Total decay and transition rates from LQCD

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CRC-TR 211 Strong-interaction matter under extreme conditions

DFG

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Intro: Some problems with particle widths on the lattice…

- Finite volume does not allow a definition of in/out states.
- Finite volume energy-levels are discrete.
- Multi particle states in finite volume have power-law corrections instead of exponential. *Lüscher '91*
- As the energy is increased, the density of finite volume level is very high. No possible resolving of those levels.
- Minkowski real time is not Euclidean imaginary time. Real-time evolution studies are "in principle" not possible. (Bedaque talk on monday)
- In many cases a lot of multi particle channels are open with more than two particles in the final state. At some point Lüscher approach becomes inapplicable.

Total widths from LQCD

M.T. Hansen, H. Meyer, DR
arXiv:1704.08993

Suppose $\mathcal{H} = \mathcal{H}_{\rm QCD} + \mathcal{V}$ and $\mathcal{V}$ is a small perturbation.

Consider a QCD-stable single-particle state $|D, \mathbf{P}\rangle$

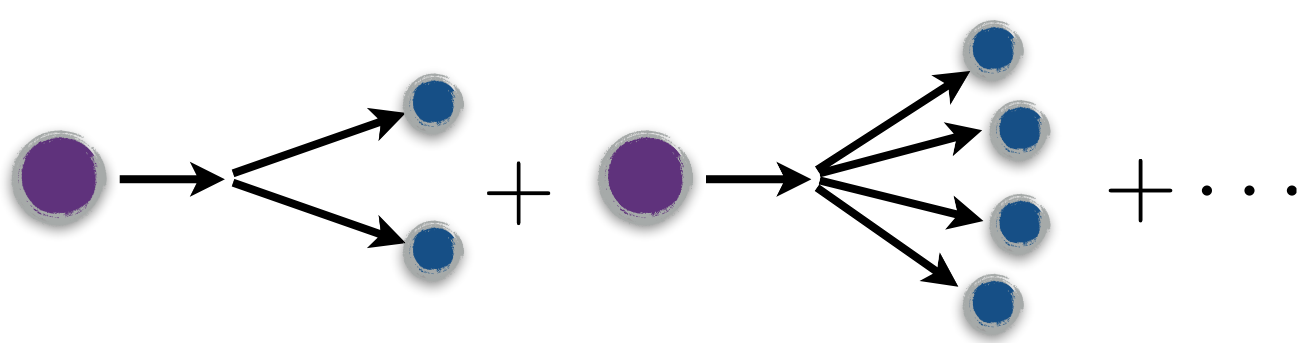
$$\left[\int d^3\mathbf{x}\mathcal{H}_{\rm QCD}(x)\right] |D, \mathbf{P}\rangle = E_D|D, \mathbf{P}\rangle$$

which can decay through $\mathcal{V}$ into a multi particle state $|E, \mathbf{p}, \alpha; \text{out}\rangle$

(imagine for now that $\mathcal{V}$ is some weak hamilton insertion)

$$\Gamma_{D\to\mathcal{Q}} \equiv \frac{1}{2M_D} \sum_\alpha \frac{1}{S_\alpha} \int d\Pi_\alpha(k_1, \cdots, k_{N_\alpha}) |\langle E_D, \mathbf{P}, \alpha; \text{out}|\mathcal{V}_\mathcal{Q}(0)|D, \mathbf{P}\rangle|^2$$

$$= \frac{1}{2M_D} \int d^4x \, \langle D, \mathbf{P}|\mathcal{V}_\mathcal{Q}(x)\mathcal{V}_\mathcal{Q}(0)|D, \mathbf{P}\rangle \,.$$



This is the total width into ALL allowed multi particle states with quantum numbers $\mathcal{Q}$.

We can generalise the previous relation:

Let $\mathcal{V}_{\mathcal{Q}} \to \mathcal{J}_{\mathcal{Q}}$ be a local current that can inject or carry away energy and/or momentum.

(This is useful for describing scattering with leptons or photons).

We define the *transition spectral function*:

$$\rho_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p}) = \int d^4x \; e^{i(E-E_N)t - i(\mathbf{p}-\mathbf{P})\cdot\mathbf{x}} \, \langle N,\mathbf{P}|\mathcal{J}^\dagger_{\mathcal{Q}}(x)\mathcal{J}_{\mathcal{Q}}(0)|N,\mathbf{P}\rangle \, .$$

and we obtain the less general result if we back-substitute:
$\mathcal{J}_{\mathcal{Q}} \to \mathcal{V}_{\mathcal{Q}}, \; E = E_N, \; \mathbf{p} = \mathbf{P}$

$$\Gamma_{N\to\mathcal{Q}} = \frac{1}{2M_N}\rho_{\mathcal{Q},\mathbf{P}}(E_N,\mathbf{P})$$

Widths can be obtained from spectral functions!

so far this was a continuum Minkowski discussion

Let's go to the lattice!!

Write the Euclidean correlator (*which is what we can calculate on the lattice*) most closely related to the previous discussion.

$$G_{\mathcal{Q},\mathbf{P}}(\tau,\mathbf{x},L) \equiv 2E_N L^6 e^{-E_N\tau + i\mathbf{P}\cdot\mathbf{x}} \lim_{\tau_f\to\infty}\lim_{\tau_i\to-\infty} \frac{\langle \hat{N}(\tau_f,\mathbf{P})\mathcal{J}^\dagger_{\mathcal{Q}}(\tau,\mathbf{x})\mathcal{J}_{\mathcal{Q}}(0)\hat{N}^\dagger(\tau_i,\mathbf{P})\rangle_{\text{conn}}}{\langle \hat{N}(\tau_f,\mathbf{P})\hat{N}^\dagger(\tau_i,\mathbf{P})\rangle},$$

$$= 2E_N L^3 e^{-E_N\tau + i\mathbf{P}\cdot\mathbf{x}} \langle N,\mathbf{P}|\mathcal{J}^\dagger_{\mathcal{Q}}(\tau,\mathbf{x})\mathcal{J}_{\mathcal{Q}}(0)|N,\mathbf{P}\rangle_L\,.$$

(its all about this 4-point function)

Take the Fourier transform and use complete set of now ***finite volume states***:

K.-F. Liu 1703.04690,
Hashimoto 1703.01881

$$\widetilde{G}_{\mathcal{Q},\mathbf{P}}(\tau,\mathbf{p},L) = 2E_N L^6 \sum_k e^{-E_k(L)\tau}|M_{k,N\to\mathcal{Q}}(\mathbf{p},L)|^2 \qquad \tau > 0\,,$$

$$M_{k,N\to\mathcal{Q}}(\mathbf{p},L) \equiv \langle E_k(L),\mathbf{p},\mathcal{Q}|\mathcal{J}_{\mathcal{Q}}(0)|N,\mathbf{P}\rangle_L\,.$$

We can rewrite it as:

$$\widetilde{G}_{\mathcal{Q},\mathbf{P}}(\tau,\mathbf{p},L) = \int_0^\infty \frac{d\omega}{2\pi}\, e^{-\omega\tau}\rho_{\mathcal{Q},\mathbf{P}}(\omega,\mathbf{p},L)\,,$$

$$\rho_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p},L) \equiv 2E_N\, L^6 \sum_k |M_{k,N\to\mathcal{Q}}(\mathbf{p},L)|^2\, 2\pi\,\delta\big(E - E_k(L)\big)\,,$$

Two big problems …

$$\Gamma_{N\to\mathcal{Q}} = \frac{1}{2M_N}\rho_{\mathcal{Q},\mathbf{P}}(E_N,\mathbf{P})$$

$$\rho_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p},L) \equiv 2E_N\, L^6 \sum_k |M_{k,N\to\mathcal{Q}}(\mathbf{p},L)|^2\, 2\pi\,\delta\big(E-E_k(L)\big)\,,$$

$L \to \infty$?!

$$\rho_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p}) \equiv \sum_\alpha \frac{1}{S_\alpha}\int d\Phi_\alpha(k_1,\cdots,k_{N_\alpha})|\langle E,\mathbf{p},\alpha;\mathrm{out}|\mathcal{J}_\mathcal{Q}(0)|N,\mathbf{P}\rangle|^2\,,$$

$$\widetilde{G}_{\mathcal{Q},\mathbf{P}}(\tau,\mathbf{p},L) = \int_0^\infty \frac{d\omega}{2\pi}\, e^{-\omega\tau}\rho_{\mathcal{Q},\mathbf{P}}(\omega,\mathbf{p},L)\,,$$

?!

