Classical binding of hadrons

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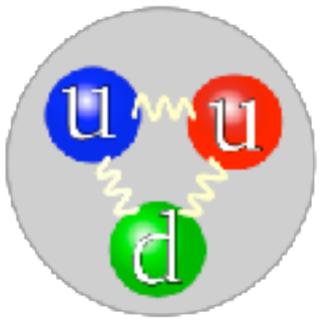
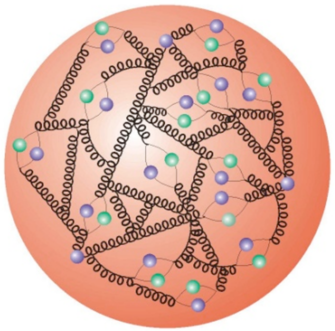


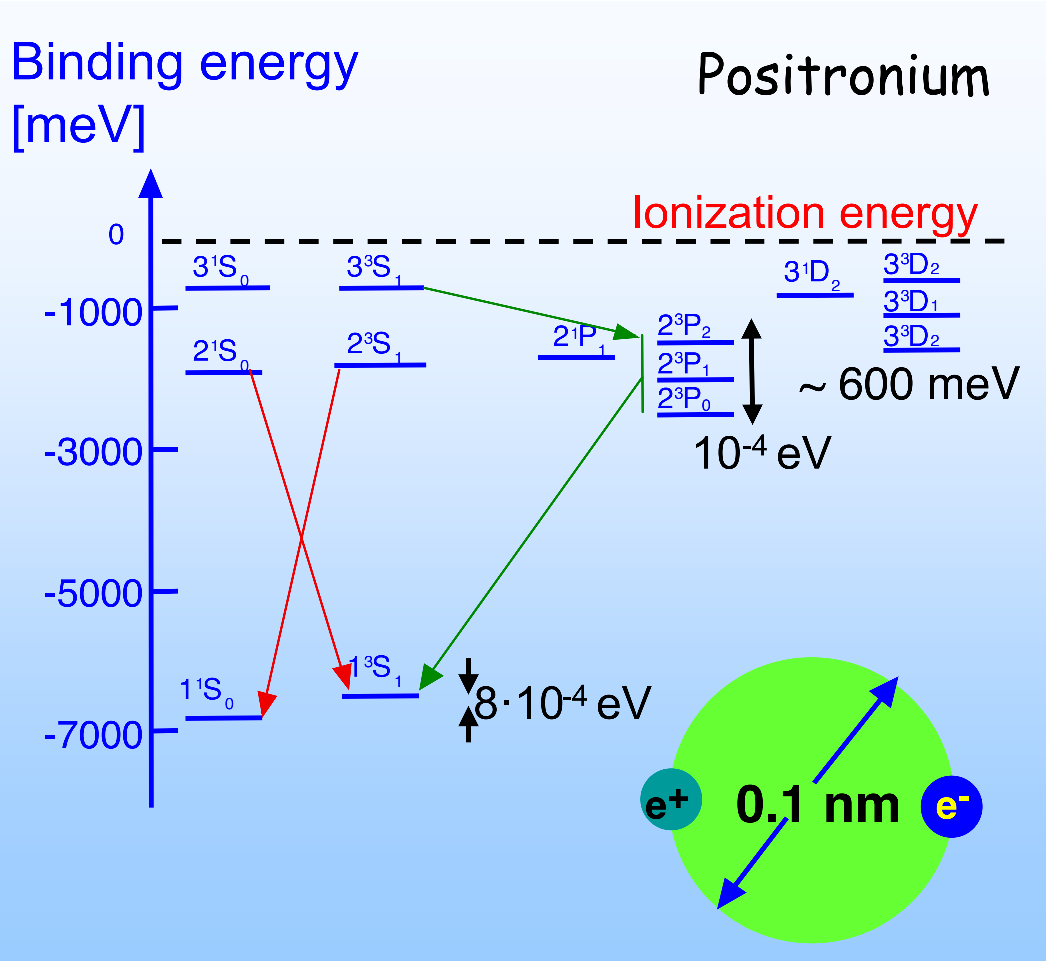


- Expand around classical field in perturbative S-matrix
- Illustration: Schrödinger atom in the $\hbar \to 0$ limit
- Λ_{QCD} from homogeneous solution of QCD field equations
- $q\bar{q}$ bound state properties
- $M = 0$ states: Spontaneous breaking of chiral symmetry

