A lattice QCD computation of quark distributions at the physical point

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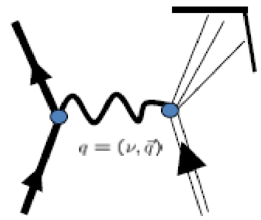
Outline

- Quark distributions and quark quasi-distributions
- Extracting quark distributions from the quasi-distributions
- Computation of the matrix elements using lattice QCD
- Renormalization
- The x dependence of the quark distributions
- Summary

Quark distributions and quase-distributions

Cross sections are measured



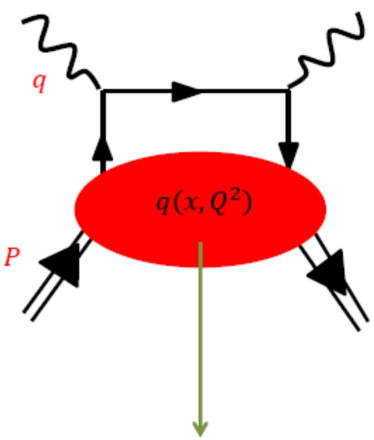
Cross sections written in terms of structure functions: $F_1(x,Q^2), F_2(x,Q^2), g_1(x,Q^2), g_2(x,Q^2), \cdots$

QCD + OPE:

$$\int_0^1 dx x^{n-2} F_2(x,Q^2) = \sum_i a_n^{(i)} C_n^{(i)}(Q^2)$$

$$\langle P | \mathcal{O}_{\mu_1 \cdots \mu_n} | P \rangle = a_n P_{\mu_1} \cdots P_{\mu_2}$$

Moments of the parton distributions: $a_n = \int dx\, x^{n-1} q(x)$



At leading order (LO) in pQCD:, $F_2(x,Q^2) = x \sum_q e_q^2 q(x,Q^2)$

Light-cone quark distributions

The most general form of the matrix element is:

$$\langle P|O^{\mu_1\mu_2\cdots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\cdots\mu_n}$$

$$\Pi^{\mu_1\mu_2\cdots\mu_n} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\cdots gP\cdots P\}_{k,j}(P^2)^j$$

We use the following four-vectors

$$P = (P_0, 0, 0, P_3) \quad \lambda = (1,0,0,-1)/\sqrt{2} \longrightarrow \boxed{\lambda\cdot P = (P_0+P_3)/\sqrt{2} = P_+}$$

$$\lambda_{\mu_1}\lambda_{\mu_2}\left\langle P\middle|O^{\mu_1\,\mu_2}\middle|P\right\rangle = 2a_n^{(0)}\left(P^+P^+ - \lambda^2\frac{M^2}{4}\right) = 2a_n^{(0)}P^+P^+$$

In general, we have

$$\lambda_{\mu_1}\cdots\lambda_{\mu_n}\Pi^{\mu_1\cdots\mu_n} = (P^+)^n \longrightarrow \boxed{\langle P|O^{+\cdots+}|P\rangle = 2a_n^{(0)}(P^+)^n}$$

Matrix elements projected on the light-cone are protected from target mass corrections

Taking the inverse Mellin transform

$$a_n^{(0)} = \int dx\, x^{n-1} q(x) \qquad\qquad q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n^{(0)}$$

Using $a_n^{(0)} = \langle P|O^{+\cdots+}|P\rangle / 2(P^+)^n$

$$q(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle$$

$$W(\xi^-,0) = e^{-ig\int_0^{\xi^-} A^+(\eta^-)d\eta^-} \quad \text{(Wilson line)}$$

- Light cone correlations
- Equivalent to the distributions in the Infinite Momentum Frame
- Light cone dominated $\xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian lattice $t^2 + z^2 \sim 0$

Quasi Distributions

X. Ji, "Parton Physics on a Euclidean Lattice," PRL 110 (2013) 262002.

Suppose we project outside of the light-cone:

$$\lambda = (0,0,0,-1) \qquad P = (P_0, 0,0, P_3) \qquad \boxed{\lambda \cdot P = P_3}$$

We take n=2

= -1

$$\langle P|O^{33}|P\rangle = 2\tilde{a}_n^{(0)}(P^3P^3 - \lambda^2 P^2/4) = 2\tilde{a}_n^{(0)}((P^3)^2 + P^2/4)$$

Mass terms contribute

In general,

$$\boxed{\langle P|O^{3\cdots 3}|P\rangle = 2\tilde{a}_{2k}^{(0)}(P_3)^{2k}\sum_{j=0}^{k}\mu^j\frac{(2k-j)!}{j!\,(2k-2j)!} \equiv 2\tilde{a}_{2k}(P_3)^{2k}}$$

with $\boxed{\mu = M^2/4(P_3)^2}$

Defining

$$\tilde{a}_n^{(0)} = \int dx\, x^{n-1} \tilde{q}^{(0)}(x) \qquad \tilde{a}_n = \int dx\, x^{n-1} \tilde{q}(x)$$

Taking the inverse Mellin transform

$$\tilde{q}^{(0)}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dn\, x^{-n} \tilde{a}_n^{(0)} \qquad \tilde{q}(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dn\, x^{-n} \tilde{a}_n$$

$$\tilde{q}(x) = \tilde{q}^{(0)}(\xi)/(1+\mu\xi^2) \; + antiquarks \qquad \xi = \frac{2x}{1+\sqrt{1+4\mu x^2}}$$

$$\tilde{q}(x, P_3) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{izk_3} \langle P | \bar{\psi}(z) \gamma^3 W(z,0) \psi(0) | P \rangle$$

$$W(z,0) = e^{-ig \int_0^z A^3(z')dz'}$$

$$k_3 = xP_3$$

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice
- Can be related to the usual distributions via a matching procedure

Extracting quark distributions from quark quasi-distributions

Infrared region untouched when going from
a finite to an infinite momentum

Infinite momentum: $p_3 \to \infty$ (before integrating over the quark transverse momentum k_T)

$$q(x,\mu) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 \Gamma\left(\frac{x}{y},\mu\right) q_{bare}(y)\,\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

