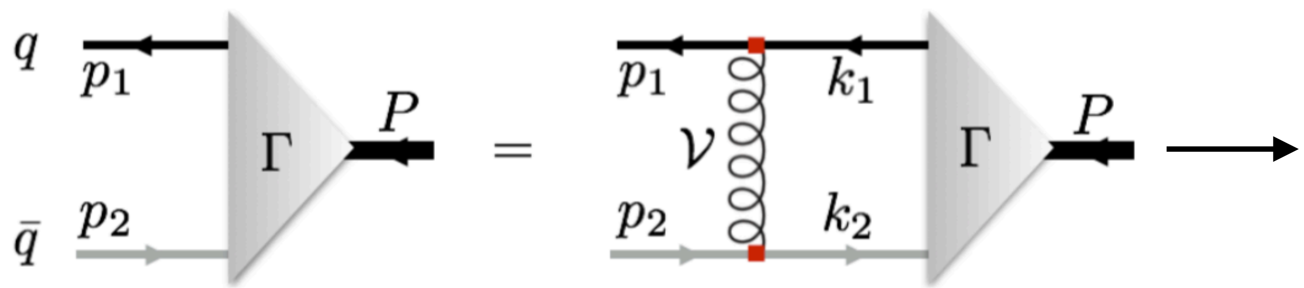
We use the theoretical framework of **Covariant** Spectator Theory (CST).

- **Strong features:**
  - Flexibility to *explore* different phenomenological kernels (more easily than in other approaches).
  - Linearly rising kernel that preserves covariance - "*learn*" about the Lorentz structure of the confining interaction;
  - It has both the correct one-body and nonrelativistic limits
    - well-suited to describe heavy-light systems,
    - "*relativistic wave functions"* which become proper nonrelativistic wave functions in the nonrelativistic limit.
- **Some limitations:**
  - We do not have a "systematic renormalization scheme" - equations regularized with form factors (cut-off parameters fitted to data).

- *Ultimately*, **test CST by comparing the theoretical predictions with experimental data.**

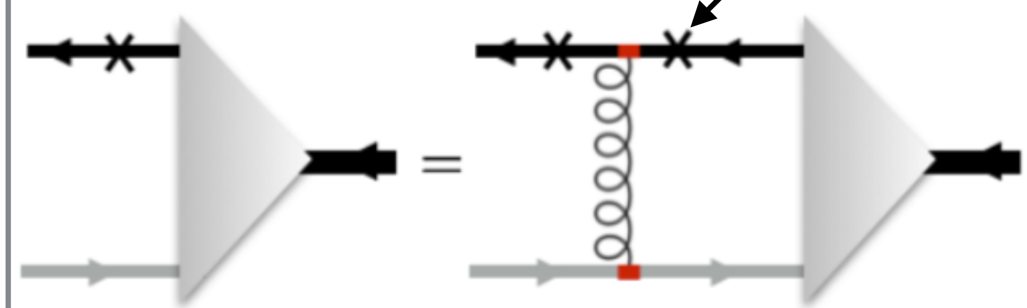
CST model for heavy and heavy-light mesons

- **CST equations for quark-antiquark bound-state** - similar to the Bethe-Salpeter equation, but keeping only the contribution from the dominant poles - 3D covariant integral equation.



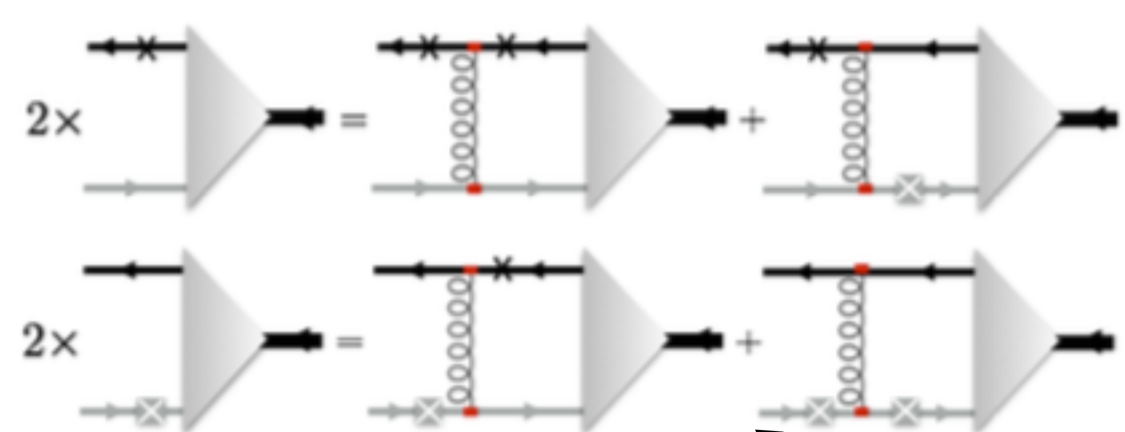
(More details on the formalism - **Elmar Biernat's talk**)

quark is on mass-shell (pos. energy)



One-channel Spectator Equation (1CSE)

- **Features** of **1CSE** -
  - ➡ Particularly suited for unequal masses; ***analysis of the relative importance of the propagator poles***
  - ➡ Numerical solutions easier (fewer singularities);
  - ➡ However, it is not a charge-conjugation equation. *Next step*, solve the two-channel Spectator Equation (2CSE)



antiquark is on mass-shell (neg. energy)

Our solutions possess good J, P quantum numbers

CST model for heavy and heavy-light mesons

- **Kernel** - $\mathcal{V}(p,k) = \underbrace{\frac{3}{4}\boldsymbol{F}_1\cdot\boldsymbol{F}_2}_{=1\text{ (color singlets)}} \sum_K \underbrace{V_K(p,k)}_{\text{momenta}} \underbrace{\Theta_1^K \otimes \Theta_2^K}_{\text{Lorentz } \Theta^K = \mathbf{1},\gamma^5,\gamma^\mu}$

  - Momentum dependence: ***"Covariantized" Cornel-type of potential***

    ➡ Linear component:

$$V_{\mathrm{L}}(\hat{p}_1 - P/2, \hat{k}_1 - P/2) = -8\sigma\pi\Bigg[\left(\frac{1}{(\hat{p}_1-\hat{k}_1)^4} - \frac{1}{\Lambda^4+(\hat{p}_1-\hat{k}_1)^4}\right) - \frac{E_{ip}}{m_i}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\int_{\mathbf{k}'_i}\left(\frac{1}{(\hat{p}_1-\hat{k}'_1)^4} - \frac{1}{\Lambda^4+(\hat{p}_1-\hat{k}'_1)^4}\right)\Bigg]$$

    ➡ OGE component (Feynman gauge):

$$V_{\mathrm{OGE}}(\hat{p}_1 - P/2, \hat{k}_1 - P/2) = -4\pi\alpha_s\left(\frac{1}{(\hat{p}_1-\hat{k}_1)^2} - \frac{1}{(\hat{p}_1-\hat{k}_1)^2-\Lambda^2}\right)$$

    ➡ "Constant" component:

$$V_{\mathrm{C}}(\hat{p}_1 - P/2, \hat{k}_1 - P/2) = (2\pi)^3\frac{E_{ik}}{m_i}C\delta^3(\mathbf{p}-\mathbf{k})$$

  - Lorentz structure:

Ward-Takahashi Identities

$$\mathcal{V}(\hat{p},\hat{k}) = [(1-y)(\underline{\mathbf{1}_1\otimes\mathbf{1}_2+\gamma_1^5\otimes\gamma_2^5}) - y\gamma_1^\mu\otimes\gamma_{\mu 2}]V_{\mathrm{L}}(\hat{p},\hat{k}) - \gamma_1^\mu\otimes\gamma_{\mu 2}[V_{\mathrm{G}}(\hat{p},\hat{k})+V_{\mathrm{C}}(\hat{p},\hat{k})]$$

mixing parameter y in V_{L}: $y=0$ (pure S+PS)... $y=1$ (pure V)

Data sets used in the fits

	State	$J^{P(C)}$	Mass (MeV)	Data set S1	S2	S3
$b\bar{b}$	$\Upsilon(4S)$	1^{--}	10579.4 ± 1.2		•	•
	$\chi_{b1}(3P)$	1^{++}	10512.1 ± 2.3			•
	$\Upsilon(3S)$	1^{--}	10355.2 ± 0.5		•	•
	$\eta_b(3S)$	0^{-+}	10337			
	$h_b(2P)$	1^{+-}	10259.8 ± 1.2			•
	$\chi_{b1}(2P)$	1^{++}	$10255.46\pm0.22\pm0.50$			•
	$\chi_{b0}(2P)$	0^{++}	$10232.5\pm0.4\pm0.5$		•	•
	$\Upsilon(1D)$	1^{--}	10155			
	$\Upsilon(2S)$	1^{--}	10023.26 ± 0.31		•	•
	$\eta_b(2S)$	0^{-+}	9999 ± 4	•	•	•
	$h_b(1P)$	1^{+-}	9899.3 ± 0.8			•
	$\chi_{b1}(1P)$	1^{++}	$9892.78\pm0.26\pm0.31$			•
	$\chi_{b0}(1P)$	0^{++}	$9859.44\pm0.42\pm0.31$		•	•
	$\Upsilon(1S)$	1^{--}	9460.30 ± 0.26		•	•
	$\eta_b(1S)$	0^{-+}	9399.0 ± 2.3	•	•	•
$b\bar{c}$	$B_c(2S)^\pm$	0^-	6842 ± 6			•
	B_c^+	0^-	6275.1 ± 1.0	•	•	•
$b\bar{s}$	$B_{s1}(5830)$	1^+	5828.63 ± 0.27			•
$b\bar{q}$	$B_1(5721)^{+,0}$	1^+	5725.85 ± 1.3			•
$b\bar{s}$	B_s^*	1^-	5415.8 ± 1.5		•	•
	B_s^0	0^-	5366.82 ± 0.22	•	•	•
$b\bar{q}$	B^*	1^-	5324.65 ± 0.25		•	•
	$B^{\pm,0}$	0^-	5279.45	•	•	•

	State	$J^{P(C)}$	Mass (MeV)	Data set S1	S2	S3
$c\bar{c}$	$X(3915)$	0^{++}	3918.4 ± 1.9		•	•
	$\psi(3770)$	1^{--}	3773.13 ± 0.35		•	•
	$\psi(2S)$	1^{--}	3686.097 ± 0.010		•	•
	$\eta_c(2S)$	0^{-+}	3639.2 ± 1.2	•	•	•
	$h_c(1P)$	1^{+-}	3525.38 ± 0.11			•
	$\chi_{c1}(1P)$	1^{++}	3510.66 ± 0.07			•
	$\chi_{c0}(1P)$	0^{++}	3414.75 ± 0.31		•	•
	$J/\Psi(1S)$	1^{--}	3096.900 ± 0.006		•	•
	$\eta_c(1S)$	0^{-+}	2983.4 ± 0.5	•	•	•
$c\bar{s}$	$D_{s1}(2536)^\pm$	1^+	2535.10 ± 0.06			•
	$D_{s1}(2460)^\pm$	1^+	2459.5 ± 0.6			•
$c\bar{q}$	$D_1(2420)^{\pm,0}$	1^+	2421.4			•
	$D_0^*(2400)^0$	0^+	2318 ± 29		•	•
$c\bar{s}$	$D_{s0}^*(2317)^\pm$	0^+	2317.7 ± 0.6		•	•
	$D_s^{*\pm}$	1^-	2112.1 ± 0.4		•	•
$c\bar{q}$	$D^*(2007)^0$	1^-	2008.62			•
$c\bar{s}$	$D_s^\pm$	0^-	1968.27 ± 0.10	•	•	•
$c\bar{q}$	$D^{\pm,0}$	0^-	1867.23	•	•	•

S1: 9 PS mesons

S2: 25 PS+V+S mesons

S3: 39 PS+V+S+AV mesons

q represents a light quark (u or d)

We use $m_u = m_d \equiv m_q$

Global fits with fixed quark masses and *y*=0

- **First step**: we perform global fits to the heavy + heavy-light meson spectrum.

