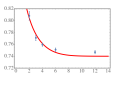
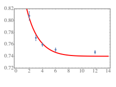


Pseudo-PDFs and Quasi-PDFs Structure

A.V. Radyushkin (ODU/Jlab)

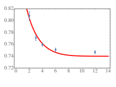
Light Cone 2018
May 14, 2018



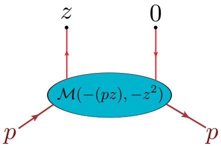
Parton Densities

- May 11, 1918 – R.P. Feynman's birthday
- Feynman diagrams, propagator, path integrals, parton model ...
- Original Feynman approach to PDFs $f(x)$: infinite momentum $P_3 \to \infty$ limit of $k_3 = xP_3$ momentum distributions ($\sim$ quasi-PDFs $Q(x, P_3)$)
- $f(x)$ were treated as $k_\perp$-integrals of more detailed $f(x, k_\perp)$ distributions
- From the start it was understood that $Q(x, P_3 \to \infty) \to f(x)$ limit exists only if $f(x, k_\perp)$ rapidly decreases with $k_\perp$
- "Transverse momentum cut-off", $\langle k_\perp^2 \rangle \sim 1/R_{\rm hadr}^2$
- Question 1: why $Q(x, P_3)$ differs from $f(x)$?
- Question 2: how does $Q(x, P_3)$ convert into $f(x)$ when $P_3 \to \infty$?
- Qualitative answer: yP_3 comes from two sources:
  from the motion of the hadron as a whole (xP_3) and
  from Fermi motion of quarks inside the hadron $(y - x)P_3 \sim 1/R_{\rm hadr}$

- $(y - x)P_3 \sim 1/R_{\rm hadr}$ part has the same origin as transverse momentum
- $\Rightarrow$ One should be able to relate quasi-PDFs to TMDs



Pseudo-distributions and PDFs



- Basic matrix element (ignoring spin)

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2)$$

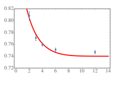
- Lorentz invariance: $\mathcal{M}$ depends on z through $(pz) \equiv -\nu$ and z^2
- Ioffe time ν: $\mathcal{M}(\nu, -z^2)$ = Ioffe time pseudo-distribution (pseudo-ITD)
- Pseudo $\equiv$ off the light cone
- For any Feynman diagram, for arbitrary z^2 and arbitrary p^2

$$\mathcal{M}(\nu, -z^2) = \int_{-1}^{1} dx\, e^{ix\nu}\, \mathcal{P}(x, -z^2)$$

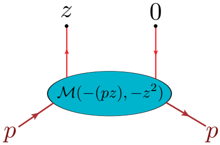
- Limits $-1 \leq x \leq 1$, negative x correspond to anti-particles
- On the light cone: usual ITD and usual PDF $\mathcal{P}(x, 0) = f(x)$
- If $z^2 \to 0$ limit is singular, regularization (like $\overline{\rm MS}$) is needed, $f(x) \to f(x, \mu^2)$

$$\mathcal{M}(\nu, 0)|_{\mu^2} \equiv \mathcal{I}(\nu, \mu^2) = \int_{-1}^{1} dx\, e^{ix\nu}\, f(x, \mu^2)$$

- Pseudo-PDF $\mathcal{P}(x, -z^2)$: Fourier transform of pseudo-ITD with respect to ν for fixed z^2



Transverse momentum dependence and quasi-PDFs



- Basic matrix element (ignoring spin)

$$\langle p|\phi(0)\phi(z)|p\rangle = \mathcal{M}(-(pz), -z^2)$$

- Lorentz invariance: $\mathcal{M}$ depends on z through $(pz) \equiv -\nu$ and z^2

- Take $z = (z_+ = 0, z_-, z_1, z_2)$. Then $\nu = -p^+ z^-$ and $-z^2 = z_1^2 + z_2^2$
- Introduce TMD $\mathcal{F}(x, k_1^2 + k_2^2))$:

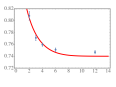
$$\mathcal{M}(\nu, z_1^2 + z_2^2) = \int_{-1}^{1} dx\ e^{ix\nu} \int_{-\infty}^{\infty} dk_1 dk_2 e^{i(k_1 z_1 + k_2 z_2)} \mathcal{F}(x, k_1^2 + k_2^2)$$