Finite momentum: p_3 fixed

$$\tilde{q}(x,P_3) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} \tilde{Z}_F\,(P_3)\right\} + \frac{\alpha_s}{2\pi}\int_{x/y_c}^1 \tilde{\Gamma}\left(\frac{x}{y},P_3\right) q_{bare}(y)\,\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

$$\tilde{q}(\pm y_c) = 0$$

In principle, $y_c \to \infty$

The two equations can be solved for the quark distributions, resulting in a matching equation:

$$q(x,\mu) = \tilde{q}(x,p_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,p_3)\delta Z_F\left(\frac{\mu}{p_3}, x_c\right) - \frac{\alpha_s}{2\pi}\int_{-x_c}^{-|x|/y_c} \delta\Gamma\left(y, \frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y}, p_3\right)\frac{dy}{|y|} - \frac{\alpha_s}{2\pi}\int_{+|x|/y_c}^{+x_c} \delta\Gamma\left(y, \frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y}, p_3\right)\frac{dy}{|y|}$$

Matching equation

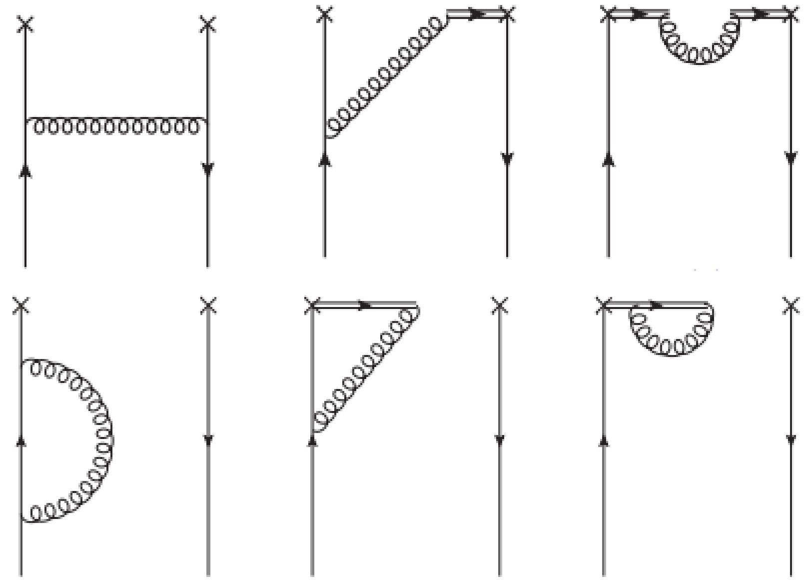
Where $\delta\Gamma = \tilde{\Gamma} - \Gamma$

$\delta Z_F = \tilde{Z}_F - Z_F$ are calculated using perturbation theory in the continuum

The integral in x in the quasi-quark self-energy, $\tilde{Z}_F$, is left unintegrated, hence the dependence on the limits of integration, $\pm x_c$. At the end, $x_c \to \infty$ in δZ_F.

Because quasi-quark vertex correction, $\tilde{\Gamma}$, only vanishes at the infinity, the range of integration in the vertex also extends to zero in the convolution as $y_c \to \infty$

Perturbative QCD in the continuum



Vertex: $\boldsymbol{\Gamma}$ **or** $\tilde{\boldsymbol{\Gamma}}$

Self-energy: $\boldsymbol{Z_F}$ **or** $\tilde{\boldsymbol{Z}}_{\boldsymbol{F}}$

First computed in a cut-off scheme: X. Xiong, X. Ji, J. H. Zhang and Y. Zhao, PRD 90 014051 (2014)

Then computed using $\overline{MS}$: W. Wang, S. Zhao and R. Zhu, Eur. Phys. J. C78 (2018) 147; I. W. Stewart, Y. Zhao, PRD 97 054512 (2018

Divergence associated with $x \to \pm\infty$ not treated

Renormalization of qPDF in the continuum: X. Ji, J. H. Zhang and Y. Zhao, PRD 92, 034006 (2015); J. W. Chen, X. Ji, J. H. Zhao, NPB 915 (2017) 1; X. Ji, J. H. Zhao, Y. Zhao, PRL 120 (2018) 112001

Renormalization of qPDF to all orders in QCD perturbation theory in configuration space: T Ishikawa, Yan-Qing Ma, Jian-Wei Qiu, S. Yoshida, PRD 96, 094019 (2017)

One can compute the unpolarized distribution with the Wilson line parallel (γ_3) or perpendicular (γ_0) to the direction of the current

For the parallel case, the computation was done in the previous references.
For the perpendicular case, γ_0, in the $\overline{MS}$ scheme ($x_c \to \infty$) the kernels of the matching are:

$$\delta\Gamma\left(y, \frac{\mu}{p_3}\right) = \begin{cases} -\frac{1+y^2}{1-y} ln \frac{y-1}{y} + 1 & y > 1 \\ -\frac{1+y^2}{1-y} ln \frac{\mu^2}{4p_3^2 y(1-y)} - \frac{y+y^2}{1-y} & 0 < y < 1 \\ -\frac{1+y^2}{1-y} ln \frac{y}{y-1} - 1 & y < 0 \end{cases}$$

$$\delta Z_F\left(\frac{\mu}{p_3}, x_c\right) = \begin{cases} \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta-1}{\eta} - 1\right) & \eta > 1 \\ \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\mu^2}{4p_3^2 \eta(1-\eta)} + \frac{\eta+\eta^2}{1-\eta}\right) & 0 < \eta < 1 \\ \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta}{\eta-1} + 1\right) & \eta < 0 \end{cases}$$

The above expressions have a problem: they have infinities associated with the limit $x_c \to \infty$ when doing the convolution and calculating the x dependence of the integrals;

This infinity has its origin in the self-energy of the quasi-PDF outside the physical region;

A possible solution is to renormalize these infinities, having in mind that the Ward identity is always respected: particle number is always preserved