- Adjustable model parameters: $\sigma, \alpha_s, C, \qquad \Lambda = 2m_1$

- Model parameters **not adjusted** in the fits:

  Constituent quark masses (in GeV): $m_b = 4.892, m_c = 1.600, m_s = 0.448, m_q = 0.346$
  Scalar + pseudoscalar confinement: $y = 0$

- **Models**

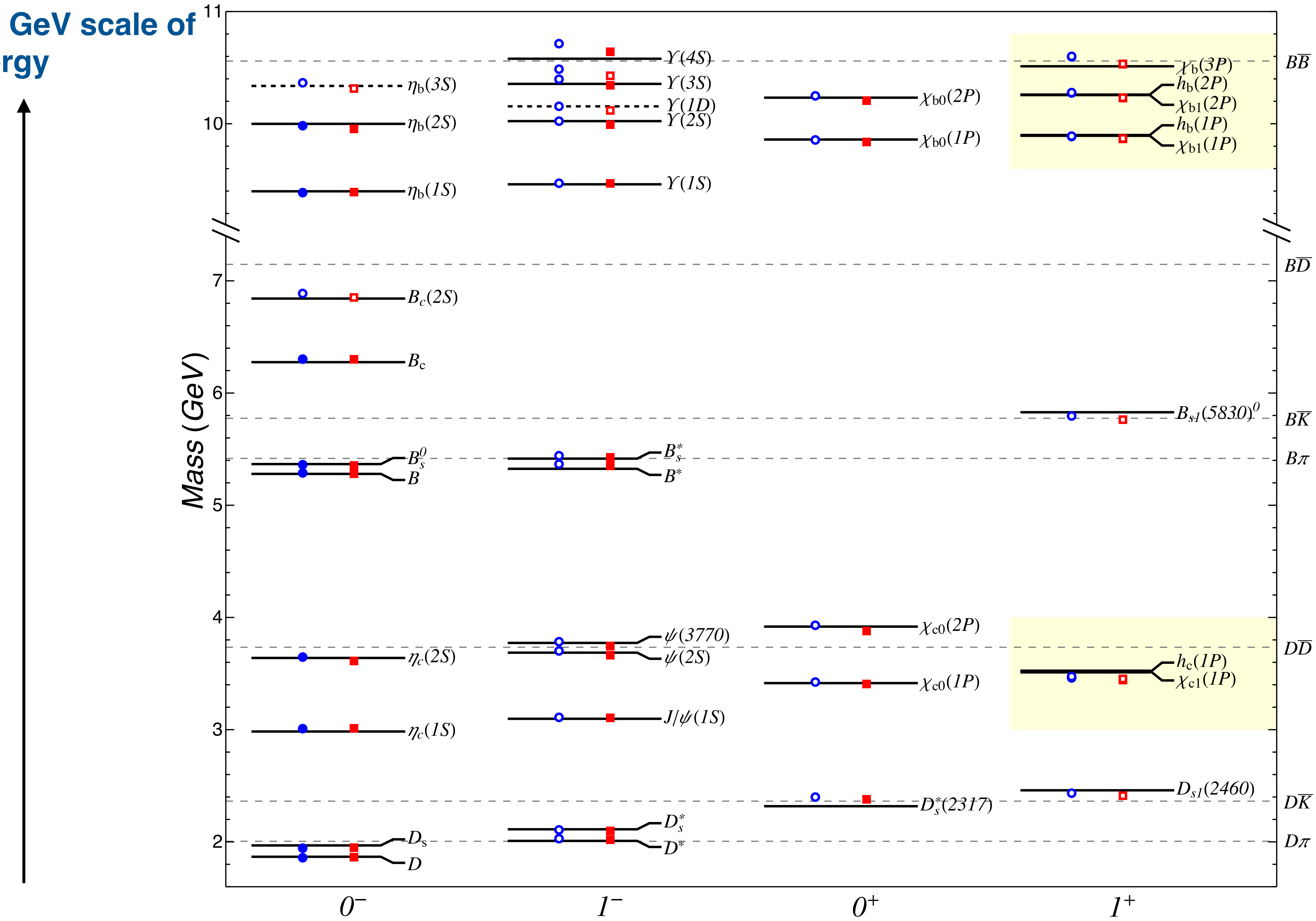
  - Model $\mathrm{M0}_{\mathrm{S1}}$: fitted to 9 pseudoscalar meson masses only (set S1)

  - Model $\mathrm{M0}_{\mathrm{S2}}$: fitted to 25 pseudoscalar, vector and scalar meson masses (set S2)

SL *et al.*, PLB **764**, 38 (2017)

Spectroscopy with fixed constituent quark masses

~10 GeV scale of energy



Global fits with fixed quark masses and *y*=0

- The results of the two fits are remarkably similar!

rms differences to experimental masses (set S3):

Model	σ [GeV^2]	α_s	C [GeV]
M0$_{S1}$	0.2493	0.3643	0.3491
M0$_{S2}$	0.2247	0.3614	0.3377

→

Model	$\Delta_{\rm rms}$ [GeV]
M0$_{S1}$	0.037
M0$_{S2}$	0.036

- Kernel parameters are already well determined through **pseudoscalar states**

Almost 100% L=0, S=0
(S-wave, spin singlet)

$\langle 0^- | \mathbf{L}\cdot\mathbf{S} | 0^- \rangle = 0$ Spin-orbit force vanishes

$\langle 0^- | S_{12} | 0^- \rangle = 0$ Tensor force vanishes

$\langle 0^- | \mathbf{S}_1\cdot\mathbf{S}_2 | 0^- \rangle = -3/4$ Spin-spin force acts in singlet only

Good test for a covariant kernel:

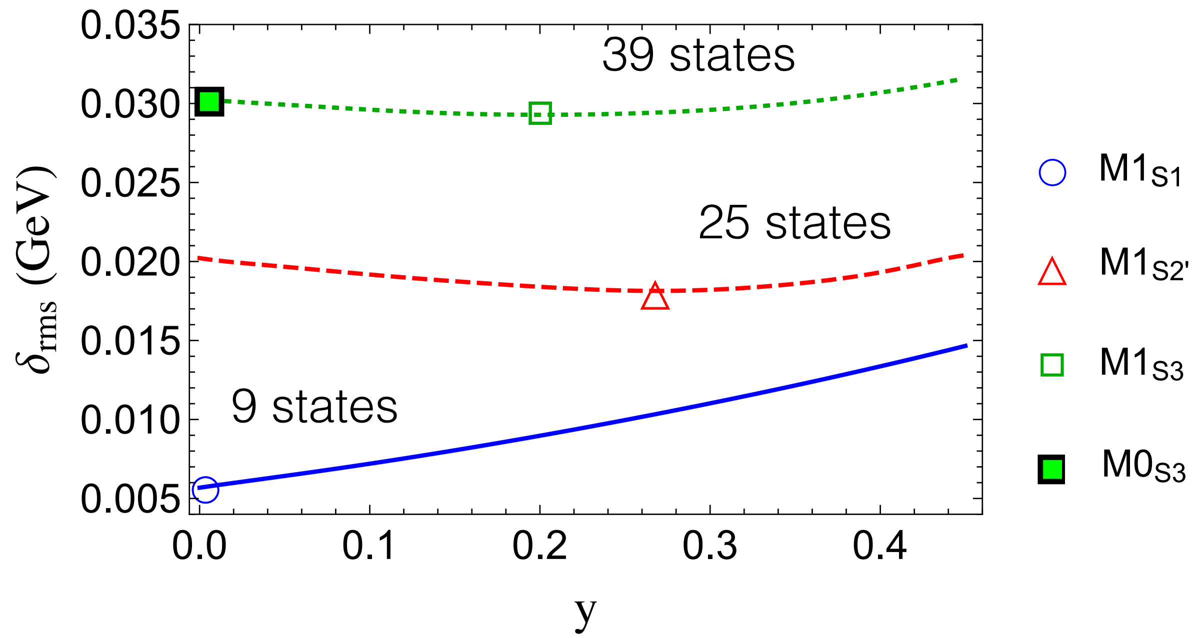
- Pseudoscalar states do not constrain spin-orbit and tensor forces, and cannot separate spin-spin from central force.
- But they should be determined through covariance.
- Model M0$_{S1}$ indeed predicts spin-dependent forces correctly!

SL *et al.*, PLB **764**, 38 (2017)

Fits with variable quark masses and confinement (S+PS)-V mixing *y*

- In a **new series** of fits we treat quark masses and mixing parameter *y* as **adjustable parameters**.

Model	σ (GeV^2)	α_s	C (GeV)	y	m_b (GeV)	m_c (GeV)	m_s (GeV)	m_q (GeV)	N_s	$\delta_{\rm rms}$	$\Delta_{\rm rms}$ (GeV)
$\mathrm{M0_{S1}}$	0.2493	0.3643	0.3491	**0.0000**	**4.892**	**1.600**	**0.4478**	**0.3455**	9	0.017	0.037
$\mathrm{M1_{S1}}$	0.2235	0.3941	0.0591	0.0000	4.768	1.398	0.2547	0.1230	9	0.006	0.041
$\mathrm{M0_{S2}}$	0.2247	0.3614	0.3377	**0.0000**	**4.892**	**1.600**	**0.4478**	**0.3455**	25	0.028	0.036
$\mathrm{M1_{S2}}$	0.1893	0.4126	0.1085	0.2537	4.825	1.470	0.2349	0.1000	25	0.022	0.033
$\mathrm{M1_{S2'}}$	0.2017	0.4013	0.1311	0.2677	4.822	1.464	0.2365	0.1000	24	0.018	0.033
$\mathrm{M1_{S3}}$	0.2022	0.4129	0.2145	0.2002	4.875	1.553	0.3679	0.2493	39	0.030	0.030
$\mathrm{M0_{S3}}$	0.2058	0.4172	0.2821	**0.0000**	4.917	1.624	0.4616	0.3514	39	0.031	0.031



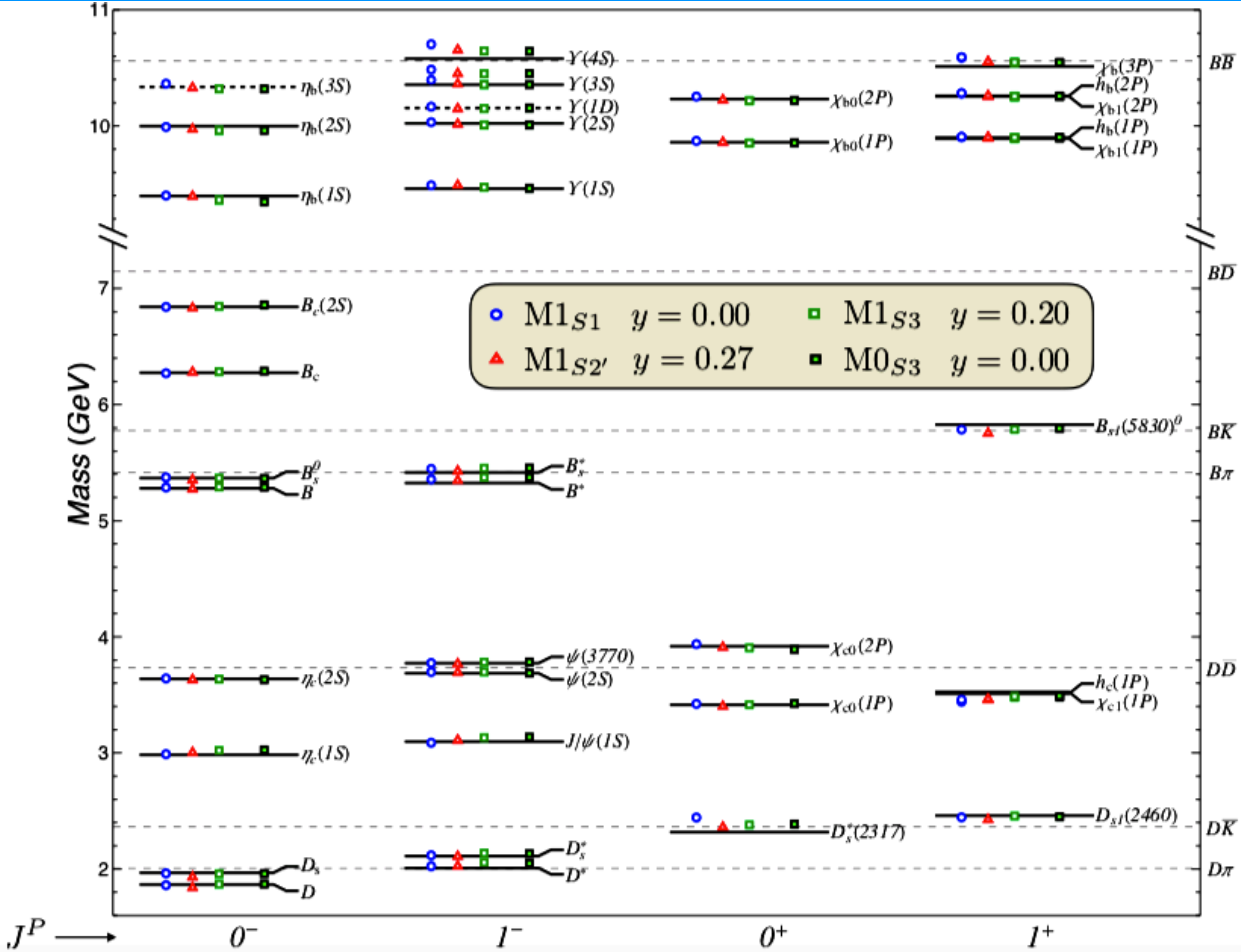
- The **quality** of fits in not much improved.
- Best model M1S3 has y=0 (but minimum very shallow).