- Take $z = (0, 0, 0, z_3)$, then $-(pz) \equiv \nu = Pz_3$ and $-z^2 = z_3^2$
- Introduce quasi-PDF (Ji,2013)

$$\mathcal{M}(Pz_3, z_3^2) = \int_{-\infty}^{\infty} dy\, e^{iyPz_3}\, Q(y, P)$$

- Inverse transformation (using $z_3 = \nu/P$ on the second step)

$$Q(y, P) = \frac{P}{2\pi} \int_{-\infty}^{\infty} dz_3\, e^{-iyPz_3}\, \mathcal{M}(Pz_3, z_3^2) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\ e^{-iy\nu}\, \mathcal{M}(\nu, \nu^2/P^2)$$



qPDF/TMD relation

$$Q(y,P) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi}\, e^{-iy\nu}\, \mathcal{M}(\nu, \nu^2/P^2)$$

$$\mathcal{M}(\nu, z_1^2+z_2^2) = \int_{-1}^{1} dx\, e^{ix\nu} \int_{-\infty}^{\infty} dk_1 dk_2 e^{i(k_1 z_1 + k_2 z_2)} \mathcal{F}(x, k_1^2+k_2^2)$$

- Take $z_1 = 0, z_2 = \nu/P$ and use for qPDF

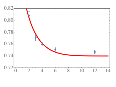
$$Q(y,P) = P \int_{-1}^{1} dx \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2 + (y-x)^2 P^2)$$

- qPDF variable y has the $-\infty < y < \infty$ support, since $-\infty < k_2 < \infty$
- Relation between pseudo-PDF and TMD

$$\mathcal{P}(x, z_\perp^2) = \int d^2\mathbf{k}_\perp e^{i(\mathbf{k}_\perp \mathbf{z}_\perp)} \mathcal{F}(x, k_\perp^2)$$

- Quasi-PDF to pseudo-PDF relation

$$Q(y,P) = \frac{|P|}{2\pi} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dz_3\, e^{-i(y-x)Pz_3}\, \mathcal{P}(x, z_3^2)$$



Pseudo-PDFs, quasi-PDFs and TMDs

$$\mathcal{P}(x, z_\perp^2) = \int d^2\mathbf{k}_\perp e^{i(\mathbf{k}_\perp \mathbf{z}_\perp)} \mathcal{F}(x, k_\perp^2)$$

- When $\mathcal{F}(x, k_\perp^2)$ rapidly vanishes with $k_\perp$, pseudo-PDF and pseudo-ITD are regular for $z^2 = 0$, and $\mathcal{P}(x, 0) = f(x)$
- Recall quasi-PDF to pseudo-PDF relation

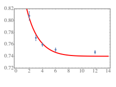
$$Q(y, P) = \frac{|P|}{2\pi} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dz_3\, e^{-i(y-x)Pz_3}\, \mathcal{P}(x, z_3^2)$$

- Expand $\mathcal{P}(x, z_3^2)$ in z_3^2

$$\mathcal{P}(x, z_3^2) = \sum_{l=0}^{\infty} (z_3^2 \Lambda^2)^l\, \mathcal{P}_l(x)$$

- $Q(y, P)$ approaches $f(y)$ like

$$Q(y, P) = f(y) + \sum_{l=1}^{\infty} \left(\frac{\Lambda^2}{P^2}\right)^l \frac{\partial^{2l}}{\partial y^{2l}} \mathcal{P}_l(y)$$



Problems with $1/P^{2l}$ expansion

$$Q(y,P) = f(y) + \sum_{l=1}^{\infty} \left(\frac{\Lambda^2}{P^2}\right)^l \frac{\partial^{2l}}{\partial y^{2l}} \mathcal{P}_l(y)$$

- Support mismatch: $-\infty < y < \infty$ for quasi-PDF $Q(y,P)$, while $\mathcal{P}_l(y)$'s vanish outside $-1 \le y \le 1$
- Do not take this expansion too literally
- Innocently-looking derivatives of $\mathcal{P}_l(y)$ generate infinite tower of singular functions like $\delta(y)$, $\delta(y \pm 1)$ and their derivatives
- Recall: even if a function $f(y)$ has a nontrivial support Ω (say, $-1 \le y \le 1$), one may formally represent it by a series