And the final matching can be written taking $x_c \to \infty$

$$\delta\Gamma^R\left(y,\frac{\mu}{p_3}\right) = \begin{cases} -\frac{1+y^2}{1-y} ln\frac{y-1}{y} + 1 + \frac{3}{2y} & y > 1 \\ -\frac{1+y^2}{1-y} ln\frac{\mu^2}{4p_3^2 y(1-y)} - \frac{y+y^2}{1-y} & 0 < y < 1 \\ -\frac{1+y^2}{1-y} ln\frac{y}{y-1} - 1 + \frac{3}{2(1-y)} & y < 0 \end{cases}$$

The terms in red have their origin in the renormalization of the infinities associated with the momentum fraction of the qPDFs

$$\delta Z_F^R\left(\frac{\mu}{p_3}\right) = \begin{cases} \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln\frac{\eta-1}{\eta} - 1 - \frac{3}{2\eta}\right) & \eta > 1 \\ \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln\frac{\mu^2}{4p_3^2 \eta(1-\eta)} + \frac{\eta+\eta^2}{1-\eta}\right) & 0 < \eta < 1 \\ \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln\frac{\eta}{\eta-1} + 1 - \frac{3}{2(1-\eta)}\right) & \eta < 0 \end{cases}$$

Renormalizes the whole momentum fraction in the unphysical region;

Automatically preserves quark number in all stages of the computation;

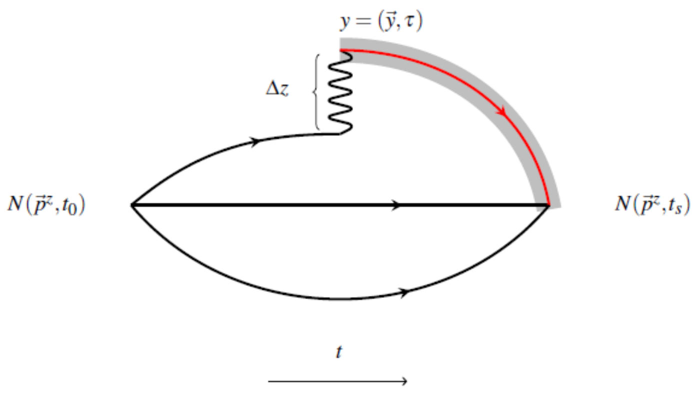
Quark number is not scale dependent;

For the helicity case, similar as above, obtained by adding a factor $2\,(1+y)$ in the physical region;

For an alternative prescription, see T. Izubuchi et al., 1801.03917

Computation of matrix elements using the lattice QCD

$$\frac{C^{3pt}(T_s,\tau,0;P_3)}{C^{2pt}(T_s,0;P_3)} \propto h(P_3,z), \qquad 0 \ll \tau \ll T_s$$



With the 3 point function given by:

$$C^{3pt}(t,\tau,0) = \langle N_\alpha(\vec{P},t)\mathcal{O}(\tau)\overline{N_\alpha}(\vec{P},0)\rangle$$

And

$$\mathcal{O}(z,\tau,Q^2=0) = \sum_{\vec{y}} \bar{\psi}(y+z)\Gamma\, W\,(y+z,y)\psi(y)$$

Where the matrix elements (ME) are: $h(P_3,z) = \langle P|\bar{\psi}(z)\Gamma\, W\,(z,0)\psi(0)|P\rangle$

Setup: $N_f = 2, \qquad \beta = \frac{6}{g_0^2} = 2.10, \qquad a = 0.0938(3)(2)\,fm$

$48^3 \times 96, \qquad L = 4.5 fm, \qquad m_\pi = 0.1304(4)\ GeV, \qquad m_\pi L = 2.98(1)$

$P_3 = \frac{6\pi}{L}, \frac{8\pi}{L}, \frac{10\pi}{L} = 0.84, 1.11, 1.38$ GeV

The signal for the computation of C^{2pt} and C^{3pt} decays exponentially with the nucleon momentum P_3.

This decay can be attenuated, although not eliminated, by using momentum smearing in the quark fields:

$$S_M(k)\psi(x) = \frac{1}{1+8\kappa}\left[\psi(x) + \kappa \sum e^{ik\hat{j}} U_j(x)\psi(x+\hat{j})\right]$$

The signal also decays exponentially with the source-sink separation T_s, which, however, has to be large enough to prevent excited states contamination.

We have then a problem:

1) if T_s is kept small, one can go to high P_3, but the final result is meaningless because of contamination from excited states;

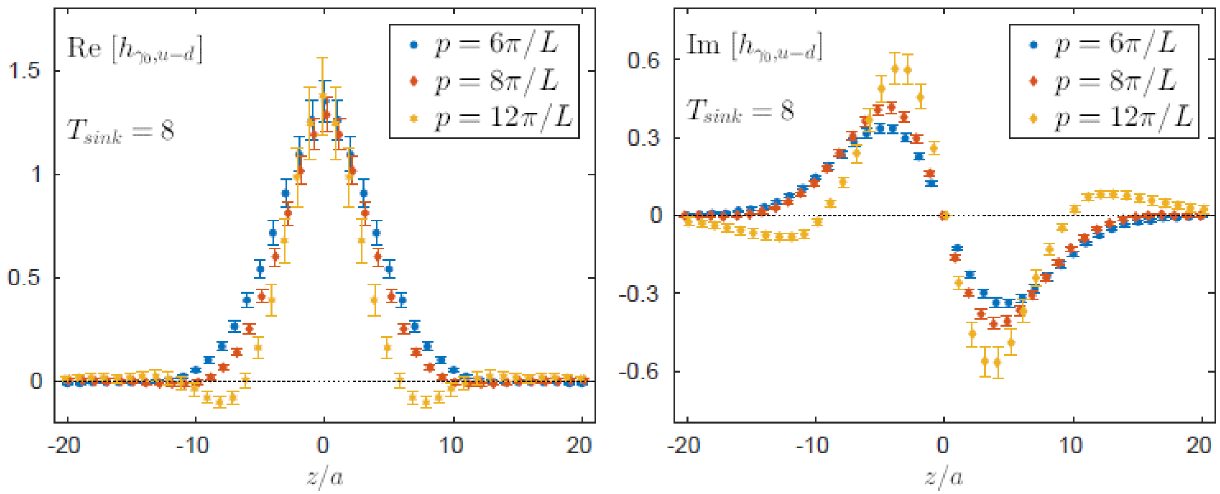
2) if T_s is large enough to make sure that there is no excited states contamination, then P_3 cannot be too large, assuming reasonable computer time;

3) a compromise has to be built: control over excited states is the priority!