The mass spectrum alone does not constrain much the parameter y and the quark constituent masses!

➡ *To investigate* wether or not other physical observables (e.g. decay constants) are more stringent with respect to y and the masses.

➡ *Vector contributions to the linear confining interaction between 0% and ~30% lead essentially to the same agreement with the data.*

Fits with variable quark masses and confinement (S+PS)-V mixing *y*



CST amplitudes *or* “wave-functions”

- Equation for the vertex function,

$$\Gamma_{1CS}(\hat{p}_1,p_2) = -\int \frac{d^3k}{(2\pi)^3}\frac{m_1}{E_{1k}}\sum_K V_K(\hat{p}_1,\hat{k}_1)\Theta_1^{K(\mu)}\frac{m_1+\hat{\not k}_1}{2m_1}\Gamma_{1CS}(\hat{k}_1,k_2)\frac{m_2+\not k_2}{m_2^2-k_2^2-i\epsilon}\Theta^K_{2(\mu)}$$

we solve it for the following matrix elements

CST vertex functions

$$\Gamma^{+\rho'}_{\lambda\lambda'}(p) \equiv \bar{u}_1^{+}(\mathbf{p},\lambda)\Gamma(p)u_2^{\rho'}(\mathbf{p},\lambda')$$

CST “wave functions”

$$\Psi^{+\rho}_{1,\lambda_1\lambda_2}(k) \equiv \sqrt{\frac{m_1m_2}{E_{1k}E_{2k}}}\frac{\rho}{E_{2k}-\rho(E_{1k}-\mu)}\Gamma^{+\rho}_{\lambda_1\lambda_2}(k)$$

- We expand in a basis

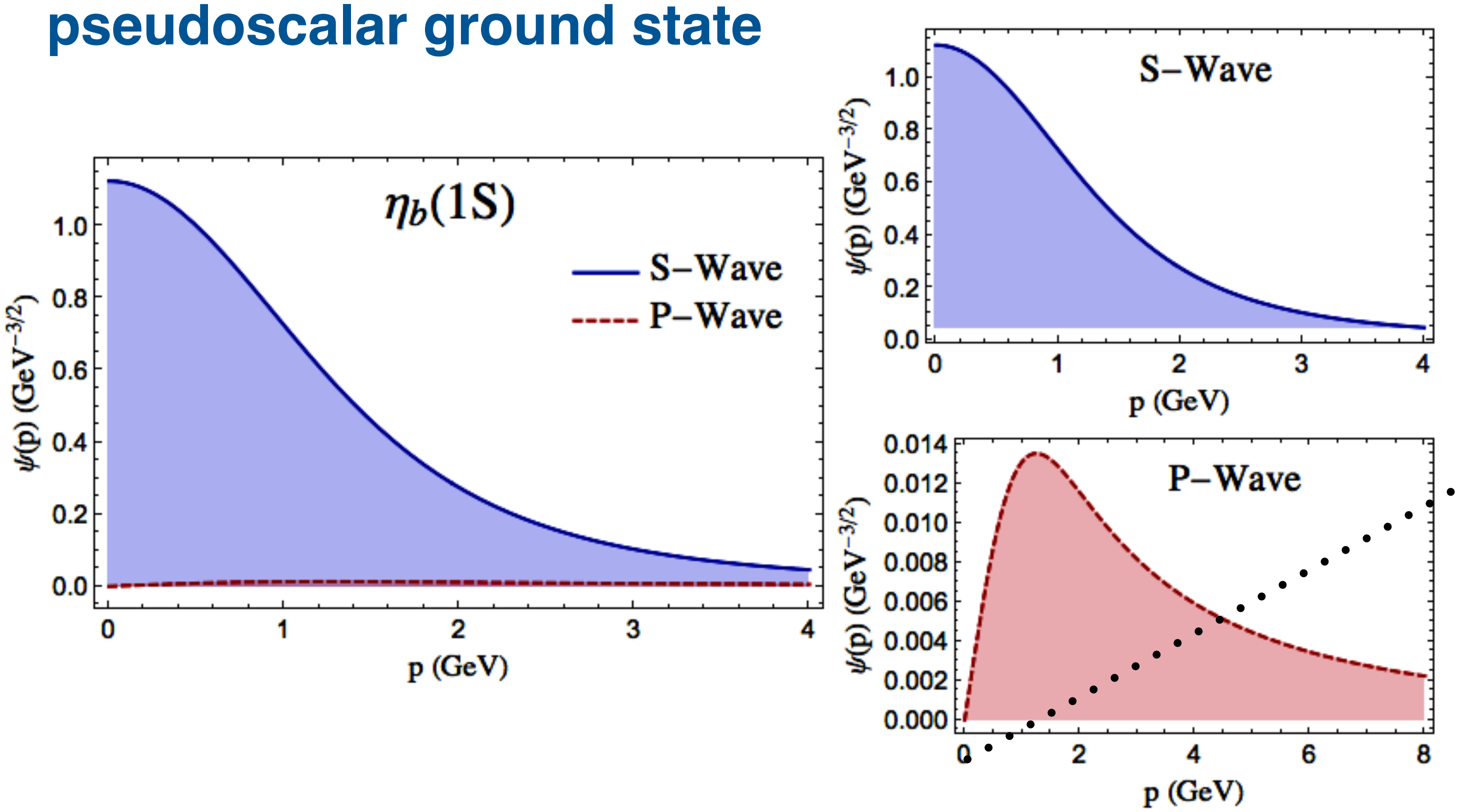
$$\Psi^{+\rho}_{1,\lambda\lambda'}(p) = \sum_j \underline{\psi_j^\rho(p)}\chi^\dagger_\lambda(\hat{\mathbf{p}})\,K_j^\rho(\hat{\mathbf{p}})\,\chi_{\lambda'}(\hat{\mathbf{p}})$$

J^P	$K_1^-(\hat{\mathbf{p}})$	Wave	$K_2^-(\hat{\mathbf{p}})$	Wave	$K_1^+(\hat{\mathbf{p}})$	Wave	$K_2^+(\hat{\mathbf{p}})$	Wave
0^-	$\mathbf{1}$	S	-	-	$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}$	P	-	-
0^+	$\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}$	P	-	-	$\mathbf{1}$	S	-	-
1^-	$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}$	S	$\frac{1}{\sqrt{2}}\left(3\boldsymbol{\xi}\cdot\hat{\mathbf{p}}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}\right)$	D	$\sqrt{3}\boldsymbol{\xi}\cdot\hat{\mathbf{p}}$	P_s	$\sqrt{\frac{3}{2}}\left(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}-\boldsymbol{\xi}\cdot\hat{\mathbf{p}}\right)$	P_t
1^+	$\sqrt{3}\boldsymbol{\xi}\cdot\hat{\mathbf{p}}$	P_s	$\sqrt{\frac{3}{2}}\left(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}-\boldsymbol{\xi}\cdot\hat{\mathbf{p}}\right)$	P_t	$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}$	S	$\frac{1}{\sqrt{2}}\left(3\boldsymbol{\xi}\cdot\hat{\mathbf{p}}\,\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}-\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\xi}}\right)$	D

which **explicitly** displays its orbital-angular-momentum content.

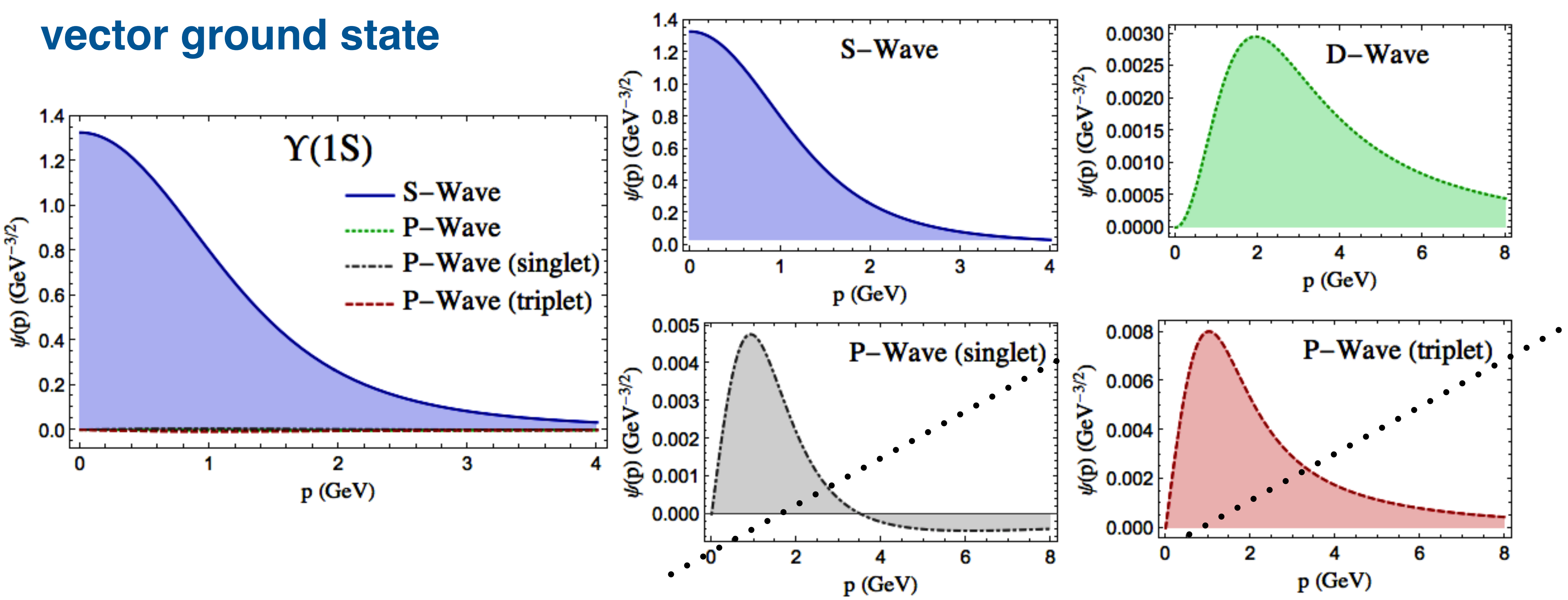
CST amplitudes *or* "wave-functions"