PH 1605.01532, 1711.10851 and to appear

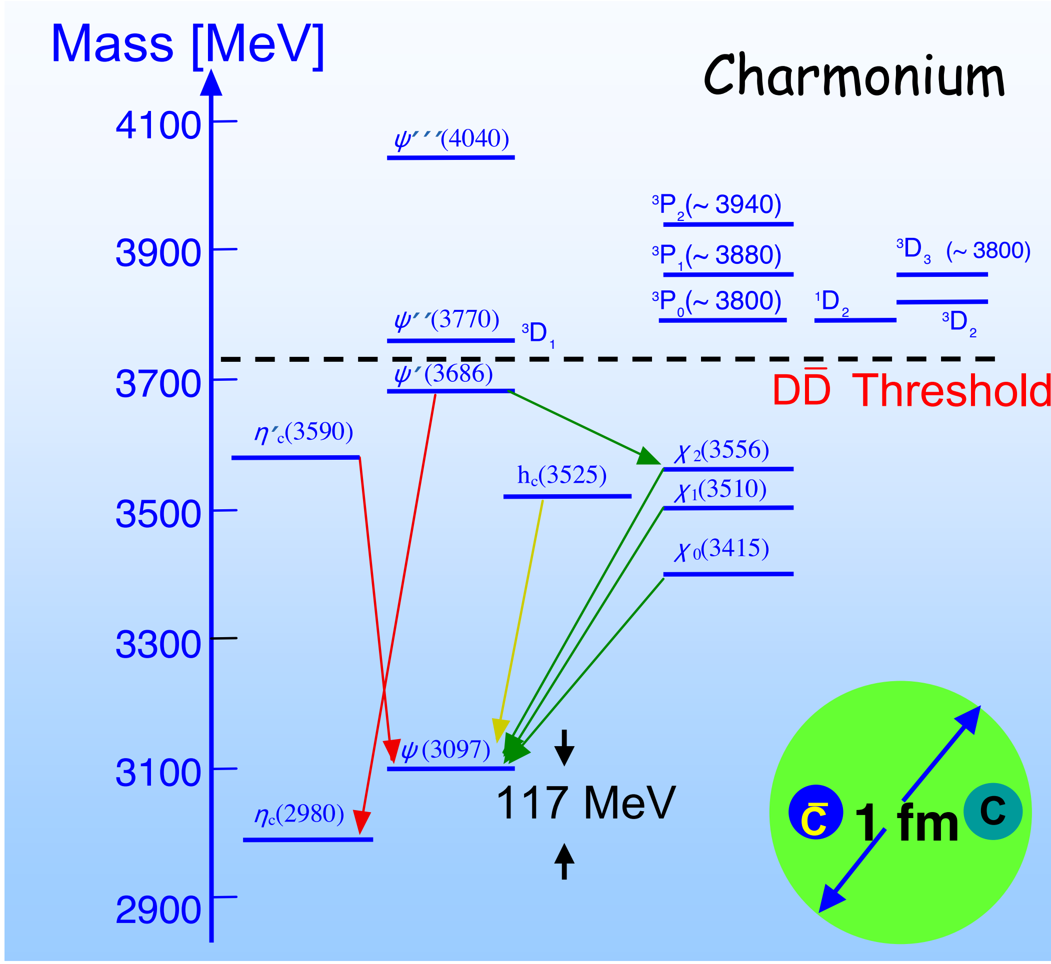
"The J/ψ is the Hydrogen atom of QCD"

QED



$$V(r) = -\frac{\alpha}{r}$$

QCD



$$V(r) = c\,r - \frac{4}{3}\frac{\alpha_s}{r}$$

Perturbative S-matrix

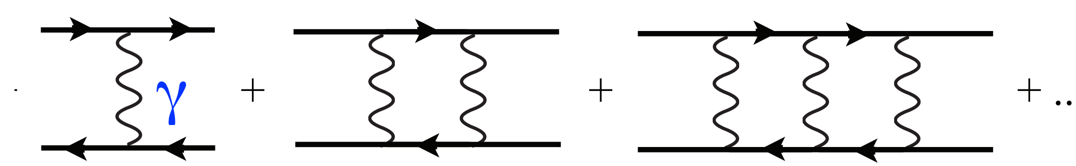
$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \qquad \mathcal{H}_0 \left|i\right\rangle_{in} = E_i \left|i\right\rangle_{in}$$

$$S_{fi} = {}_{out}\langle f, t \to \infty | \left\{ \mathrm{T} \exp \left[-\, i \int_{-\infty}^{\infty} dt\, \mathcal{H}_I(t) \right] \right\} | i, t \to -\infty \rangle_{in}$$

Formally exact IP expression, provided the *in*- and *out*-states have a non-vanishing overlap with the the physical i, f states.

Bound states have no overlap with free *in*- and *out*-states at $t = \pm\infty$

No Feynman diagram has a bound state pole.



Expanding around free states is inadequate for bound states.

Expanding around a stationary action

A stationary action implies a classical gauge field:

$$\frac{\delta \mathcal{S}[A^\mu]}{\delta A^\mu} = 0 \qquad \int [dA^\mu] \exp\left(iS[A^\mu]/\hbar\right) \quad \Longrightarrow \quad \hbar \to 0$$

We should expand around *in* and *out* states with their classical gauge field

Positronium is bound by its classical potential $V(r) = -\alpha/r$

The $\hbar \to 0$ limit selects an optimal expansion for bound states.