Importance of a correct source-sink separation

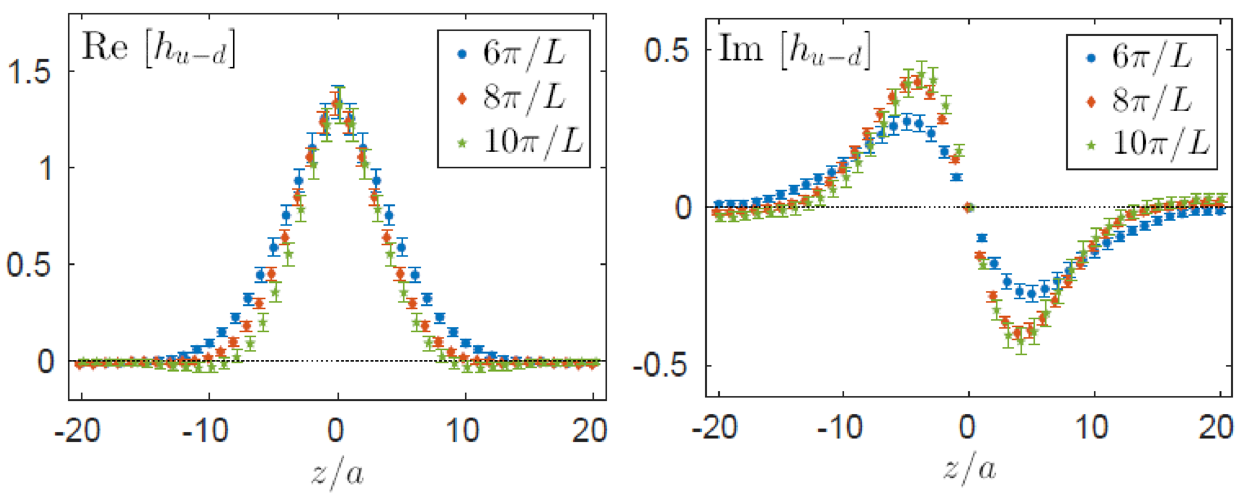
Real and imaginary parts of $h(P_3, z)$

$T_s = 8a \approx 0.75$ fm



Large excited states contamination at larger values of P_3

$T_s = 12a \approx 1.13$ fm

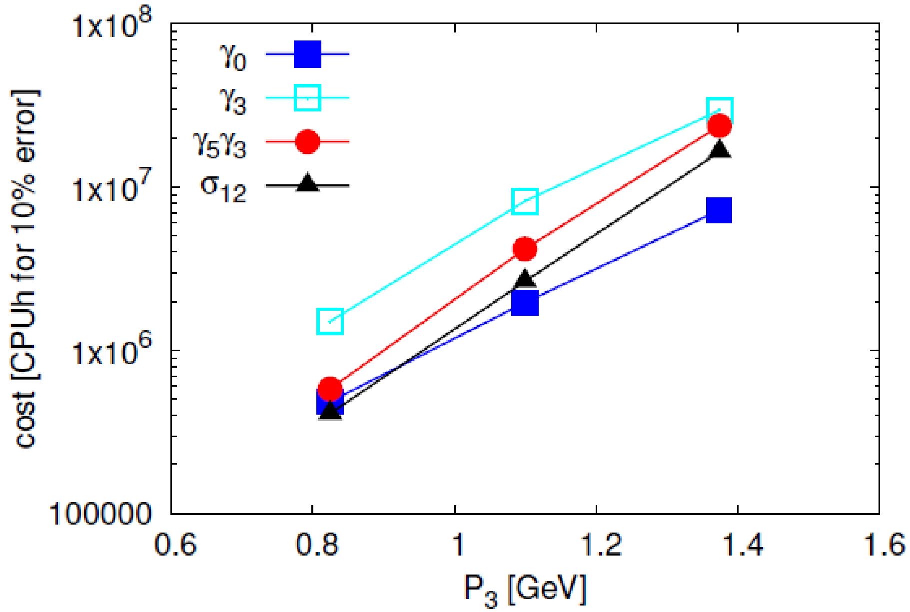


Excited states seem under control

We need to go to large T_s to avoid excited state contamination. $T_s = 12a$ seems to be the lowest safe choice;

We show below the computational cost, for the values of P_3 used in our simulations, for the various Dirac structures

$T_s = 12a \approx 1.13\,\text{fm}$



It is clear from the plot that γ_0 cheaper than γ_3 for the computation of the unpolarized PDFs

The γ_0 insertion also avoids mixing, and it has thus smaller statistical uncertainties associated with it than γ_3

Computation made for unpolarized (γ_0) and helicity ($\gamma_5\gamma_3$) distributions