- **pseudoscalar ground state**

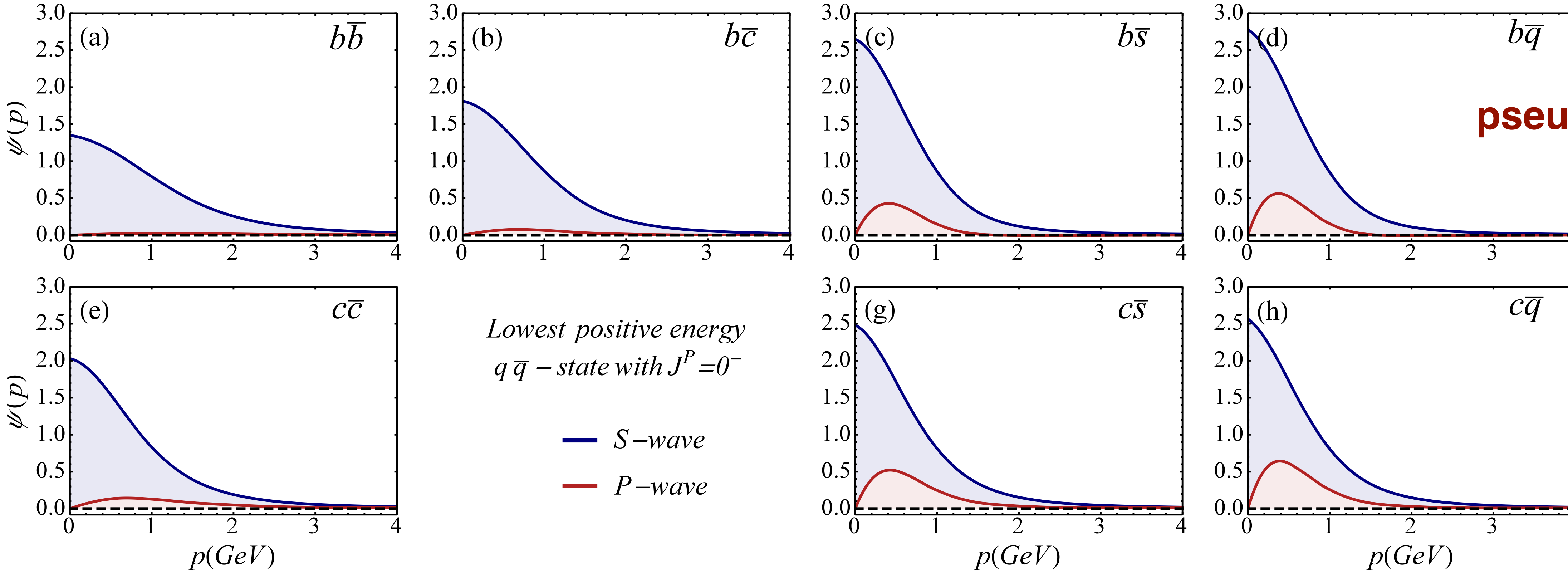


these components would vanish in the NR limit

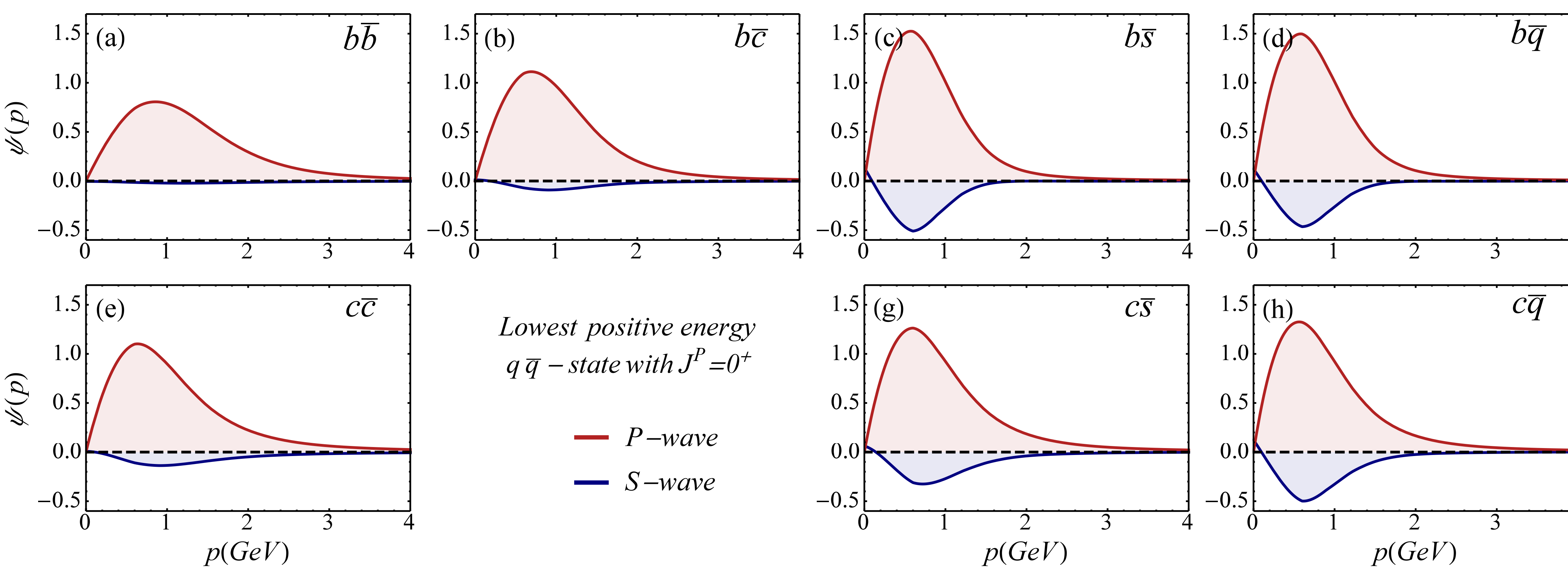
- **vector ground state**



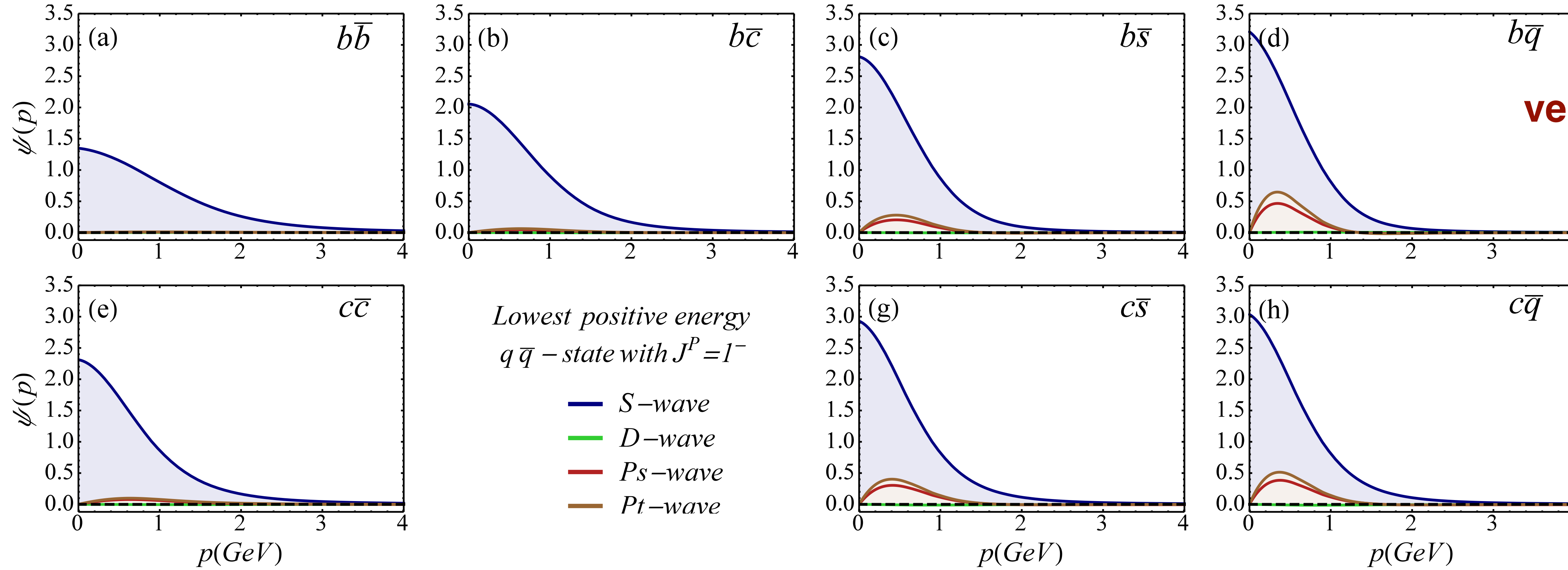
Importance of the relativistic components