The "Potential Picture"

$$\mathcal{H} = \mathcal{H}_V + \mathcal{H}_I \qquad \mathcal{H}_V = \mathcal{H}_0 + \mathcal{H}_I(A_{cl})$$

$$S_{fi} = {}_V\langle f, t \to \infty | \left\{ \mathrm{T}\exp\left[-i\int_{-\infty}^{\infty} dt\, \mathcal{H}_I(t) \right] \right\} | i, t \to -\infty \rangle_V$$

$$\mathcal{H}_V \, |i\rangle_V = E_i \, |i\rangle_V$$

Particles will propagate in the classical field, as appropriate for bound states.

Can provide a unique framework for bound state calculations.

Now: Stay at $(\mathcal{H}_I)^0$ (Born) level. Consider bound asymptotic states.

To do: Derivation of and higher order contributions to the PP.

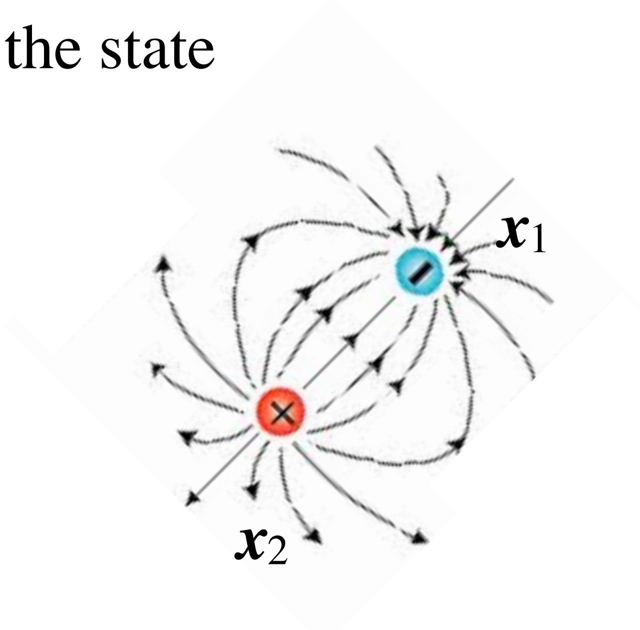
The classical field for Positronium

$$\frac{\delta \mathcal{S}_{QED}}{\delta \hat{A}^0(t,\boldsymbol{x})} = 0 \quad \Rightarrow \quad -\boldsymbol{\nabla}^2 \hat{A}^0(t,\boldsymbol{x}) = e\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$$

$$\hat{A}^0(t,\boldsymbol{x}) = \int d^3\boldsymbol{y}\,\frac{e}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\psi^\dagger\psi(t,\boldsymbol{y})$$

The classical field is the expectation value of $\hat{A}^0$ in the state

$$|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = \bar{\psi}(t,\boldsymbol{x}_1)\psi(t,\boldsymbol{x}_2)\,|0\rangle$$



$$\frac{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\,e\hat{A}^0(\boldsymbol{x})\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle}{\langle \boldsymbol{x}_1,\boldsymbol{x}_2|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle} = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

$$\equiv eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2)$$

Note:
- A^0 is determined instantaneously for all $\boldsymbol{x}$
- It depends on $\boldsymbol{x}_1, \boldsymbol{x}_2$
- $eA^0(\boldsymbol{x}_1) = -eA^0(\boldsymbol{x}_2) = -\dfrac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$ is the classical $-\alpha/r$ potential

The Schrödinger equation

$$\mathcal{H}_V(t;\boldsymbol{x}_1,\boldsymbol{x}_2) = \int d\boldsymbol{x}\,\psi^\dagger(t,\boldsymbol{x})\big[-i\boldsymbol{\nabla}\cdot\boldsymbol{\alpha}+m\gamma^0+\tfrac{1}{2}eA^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2)\big]\psi(t,\boldsymbol{x})$$

$$|M\rangle_V = \int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\bar\psi(\boldsymbol{x}_1)\,\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi(\boldsymbol{x}_2)\,|0\rangle$$

$\mathcal{H}_V\,|M\rangle_V = M\,|M\rangle_V$ gives the bound state equation for $\Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)$:

$$\big[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}}+m\gamma^0\big]\Phi(\boldsymbol{x})+\Phi(\boldsymbol{x})\big[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}}-m\gamma^0\big] = \big[M-V(|\boldsymbol{x}|)\big]\Phi(\boldsymbol{x})$$

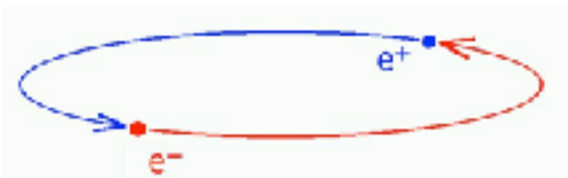
with $V(|\boldsymbol{x}|) = -\dfrac{\alpha}{|\boldsymbol{x}|}$

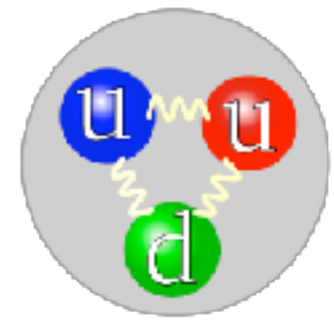
This BSE reduces to the Schrödinger equation for non-relativistic kinematics.

