Full Nucleon Structure Functions from the Lattice

G. Schierholz

Deutsches Elektronen-Synchrotron DESY

With

A. Chambers, R. Horsley, Y. Nakamura, H. Perlt, P. Rakow, A. Schiller, K. Somfleth, R. Young, J. Zanotti

QCDSF Collaboration

Phys. Rev. Letters 118 (2017) 242001 [arXiv:1703.01153]

Classical Approach

- Moments

$$\mu_n(q^2) = f \int_0^1 dx\, x^n\, F_1(x, q^2)$$

Mellin transform

$$F_1(x, q^2) = \frac{1}{2\pi i f} \int_{c-i\infty}^{c+i\infty} ds\, x^{-s-1} \mu_s(q^2)$$

- OPE

$$\mu_1(q^2) = c_2(q^2 a^2)\langle N|\mathcal{O}_2(a)|N\rangle + \frac{c_4(q^2 a^2)}{q^2}\langle N|\mathcal{O}_4(a)|N\rangle + \cdots$$

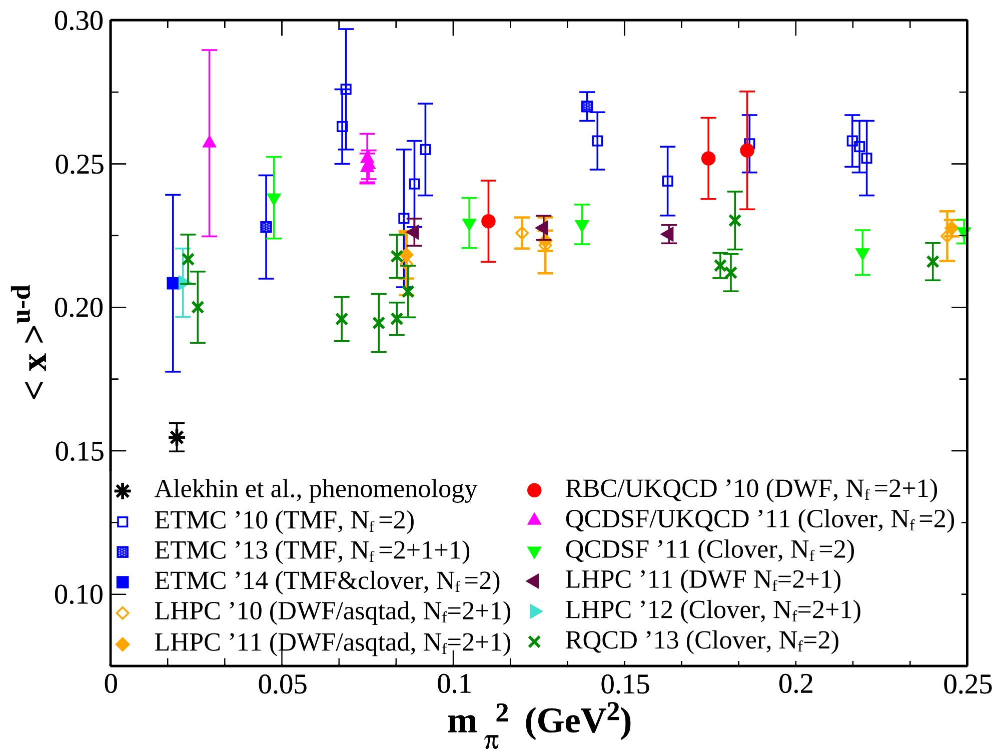
Lattice: $q^2 \sim \dfrac{1}{a^2}$

$$\mu_2(q^2) = c_3(q^2 a^2)\langle N|\mathcal{O}_3(a)|N\rangle + \frac{c_5(q^2 a^2)}{q^2}\langle N|\mathcal{O}_5(a)|N\rangle + \cdots$$

$$\vdots$$

- The computations are limited to a few lower moments, due to issues of operator mixing and renormalization. Even so, the uncertainties are at least comparable to the magnitude of the power corrections

Martinelli & Sachrajda



Constantinou [arXiv:1511.00214]

OPE without OPE

Compton amplitude & OPE: theoretical basis for calculation of deep-inelastic structure functions

$$T_{\mu\nu}(p,q) = \int \mathrm{d}^4x \, \mathrm{e}^{iqx} \langle p, s | T J_\mu(x) J_\nu(0) | p, s \rangle$$

$$= \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \mathcal{F}_1(\omega, q^2) + \left(p_\mu - \frac{pq}{q^2} q_\mu\right) \left(p_\nu - \frac{pq}{q^2} q_\nu\right) \frac{1}{pq} \mathcal{F}_2(\omega, q^2)$$

$$+ \ \epsilon_{\mu\nu\lambda\sigma} \, q_\lambda \, s_\sigma \frac{1}{pq} \mathcal{G}_1(\omega, q^2) + \ \epsilon_{\mu\nu\lambda\sigma} \, q_\lambda \, (pq \, s_\sigma - sq \, p_\sigma) \ \frac{1}{(pq)^2} \mathcal{G}_2(\omega, q^2)$$