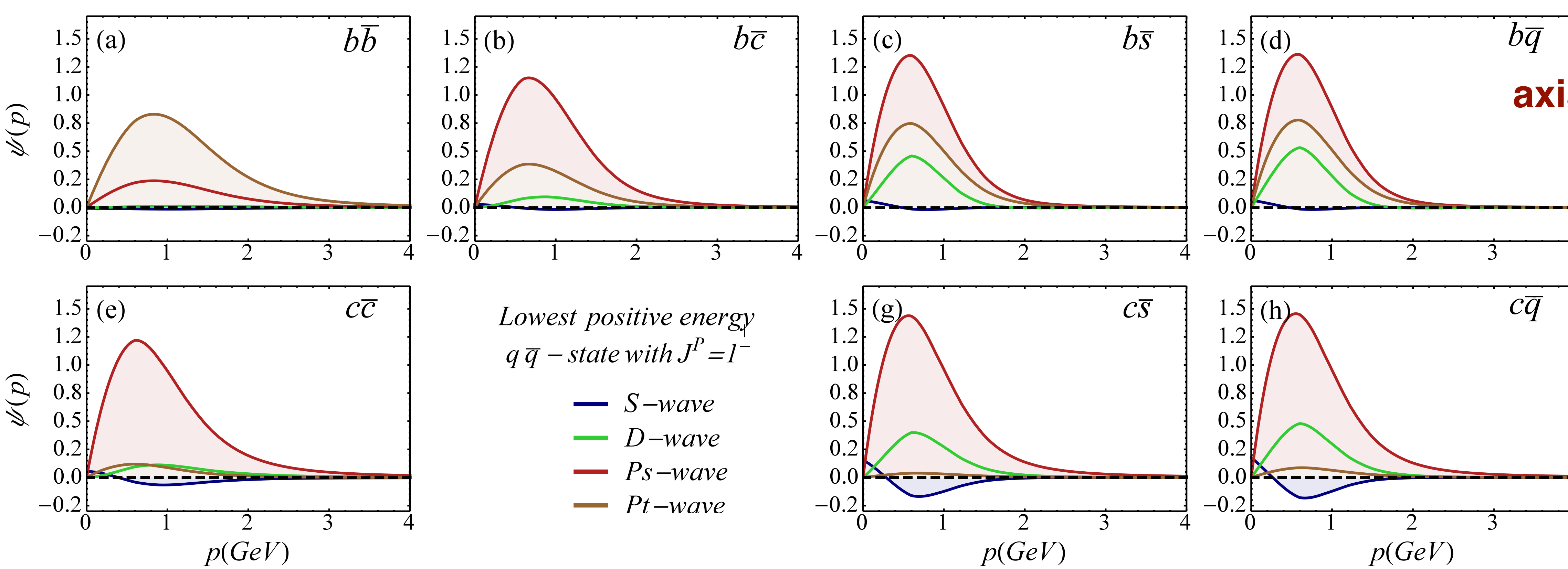
pseudoscalar



scalar

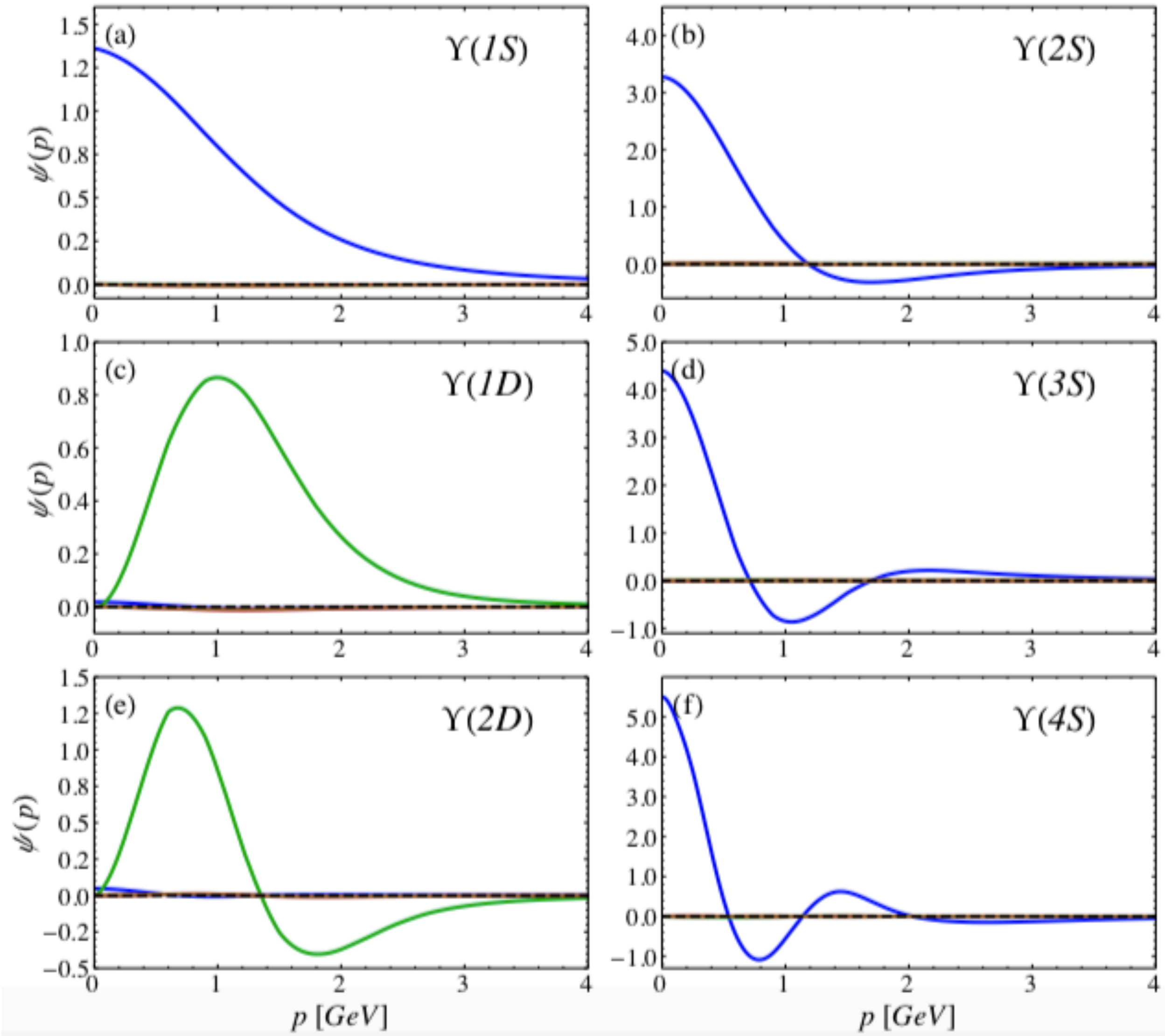


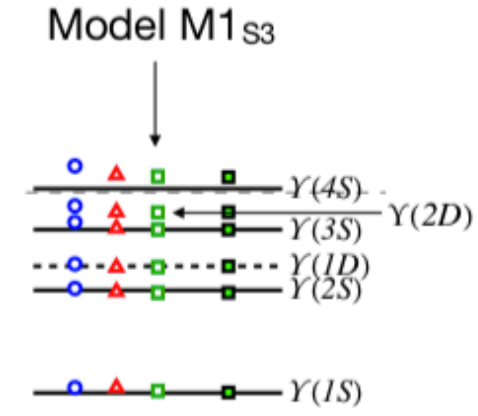
vector

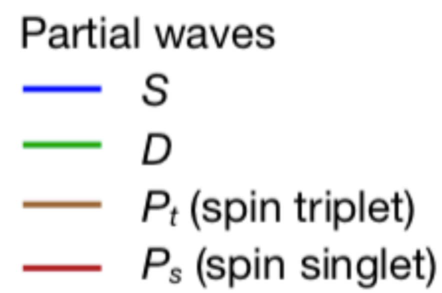


axial vector

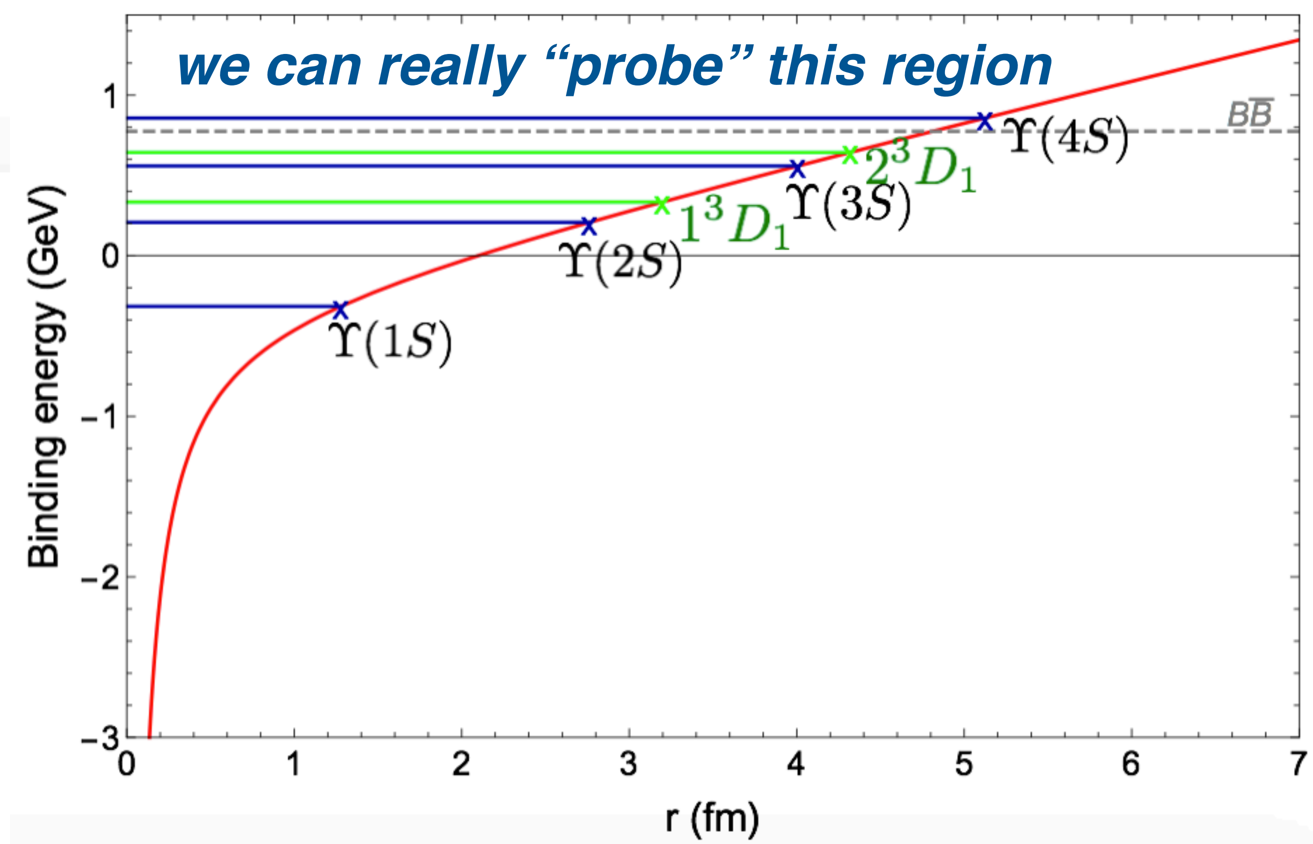
- **Vector bottomonium**

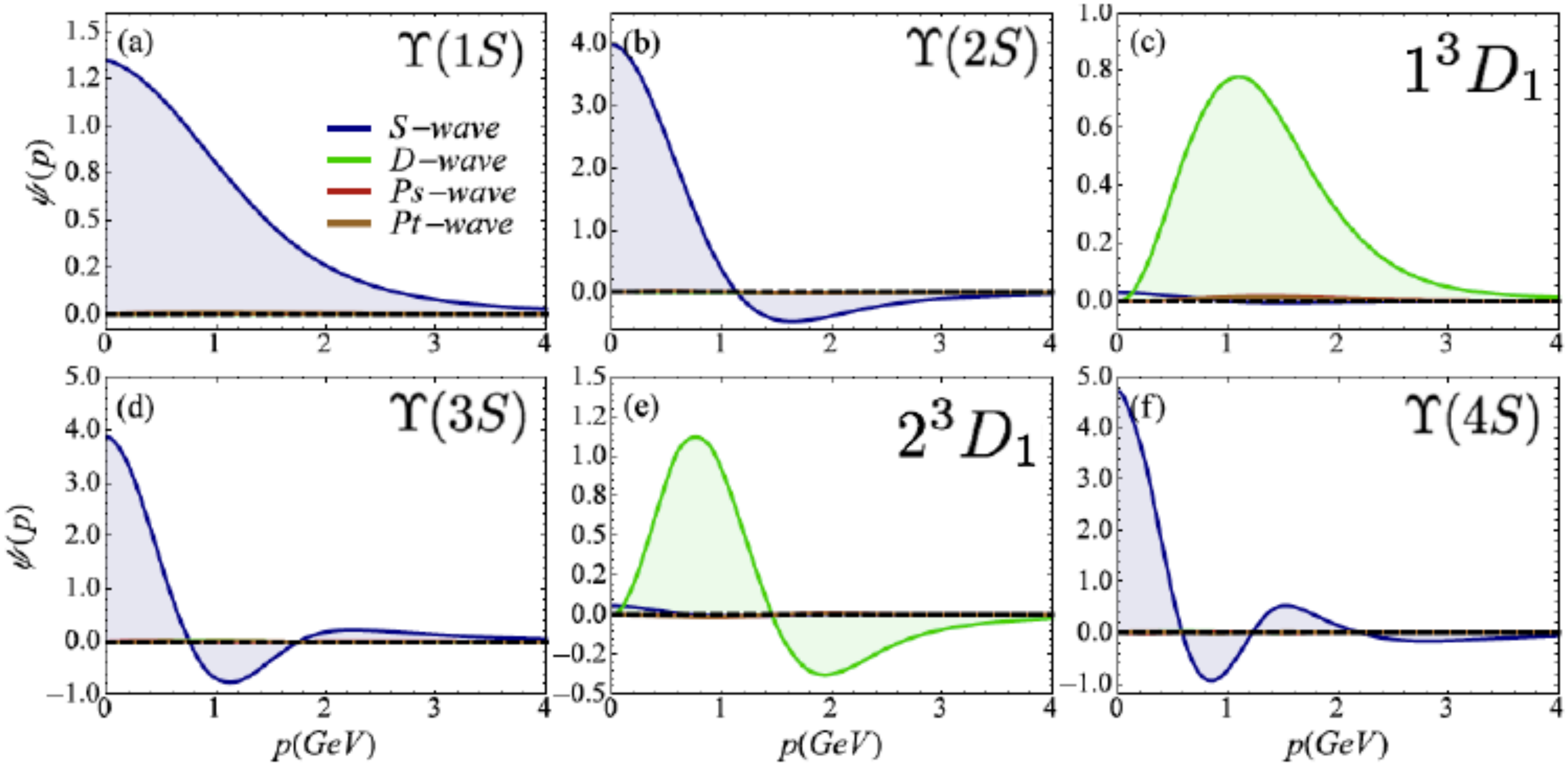






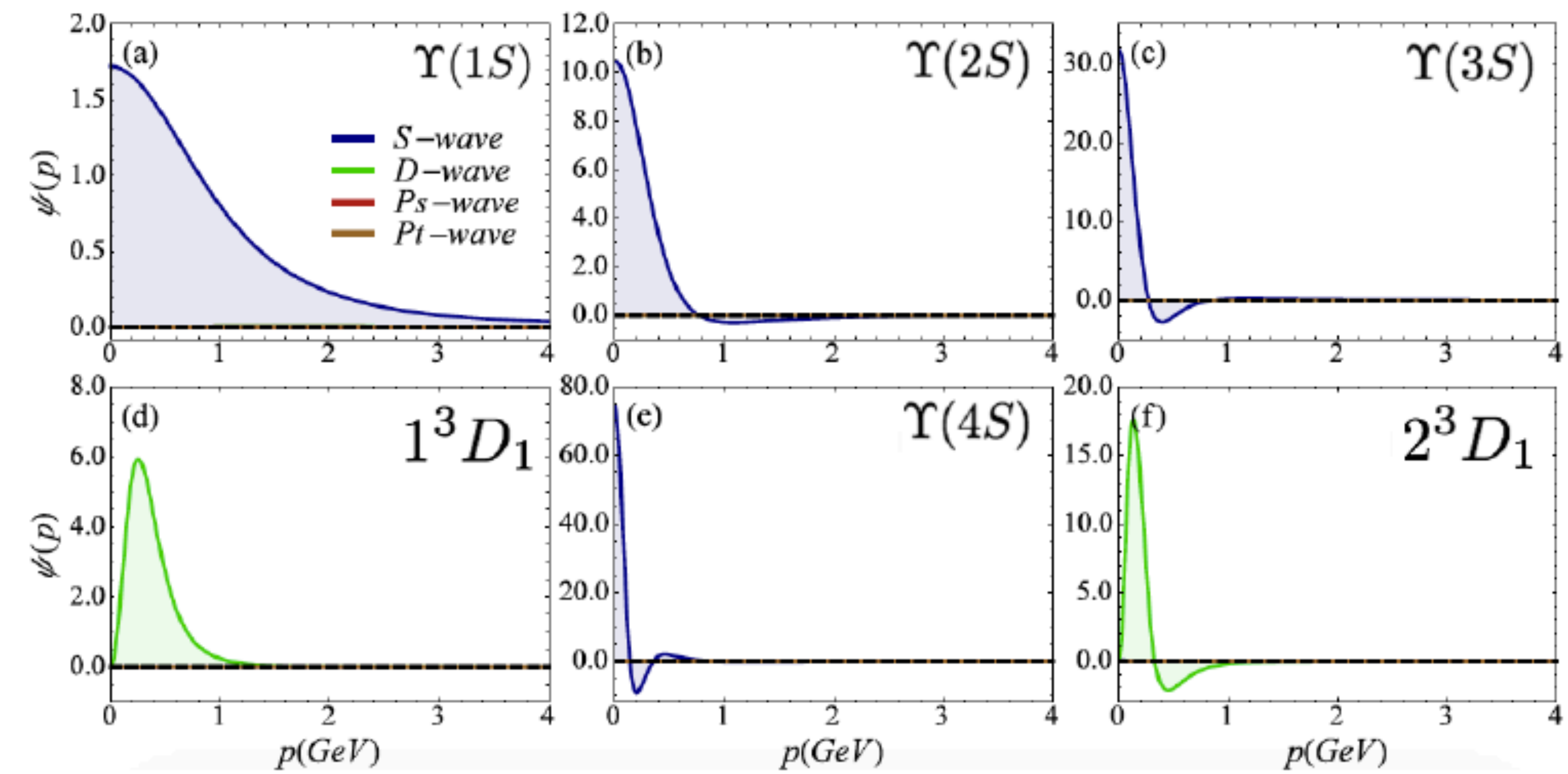
➡ Advantage of our numerical implementation: solve for a whole tower of radially excited states *simultaneously*!





➡ We observed that, without the linear confining piece, the level ordering between S and D states would be interchanged!

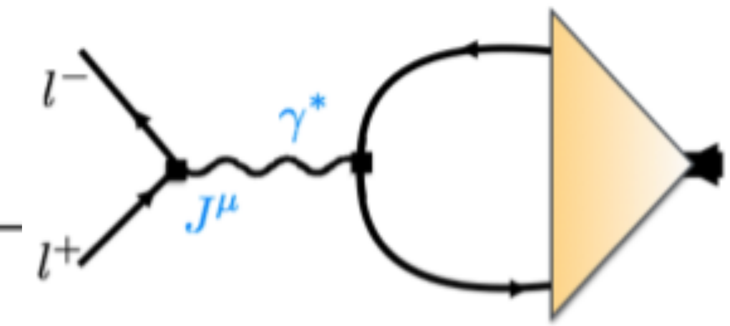
A linear piece in the CST kernel seems to be important to get the radial excited states right!



Decay constants for heavy-quarkonia

(in preparation…)

Quark content	n	Meson	$J^{P(C)}$	PDG	Lattice	DSE I	DSE II	BLFQ	$\mathrm{M}_{Q\bar{Q}}\Lambda_{\mathrm{OGE}}$ (this work)
	1	$\eta_b(1S)$	0^{-+}	–	667^{+6}_{-6}	773	756	589	795
	2	$\eta_b(2S)$	0^{-+}	–	–	419(8)	285	427	596
	3	$\eta_b(3S)$	0^{-+}	–	–	534(57)	333	331	536
	4	$\eta_b(4S)$	0^{-+}	–	–	–	40(15)	–	503
	1	$\Upsilon(1S)$	1^{--}	689^{+5}_{-5}	649^{+31}_{-31}	768	707	689	703
$b\bar{b}$	2	$\Upsilon(2S)$	1^{--}	479^{+4}_{-4}	481^{+39}_{-39}	467(17)	393	484	573
	3	1^3D_1	1^{--}	–	–	41(7)	371(2)	4.2	26
	4	$\Upsilon(3S)$	1^{--}	414^{+4}_{-4}	–	–	9(5)	366	536
	5	2^3D_1	1^{--}	–	–	–	165(50)	–	38
	6	$\Upsilon(4S)$	1^{--}	328^{+17}_{-18}	–	–	20(15)	–	518
	1	$\eta_c(1S)$	0^{-+}	330^{+13}_{-13}	393^{+9}_{-9}	401	378	368	547
	2	$\eta_c(2S)$	0^{-+}	211^{+35}_{-42}	–	244(12)	82	280	461
	3	$\eta_c(3S)$	0^{-+}	–	–	145(145)	206	–	417
$c\bar{c}$	4	$\eta_c(4S)$	0^{-+}	–	–	–	87	–	387