The $\hbar \to 0$ limit is required for its derivation.

Classical field in QCD

Global gauge invariance allows classical gauge field for neutral atoms, but not for color singlet hadrons in QCD





$$A^0 = \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{\alpha}{|\boldsymbol{x}-\boldsymbol{x}_2|}$$

$$A_a^0(\boldsymbol{x}) = 0$$

Positronium
QED

Proton
QCD

However, a classical gluon field is allowed for quarks of fixed colors C:

$$A_a^0(\boldsymbol{x};C) \neq 0 \qquad \sum_C A_a^0(\boldsymbol{x};C) = 0$$

Three consequences of $\hbar \to 0$ in QCD

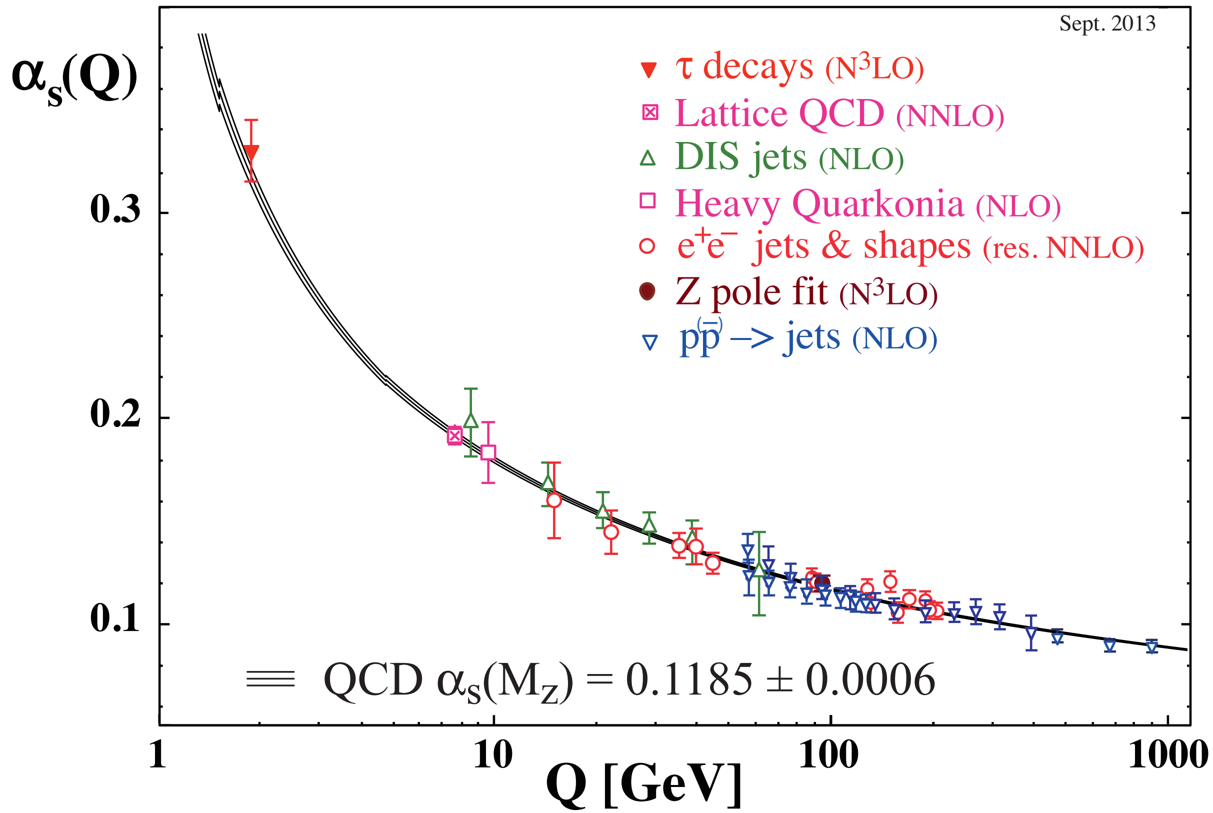
1. The suppression of loops,
   stops the running of α_s

   Estimates for the frozen coupling indicate

   $\alpha_s(0)/\pi \approx 0.14$

$\Rightarrow$ PQCD corrections to $\mathcal{O}(\hbar^0)$ can be relevant.