Crossing symmetry, $T_{\mu\nu}(p,q) = T_{\nu\mu}(p,-q)$, implies

$$\mathcal{F}_{1,2}(-\omega, q^2) = \pm \, \mathcal{F}_{1,2}(\omega, q^2) \, , \quad \mathcal{G}_{1,2}(-\omega, q^2) = - \, \mathcal{G}_{1,2}(\omega, q^2) \qquad \omega = \frac{1}{x} = \frac{2pq}{q^2}$$

In the physical region $1 \leq |\omega| \leq \infty$

$$\mathrm{Im} \, \mathcal{F}_{1,2}(\omega, q^2) = 2\pi F_{1,2}(\omega, q^2) \, , \quad \mathrm{Im} \, \mathcal{G}_{1,2}(\omega, q^2) = 2\pi g_{1,2}(\omega, q^2)$$

Dispersion relations

$$\mathcal{F}_1(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{F_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} - \frac{F_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right] + \mathcal{F}_1(0, q^2)$$

$$\mathcal{F}_2(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{F_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{F_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$$\mathcal{G}_1(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{g_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{g_1(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$$\mathcal{G}_2(\omega, q^2) = 2\omega \int_1^\infty d\bar{\omega} \left[\frac{g_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} - \omega)} + \frac{g_2(\bar{\omega}, q^2)}{\bar{\omega}\,(\bar{\omega} + \omega)} \right]$$

$\uparrow$

polarizability

For $p_3 = q_3 = q_0 = 0$, substituting $\bar{\omega}$ by $1/x$

$$T_{33}(p,q) = \mathcal{F}_1(\omega, q^2) = 4\omega \int_0^1 dx\, \frac{\omega x}{1-(\omega x)^2}\, F_1(x,q^2) + \mathcal{F}_1(0,q^2)$$

$$= \sum_{n=2,4,\cdots}^{\infty} 4\omega^n \int_0^1 dx\, x^{n-1} F_1(x,q^2) + \mathcal{F}_1(0,q^2)$$

$$T_{03}(p,q) \stackrel{\vec{s}\|\vec{p}}{=} \frac{(\vec{q}\times\vec{s})_3}{pq}\, \mathcal{G}_1(\omega,q^2) = \frac{(\vec{q}\times\vec{s})_3}{pq}\, 4\omega \int_0^1 dx\, \frac{1}{1-(\omega x)^2}\, g_1(x,q^2)$$

$$= \frac{(\vec{q}\times\vec{s})_3}{pq} \sum_{n=1,3,\cdots}^{\infty} 4\omega^n \int_0^1 dx\, x^{n-1} g_1(x,q^2)$$

$$T_{03}(p,q) \stackrel{\vec{s}\|\vec{q}}{=} -\frac{(\vec{p}\times\vec{q})_3\, \vec{s}\vec{q}}{(pq)^2}\, \mathcal{G}_2(\omega,q^2) = -\frac{(\vec{p}\times\vec{q})_3\, \vec{s}\vec{q}}{(pq)^2}\, 4\omega \int_0^1 dx\, \frac{1}{1-(\omega x)^2}\, g_2(x,q^2)$$

$$= -\frac{(\vec{p}\times\vec{q})_3\, \vec{s}\vec{q}}{(pq)^2} \sum_{n=1,3,\cdots}^{\infty} 4\omega^n \int_0^1 dx\, x^{n-1} g_2(x,q^2)$$

includes power corrections

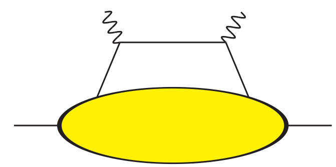
From T_{33} to μ_n and $F_1(x,q^2)$

The Compton amplitude can be computed most efficiently, including singlet (disconnected) matrix elements, by the Feynman-Hellmann technique. By introducing the perturbation to the Lagrangian

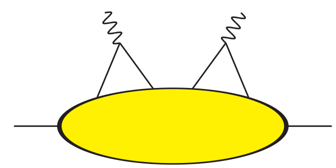
$$\mathcal{L}_{\rm QCD}(x) \to \mathcal{L}_{\rm QCD}(x) + \lambda \mathcal{J}_3(x)\,, \quad \mathcal{J}_3(x) = Z_V \cos(\vec{q}\vec{x})\; e_q\, \bar{q}(x)\gamma_3 q(x)$$

and taking the second derivative of $\langle N(\vec{p},t)\bar{N}(\vec{p},0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\,t}$ with respect to λ on both sides, we obtain

$$-2E_\lambda(p,q)\,\frac{\partial^2}{\partial\lambda^2}E_\lambda(p,q)\,\big|_{\lambda=0} = T_{33}(p,q)$$



The amplitude encompasses the dominating ‘handbag’ diagram as well as the power-suppressed ‘cats ears’ diagram. Varying q^2 will allow to test the twist expansion. No further renormalization is needed



Implementation

All we need to compute are nucleon two-point functions, from which we derive the energy levels E_λ. If that has been done successfully, we can resort to continuum language

Valence quark distribution functions

- Computationally cheap. No extra background (vacuum) gauge field configurations have to be generated
- The electromagnetic current needs to be inserted in quark propagators of nucleon two-point function only
- Propagators can be used to compute a variety of other observables, including form factors and Compton amplitudes of all stable particles

