Recursion relations for off-shell amplitudes on the light-front and Wilson lines

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In collaboration with Christian Cruz-Santiago, Piotr Kotko, Mirko Serino

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Introduction

- Feynman diagram technique is a powerful method for computing scattering amplitudes. However, with increase of external states, many graphs contribute.
- In some cases simple results follow after combining many diagrams, eg. MHV amplitudes. This indicates that Feynman graphs may not be the most efficient way to capture all the hidden aspects of the structure of the theory.
- Other techniques for scattering amplitudes: Britto-Cachazo-Feng-Witten relations, Cachazo-Svrcek-Witten vertices, twistors.
- Can the light-front techniques reveal more about scattering amplitudes?
- Turns out some variables are natural on the light-front and hidden Wilson lines appear (see also next talk by Piotr Kotko).

Helicity amplitudes

Consider amplitudes with well defined helicities. One can sum incoherently the squares of the amplitudes which contribute to the given process.
Gauge invariant color decomposition:

$$\mathcal{M}^{a_1 \ldots a_n}(\varepsilon_1^{\lambda_1} \ldots \varepsilon_n^{\lambda_n}) \sum_{non-cyclic\ perm} \mathrm{Tr}(t^{a_1} \ldots t^{a_n}) \mathcal{M}(1^{\lambda_1} \ldots n^{\lambda_n})$$

λ_i polarizations of gluons $\quad [t^a, t^b] = i\sqrt{2} f^{abc} t^c, \quad \mathrm{Tr}(t^a t^b) = \delta^{ab}$
sum over non-cyclic permutations of the indices $1, 2, \ldots, n$.

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- Vanishing amplitudes: $\mathcal{M}(\pm, +, \cdots +) = 0$ (tree level)
- $\mathcal{M}(-, -, +, \cdots +)$ Maximally Helicity Violating amplitudes (MHV)
- $\mathcal{M}(-, -, -, +, \cdots +)$ Next-to-Maximally Helicity Violating amplitudes (NMHV)
- Next-to-next-to Maximally Helicity Violating (NNMHV) amplitudes, etc...

Spinor algebra

States with definite helicities using projectors

$$u_\pm(k) = P_\pm u(k), \quad v_\pm(k) = P_\mp v(k)$$

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

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Introduce the notation

$$|k_i \pm\rangle = u_{\pm}(k_i), \quad \langle k_i \pm | = \overline{u_{\pm}(k_i)}$$

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From these definitions one can construct the spinor products

$$\langle ij\rangle \equiv \langle i - |j+\rangle = \overline{u_{-}(k_i)}u_{+}(k_j), \quad [ij] \equiv \langle i + |j-\rangle = \overline{u_{+}(k_i)}u_{-}(k_j)$$

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Spinor products are complex square roots of Lorentz product

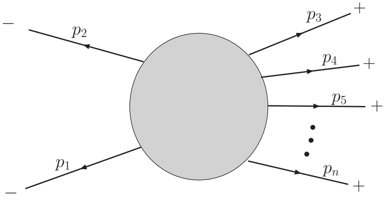
$$\langle ij \rangle [ji] = s_{ij} = 2 k_i \cdot k_j$$

MHV amplitudes

Maximally Helicity Violating - MHV amplitudes : Parke - Taylor formula for the kinematical part of the amplitude

$$\mathcal{M}_n(1^-,2^-,3^+,\ldots,n^+) = ig^{n-2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

Parke,Taylor, Phys.Rev.Lett. 56 (1986) 2459



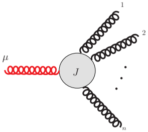
Proven using recursive techniques
Berends, Giele, Nucl. Phys. B306 (1988) 759

Calculation of MHV on the light-front

Calculate currents on the light-front : amplitudes with one leg off-shell

$$J^{\mu}\left(k_{1\ldots N};\varepsilon_1^{\lambda_1},\ldots,\varepsilon_N^{\lambda_N}\right)\equiv J^{\mu\,(\lambda_1\ldots\lambda_N)}\left(k_{1\ldots N}\right)$$