This inverse Laplace transformation is an numerically ill-defined problem.

If we aim at **total** decay rates ….

The Backus-Gilbert Method solves both problems at once!

$$\widehat{\rho}_{\mathcal{Q},\mathbf{P}}(\overline{\omega},\mathbf{p},L,\Delta) \equiv \int_0^\infty d\omega\;\widehat{\delta}_\Delta(\overline{\omega},\omega)\,\rho_{\mathcal{Q},\mathbf{P}}(\omega,\mathbf{p},L)\,.$$

$$\int_0^\infty d\omega\,\widehat{\delta}_\Delta(\overline{\omega},\omega) = 1, \qquad \lim_{\Delta\to 0}\int_0^\infty d\omega\,\widehat{\delta}_\Delta(\overline{\omega},\omega)\phi(\omega) = \phi(\overline{\omega})\,,$$

$$\rho_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p}) = \lim_{\Delta\to 0}\lim_{L\to\infty}\widehat{\rho}_{\mathcal{Q},\mathbf{P}}(E,\mathbf{p},L,\Delta)\,,$$

order of limits is crucial!

(

(open big parenthesis …)

Some remarks on the BG method

$$G(\tau_i) = \int_0^\infty d\omega \;\phi(\omega) K(\omega, \tau_i)$$

- Developed in 1967 by geophysicists Backus and Gilbert to study the propagation of earthquakes on the Earth.
- It is a **linear method**: $\widehat{\delta}_\Delta(\overline{\omega}, \omega) = \sum_j C_j(\overline{\omega}, \Delta)\; e^{-\omega \tau_j}$,

$$\widehat{\phi}(\overline{\omega}, \Delta) = \int_0^\infty d\omega \, \widehat{\delta}_\Delta(\overline{\omega}, \omega) \phi(\omega) = \sum_j C_j(\overline{\omega}, \Delta)\; \widetilde{G}(\tau_j)\,.$$

- It **converges to the exact solution**:

$$\lim_{\Delta \to 0} \int_0^\infty d\omega \, \widehat{\delta}_\Delta(\overline{\omega}, \omega) \phi(\omega) = \phi(\overline{\omega})$$

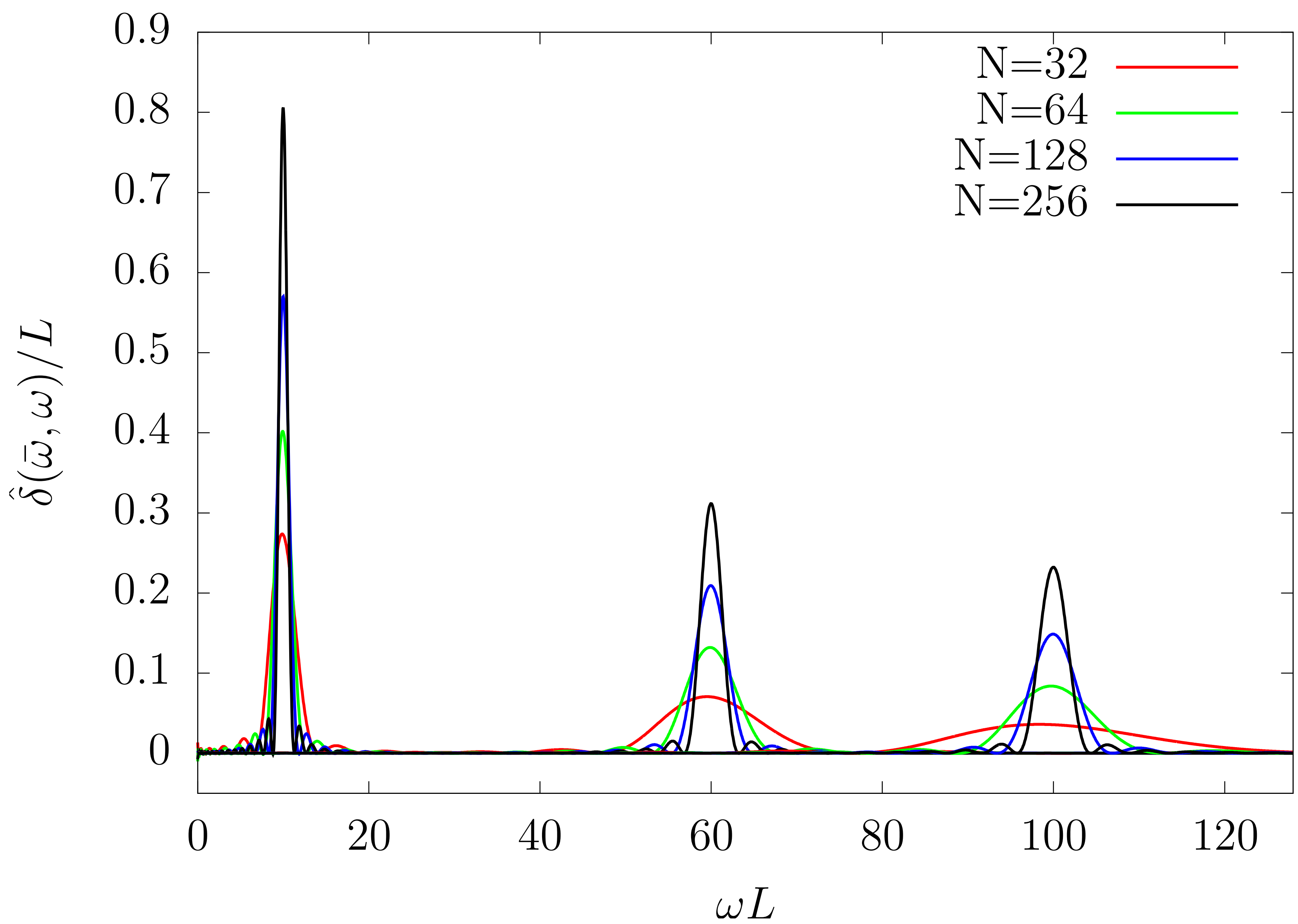
- No a priori Ansatz. **Model Independent estimator**.
- **No free lunch:** The regularisation of the problem is translated into a trade off between resolving power and error estimation.