$\alpha_s^{crit} \approx 0.43$ ★ ← Gribov hep-ph/9902279



2. In the absence of loops, the QCD scale Λ_{QCD} cannot arise from renormalization.

3. Poincaré invariance, unitarity etc. should hold at each power of $\hbar$

The QCD scale Λ_{QCD}

At $O(\hbar^0)$ (no loops) the QCD scale can arise only via a boundary condition

$$\frac{\delta}{\delta A_a^0} S_{QCD} = 0 \qquad \Rightarrow \qquad \partial_i F_a^{i0} = -g f_{abc} A_b^i F_c^{i0} + g\psi_A^\dagger T_a^{AB} \psi_B$$

A homogeneous, $\mathcal{O}\left(\alpha_s{}^0\right)$ solution with $\hat{A}_a^i = 0$ and hence $\boldsymbol{\nabla}^2 \hat{A}_a^0 = 0$

$$\hat{A}_a^0(\boldsymbol{x}) = \kappa \sum_{B,C} \int d\boldsymbol{y}\, (\boldsymbol{x}\cdot\boldsymbol{y})\, \psi_B^\dagger(\boldsymbol{y})\, T_a^{BC} \psi_C(\boldsymbol{y})$$ appears unique:

- Linear in $\boldsymbol{x}$ for translation invariance: $\hat{A}_a^0(\boldsymbol{x}_1) - \hat{A}_a^0(\boldsymbol{x}_2) \neq f(\boldsymbol{x}_1 + \boldsymbol{x}_2)$
- $\boldsymbol{x}\cdot\boldsymbol{y}$ for rotational invariance
- $\boldsymbol{x}$-independent field energy density $\sum_a |\boldsymbol{\nabla} \hat{A}_a^0(\boldsymbol{x})|^2$ must be universal

$\Rightarrow$ determines $\varkappa$ up to a scale Λ [GeV]

Classical color field for mesons

$$|M\rangle = \sum_{A,B} \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}^A(\boldsymbol{x}_1)\, \Phi^{AB}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi^B(\boldsymbol{x}_2)\, |0\rangle$$

$$\Phi^{AB}(\boldsymbol{x}) = \frac{1}{\sqrt{N_C}}\, \delta^{AB}\, \Phi(\boldsymbol{x})$$

$$\hat{A}^0_a(\boldsymbol{x}) = \kappa \sum_{B,C} \int d\boldsymbol{y}\, (\boldsymbol{x}\cdot\boldsymbol{y})\, \psi^\dagger_B(\boldsymbol{y})\, T_a^{BC}\, \psi_C(\boldsymbol{y})$$

$$\frac{\langle \boldsymbol{x}_1^A, \boldsymbol{x}_2^A |\, \hat{A}^0_a(\boldsymbol{x})\, | \boldsymbol{x}_1^A, \boldsymbol{x}_2^A \rangle}{\langle \boldsymbol{x}_1^A, \boldsymbol{x}_2^A | \boldsymbol{x}_1^A, \boldsymbol{x}_2^A \rangle} = \kappa(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \boldsymbol{x}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, T_a^{AA}$$

for each quark color A

$$\Rightarrow \quad A^0_a(\boldsymbol{x}; \boldsymbol{x}_1, \boldsymbol{x}_2, A) = \left[\boldsymbol{x} - \tfrac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2)\right] \cdot \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}\, T_a^{AA}\, 6\Lambda^2 \qquad \mathcal{O}\left(\alpha_s^0\right)$$

$$\sum_a \left[\boldsymbol{\nabla}_x A^0_a(\boldsymbol{x}; \boldsymbol{x}_1, \boldsymbol{x}_2, A)\right]^2 = 12\Lambda^4$$

Universal field energy

$$\sum_A A^0_a(\boldsymbol{x}; \boldsymbol{x}_1, \boldsymbol{x}_2, A) \propto \mathrm{Tr}\, T^{AA} = 0$$

Another hadron feels
no field at any $\boldsymbol{x}$

$$V(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \tfrac{1}{2} g \sum_a T_a^{AA} \left[A^0_a(\boldsymbol{x}_1; \boldsymbol{x}_1, \boldsymbol{x}_2, A) - A^0_a(\boldsymbol{x}_2; \boldsymbol{x}_1, \boldsymbol{x}_2, A)\right] = g\Lambda^2 |\boldsymbol{x}_1 - \boldsymbol{x}_2|$$

Linear potential, independent of quark color component A

Classical color field for baryons

$$|M\rangle = \sum_{A,B,C} \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, d\boldsymbol{x}_3\, \psi_A^\dagger(\boldsymbol{x}_1)\psi_B^\dagger(\boldsymbol{x}_2)\psi_C^\dagger(\boldsymbol{x}_3)\, \Phi^{ABC}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\,|0\rangle \qquad \Phi^{ABC} = \epsilon^{ABC}\Phi$$