$k_{1\ldots N}=k_1+\cdots+k_N$



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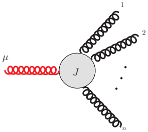

On-shell amplitude can be obtained by reduction formula

$$\mathcal{M}^{(\lambda_0\lambda_1\ldots\lambda_N)}\left(k_0,k_1,\ldots,k_N\right)=ik_{1\ldots N}^2\,\varepsilon_0^{\lambda_0}\cdot J^{(\lambda_1\ldots\lambda_N)}\left(k_{1\ldots N}\right)\Big|_{k_{1\ldots N}=-k_0},\qquad k_0^2=0$$

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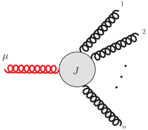

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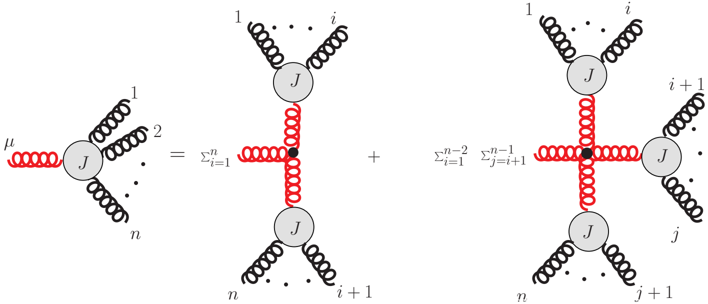
Current in light-cone gauge

$$J^{(\lambda_{1\dots i}\lambda_1\dots\lambda_i)}\left(k_{1\dots i}\right)=\varepsilon_{1\dots i}^{\lambda_{1\dots i}\,*\,\nu}\,\frac{-i}{k_{1\dots i}^2}\,J_{(\mathrm{amp})\,\nu}^{(\lambda_1\dots\lambda_i)}\left(k_{1\dots i}\right)$$

When the propagator $k_{1\dots i}^2$ is amputated, we have an off-shell amplitude.

Berends-Giele recursion

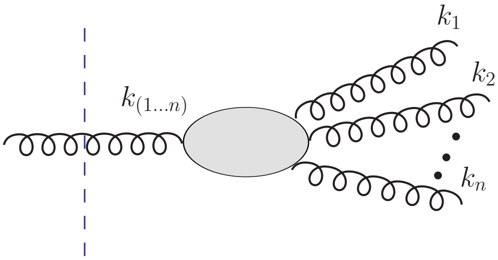
Berends, Giele, Nucl. Phys. B306 (1988) 759



off-shell
on-shell

Recursion relations on the light-front

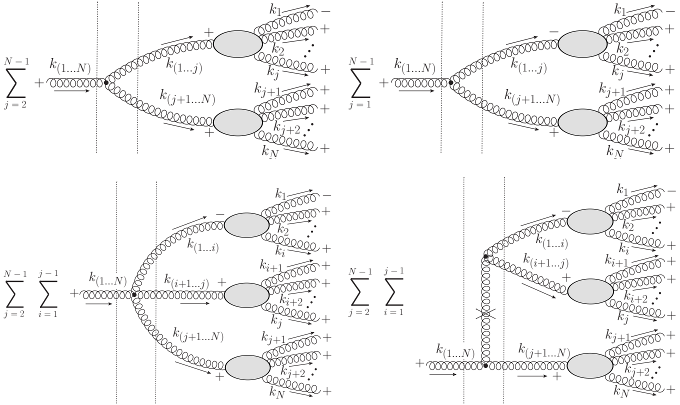
Use Berends-Giele for the light-front current. Initial leg off-shell, energy denominator in the initial state. Final states are on-shell.