Some remarks on the BG method

- The optimal coefficients $C_j(\overline{\omega}, \Delta)$ are calculated by minimizing

$$\Delta = \int_0^\infty d\omega \, (\bar{\omega} - \omega)^2 \, \widehat{\delta}_\Delta(\overline{\omega}, \omega)^2 \quad \text{subject to} \quad \int_0^\infty d\omega \, \widehat{\delta}_\Delta(\overline{\omega}, \omega) = 1$$

In principle one can make the width very small if enough points are available…



… but the scaling is logarithmic…

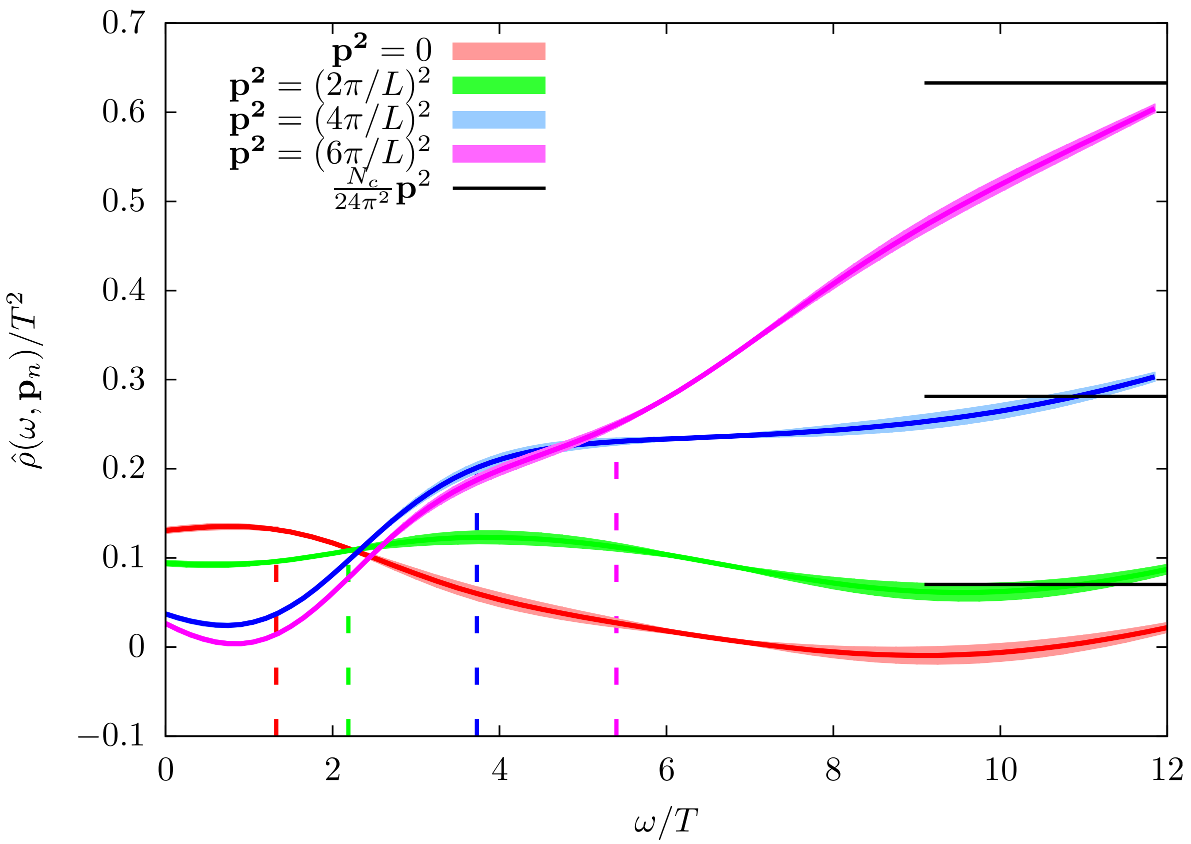
Extensions of the Backus-Gilbert

Finite-T applications:

Phys. Rev. D **92** 094510 (2015)
Brandt, Meyer, Francis, DR.

$$G(\tau_i, T, \mathbf{p}) = \int_0^\infty d\omega \left(\frac{\rho_{\mathrm{A}}(\omega, \mathbf{p})}{\tanh(\omega/2)} \right) \underbrace{\left(\frac{\cosh(\omega(\beta/2 - \tau_i))}{\cosh(\omega\beta/2)} \right)}_{\doteq K(\omega, \tau_i)}$$

$$\widehat{\delta}_\Delta(\overline{\omega}, \omega) = \sum_j C_j(\overline{\omega}, \Delta) \left(\frac{\cosh(\omega(\beta/2 - \tau_j))}{\cosh(\omega\beta/2)} \right)$$



its exact if the target is constant!!

EPJ Web of Conferences **175** 13021 (2018) Meyer, Hansen, DR.

- Modify threshold
- Subtract known contributions
- Magic omegas
- Combination with Lüscher appr.
- …

)

(… you better close it)

Back to our decay widths

So our strategy is:

1. Calculate the Euclidean 4-point function in question

$$\frac{\langle \hat{N}(\tau_f, \mathbf{P}) \mathcal{J}_{\mathcal{Q}}^{\dagger}(\tau, \mathbf{x}) \mathcal{J}_{\mathcal{Q}}(0) \hat{N}^{\dagger}(\tau_i, \mathbf{P}) \rangle}{\langle \hat{N}(\tau_f, \mathbf{P}) \hat{N}^{\dagger}(\tau_i, \mathbf{P}) \rangle},$$

This by itself can be quite challenging!

Boundaries, excited states contamination, signal to noise problems.

2. Run the Backus-Gilbert on it to obtain

$$\rho_{\mathcal{Q},\mathbf{P}}(E, \mathbf{p}) = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{\rho}_{\mathcal{Q},\mathbf{P}}(E, \mathbf{p}, L, \Delta),$$

$$\Gamma_{N \to \mathcal{Q}} = \frac{1}{2M_N} \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{\rho}_{\mathcal{Q},\mathbf{P}}(E_N, \mathbf{P}, L, \Delta),$$

3. Make sure you are in the right window of big volume and small Delta.

The order of the double limit

Take a scalar toy model and consider only

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g K \phi^2$$ what does the smoothing when $L \to \infty$?