Expectation value of $\hat{A}_a^0(\boldsymbol{x}) = \kappa \sum_{B,C} \int d\boldsymbol{y}\,(\boldsymbol{x}\cdot\boldsymbol{y})\,\psi_B^\dagger(\boldsymbol{y})\,T_a^{BC}\psi_C(\boldsymbol{y})$

in $\psi_A^\dagger(\boldsymbol{x}_1)\psi_B^\dagger(\boldsymbol{x}_2)\psi_C^\dagger(\boldsymbol{x}_3)\,|0\rangle$ $(A \neq B \neq C)$ determines the classical field:

$$A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC) = \left[\boldsymbol{x} - \tfrac{1}{3}(\boldsymbol{x}_1+\boldsymbol{x}_2+\boldsymbol{x}_3)\right]\cdot\left(T_a^{AA}\boldsymbol{x}_1 + T_a^{BB}\boldsymbol{x}_2 + T_a^{CC}\boldsymbol{x}_3\right)\frac{6\Lambda^2}{d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)}$$

where $d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$

$$\sum_a \left|\boldsymbol{\nabla}_x A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC)\right|^2 = 12\Lambda^4$$ Universal field energy

$$\sum_{A,B,C} \epsilon^{ABC} A_a^0(\boldsymbol{x};\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,ABC) = 0$$ No classical field for singlet state

$$V(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = g\Lambda^2 d(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)$$

Bound state equation for mesons (rest frame)

$$\mathcal{H}_V\,|M\rangle_V = M\,|M\rangle_V$$ Bound state condition implies, with $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$

$$i\boldsymbol{\nabla}\cdot\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\} + m\left[\gamma^0,\Phi(\boldsymbol{x})\right] = \left[M - V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

$$V(\boldsymbol{x}) = g\Lambda^2|\boldsymbol{x}| \equiv V'|\boldsymbol{x}|$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x}) = \sum_i \Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

we may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory	$[s=0,\ \ell=j]$:	$-\eta_P = \eta_C = (-1)^j$	$\gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$
0^{--} trajectory	$[s=1,\ \ell=j]$:	$\eta_P = \eta_C = -(-1)^j$	$\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{++} trajectory	$[s=1,\ \ell=j\pm1]$:	$\eta_P = \eta_C = +(-1)^j$	$1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{+-} trajectory	[exotic] :	$\eta_P = -\eta_C = (-1)^j$	$\gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

Example: 0^{-+} trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$\eta_P = (-1)^{j+1}$

$\eta_C = (-1)^j$

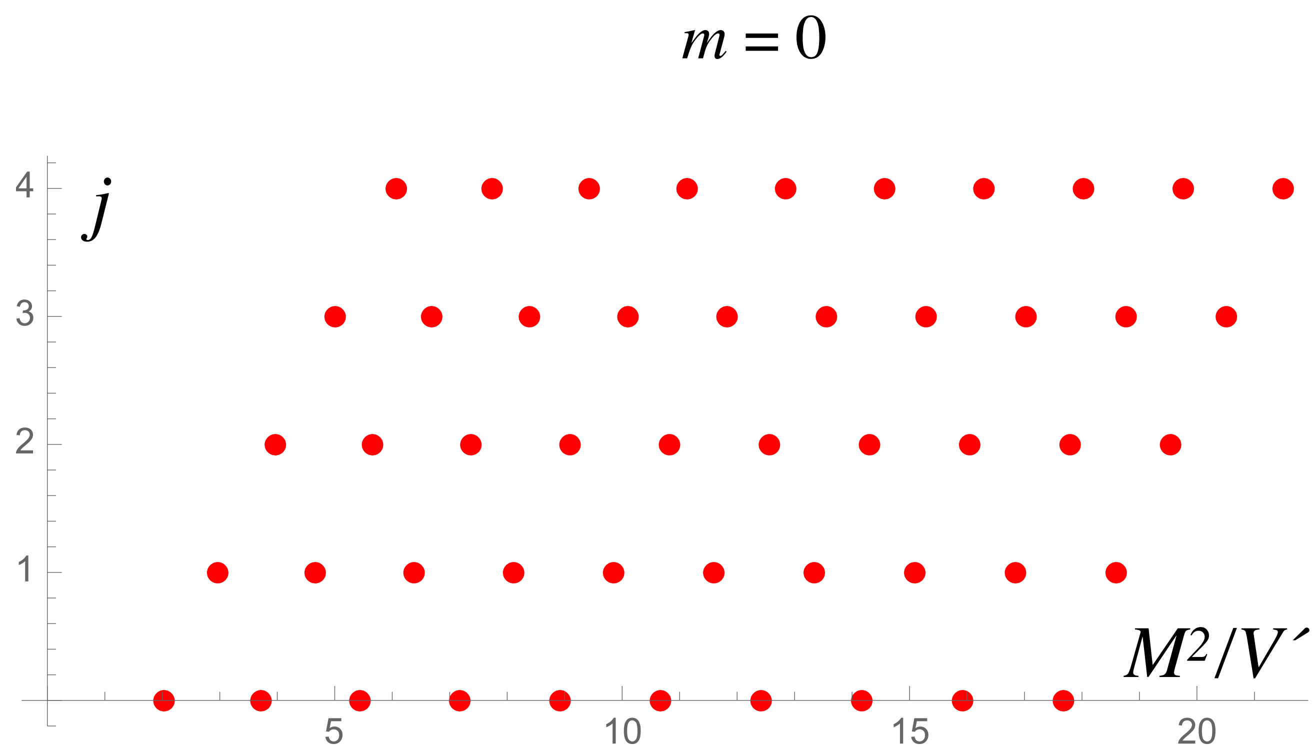
Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