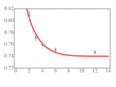
$$f(y) = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} M_N \, \delta^{(N)}(y)$$

over the functions $\delta^{(N)}(y)$ with an apparent support at one point $y = 0$ only

- M_N are moments of $f(y)$

$$M_N = \int_\Omega dy \, y^N f(y)$$

- While the difference between $Q(y,P)$ and $f(y)$ is formally given by a series in powers of $1/P^2$, its coefficients are not the ordinary functions of y



Moments of Quasi-PDFs

- In terms of TMDs:

$$Q(y,P) = f(y) + \sum_{l=1}^{\infty} \int d^2k_\perp \frac{k_\perp^{2l}}{4^l P^{2l} (l!)^2} \frac{\partial^{2l}}{\partial y^{2l}} \mathcal{F}(y, k_\perp^2)$$

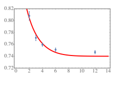
- To eliminate mismatch, take y^n moments $\langle y^n \rangle_Q$ of the quasi-PDFs

$$\langle y^n \rangle_Q \equiv \int_{-\infty}^{\infty} dy\, y^n Q(y,P) = \sum_{l=0}^{[n/2]} \frac{n!}{(n-2l)!(l!)^2} \frac{\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}}{4^l P^{2l}}$$

- $\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}$ are the combined moments of TMDs

$$\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}} \equiv \int_{-1}^{1} dx\, x^{n-2l} \int d^2k_\perp\, k_\perp^{2l}\, \mathcal{F}(x, k_\perp^2)$$

- Expansion makes sense only when $\mathcal{F}(x, k_\perp^2)$ vanishes faster than any power of $1/k_\perp^2$
- Is it possible to study the approach of $Q(y,P)$ to $f(y)$ for fixed y?



Relations between quasi-PDFs and TMDs

- z_3-dependence has the same origin as $k_\perp$ dependence of TMDs
- Quasi-PDFs can be obtained from TMDs (A.R., 2016)

$$Q(y,P)/P = \int_{-1}^{1} dx \int_{-\infty}^{\infty} dk_1 \mathcal{F}(x, k_1^2 + (y-x)^2 P^2)$$

- Or from pseudo-PDFs

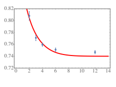
$$Q(y,P) = \frac{P}{2\pi} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dz_3 \;\; e^{i(x-y)(Pz_3)} \, \mathcal{P}(x, z_3^2)$$

- Try factorized model

$$\mathcal{P}(x, z_3^2) = f(x) I(z_3^2)$$

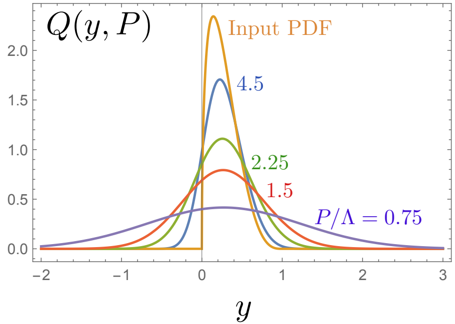
- Popular idea: Gaussian dependence $I(z_3^2) = e^{-z_3^2\Lambda^2/4}$

$$Q_G^{\text{fact}}(y,P) = \frac{P}{\Lambda\sqrt{\pi}} \int_{-1}^{1} dx \, f(x) \, e^{-(y-x)P^2/\Lambda^2}$$

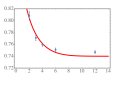


Numerical results for Gaussian model

- Take PDF $f(x) = u_v(x) - d_v(x) = \frac{315}{32}\sqrt{x}(1-x)^3\theta(0 \le x \le 1)$ obtained by pseudo-PDF method (Orginos et al. 2017)



- Curves for $P/\Lambda = 0.75, 1.5, 2.25$ are close to qPDFs obtained by Lin et al (2016), upper momentum $P = 1.3$ GeV, effective $\Lambda \approx 600$ MeV
- Need $P \sim 4.5\,\Lambda \approx 2.7$ GeV to get reasonably close to input PDF
- Note a lot of dirt for negative y, even for $P/\Lambda = 4.5$