Sea quark and gluon distribution functions

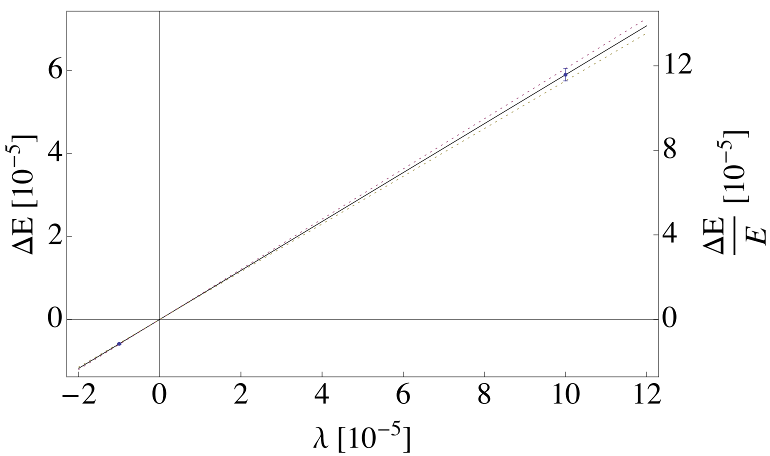
- Need to generate new background gauge field configurations with the electromagnetic current being attached to the sea quarks
- As before, the new configurations lend themselves to the calculation of many other observables, besides the nucleon Compton amplitude

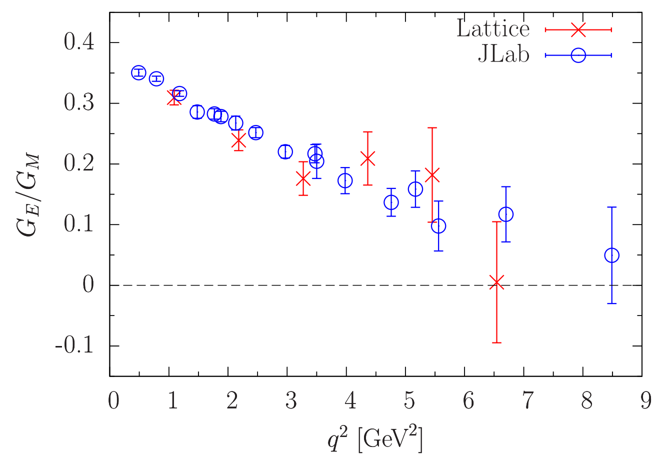
Example: Nucleon form factors at large q^2

$$G_{\mathrm{E,M}}(q^2) \propto \partial E_\lambda(p,p')/\partial\lambda\big|_{\lambda=0}$$

$\vec{p} + \vec{p}' = 0$ Breit frame

$$\Delta E_\lambda = E_\lambda - E_0$$





Powerful method

Phys. Rev. D96 (2017) 114509 [arXiv:1702.01513]

Moments

The lowest M [even] moments [odd moments need $\langle p,\ s|T J_\mu(x) J_\nu^5(0)|p,\ s\rangle$]

$$\mu_{2m-1} = \int_0^1 dx\, x^{2m-1} F_1(x)$$

can be computed from a finite number of sampled points $\{T_{33}(\omega_n)\}$ of the Compton amplitude

$$t_n = T_{33}(\omega_n)\,,\ \ n = 1, \cdots, N$$

by the set of linear equations

$$\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{pmatrix} = \begin{pmatrix} 4\omega_1^2 & 4\omega_1^4 & \cdots & 4\omega_1^{2M} \\ 4\omega_2^2 & 4\omega_2^4 & \cdots & 4\omega_2^{2M} \\ \vdots & \vdots & \vdots & \vdots \\ 4\omega_N^2 & 4\omega_N^4 & \cdots & 4\omega_N^{2M} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_3 \\ \vdots \\ \mu_{2M-1} \end{pmatrix} \quad \text{Vandermonde M}$$

Solutions are well documented in the literature. Alternatively, we can fit the Compton amplitude by the interpolating polynomial

$$T_{33}(\omega) = 4\left(\omega^2 \mu_1 + \omega^4 \mu_3 + \cdots + \omega^{2M} \mu_{2M-1}\right)$$

Structure function

The structure function $F_1(x)$ can be computed from $\{T_{33}(\omega_n)\}$ by discretizing the integral equation

$$t_n = \epsilon \sum_{m=1}^{M} K_{nm} f_m \,, \quad n = 1, \cdots, N$$

[here: points equidistant with step size ϵ]

with

$$f_m = F_1(x_m) \,, \quad K_{nm} = \frac{4\,\omega_n^2 x_m}{1 - (\omega_n x_m)^2} \,, \quad N < M$$

The $N \times M$ matrix K is written

$$K = U \; [\mathrm{diag}(w_1, \cdots, w_N)] \; V^T$$

where W is singular: $w_l \approx 0, L < l \leq N$. Solution by singular value decomposition (SVD)

$$f_m = \sum_{n=1}^{N} K^{-1}_{mn} \epsilon^{-1} \, t_n$$

with K^{-1} being the pseudoinverse

$$K^{-1} = V \; [\mathrm{diag}(1/w_1, \cdots, 1/w_K, 0, \cdots, 0)] \; U^T$$

Mathematica

Alternative techniques

Educated fit

$$x\,F_1(x) = A_q\,x^{\alpha}(1-x)^{\beta} \quad \rightarrow \quad T_{33}(\omega) = 4A_q\,\Gamma(1+\beta)\sum_n \frac{\Gamma(\alpha+n)}{\Gamma(1+\alpha+\beta+n)}\,\omega^{n+1}$$