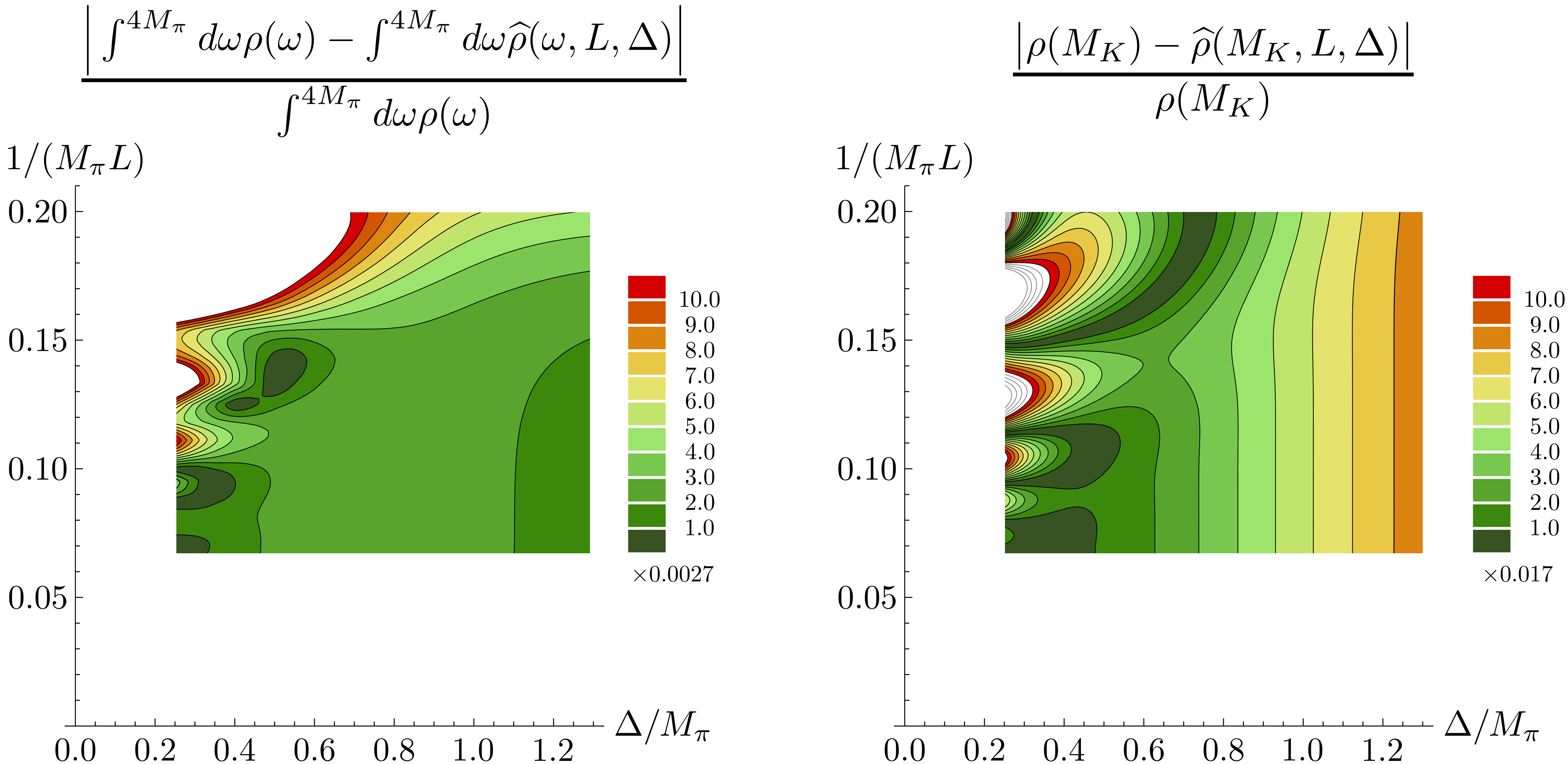
$$\rho_{\mathcal{Q},\mathbf{0}}(E,\mathbf{0},L) \equiv 2M_K\, L^6 \sum_k |M_{k,K\to\phi\phi}(\mathbf{0},L)|^2\, 2\pi\, \delta\big(E - E_k(L)\big)\,,$$

$$|M_{k,K\to\phi\phi}(\mathbf{0},L)|^2 = g^2 M_K^2 \frac{\nu_k}{4M_K E_k(L)^2 L^9}\bigg|_{E_k(L)=2\sqrt{M_\phi^2+(2\pi/L)^2\mathbf{q}_k^2}}\,,$$

$$\widehat{\rho}_{\mathcal{Q},\mathbf{0}}(\overline{\omega},\mathbf{0},L,\Delta) = \int_0^\infty d\omega\; \widehat{\delta}_\Delta(\overline{\omega},\omega)\rho_{\mathcal{Q},\mathbf{0}}(\omega,\mathbf{0},L)$$

for now we consider a normalised gaussian resolution function to study the effect of smoothing and the double limit <u>INSTEAD</u> of the resolution functions that come from BG.

$$\widehat{\delta}_\Delta(\overline{\omega},\omega) = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-(\omega-\overline{\omega})/(2\Delta^2)}$$



$\frac{\left|\int^{4M_\pi} d\omega \rho(\omega) - \int^{4M_\pi} d\omega \widehat{\rho}(\omega, L, \Delta)\right|}{\int^{4M_\pi} d\omega \rho(\omega)}$
$1/(M_\pi L)$
Δ/M_π
×0.0027
$\frac{\left|\rho(M_K) - \widehat{\rho}(M_K, L, \Delta)\right|}{\rho(M_K)}$
$1/(M_\pi L)$
Δ/M_π
×0.017

A more general toy case

- Consider the total decay into two open channels: a 2-particle decay + 3-particle decay.

  (our method could do it easily…)

$$3M_\pi < 2M_K < M_\phi$$

$$\mathcal{L}_{\rm int} = \frac{\lambda}{6}\phi(x)\pi(x)^3 + \frac{gM_\phi}{2}\phi(x)K(x)^2$$

- Work out the infinite volume transition spectral function:

$$\frac{1}{2M_\phi M_\pi}\rho_{\mathcal{Q},\mathbf{0}}(\omega,\mathbf{0}) = \frac{\lambda^2}{3072\pi^3}\left(\frac{\omega}{M_\pi}\right)^2\frac{M_\pi}{M_\phi}\mathcal{F}(\omega/M_\pi)\theta(\omega-3M_\pi) + \frac{g^2}{32\pi}\frac{M_\phi}{M_\pi}\sqrt{1-\frac{4M_K^2}{\omega^2}}\theta(\omega-2M_K)\,,$$

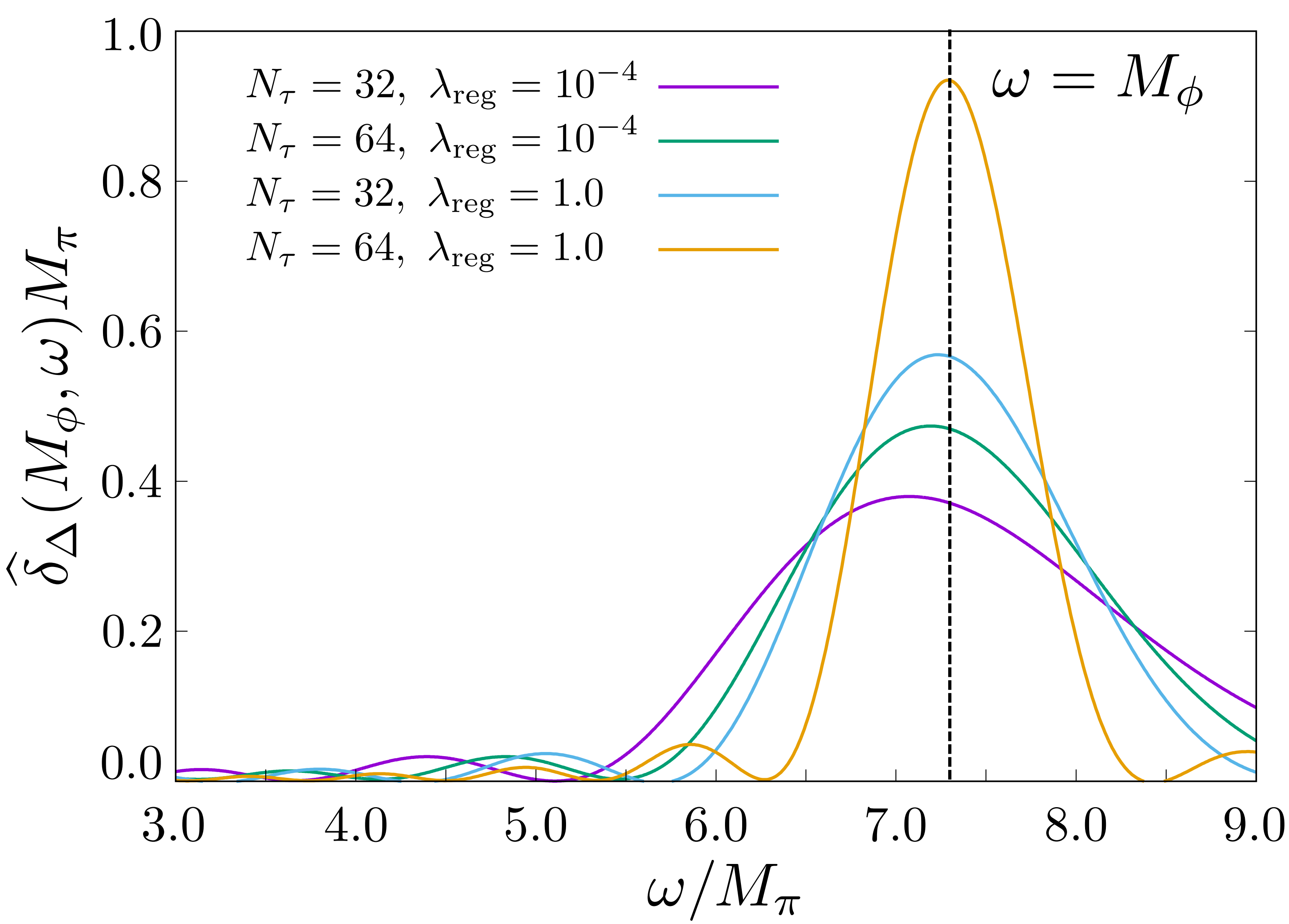
- Obtain the same from the BG output and compare to the exact result. In particular the width can be then extracted:

$$\rho_{\mathcal{Q},\mathbf{0}}(\overline{\omega},\mathbf{0}) = \lim_{\Delta\to 0}\lim_{L\to\infty}\widehat{\rho}_{\mathcal{Q},\mathbf{0}}(\bar{\omega},\mathbf{0},L,\Delta),$$

$$\frac{\Gamma_{\phi\to KK}}{M_\pi} + \frac{\Gamma_{\phi\to\pi\pi\pi}}{M_\pi} = \frac{1}{2M_\phi M_\pi}\lim_{\Delta\to 0}\lim_{L\to\infty}\widehat{\rho}_{\mathcal{Q},\mathbf{0}}(M_\phi,\mathbf{0},L,\Delta)\,.$$

Be more realistic

- We assume uncertainty on the finite volume Euclidean correlator (as it would be the situation on the lattice).
- This implies regulating the problem in order to give an error estimate on $\widehat{\rho}_{\mathcal{Q},\mathbf{0}}(\bar{\omega}, \mathbf{0}, L, \Delta)$



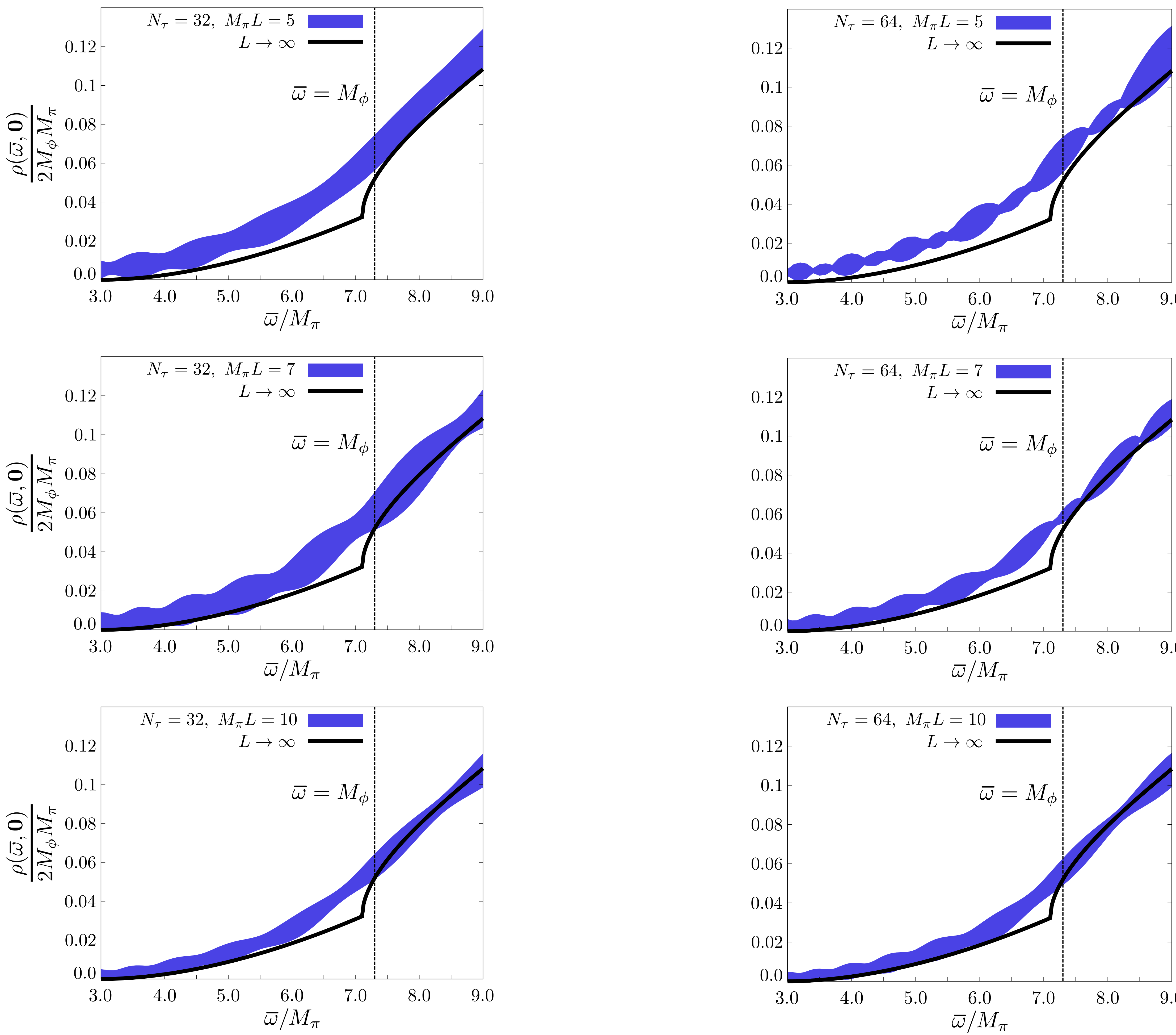
$$W_{ij}(\overline{\omega}) = \int_0^\infty d\omega e^{-\omega\tau_i}(\omega - \overline{\omega})^2 e^{-\omega\tau_j},$$

$$W_{ij}(\overline{\omega}) \to \lambda_{\rm reg} W_{ij}(\overline{\omega}) + (1 - \lambda_{\rm reg}) S_{ij}$$