Local normalizability at $r = 0$ and at $V(r) = M$ determines the discrete M

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models

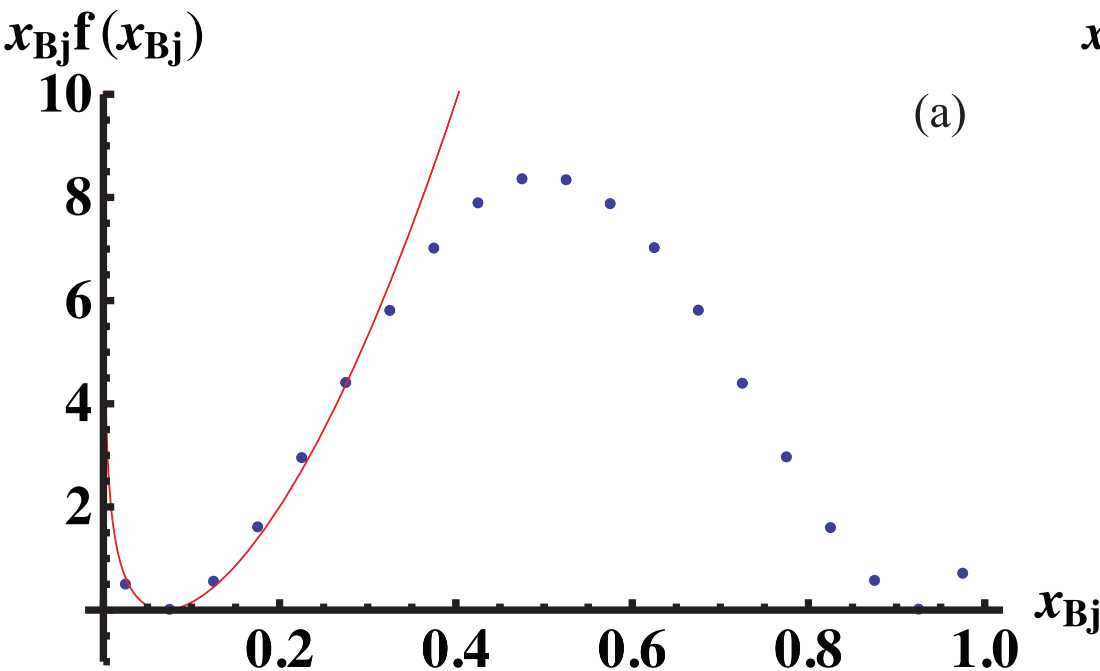


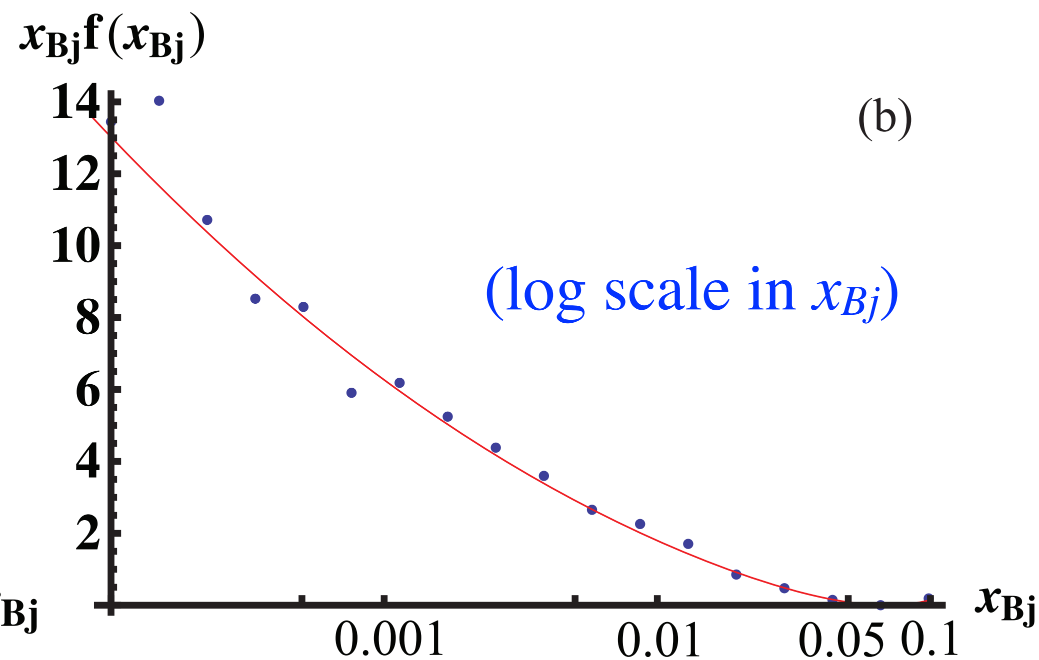
Parton distributions have a sea component

In D=1+1 dimensions the sea component is prominent at low m/e :

$$m/e = 0.1$$

D. D. Dietrich, PH, M. Järvinen
arXiv 1212.4747





The red curve is an analytic approximation, valid in the $x_{Bj} \to 0$ limit.