Fredholm integral equation of the first kind

Solution [unique]

- Discrete Galerkin method [Hölder continuous functions]

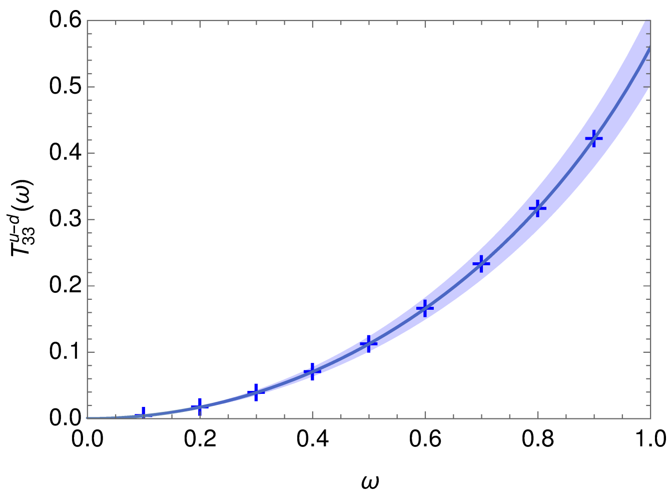
- Chebyshev and Legendre wavelet collocation

- Inverse generalized Laplace–Stieltjes transform

Proof of Concept

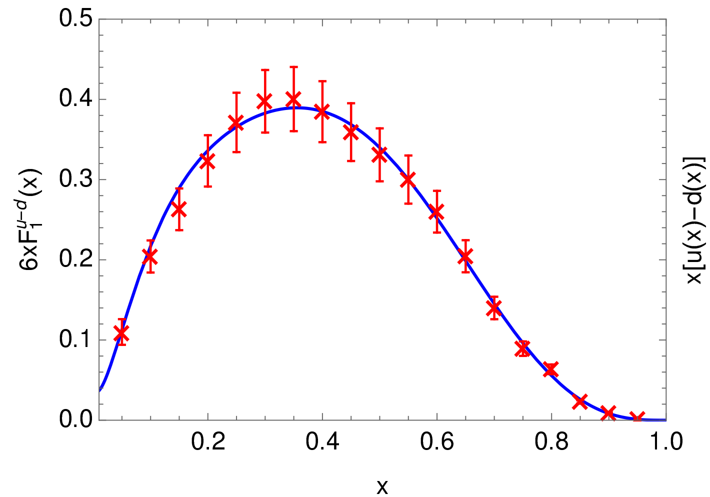
In

$$T_{33}^{u-d}(\omega) = 4\omega \int_0^1 dx \, \frac{\omega x}{1-(\omega x)^2} F_1^{u-d}(x)$$



Out

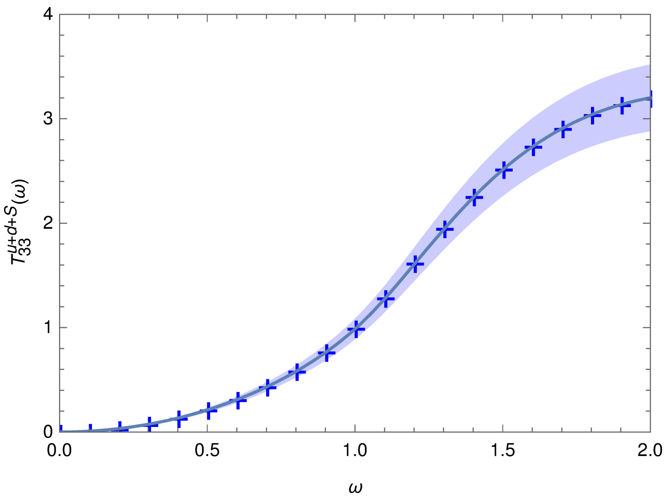
$$2xF_1^{u-d}(x) = \frac{1}{3}\, x[u(x) - d(x)]$$



MSTW-lo

In

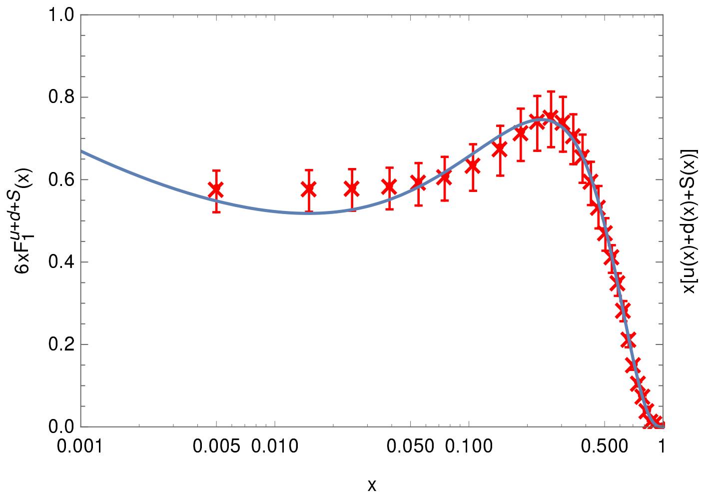
$$T_{33}^{u+d+S}(\omega) = 4\omega\, \mathrm{P}\int_0^1 dx\, \frac{\omega x}{1-(\omega x)^2} F_1^{u+d+S}(x)$$