<u>Error estimation:</u>

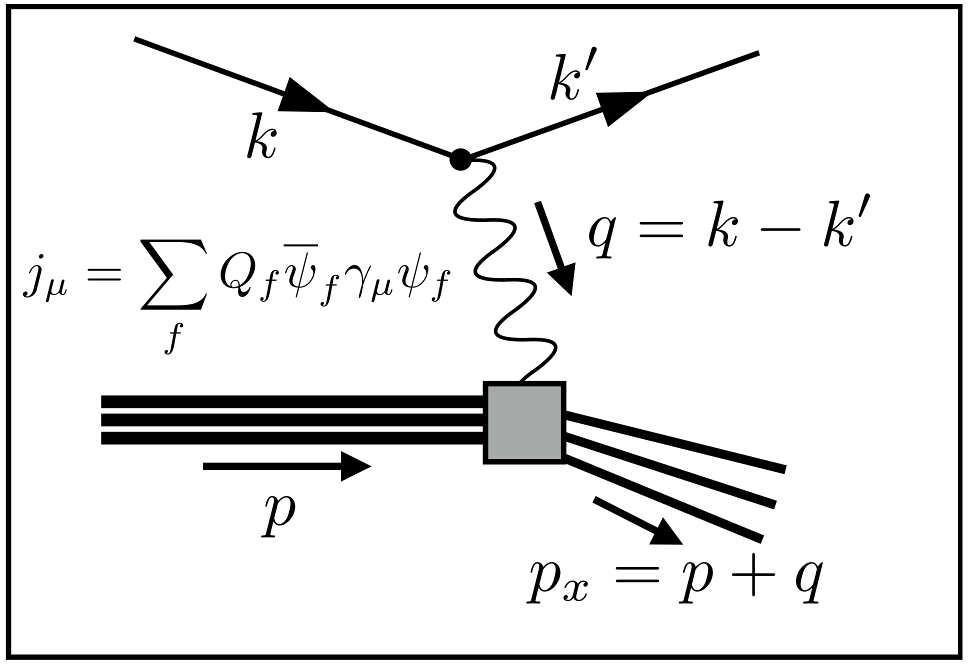
$$\Delta\widehat{\rho}_{\mathcal{Q},\mathbf{0}}(\bar{\omega}, \mathbf{0}, L, \Delta) = \sqrt{C_i(\overline{\omega}, \Delta) S_{ij} C_j(\overline{\omega}, \Delta)}$$

Think of $\lambda_{\rm reg}$ by asking:
What final uncertainty I want?
Can I achieve a good enough resolution with the number of points that I have?



$N_\tau = 32,\ M_\pi L = 5$
$L \to \infty$
$\overline{\omega} = M_\phi$
$\frac{\rho(\overline{\omega}, \mathbf{0})}{2M_\phi M_\pi}$
$\overline{\omega}/M_\pi$
$N_\tau = 64,\ M_\pi L = 5$
$N_\tau = 32,\ M_\pi L = 7$
$N_\tau = 64,\ M_\pi L = 7$
$N_\tau = 32,\ M_\pi L = 10$
$N_\tau = 64,\ M_\pi L = 10$

DIS ($e + p \longrightarrow e' + \text{hadrons}$)



"spin-averaged hadronic tensor:"

$$W_{\mu\nu}(p,q) = \frac{1}{4\pi\, n_\lambda} \sum_\lambda \int d^4x\, e^{iq\cdot x} \langle N, \mathbf{p}, \lambda | j_\mu(x) j_\nu(0) | N, \mathbf{p}, \lambda \rangle \,,$$

structure functions depend only on invariants and can be projected out:

$$W_{\mu\nu} = F_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{F_2}{p\cdot q} \left(p_\mu - \frac{p\cdot q\; q_\mu}{q^2} \right) \left(p_\nu - \frac{p\cdot q\; q_\nu}{q^2} \right) ,$$

and in terms of those … $\nu = \frac{q\cdot p}{M}\,, \quad x = \frac{Q^2}{2M\nu}\,, \quad Q^2 = -q^2\,, \quad y = \frac{p\cdot q}{p\cdot k}$

$$\frac{d^2\sigma}{dx dy} = \frac{e^4 M k^0}{2\pi Q^4} \left[x y^2 F_1 + (1-y) F_2 \right],$$

following Manohar '92 hep-ph/9204208 (unpolarized cross-section)

$$M\nu = E_\mathbf{p} p_x^0 - E_\mathbf{p}^2 - \mathbf{q}\cdot\mathbf{p}\,,$$
$$Q^2 = \mathbf{q}^2 - (p_x^0 - E_\mathbf{p})^2 .$$

$$\{\nu, Q^2\} \Longleftrightarrow \{\mathbf{p}^2, \mathbf{q}^2, \mathbf{p}\cdot\mathbf{q}\,, p_x^0\}$$

this redundancy can be exploited in our advantage for building the optimal resolution function!

DIS ($e + p \longrightarrow e' + \text{hadrons}$) on the lattice

1. Calculate the relevant four point function

$$\widetilde{G}_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) \equiv 2E_\mathbf{p} L^6 e^{-E_\mathbf{p}\tau} \int d^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \lim_{\tau_f\to\infty} \lim_{\tau_i\to-\infty} \frac{\sum_\lambda \langle \Psi_\lambda(\tau_f, \mathbf{p}) j_\mu(\tau, \mathbf{x}) j_\nu(0) \Psi_\lambda^\dagger(\tau_i, \mathbf{p})\rangle_\mathrm{conn}}{\sum_\lambda \langle \Psi_\lambda(\tau_f, \mathbf{p}) \Psi_\lambda^\dagger(\tau_i, \mathbf{p})\rangle},$$

2. It will contain the desired contribution.

$$\begin{aligned}\widetilde{G}_{\mu\nu,\mathbf{p}}(\tau, \mathbf{p}_x, L) &= e^{-E_\mathbf{p}\tau} \langle N, \mathbf{p} | j_\mu(\tau, \mathbf{q}) j_\nu(0) | N, \mathbf{p}\rangle_L + \dots \\ &= \int_0^\infty dp_x^0\, W_{\mu\nu}(p_x^0, \mathbf{p}_x; L) e^{-p_x^0 \tau}\end{aligned}$$

all final states are taken into account!!

3. Apply the Backus-Gilbert method to obtain:

$$\widehat{W}_{\mu\nu}(p_x^0, \mathbf{p}_x, L, \Delta) = \int_0^\infty d\omega\, \widehat{\delta}_\Delta(p_x^0, \omega) W_{\mu\nu}(\omega, \mathbf{p}_x, L)$$