	1	J/ψ	1^{--}	407^{+5}_{-5}	405^{+6}_{-6}	450	411	404	525
	2	$\psi(2S)$	1^{--}	290^{+2}_{-2}	–	30(3)	155	290	531
	3	$\psi(3770)$	1^{--}	97.7^{+3}_{-3}	–	118(91)	45	0.9	98



- The obtained **decay constants** are in reasonable agreement with experiment and other theoretical approaches.

➡ extremely sensitive to the UV asymptotic behavior,

➡ Fitting the decay constants, we may infer what is the "*correct*" fall-off of the wave-functions.

Part II - Comparison of two Minkowski-space approaches

- **Goals of this work:**

  (i) to compare two fully relativistic methods in the case of heavy quarkonia: CST and Basis Light-Front Quantization (BLFQ). Yang Li, James P. Vary, Pieter Maris *et al.*

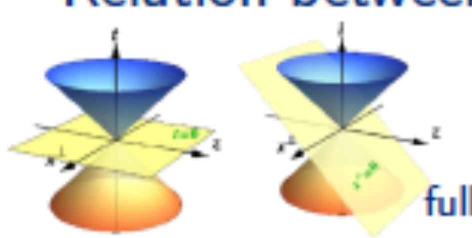
  (ii) to establish a connection between equal-time wave functions and light-front wave functions.

- ***Why*?**
  - ➡ Comparative studies are both important and needed.

    robustness tests, better control of model dependencies

  - ➡ It would be very useful to import the knowledge from equal-time methods towards the determination of Light-front wave functions, as they are used in a variety of high-energy hadronic processes and experimentally observable quantities.

Relation between equal-time and light-front wave functions?

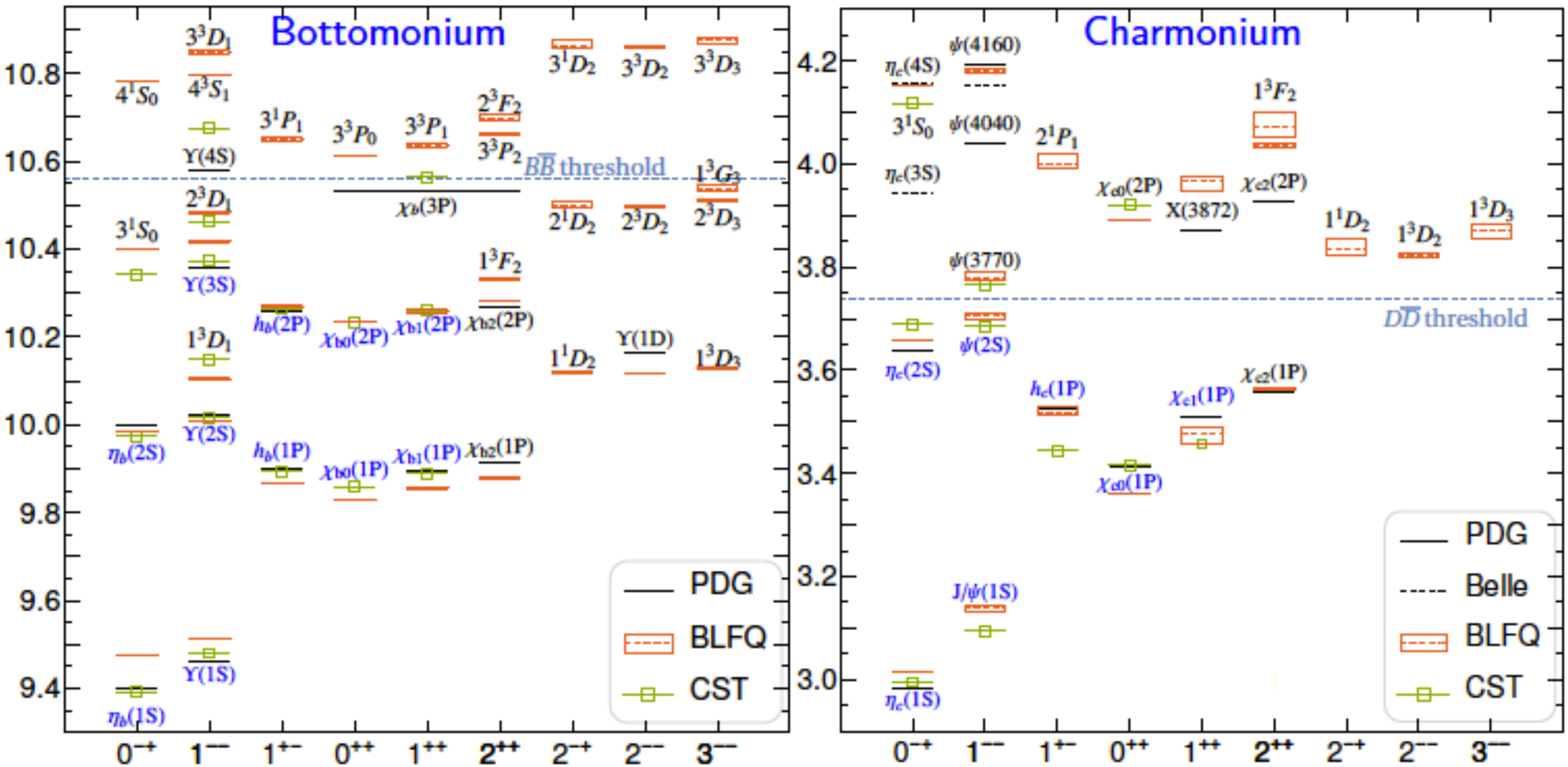


$$\underbrace{\Psi(k;P)}_{\text{full Bethe-Salpeter wave function}} \begin{cases} \text{integrate over } k^0 \rightarrow \Psi_{IF}(\boldsymbol{k};P) \\ \text{integrate over } k^- \rightarrow \Psi_{LF}(\boldsymbol{k}_\perp, x; P) \end{cases}$$

- **Advantage:** Both CST and BLFQ are formulated directly in Minkowski-space!