$\omega \in [0, 2]$

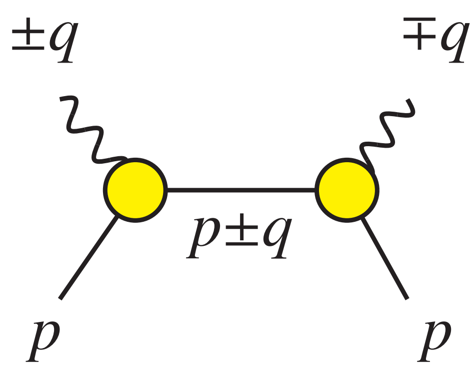
Out

$$2xF_1^{u+d+S}(x) = \frac{1}{3}x[u(x)+d(x)+S(x)]$$



MSTW-lo

Note that intermediate states of the (semi-)elastic Compton amplitude $T_{\mu\nu}(\omega, q^2)$ can go on-shell for $\omega \geq 1$



For example: nucleon pole

$$p^2 = -E^2 + \vec{p}^2 = -m_N^2$$

$$(p \pm q)^2 = p^2 \pm 2pq + q^2 = -m_N^2 + q^2(1 \pm \omega)$$
$$= -m_N^2 \quad \text{for } \omega = \mp 1$$

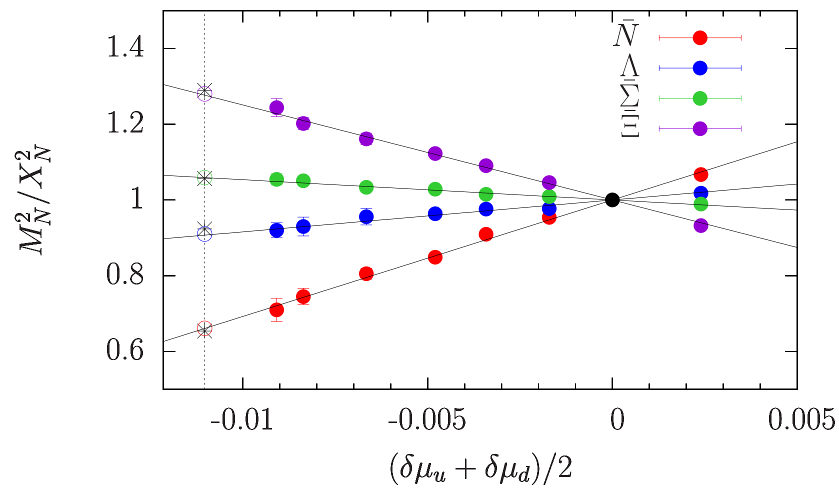
However, this contribution is power suppressed by the product of nucleon form factors, $[F_1(q^2)]^2$. In our example (see next slides) $q^2 \approx 4\,\text{GeV}^2$, which leads to a suppression factor of $\approx 1/500$

Lattice Study

SU(3) symmetric point

$M_\pi = M_K \approx 420\,\text{MeV}$

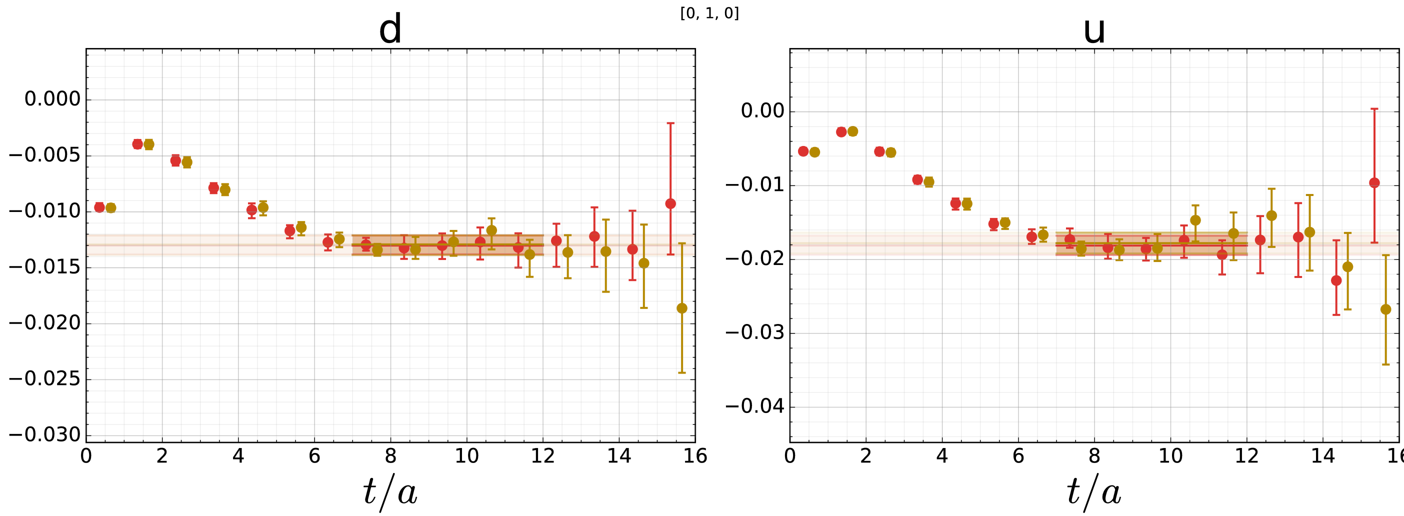
V	a [fm]	q^2 [GeV^2]
$32^3 \times 64$	0.074	1.37
$32^3 \times 64$	0.074	2.74
$32^3 \times 64$	0.074	3.56
$32^3 \times 64$	0.074	4.66
$32^3 \times 64$	0.074	5.48
$32^3 \times 64$	0.074	6.85
$48^3 \times 96$	0.068	1.44