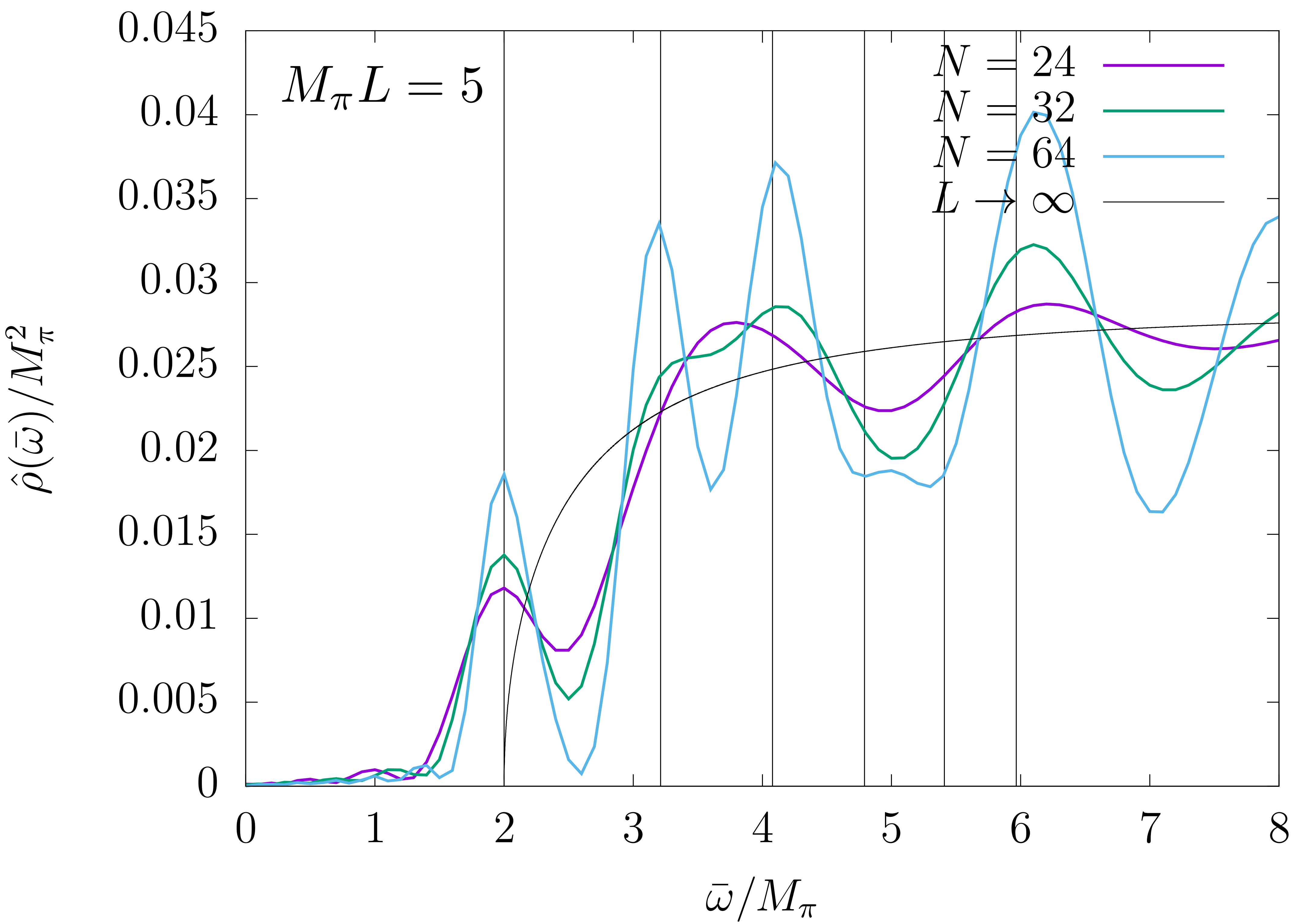
$$W_{\mu\nu}(p_x^0, \mathbf{p}_x) = \lim_{\Delta\to 0} \lim_{L\to 0} \widehat{W}_{\mu\nu}(p_x^0, \mathbf{p}_x, L, \Delta)$$

Conclusions

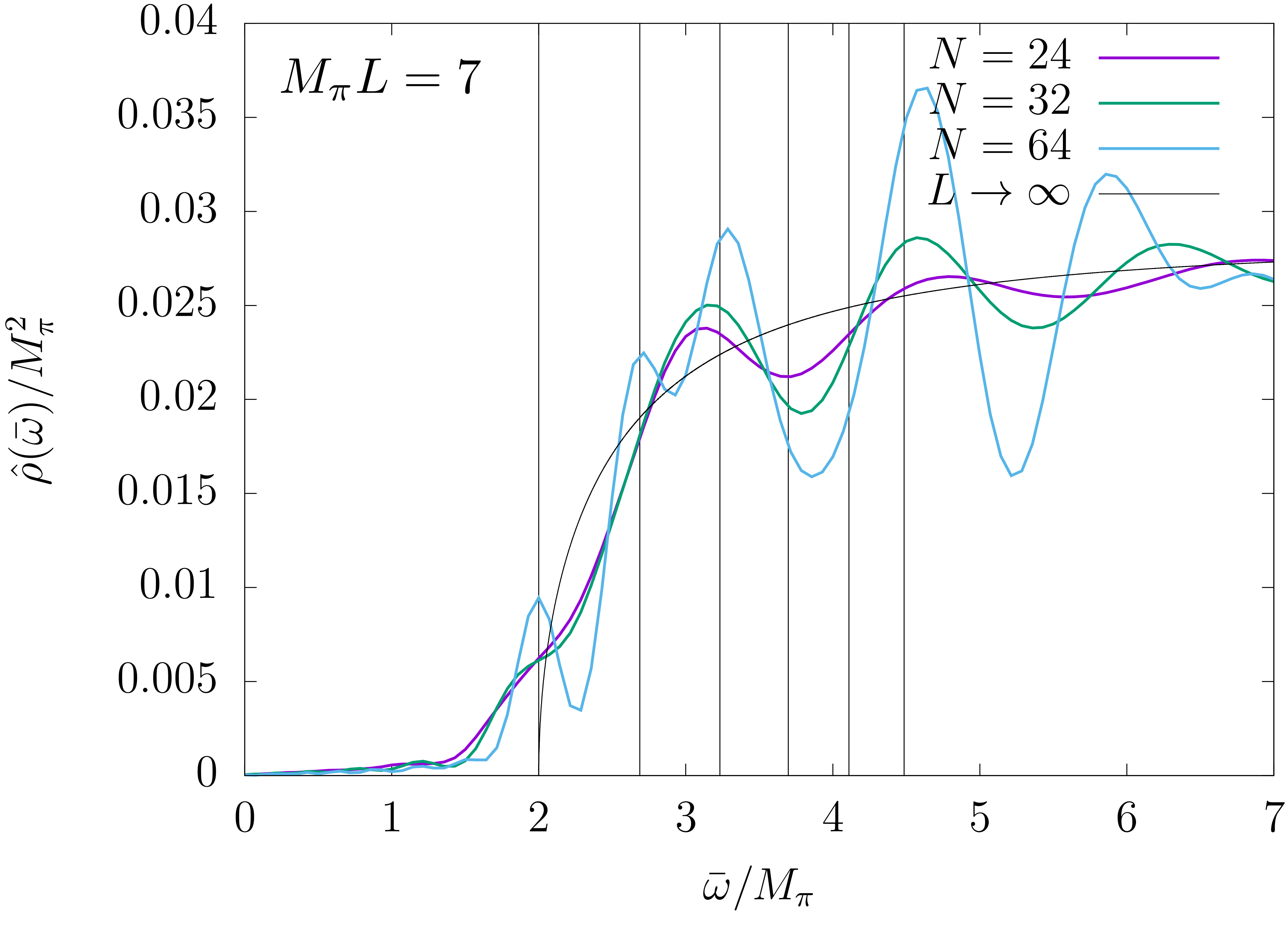
- We have found that it may be feasible to calculate **total widths** from the lattice for QCD-stable states.
- We have made progress on understanding finite volume effects when estimating spectral functions.
- The method allows for a lot of extensions and improvements.
- The approach is complementary to other methods like LL. (density of states and subtraction of channels.) Best when slowly varying spectral function.
- As the energy is increased and more and more channels are open, it may be the way to go.