SL *et al.*, EPJC **77**, 696 (2017)

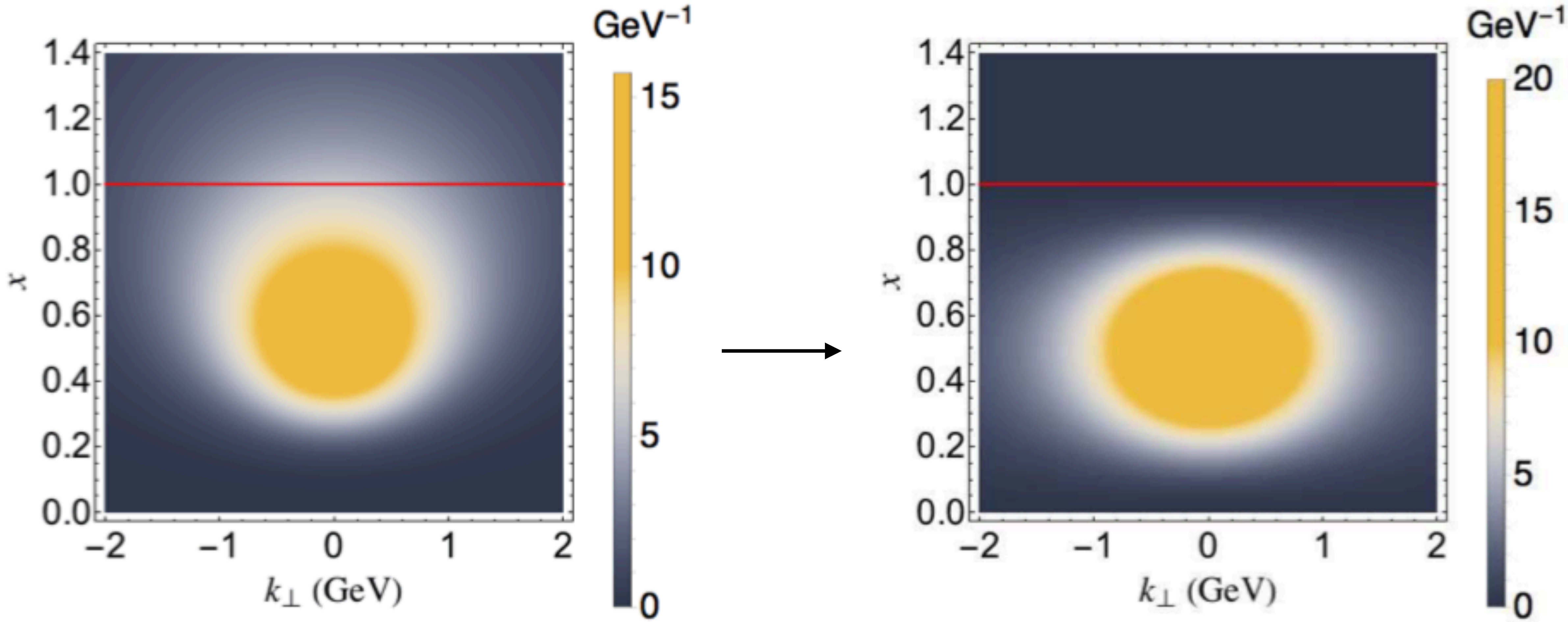
Comparing the spectroscopy



CST light-front wave functions

- CST solutions in the **meson rest frame** $P = (M, \mathbf{0})$

wave function of J/ψ (1S) with λ=0



Brodsky-Huang-Lepage prescription

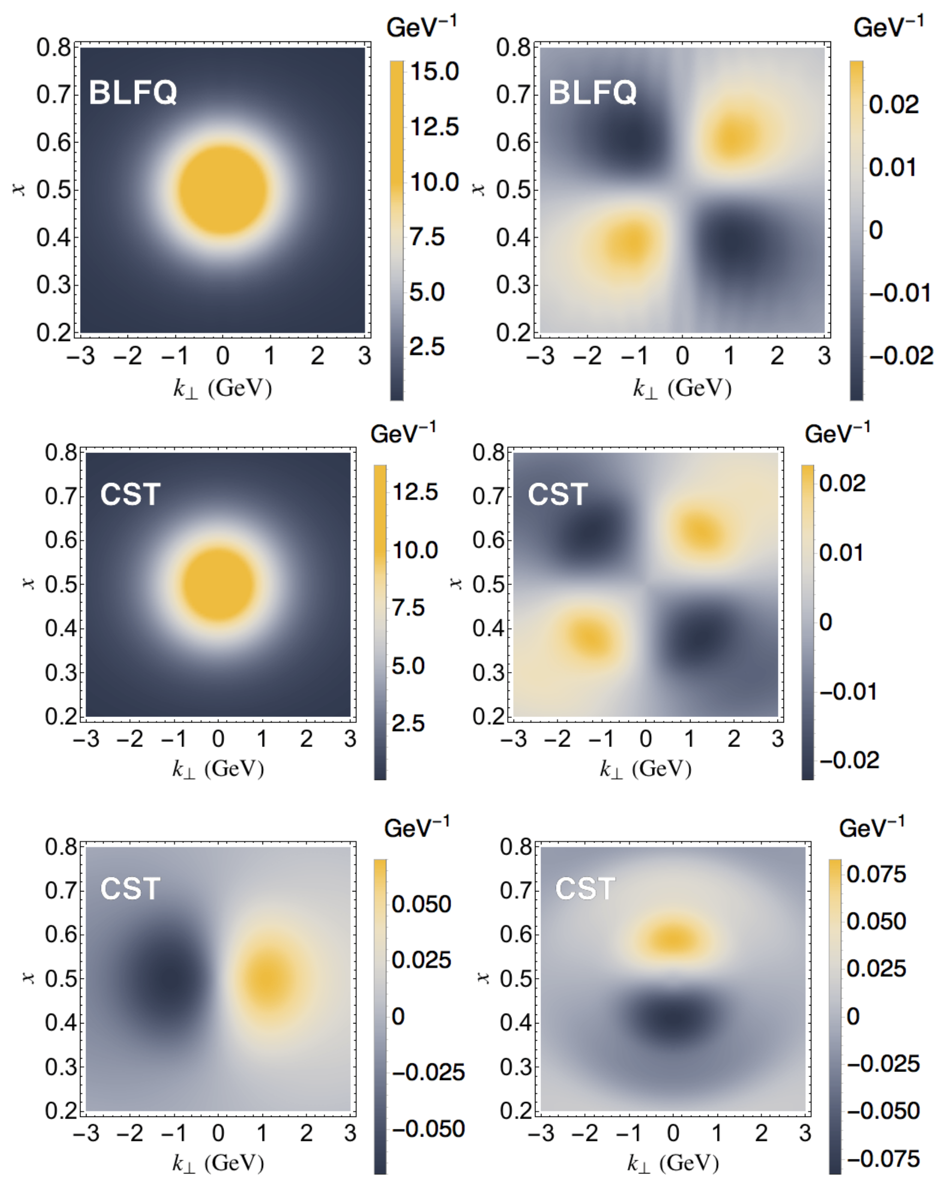
$$x = \frac{k^+}{P^+} = \frac{E_k + k^3}{M} = \frac{\sqrt{m^2 + \mathbf{k}_\perp^2 + (k^3)^2} + k^3}{M}$$

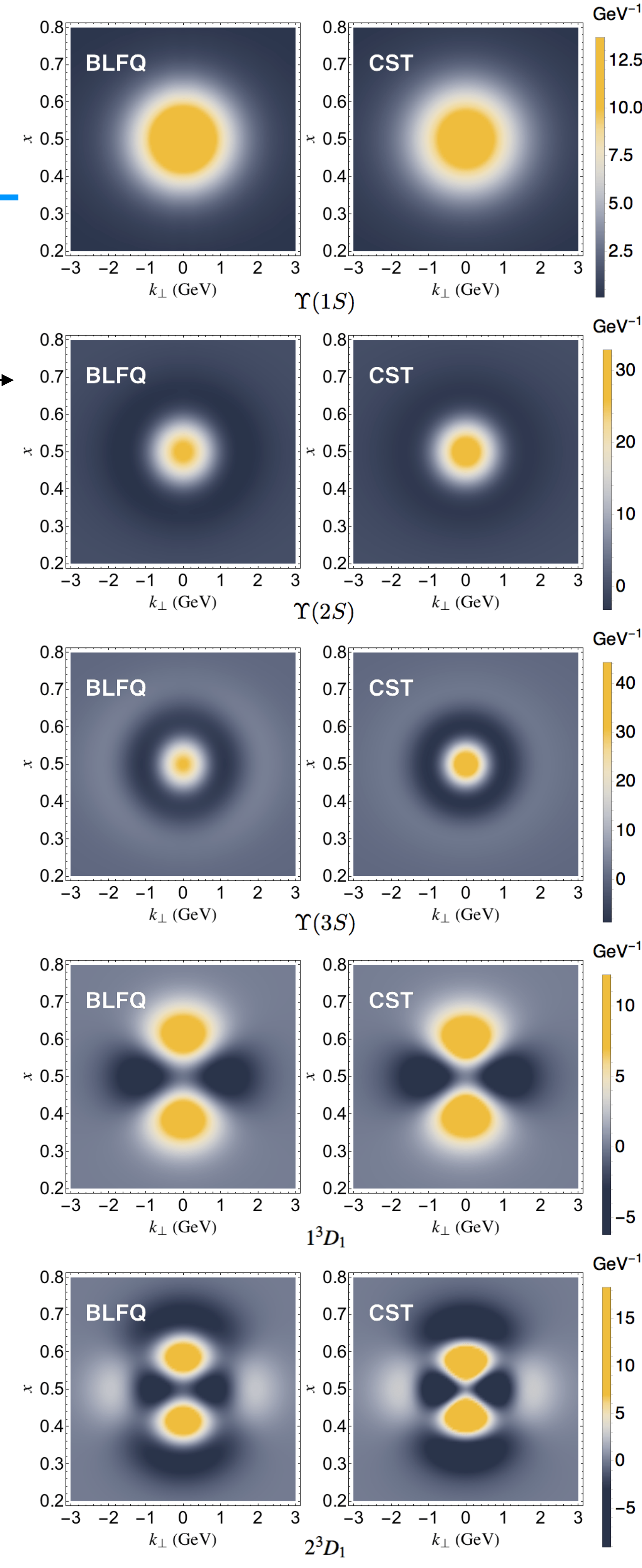
$$x = \frac{k^+}{P^+} \equiv \frac{E_k + k^3}{2E_k} = \frac{1}{2} + \frac{k^3}{2\sqrt{\mathbf{k}_\perp^2 + (k^3)^2 + m^2}}$$

- **Idea**: Investigate to what extent this *prescription* can be applied to heavy quarkonia!

Comparing LFWFs

- For the **dominant components**: qualitative features in perfect agreement, even for the higher radial excitations
- For the **subdominant components**: extra components emerge from the CST amplitude's and are absent in BLFQ because there positive energy and negative energy states do not mix.