Valence quark insertion

$$\Delta E_\lambda = E_\lambda - E_0 \qquad\qquad \langle N(\vec{p},t)\bar{N}(\vec{p},0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\,t}$$

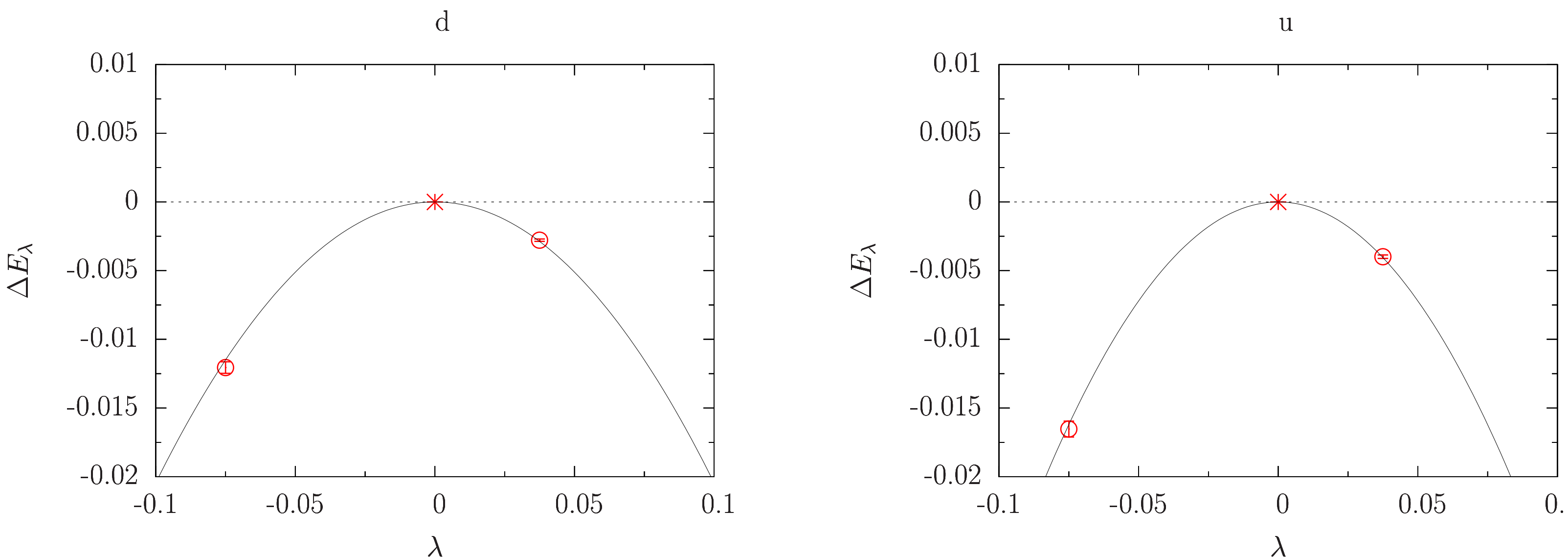
$$\omega = 0.3$$



$\longrightarrow$ continuum physics

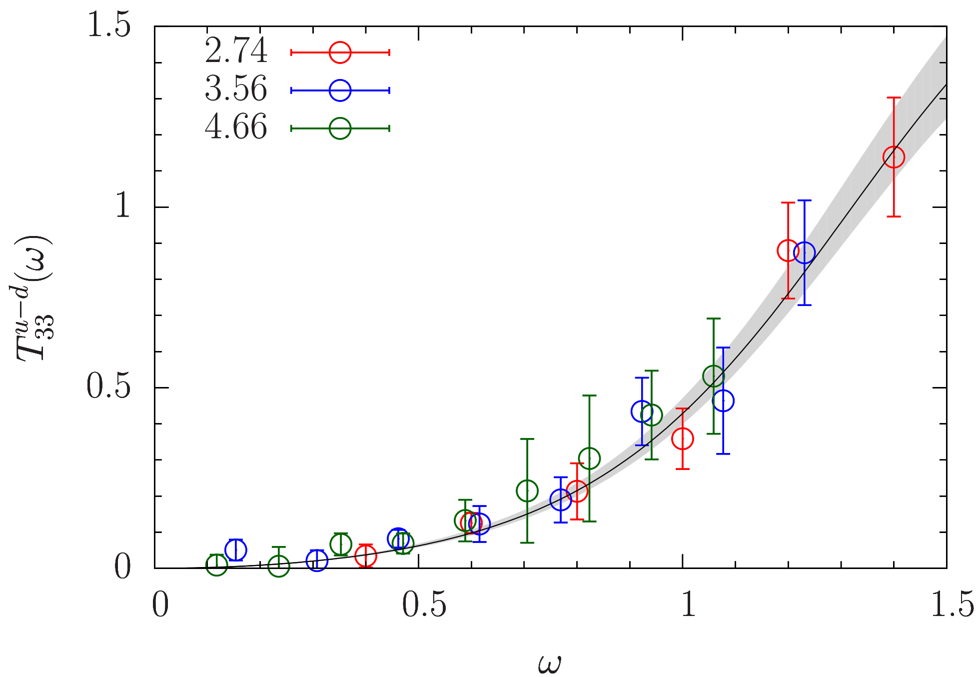
$$\Delta E_\lambda = E_\lambda - E_0 \propto \lambda^2$$