(also referred as the 'fragmentation function' Motyka, AMS, Phys. Rev. D79 (2009) 085016)

Recursion relations on the light-front

Berends-Giele recursion analog for the light front currents (light-cone gauge) $(- - + \cdots +)$: MHV configuration



Energy denominators instead of propagators.

Recursion relations on the light-front

Berends-Giele recursion analog for the light front currents (light-cone gauge)

$$J^{(\lambda_{1\ldots N}\lambda_1\ldots\lambda_N)}(k_{1\ldots N}) = \frac{-i}{k^2_{1\ldots N}}$$

$$\Bigg\{ \sum_{\lambda,\lambda'=\pm} \sum_{i=1}^{N-1} V_3^{(\lambda_{1\ldots N}\lambda\lambda')}(k_{1\ldots i}, k_{(i+1)\ldots N}) J^{(\lambda\lambda_1\ldots\lambda_i)}(k_{1\ldots i}) J^{(\lambda'\lambda_{i+1}\ldots\lambda_N)}(k_{(i+1)\ldots N})$$

$$+ \sum_{\lambda,\lambda',\lambda''=\pm} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} V_{4(\mathrm{LC})}^{(\lambda_{1\ldots N}\lambda\lambda'\lambda'')}(k_{1\ldots i}, k_{(i+1)\ldots j}, k_{(j+1)\ldots N})$$

$$\times J^{(\lambda\lambda_1\ldots\lambda_i)}(k_{1\ldots i}) J^{(\lambda'\lambda_{i+1}\ldots\lambda_j)}(k_{(i+1)\ldots j}) J^{(\lambda''\lambda_{j+1}\ldots\lambda_N)}(k_{(j+1)\ldots N}) \Bigg\}$$

$V_{4(\mathrm{LC})}$ contains the contributions from the 4-gluon vertex and terms which appear because of light-cone gauge.

Solution to the recursion relation for the MHV-type helicity choice $(- - + \cdots +)$ C. Cruz-Santiago, AS, Nucl. Phys. B 875 (2013) 368

$$J^{(--+\cdots+)}(k_{1\dots N}) = \tilde{J}^{(--+\cdots+)}(k_{1\dots N}) - ig \sum_{i=2}^{N-1} \tilde{J}^{(--+\cdots+)}(k_{1\dots i}) \frac{k^+_{1\dots N}}{k^+_{(i+1)\dots N}\, \tilde{v}_{(1\dots i)(i+1)}} J^{(-+\cdots+)}(k_{(i+1)\dots N})$$

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New object appears

$$\tilde{J}^{(--+\cdots+)}(k_{1\ldots i}) = \frac{2g^{i-1}}{k^2_{1\ldots i}} \left(\frac{k^{+}_{1\ldots i}}{k^{+}_{1}}\right)^2 \frac{\tilde{v}^3_{1(1\ldots i)}}{\tilde{v}_{(1\ldots i)i}\,\tilde{v}_{i(i-1)}\ldots\tilde{v}_{32}\,\tilde{v}_{21}}$$

where

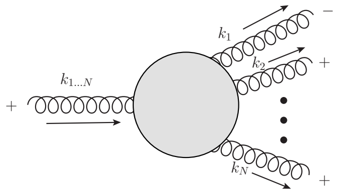
$$\tilde{v}_{ij} = -p^{\bullet}_{i} + p^{+}_{i}\frac{p^{\bullet}_{j}}{p^{+}_{j}}, \qquad p^{\bullet} = p\cdot\varepsilon^{+}_{\perp}, \qquad \varepsilon^{\pm}_{\perp} = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$$

is proportional to an off-shell continuation of $\langle ij\rangle$. Thus $\tilde{J}^{(--+\cdots+)}(k_{1\ldots i})$ up to the propagator this is off-shell continuation of the MHV amplitude.