6 directions of Wilson line: $\pm x, \pm y, \pm z$

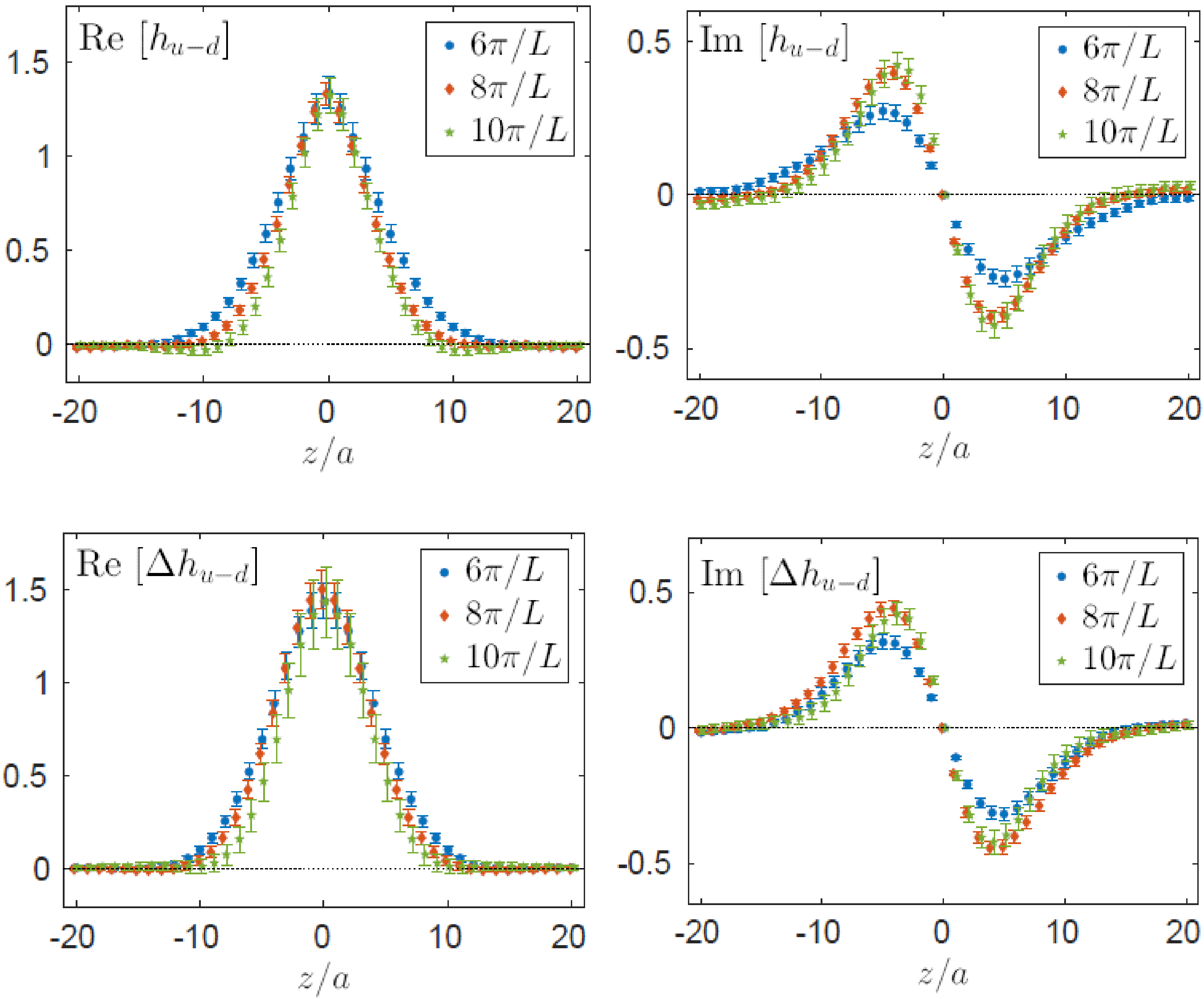
16 source positions

Separation $T_s \approx 1.1$ fm as the lowest safe choice

$P_3 = \frac{6\pi}{L}$			$P_3 = \frac{8\pi}{L}$			$P_3 = \frac{10\pi}{L}$		
Ins.	N_{conf}	N_{meas}	Ins.	N_{conf}	N_{meas}	Ins.	N_{conf}	N_{meas}
γ_0	50	4800	γ_0	425	38250	γ_0	655	58950
$\gamma_5\gamma_3$	65	6240	$\gamma_5\gamma_3$	425	38250	$\gamma_5\gamma_3$	655	58950

With these configurations, we compute the corresponding matrix elements

C. Alexandrou et al., 1803.02685



Unpolarized

Helicity

The bare matrix elements $h(P_3, z) = \langle P|\bar{\psi}(z)\Gamma W(z,0)\psi(0)|P\rangle$, however, contain divergences:

Renormalization is necessary!

Renormalization

Before 2017, only the bare ME were available: a direct comparison with data was compromised

In 2017, renormalization of the ME became available, both:

1) Perturbatively, where it was shown that: M. Constantinou, H. Panapaulos, PRD (2017)054506

Mixing between the vector γ_3, and scalar operators happen. For the case of the vector γ_0, which is perpendicular to the Wilson line, such mixing does not happen;

A linear divergence, $\propto \frac{|z|}{a}$, appears in the tadpole diagrams. It resums, in all orders of PT, to an exponential form $Exp\left(-\frac{c|z|}{a}\right)$ Dotsenko et al. NPB 169 (1980) 527

2) Nonperturbatively:

Using the RI'-MOM to remove the linear divergence, resumed into the exponential, plus the log divergence with respect to the regulator a

C. Alexandrou et al., NPB 923 (2017) 394 (Frontier Article)
J-W. Chen et al., PRD 97 014505 (2018)

Using the auxiliary field approach J. Green, K. Jansen, FS 1707.07152

We present results for the RI'-MOM scheme

We need however, the distributions in the $\overline{MS}$ scheme.

Two options to obtain the quark distributions from the ME renormalized in the RI'-MOM scheme:

a) Compute the qPDFs in the RI'-MOM. Then compute a matching kernel relating qPDFs in the RI'-MOM scheme directly to PDFs in the $\overline{MS}$ scheme;

Stewart & Zhao, PRD 97 054512 (2018)

b) Or go through a two step process:

1) Convert the ME from RI'-MOM to $\overline{MS}$ using perturbation theory

M. Constantinou, H. Panapaulos, PRD (2017)054506

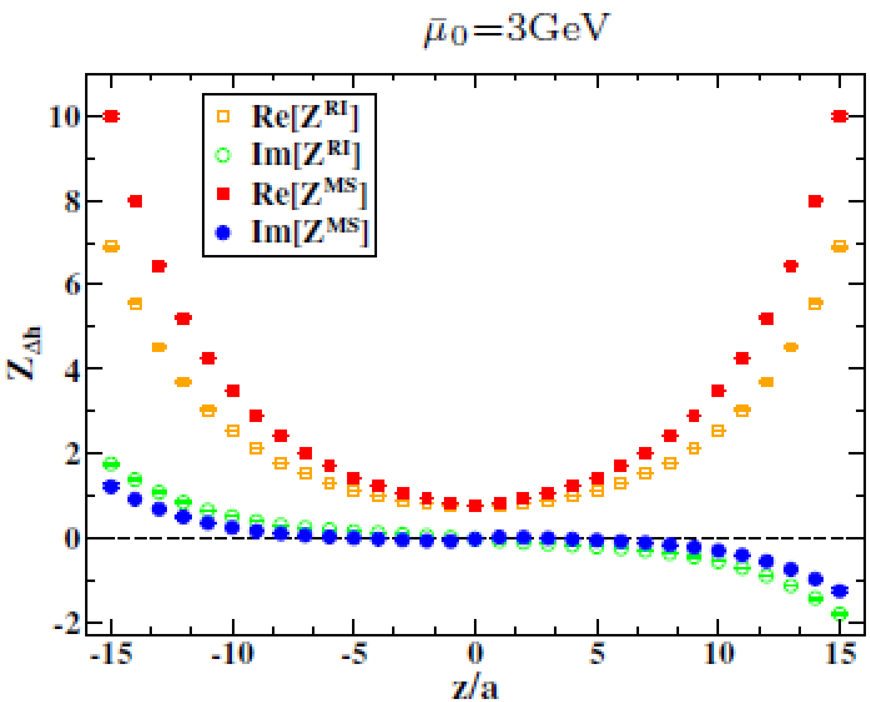
2) Compute the qPDF in the $\overline{MS}$ scheme, and then match the $\overline{MS}$ qPDF to the $\overline{MS}$ PDF

C. Alexandrou et al., NPB 923 (2017) 394 (Frontier Article)
C. Alexandrou et al., 1803.02685

Renormalization factor for helicity

RI'-MOM scheme at the scale $\bar{\mu}_0 = 3$ GeV

Perturbative conversion to $\overline{MS}$ scheme at the scale 2 GeV



$$Z_q^{-1}\, Z_{\mathcal{O}} \frac{1}{12} Tr[\nu(p,z)(\nu^{Born}(p,z))^{-1}]|_{p^2=\bar{\mu}_0^2} = 1$$

$$Z_q = \frac{1}{12} Tr[\left(S(p)\right)^{-1} S^{Born}(p)]|_{p^2=\bar{\mu}_0^2}$$

The vertex function ν contains the same divergences as the nucleon matrix elements

The factor $Z_{\mathcal{O}}$ subtracts both the linear and log divergences.

The linear divergence associated with the Wilson line makes $Z_{\mathcal{O}}$ to grow very fast for large z;

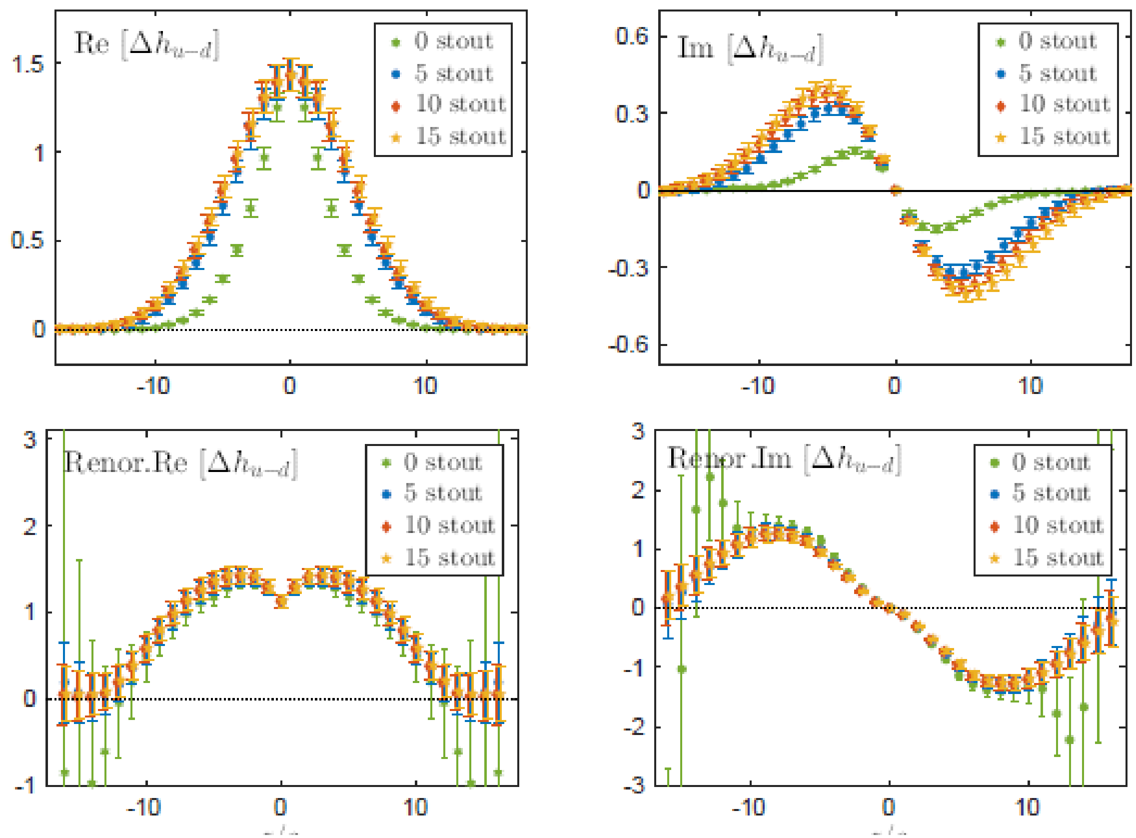
That makes the renormalized ME to have amplified errors at large z;

We thus apply stout smearing to the Wilson lines only in order to smooth the divergence;

In the end, if the procedure is consistent, the resulting renormalized ME should be the same, independent of the smearing applied

Renormalized ME for the helicity case

$P_3 \approx 0.83\ GeV$



Bare ME

Renormalized ME

ME sit on top of each other after renormalization

Renormalization is doing its job!