$$\omega = 0.3$$



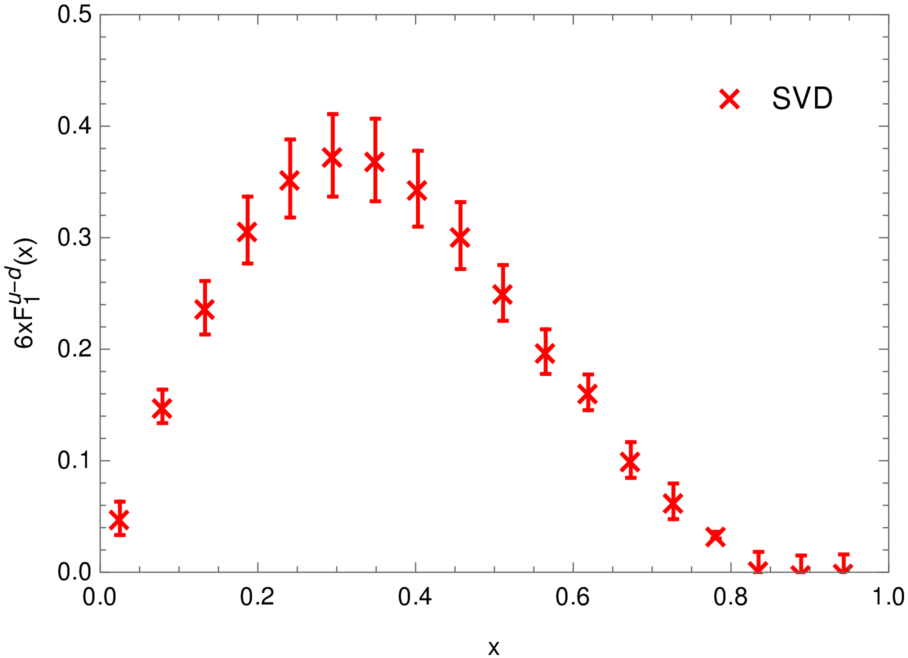
q^2 [GeV^2]