Light-cone distributions

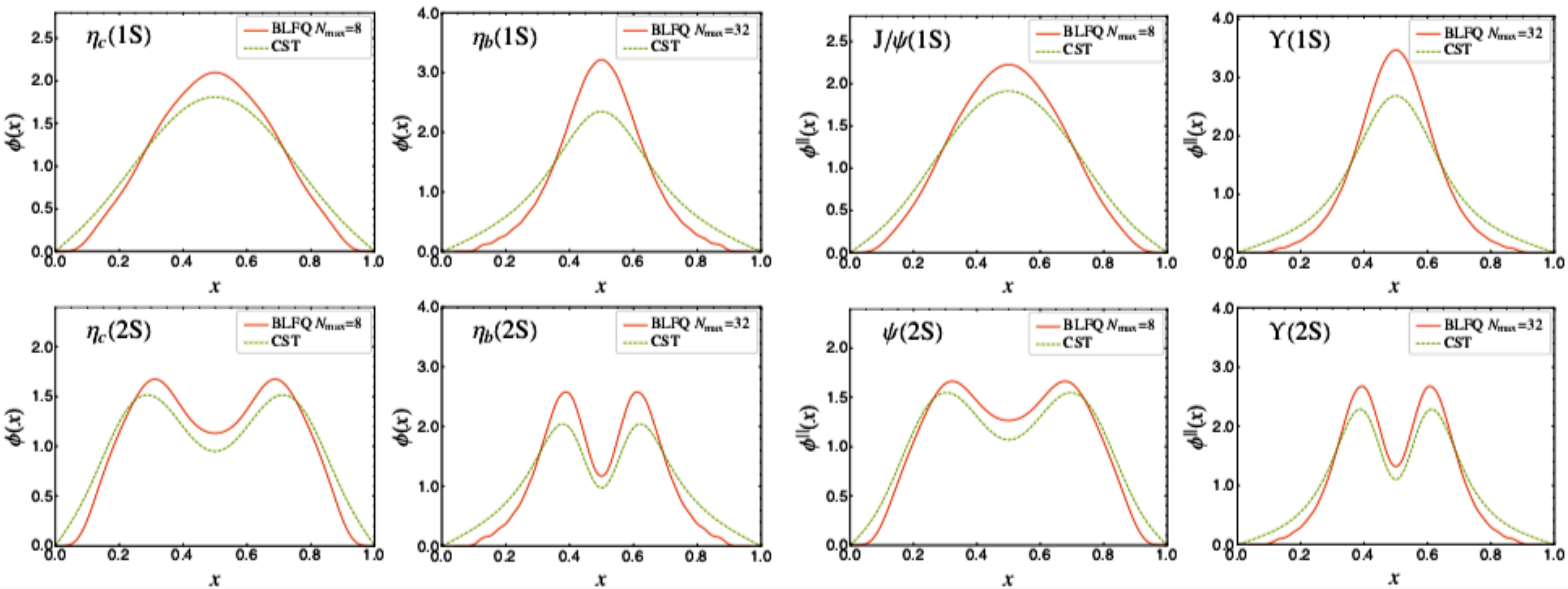
- Leading twist **Distribution amplitudes** (**DAs**)

$$\frac{f_{P,V}}{2\sqrt{2Nc}}\phi_{P,V^{||}}(x;\mu) = \frac{1}{\sqrt{x(1-x)}}\int_0^{k_\perp\leq\mu}\frac{d^2\mathbf{k}_\perp}{2(2\pi)^3}\psi^{\lambda=0}_{\uparrow\downarrow\mp\downarrow\uparrow}(\mathbf{k}_\perp,x)$$

- PS
+ V

Pseudoscalar quarkonia — **Vector quarkonia**

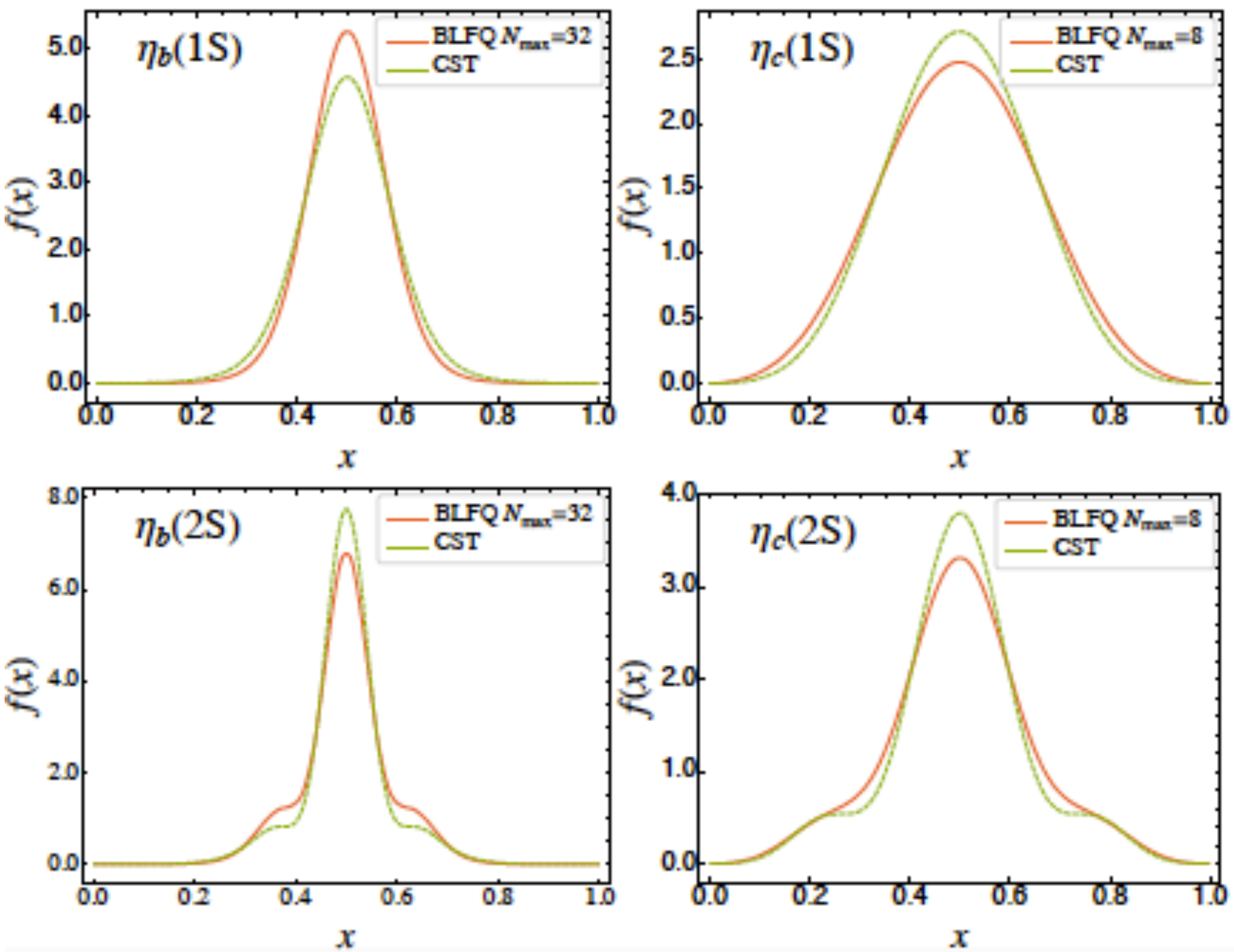
Charmonium | Bottomonium | Charmonium | Bottomonium



Light-cone distributions

- Similar to the DAs, **Parton distribution functions** (**PDFs**) depend on the UV cut-off scale.

$$f(x;\mu) = \frac{1}{2x(1-x)} \sum_{s,\bar{s}} \int_0^{\leq \mu^2} \frac{d^2\mathbf{k}_\perp}{(2\pi)^3} |\psi_{s\bar{s}}(\mathbf{k}_\perp, x)|^2$$

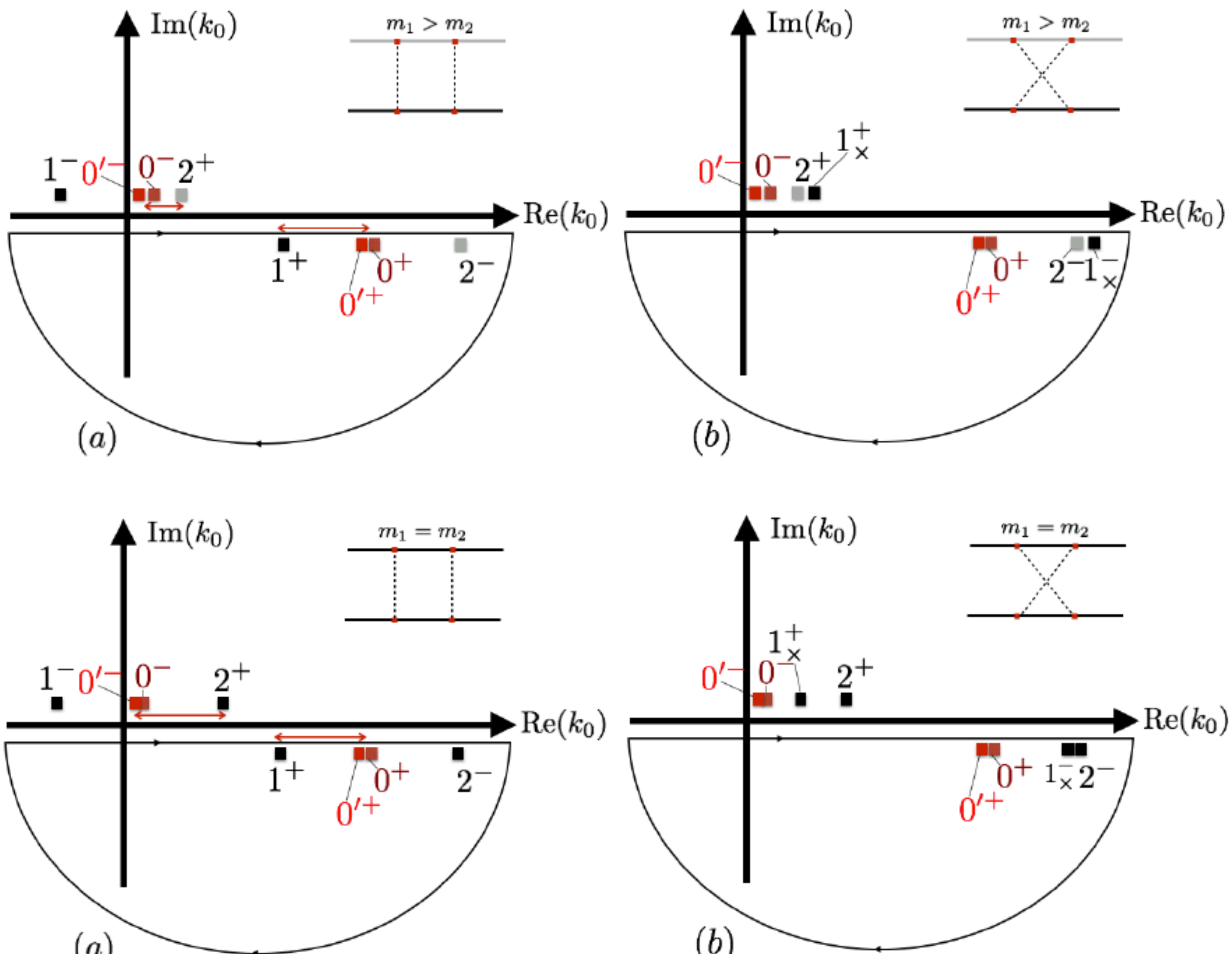


Summary and Outlook

- We have used the **CST approach** to construct a **covariant interaction model** that provides an **accurate global description** of **heavy** and **heavy-light mesons.**
    - ➡ Several phenomenological aspects can be observed with the analysis of the wave functions.
    - ➡ To do: Use a running coupling constant and replace the fixed constituent quark mass by a running quark mass.
    - ➡ Use the obtained wave functions in the calculation of other experimentally observable quantities (elastic and transition form factors, etc…) .
- Showed **some of the advantages** of performing comparative studies
    - ➡ Using the BHL map, we were able to observe the same overall features as in *genuine* LFWF (the solutions from BLFQ)
    - ➡ To do: Test the validity of the BHL map in the heavy-light case. Compare the CST LFWFs with other phenomenological LFWFs (e.g. LFWFs motivated by the soft-wall AdS/QCD model).
    - ➡ Near future: Explore the established methods of the light-front field theory, and apply them using the new CST LFWFs. Compute relevant scattering processes.

Thank you for your attention!

Analysis of the poles



Complete set of CST equations