The x dependence of the quark distributions

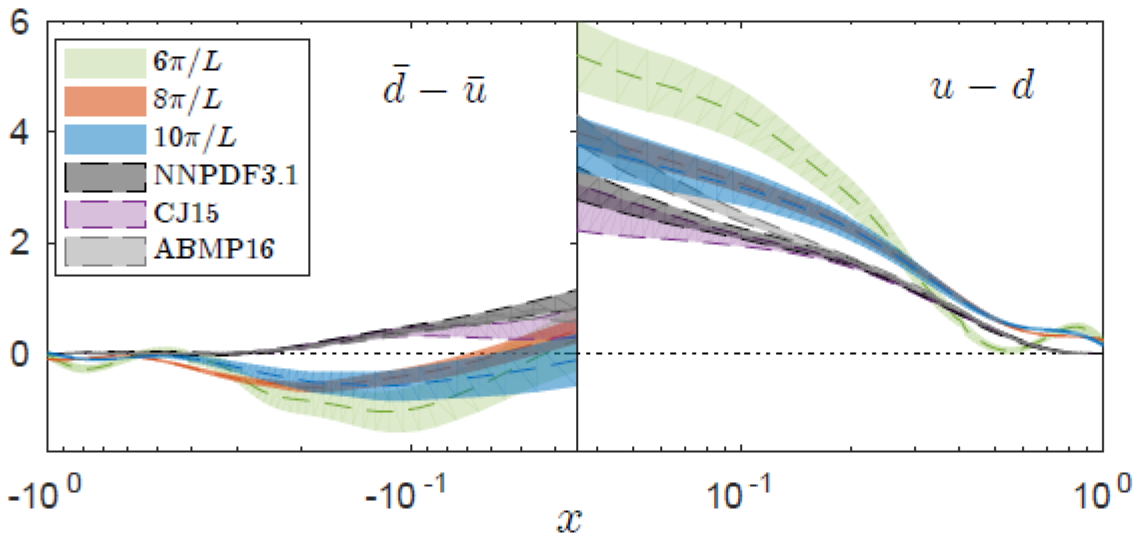
Once we have the ME, we compute the qPDF:

$$\tilde{q}(x,\mu^2,P_3) = \int \frac{dz}{4\pi} e^{-ixP_3z} \langle P | \bar{\psi}(z) \Gamma \mathrm{W}(\mathrm{z},0) \psi(0) | P \rangle$$

And then apply the matching plus target mass corrections to obtain the light-cone PDF:

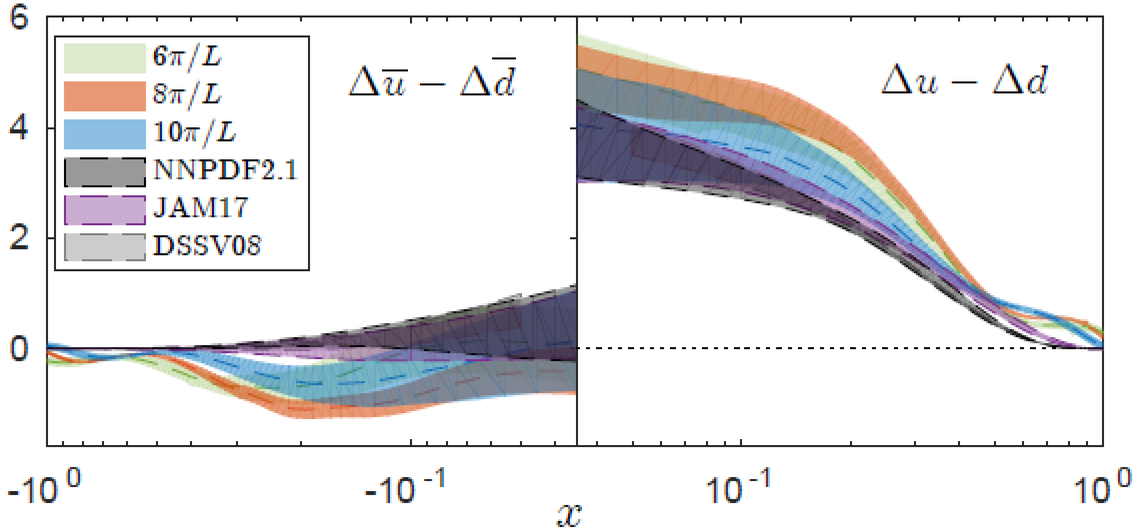
$$q(x,\mu) = \int_{-\infty}^{+\infty} \frac{d\xi}{\xi} C\left(\xi, \frac{\mu}{xP_3}\right) \tilde{q}\left(\frac{x}{\xi}, \mu, P_3\right)$$