$$J^{(--+\cdots+)}(k_{1\ldots N}) = \tilde{J}^{(--+\cdots+)}(k_{1\ldots N})$$
$$- ig\sum_{i=2}^{N-1}\tilde{J}^{(--+\cdots+)}(k_{1\ldots i})\frac{k^+_{1\ldots N}}{k^+_{(i+1)\ldots N}\,\tilde{v}_{(1\ldots i)(i+1)}}J^{(-+\cdots+)}(k_{(i+1)\ldots N})$$

Amputating and taking on-shell limit one recovers on-shell MHV amplitude.

$$\left(J^{(--+\cdots+)}(k_{1\ldots N})\,k^2_{1\ldots N}\right)\Big|_{\mathrm{onshell}} \longrightarrow ig^{N-1}\frac{\langle 1\,(1\ldots N)\rangle^3}{\langle (1\ldots N)N\rangle\langle N\,(N-1)\rangle\ldots\langle 32\rangle\,\langle 21\rangle}$$



Second term in the solution vanishes in the on-shell limit.

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$$J^{(--+\cdots+)}(k_{1\ldots N}) = \tilde{J}^{(--+\cdots+)}(k_{1\ldots N})$$

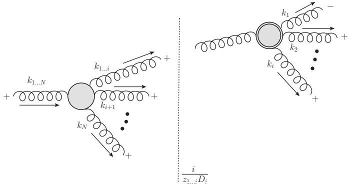
$$- ig \sum_{i=2}^{N-1} \tilde{J}^{(--+\cdots+)}(k_{1\ldots i}) \frac{k^+_{1\ldots N}}{k^+_{(i+1)\ldots N}\, \tilde{v}_{(1\ldots i)(i+1)}} J^{(-+\cdots+)}\left(k_{(i+1)\ldots N}\right)$$

Solution to the recursion relation for the MHV-type helicity choice $(--+\cdots+)$

$$J^{(--+\cdots+)}(k_{1\ldots N}) = \tilde{J}^{(--+\cdots+)}(k_{1\ldots N}) - ig\sum_{i=2}^{N-1} \tilde{J}^{(--+\cdots+)}(k_{1\ldots i})\,\frac{k^+_{1\ldots N}}{k^+_{(i+1)\ldots N}\,\tilde{v}_{(1\ldots i)(i+1)}}\,J^{(-+\cdots+)}(k_{(i+1)\ldots N})$$

Representation of the second term:

$J^{(-+\cdots+)}(k_{(i+1)\ldots N})$ $\qquad$ $\tilde{J}^{(--+\cdots+)}(k_{1\ldots i})$



It turns out that: the first term is the gauge invariant off-shell current, and the second term is needed to satisfy the gauge invariance. It contains Wilson line.

Matrix elements of Wilson lines

Gauge invariant formulation of off-shell amplitudes. Take the case of one off-shell leg. In general can consider matrix element of Wilson line.

$$\mathfrak{M} = \int d^4x\, e^{ik\cdot x} \left\langle 0 \left| \mathcal{T}\left\{ R^a_{\mathfrak{e}}(x)\, e^{iS_{\text{Y-M}}} \right\} \right| k_1, \lambda_1, a_1; \ldots; k_N, \lambda_N, a_N \right\rangle_{\text{c}}$$

where $|k_i, \lambda_i\rangle$ is the on-shell gluon state and $R^a_{\mathfrak{e}}(x)$ corresponds to the Wilson line

$$R^{a_{1\ldots N}}_{\mathfrak{e}_{1\ldots N}}(x) = \frac{1}{g}\,\mathrm{Tr}\left[t^{a_{1\ldots N}} \mathcal{P}\exp\left(ig \int_{-\infty}^{+\infty} ds\, A^b_\mu\left(x + s\,\mathfrak{e}_{1\ldots N}\right) \mathfrak{e}^\mu_{1\ldots N} t^b \right) \right],$$