Backup

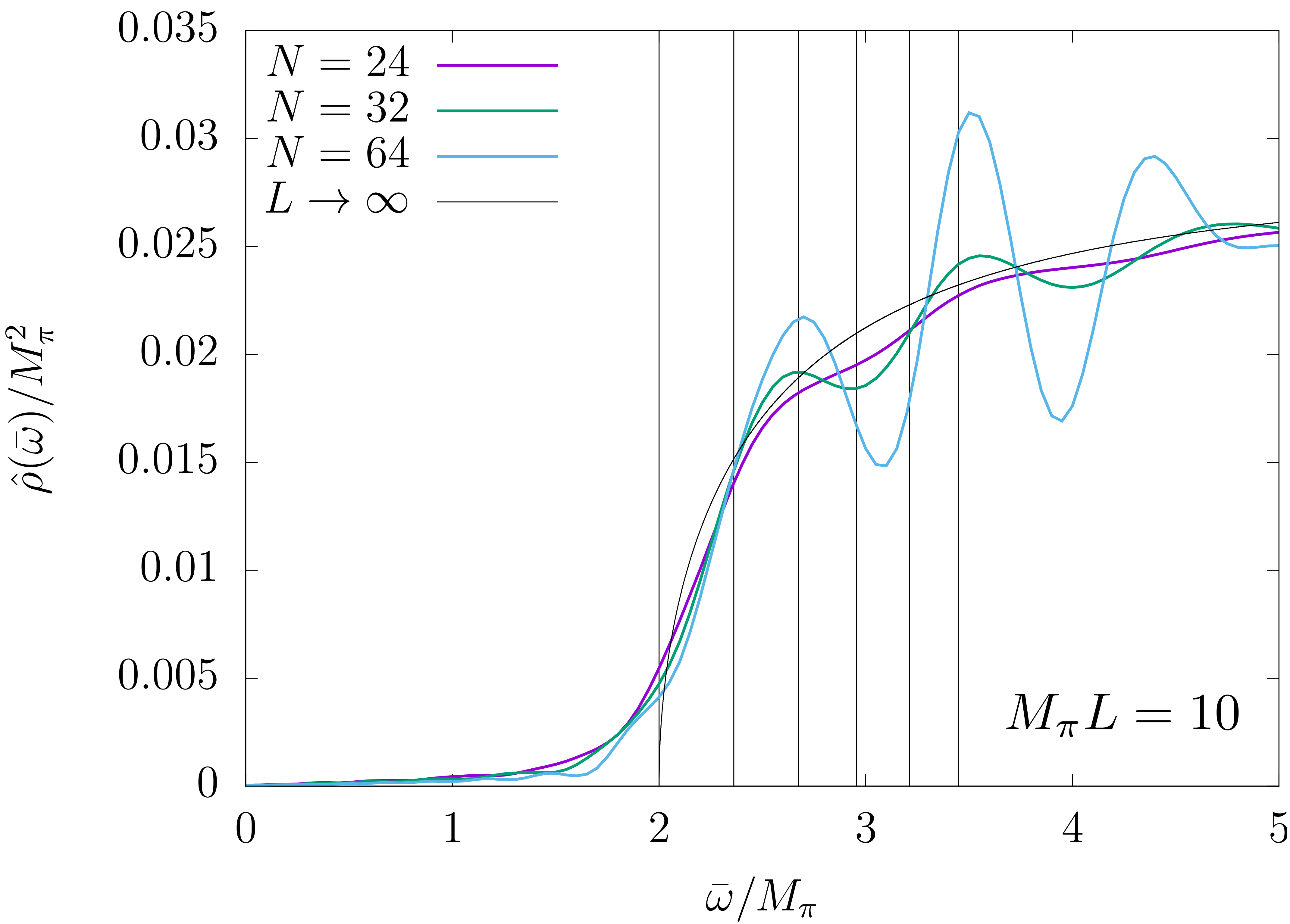
$$\widetilde{G}_{K\to\phi\phi}(\tau,\mathbf{0},L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K\to\phi\phi}(\mathbf{0},L)|^2 \,,$$



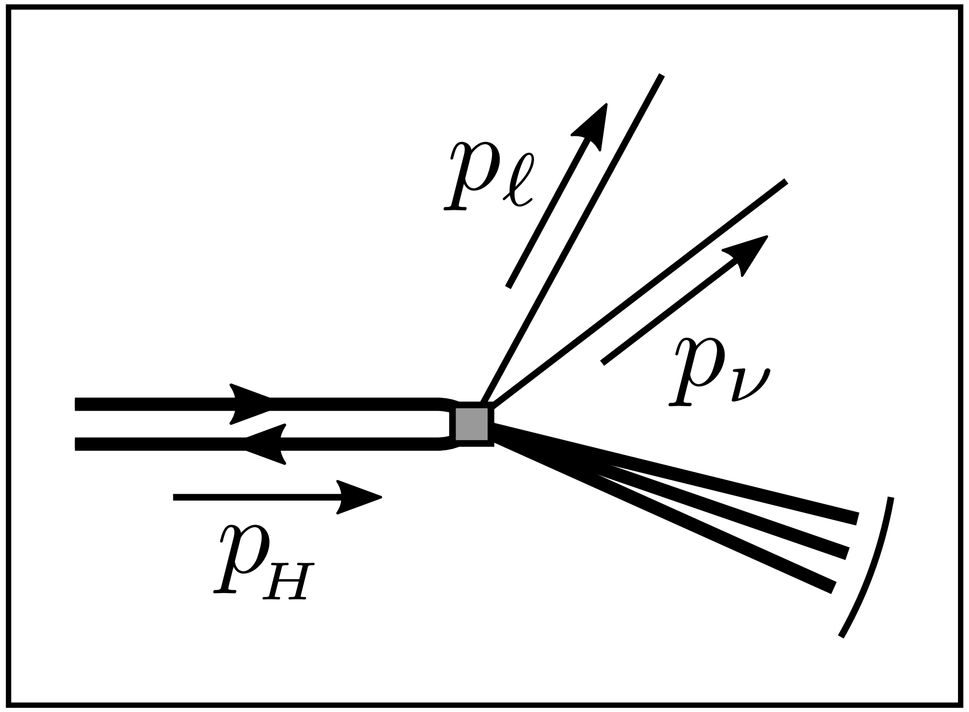
$$\widetilde{G}_{K\to\phi\phi}(\tau,\mathbf{0},L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K\to\phi\phi}(\mathbf{0},L)|^2 \,,$$



$$\widetilde{G}_{K\to\phi\phi}(\tau,\mathbf{0},L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K\to\phi\phi}(\mathbf{0},L)|^2 \,,$$



other possible decays one can study…



- Semi-leptonic weak decays:

$$W_{\mu\nu}^{H_Q\to X}(v,q) = \frac{1}{2M_{H_Q}} \int d^4x\, e^{-iq\cdot x} \langle H_Q, \mathbf{p}|\mathcal{J}_\mu^\dagger(x)\mathcal{J}_\nu(0)|H_Q,\mathbf{p}\rangle\,,$$

flavour changing current: $\mathcal{J}_\mu = \bar{q}\gamma_\mu(1-\gamma_5)Q$

$$q^\mu = p_\ell^\mu + p_\nu^\mu, \qquad v^\mu = p_H^\mu/M_H$$

- Purely hadronic decays: $c \to su\bar{d}$

$$\mathcal{H}_\mathcal{Q}(x) = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \left[\bar{s}(x)\gamma_\mu(1-\gamma_5)c(x)\right]\left[\bar{u}(x)\gamma_\mu(1-\gamma_5)d(x)\right]$$ D-meson decay

The approach is completely equivalent in both cases to DIS!

There does not seem to be any conceptual problem.