Renormalizable theories and hard term

- In QCD $\mathcal{F}(x, k_\perp^2)$ has $1/k_\perp^2$ hard part and moments $\langle x^{n-2l} k_\perp^{2l} \rangle_{\mathcal{F}}$ diverge
- In the $l = 0$ case, the divergence is logarithmic
- Reflects the perturbative evolution of quasi-PDFs $Q(y, P)$ for large P
- Logarithmic singularity in z_3^2 in coordinate representation. At one loop,

$$\mathcal{M}^{\rm hard}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \ln(z_3^2) \int_0^1 du\, B(u)\, \mathcal{M}^{\rm soft}(u\nu, 0)$$

- Altarelli-Parisi (AP) evolution kernel

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

- The function $\mathcal{M}(\nu, \nu^2/P^2)$ that generates the quasi-PDF gets

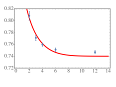
$$\mathcal{M}^{\rm hard}(\nu, \nu^2/P^2) = -\frac{\alpha_s}{2\pi} C_F \ln(\nu^2/P^2) \int_0^1 du\, B(u) \int_{-1}^1 dx\, e^{-iux\nu} f^{\rm soft}(x)$$

- Hard part of the quasi-PDF $Q(y, P)$ has a $\ln P^2$ term

$$Q^{\rm hard}(y, P) = \ln(P^2)\, \Delta(y) + \dots$$

- It is nonzero in the $-1 \le y \le 1$ region only

$$\Delta(y) = \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{du}{u} B(u) f^{\rm soft}(y/u)$$



Hard part of quasi-PDF

- $\ln z_3^2$ singularity of the ITD leads to a logarithmic perturbative evolution of the quasi-PDF $Q(y, P)$ for large P
- For TMDs, the $\ln z^2$ behavior translates into large-$k_\perp$ hard tail

$$\mathcal{F}^{\text{hard}}(x, k_\perp^2) = \frac{\Delta(x)}{\pi k_\perp^2}$$

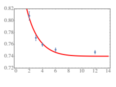
- Regularizing $1/k_\perp^2 \to 1/(k_\perp^2 + m^2)$ gives

$$\int_{-\infty}^{\infty} \frac{dk_1}{k_1^2 + (x-y)^2 P^2 + m^2} = \frac{\pi}{\sqrt{(x-y)^2 P^2 + m^2}}$$

- Determines the hard part of a quasi-distribution

$$Q^{\text{hard}}(y, P) = \int_{-1}^{1} dx \, \frac{\Delta(x)}{\sqrt{(x-y)^2 + m^2/P^2}}$$

- $Q^{\text{hard}}(y, P)$ does not vanish outside $-1 \le y \le 1$ region for finite α_s
- Shape of $Q(y, P)$ for $y > 1$ is calculable (if PDF is known)
- One should see that lattice gives it, and subtract
- Only then one gets PDF with $|x| \le 1$ support

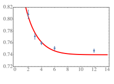


Gauge link complications

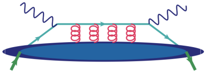
- Terms outside $|y| \le 1$ are generated by $\ln z_3^2$ term
- In QCD, there is one more source of the z^2-dependence of pseudo-ITD: gauge link $\hat{E}(0, z; A)$
- It has specific ultraviolet divergences
- Use Polyakov regularization $1/z^2 \to 1/(z^2 - a^2)$ for gluon propagator in coordinate space
- Effect of the UV cut-off a is similar to that of the lattice spacing
- At one loop, link-related UV singular terms have the structure

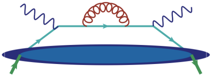
$$\Gamma_{\rm UV}(z_3, a) \sim -\frac{\alpha_s}{2\pi} C_F \left[2\frac{|z_3|}{a} \tan^{-1}\left(\frac{|z_3|}{a}\right) - 2 \ln\left(1 + \frac{z_3^2}{a^2}\right)\right]$$

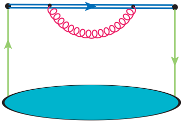
- For fixed a, these terms vanish when $z_3 \to 0$
- No violation of quark number conservation



Renormalize or exterminate?