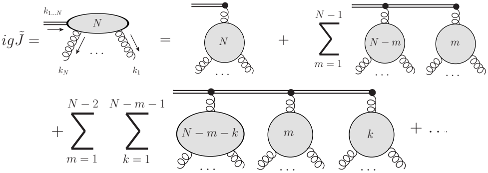
Ward identity is satisfied with respect to the on-shell legs of this matrix element

$$\tilde{\mathcal{J}}^{(\lambda_1\ldots k_i\ldots\lambda_N)}_{\mathfrak{e}_{1\ldots N}}\left(k_{1\ldots N}\right) = 0 \quad \text{for} \quad i = 1, \ldots, N\,.$$

$$\mathfrak{M} = \delta^4\left(k - k_1 - \ldots - k_N\right)\delta\left(\mathfrak{e}\cdot k\right) \tilde{J}^{(\mathfrak{e}\to\lambda_1\ldots\lambda_N)}(k)$$

P. Kotko, JHEP 1407 (2014) 128

Relation between gauge-invariant current $\tilde{J}$ and current obtained from Feynman diagrams J



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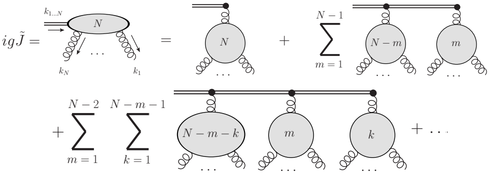

Schematically:

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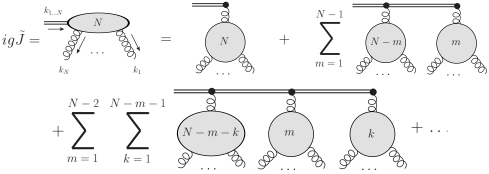

Schematically:

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The above relation for $--+\cdots+$ helicity choice is identical to the LF relation

$$J^{(--+\cdots+)}\left(k_{1\ldots N}\right) = \tilde{J}^{(--+\cdots+)}\left(k_{1\ldots N}\right)$$
$$- ig \sum_{i=2}^{N-1} \tilde{J}^{(--+\cdots+)}\left(k_{1\ldots i}\right) \frac{k^+_{1\ldots N}}{k^+_{(i+1)\ldots N}\, \tilde{v}_{(1\ldots i)(i+1)}} J^{(-+\cdots+)}\left(k_{(i+1)\ldots N}\right)$$

C. Cruz-Santiago, AS, P. Kotko, Nucl. Phys. B 895 (2015) 132; P. Kotko, M. Serino, AS, JHEP 1608 (2016) 26

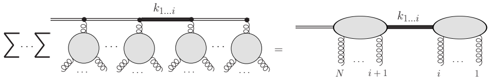
Another way of proving the equivalence between light-front and Wilson line approach. Perform BCFW like construction with a complex shift of the Wilson line slope and the sum over residues.

$$\varepsilon^{\mu} \to \varepsilon^{\mu}(z) = \varepsilon^{\mu} + z\eta^{\mu} ,$$

where $\eta = \frac{1}{\sqrt{2}}(1,0,0,-1)$ is lightlike. There will be poles coming from the Wilson line propagators

$$k_{1\dots i} \cdot \varepsilon(z) = 0 \quad \Rightarrow \quad z = -\frac{k_{1\dots i} \cdot \varepsilon}{k_{1\dots i} \cdot \eta} \equiv z_{1\dots i} .$$

Take the sum over the residues.



Recovering the previously derived relation, which indeed is the BCFW-like relation, but with the boundary term which corresponds to the off-shell current.
P. Kotko, M. Serino, AS, JHEP 1608 (2016) 26

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- Can be derived using the BCFW-like methods by complex shift of the Wilson line slope and sum over residues. Additional boundary term appear.
- These Wilson lines are also related to the more general formulation: MHV action formulation by Cachazo-Svrcek-Witten (see next talk by Piotr Kotko).