Unpolarized nonsinglet quark distribution



C. Alexandrou et al., 1803.02685

Helicity nonsinglet quark distribution



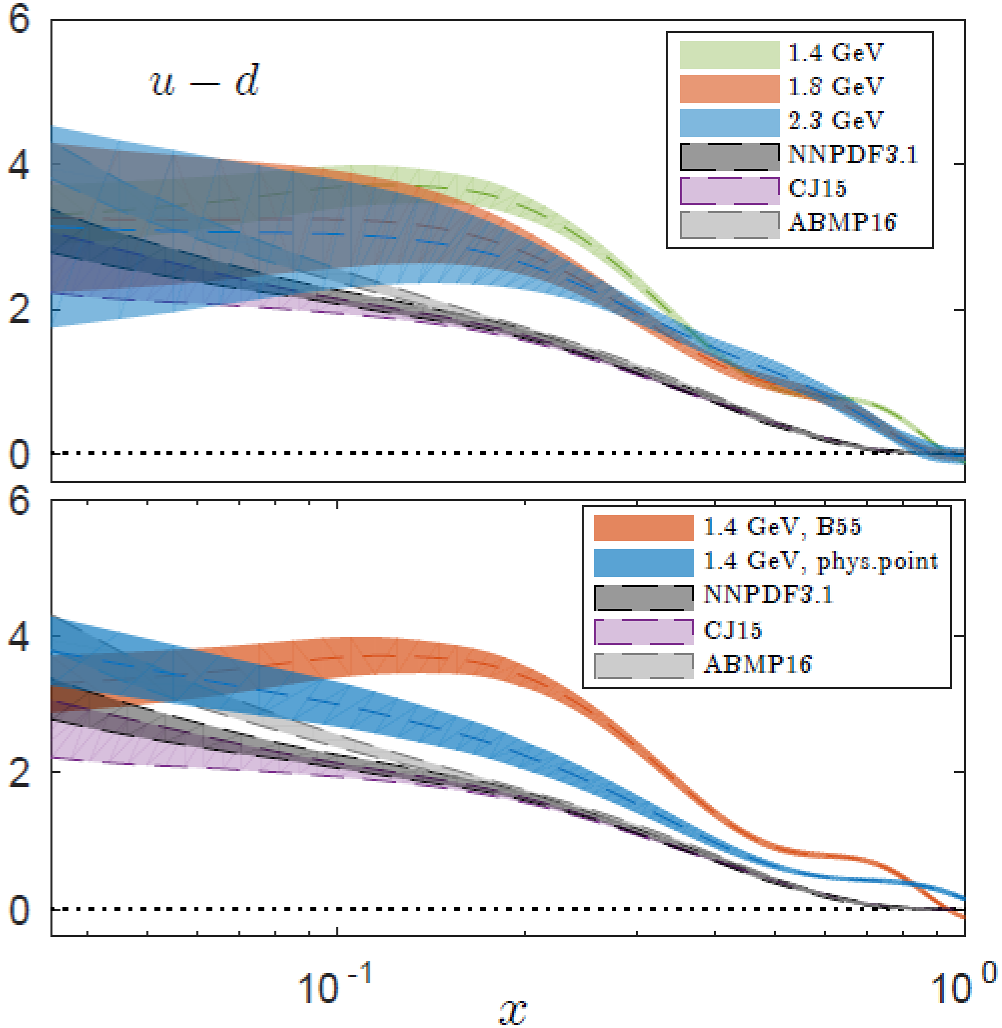
C. Alexandrou et al., 1803.02685

Remarkable qualitative agreement

For the values of P_3 used here, the ME do not decay fast enough, that is, before e^{-ixP_3z} becomes negative

When doing the Fourier transform, unphysical oscillations appear, remarkably for $x > 0.5$, and an unphysical minimum at $x \approx -0.2$

Effect of the physical pion mass



Unphysical pion mass
$m_\pi \approx 372$ **MeV**

Unphysical x Physical m_π

Summary

We have shown an *ab initio* computation of the x dependence of nonsinglet PDFs at the physical point;

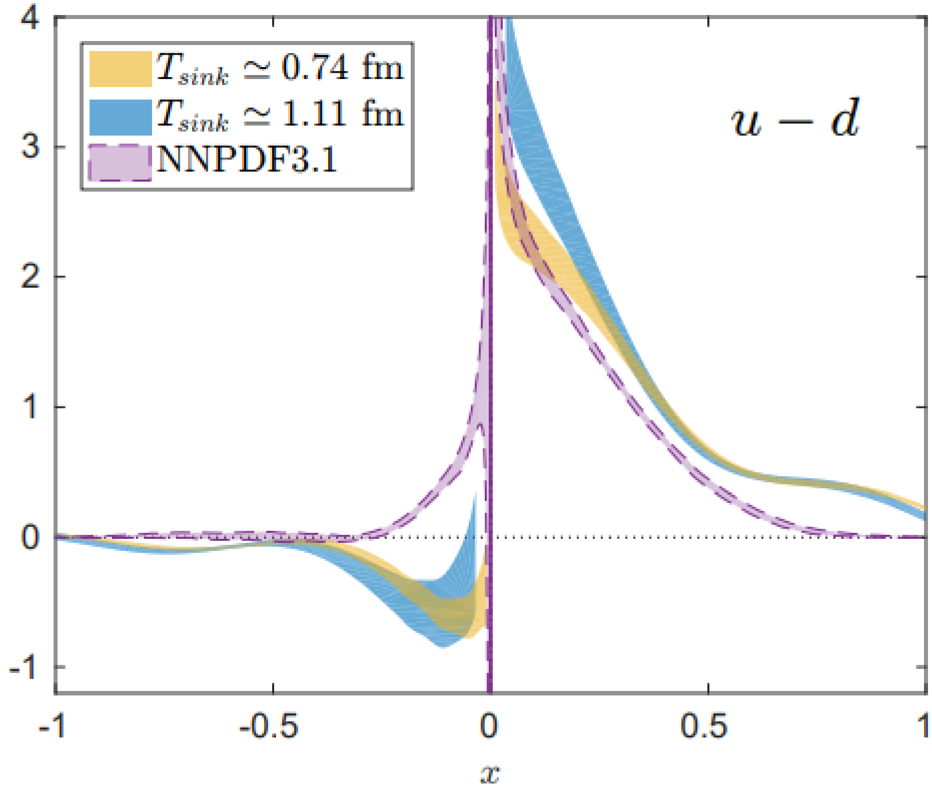
Strong pion mass dependence: the physical mass is essential to produce a shape for the PDF that is similar to the phenomenological fits;

Enormous progress over the last couple of years:

- a complete non-perturbative prescription for the ME has emerged
- a perturbative conversion from RI'-MOM and $\overline{MS}$ has been developed
- it is now possible to use an operator for the unpolarized PDF that avoids mixing
- the matching equations relating the qPDFs to the light-cone PDFs have been improved

Physical point computation also presented in J.W. Chen 1803.04393

qPDFs are intrinsically related to pseudo-PDFs (see the talk by A. Radyushkin)



$T_{sink} \simeq 0.74$ fm
$T_{sink} \simeq 1.11$ fm
NNPDF3.1
$u-d$
4
3
2
1
0
-1
-1
-0.5
0
0.5
1
x