Note: Enhancement at low x is due to bd (sea), not to $b^\dagger d^\dagger$ (valence) component.

String breaking is not included.

States with P = M = 0

We required the wave function to be normalizable at $r = 0$ and $V'r = M$

For $M = 0$ the two points coincide. Regular, massless solutions are found.

The massless 0^{++} meson "σ" may mix with the perturbative vacuum.
This spontaneously breaks chiral invariance.

$$|\sigma\rangle = \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, \bar{\psi}(\boldsymbol{x}_1)\, \Phi_\sigma(\boldsymbol{x}_1 - \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_2)\, |0\rangle \equiv \hat{\sigma}\, |0\rangle$$

For $m = 0$ and $V' = 1$:
$$\Phi_\sigma(\boldsymbol{x}) = N_\sigma \Big[J_0(\tfrac{1}{4}r^2) + \boldsymbol{\alpha} \cdot \boldsymbol{x} \frac{i}{r}\, J_1(\tfrac{1}{4}r^2) \Big]$$

where J_0 and J_1 are Bessel functions.

$\hat{P}^\mu\, |\sigma\rangle = 0$ State has *vanishing four-momentum* in any frame

It may form a non-trivial condensate.

A chiral condensate (m = 0)

Since | σ ⟩ has vacuum quantum numbers and zero momentum it can mix with the perturbative vacuum without violating Poincaré invariance

Ansatz: $|\chi\rangle = \exp(\hat\sigma)\,|0\rangle$ implies $\langle\chi|\bar\psi\psi\,|\chi\rangle = 4N_\sigma$

An infinitesimal chiral rotation of the condensate generates a pion

$$U_\chi(\beta) = \exp\left[i\beta\int d\boldsymbol{x}\,\psi^\dagger(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x})\right] \qquad U_\chi(\beta)\,|\chi\rangle = (1-2i\beta\,\hat\pi\,|\chi\rangle$$

where $\hat\pi$ is the massless 0^{-+} state with wave function $\Phi_\pi = \gamma_5\Phi_\sigma$

Small quark mass: m > 0

The massless ($M_\sigma = 0$) sigma 0^{++} state has wave function

$$\Phi_\sigma(\boldsymbol{x}) = f_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,f_2(r) + i\,\boldsymbol{\gamma}\cdot\boldsymbol{x}\,g_2(r)$$

Radial functions are Laguerre fn's

An $M_\pi > 0$ pion 0^{-+} state has rest frame wave function

$$\Phi_\pi(\boldsymbol{x}) = \big[F_1(r) + i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,F_2(r) + \gamma^0\,F_4(r)\big]\gamma_5$$

$$F_4(0) = \frac{2m}{M}F_1(0)$$

$$F_1'' + \Big(\frac{2}{r} + \frac{1}{M-r}\Big)F_1' + \big[\tfrac{1}{4}(M-r)^2 - m^2\big]F_1 = 0$$

$$\langle\chi|j_5^\mu(x)\hat{\pi}\,|\chi\rangle = iP^\mu f_\pi\,e^{-iP\cdot x} \quad\Rightarrow\quad F_4(0) = \tfrac{1}{4}iM_\pi f_\pi$$

$$\langle\chi|\bar{\psi}(x)\gamma_5\psi(x)\,\hat{\pi}\,|\chi\rangle = -i\,\frac{M^2}{2m}\,f_\pi\,e^{-iP\cdot x} \quad\Rightarrow\quad F_1(0) = i\,\frac{M^2}{8m}\,f_\pi$$

Relations are satisfied for any P.

A smooth $m \to 0$ requires $M_\pi^2 \propto m$, which is OK at lowest order in m.

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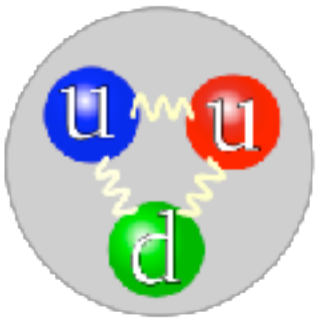
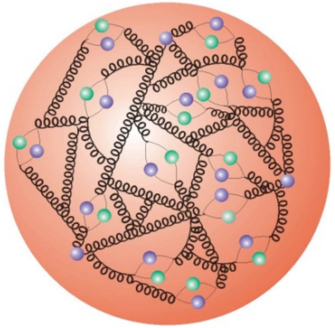


$\longleftrightarrow$



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