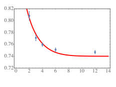


- Structure of factorization for DIS in Feynman gauge
- Gluon insertions generate gauge link $\hat{E}(0, z; A)$
- Quark self-energy diagram is not factorized as $S^c(z) \times \langle AA \rangle$
- Operator $\bar{\psi}(0)\hat{E}(0, z; A)\psi(z)$ should be accompanied by “no AA contractions”
- Link self-energy diagrams and UV-singular parts of vertex diagrams should be excluded together with associated z_3^2-dependence
- It is not sufficient just to subtract UV divergences

- Easy way out: consider reduced pseudo-ITD

$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(\nu, z_3^2)}{\mathcal{M}(0, z_3^2)}$$

- $\mathfrak{M}(\nu, z_3^2)$ has finite $a \to 0$ limit



Reduced Ioffe-time pseudo-distribution

- Reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ is a physical observable (like, say, DIS structure functions)
- No need to specify renormalization scheme, scale, etc.
- $\mathfrak{M}(\nu, z_3^2)$ is singular in $z_3 \to 0$ limit, $\ln z_3^2$ terms reflect perturbative evolution
- At one loop (with mass-type IR regularization)

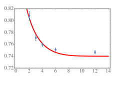
$$\mathfrak{M}(\nu, z_3^2) = \mathfrak{M}^{\rm soft}(\nu, 0) - \frac{\alpha_s}{2\pi} C_F \int_0^1 dw \left\{ \frac{1+w^2}{1-w} \left[\ln\left(z_3^2 m^2 \frac{e^{2\gamma_E}}{4} \right) + 1 \right] + 4\frac{\ln(1-w)}{1-w} \right\} \left[\mathfrak{M}^{\rm soft}(w\nu, 0) - \mathfrak{M}^{\rm soft}(\nu, 0) \right]$$

- For light-cone PDF, one should take $z^2 = 0$ and use some scheme for resulting UV divergence, say, $\overline{\rm MS}$
- Ioffe-time distribution $\mathcal{I}(\nu, \mu^2)$ is UV scheme and scale dependent

$$\mathcal{I}(\nu, \mu^2) = \int_{-1}^1 dx\, e^{ix\nu} f(x, \mu^2)$$

- Writing $\overline{\rm MS}$ ITD in terms of reduced pseudo-ITD

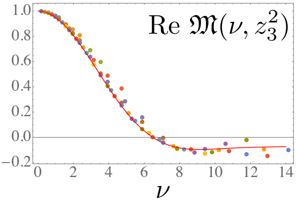
$$\mathcal{I}(\nu, \mu^2) = \mathfrak{M}(\nu, z_3^2) + \frac{\alpha_s}{2\pi} C_F \int_0^1 dw\, \mathfrak{M}(w\nu, z_3^2) \times \left\{ \frac{1+w^2}{1-w} \left[\ln\left(z_3^2 \mu^2 \frac{e^{2\gamma_E}}{4} \right) + 1 \right] + \left[4\frac{\ln(1-w)}{1-w} - 2(1-w) \right] \right\}_+$$



Evolution in lattice data

- Exploratory lattice study of reduced pseudo-ITD $\mathfrak{M}(\nu, z_3^2)$ for the valence $u_v - d_v$ parton distribution in the nucleon [Orginos et al. 2017]
- When plotted as function of ν, data both for real and imaginary parts lie close to respective universal curves
- Data show no polynomial z_3-dependence for large z_3 though z_3^2/a^2 changes from 1 to ~ 200
- Apparently no higher-twist terms in the reduced pseudo-ITD
- Real part corresponds to the cosine Fourier transform of $q_v(x) = u_v(x) - d_v(x)$

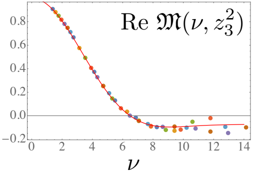
$$\mathfrak{R}(\nu) \equiv \text{Re}\,\mathfrak{M}(\nu) = \int_0^1 dx\, \cos(\nu x)\, q_v(x)$$

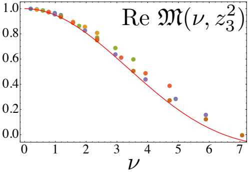


- Overall curve corresponds to the function

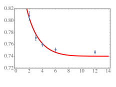
$$f(x) = \frac{315}{32}\sqrt{x}(1-x)^3$$

- Obtained by forming cosine Fourier transforms of $x^a(1-x)^b$-type functions and fitting a, b
- Shape is dominated by points with smaller values of Re $\mathfrak{M}(\nu, z_3^2)$