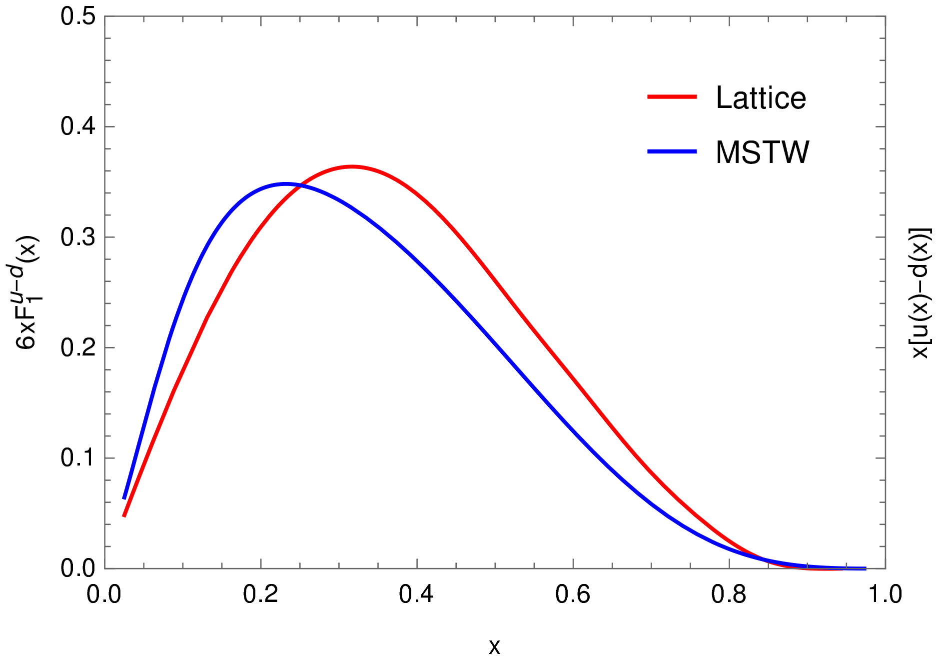
↓



Padé fit

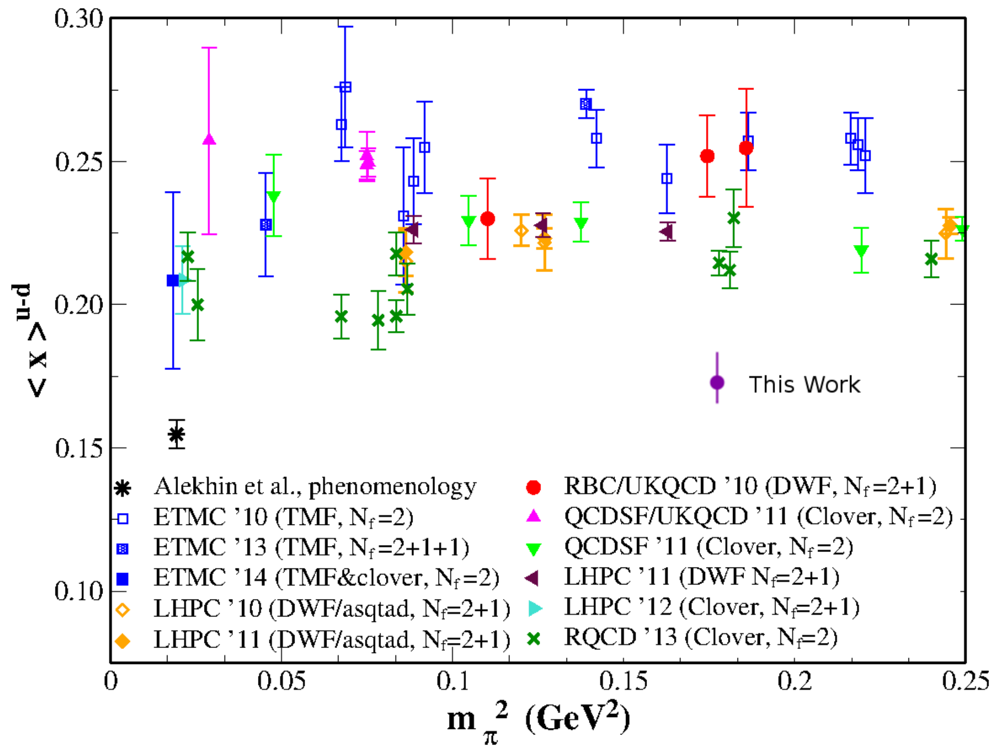
Solution





MSTW NNLO

First moment



From T_{03} to $g_1(x,q^2)$ and $g_2(x,q^2)$

The Compton amplitude $T_{03}(\omega,q^2)$ needs to be antisymmetric in the Lorentz indices, $T_{03}(\omega,q^2) = -T_{30}(\omega,q^2)$, in this case. That can be achieved by introducing the perturbation to the Lagrangian

$$\mathcal{L}_{\rm QCD}(x) \to \mathcal{L}_{\rm QCD}(x)+\lambda\mathcal{J}_{0+3}(x),\; \mathcal{J}_{0+3}(x) = Z_V\, e_q\, \bar{q}(x)(\gamma_0\cos(\vec{q}\vec{x})+\gamma_3\sin(\vec{q}\vec{x}))q(x)$$

and taking the second derivative of $\langle N(\vec{p},t)\bar{N}(\vec{p},0)\rangle_\lambda \simeq C_\lambda\, \mathrm{e}^{-E_\lambda(p,q)\,t}$ with respect to λ as before, giving

$$-2E_\lambda(p,q)\,\frac{\partial^2}{\partial\lambda^2}E_\lambda(p,q)\,\Big|_{\lambda=0} = T_{03}(p,q) - T_{30}(p,q)$$

T_{00}, T_{33} drop out

PDFs

$$F_1(x) = \sum_{i=u,d,\cdots,g} \int_x^1 \frac{dy}{y}\, c_{1,i}(x/y,\mu^2)\, f_i(y,\mu^2)$$

$$g_1(x) = \sum_{i=u,d,\cdots,g} \int_x^1 \frac{dy}{y}\, e_{1,i}(x/y,\mu^2)\, \Delta f_i(y,\mu^2)$$

$\uparrow$

perturbatively known

$$\begin{aligned} f_u(x) &= u(x) & \Delta f_u(x) &= \Delta u(x) \\ f_d(x) &= d(x) & \Delta f_d(x) &= \Delta d(x) \\ f_{\bar{u}}(x) &= \bar{u}(x) & \Delta f_{\bar{u}}(x) &= \Delta \bar{u}(x) \\ f_{\bar{d}}(x) &= \bar{d}(x) & \Delta f_{\bar{d}}(x) &= \Delta \bar{d}(x) \end{aligned}$$

Solely need to replace

$$K_{nm} = \frac{4\,\omega_n^2 x_m}{1-(\omega_n x_m)^2} \quad \to \quad K_{nm} = 2\,\omega_n^2 \int_0^1 dy\, y\, x_m \frac{c_1(y,\mu^2)}{1-(y\,\omega_n\, x_m)^2}$$

Check factorization

Outlook

- Computations can be improved in many respects
  - Apply Bayesian regression with SVD to alleviate overfitting
  - Employ momentum smearing techniques for larger values of ω
- With gradual improvements, we should be able to compute the structure functions $F_1(x, q^2)$ and $F_2(x, q^2)$, as well as $g_1(x, q^2)$ and $g_2(x, q^2)$, including contributions of higher twist, from the Compton amplitude with unprecedented accuracy
- This is possible, because the calculation skirts the issue of renormalization and operator mixing
- The method can easily be generalized to generalized parton distribution functions (GPDs) $H(x, \xi, q^2)$ and $E(x, \xi, q^2)$