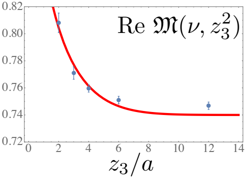




- Points corresponding to $7a \leq z_3 \leq 13a$ values
- Some scatter for points with $\nu \gtrsim 10$
- Otherwise, practically all the points lie on the universal curve based on $f(x)$.
- No z_3-evolution visible in large-z_3 data
- Points in $a \leq z_3 \leq 6a$ region
- All points lie higher than the curve based on the $z_3 \geq 7a$ data
- Perturbative evolution increases real part of the pseudo-ITD when z_3 decreases
- Conjecture that the observed higher values of $\mathrm{Re}\mathfrak{M}$ for smaller-z_3 points may be a consequence of evolution



Building $\overline{\text{MS}}$ ITD

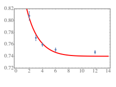


- z_3-dependence of the lattice points for "magic" Ioffe-time value $\nu = 3\pi/4$
- Eye-ball fit line has "Perturbative" $\ln(1/z_3^2)$ behavior for small z_3, rapidly tends to a constant for $z_3 > 6a$
- $\mathfrak{R}(\nu, z_3^2)$ decreases when z_3 increases

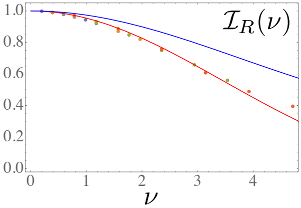
- Starts to visibly deviate from a pure logarithmic $\ln z_3^2$ pattern for $z_3 \gtrsim 5a$
- This sets the boundary $z_3 \leq 4a$ on the "logarithmic region"
- $\overline{\text{MS}}$ ITD in terms of reduced pseudo-ITD

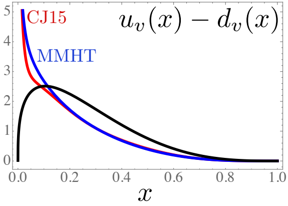
$$\mathcal{I}(\nu, \mu^2) = \mathfrak{M}(\nu, z_3^2) + \frac{\alpha_s}{2\pi} C_F \int_0^1 dw\, \mathfrak{M}(w\nu, z_3^2)$$
$$\times \left\{ \frac{1+w^2}{1-w} \left[\ln \left(z_3^2 \mu^2 \frac{e^{2\gamma_E}}{4} \right) + 1 \right] + \left[4 \frac{\ln(1-w)}{1-w} - 2(1-w) \right] \right\}_+$$

- $\mathcal{I}(\nu, \mu^2)$ should not depend on z_3
- This happens only if, for some α_s, the $\ln z_3^2$-dependence of the1-loop term cancels actual z_3^2-dependence of the data, visible as scatter in the data



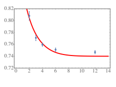
Building $\overline{\text{MS}}$ ITD





- We choose $\mu = 1/a$ which, at lattice spacing of 0.093 fm is $\approx$ 2.15 GeV
- Using $\alpha_s/\pi = 0.1$ and $z_3 \leq 4a$ data, we generate the points for $\mathcal{I}_R(\nu, (1/a)^2)$
- Upper curve corresponds to the ITD of the CJ15 global fit PDF for μ =2.15 GeV
- Evolved points are close to some universal curve with a rather small scatter
- The curve itself corresponds to the cosine transform of a normalized $\sim x^a(1-x)^b$ distribution with $a = 0.35$ and $b = 3$

- $\sim x^{0.35}(1-x)^3$ PDF compared to CJ15 and MMHT global fits for $\mu = 2.15$ GeV
- Unable to reproduce $\sim x^{-0.5}$ Regge behavior
- Possible reasons: large pion mass, quenched approximation



Summary

- Analyzed nonperturbative structure of quasi-PDFs $Q(y, P)$ using their relation to pseudo-ITDs and TMDs
- Shown that $(\Lambda^2/P^2)^n$ expansion for $Q(y, P)$ involves generalized functions
- Using factorized models for TMDs, studied rate of approach of quasi-PDFs $Q(y, P)$ to PDFs $f(y)$ when $P \to \infty$
- Analyzed perturbative structure of quasi-PDFs using their relation to pseudo-ITDs and TMDs
- Argued that link-related terms should be “exterminated”
- Proposed to use reduced pseudo-ITD
- Studied evolution of exploratory lattice data for reduced pseudo-ITD