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# Non-perturbative constraints on the matrix elements of the energy-momentum tensor


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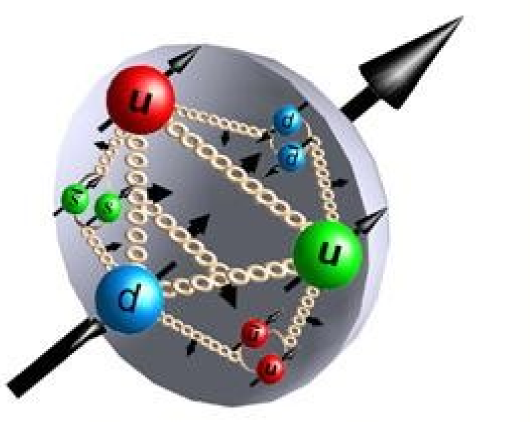
# Outline

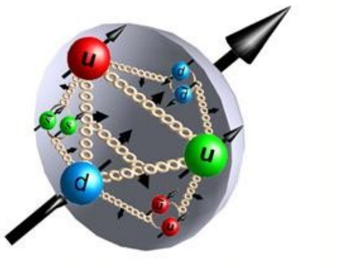
1. Form factors in local QFT
2. The form factors of $T^{\mu\nu}$
3. A distributional matching approach
4. Summary and outlook




# 1. Form factors in local QFT

- Form factors $F(q^2)$ parametrise the non-perturbative characteristics of matrix elements

  → e.g. spin structure, charge distribution, ...



- In order to fully understand the properties of form factors one therefore requires a non-perturbative approach

- **“Local QFT”** – *define a QFT using a series of physical motivated axioms*

  → axioms hold independently of the coupling regime, hence non-perturbative properties can be derived

- One of the key features of this framework is that quantised fields $\varphi(x)$ are distributions [Streater, Wightman; Haag] – this subtlety is important for consistently defining matrix elements and charges

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964)]
[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996)]

# 2. The form factors of $T^{\mu\nu}$

- Due to the properties of $T^{\mu\nu}$ the matrix elements for spin-½ momentum eigenstates can be written in the following Lorentz covariant manner:

$$\boxed{\begin{aligned}\langle p';m';M|T^{\mu\nu}(0)|p;m;M\rangle = \bar{u}_{m'}(p')\bigg[&\frac{1}{4}\gamma^{\{\mu}(p+p')^{\nu\}}A(q^2) + \frac{1}{8M}(p+p')^{\{\mu}i\sigma^{\nu\}\rho}q_\rho B(q^2) \\ &+ \frac{1}{M}\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)C(q^2)\bigg]u_m(p)\,\delta^{(+)}_M(p)\delta^{(+)}_M(p')\end{aligned}}$$

where: $|p;m;M\rangle := \delta^{(+)}_M(p)|p;m\rangle \qquad \delta^{(+)}_M(p) := 2\pi\theta(p^0)\delta(p^2-M^2)$

$$\boxed{\langle p';m';M|p;m;M\rangle = (2\pi)^4\delta^4(p'-p)\delta^{(+)}_M(p')\delta_{m'm}}$$

- Physical states have the form:

$$|\Psi^g_{M,m}\rangle = \int \frac{d^4p}{(2\pi)^4}\,\delta^{(+)}_M(p)g(p)|p;m\rangle = \int \frac{d^3p}{(2\pi)^3 2p^0}g(p)\big|_{\Gamma^+_M}|p;m\rangle$$

- For simplicity we consider massive canonical spin states, where *m* is the rest frame spin projection. Could equally well use other spin states (e.g. helicity spin)

# 3. A distributional matching approach

- Approach: Use the distributional properties of the matrix elements to impose constraints on the form factors [PL, Chiu, Brodsky, 1707.06313]

*(i)* Compute the matrix elements of $P^{\mu}$ and $J^{i}$ using the Poincaré transformation properties of the states

→ *Spacetime translations:* $e^{iP\cdot a}|p;k;M\rangle = e^{ip\cdot a}|p;k;M\rangle$

$$\boxed{\langle p';m';M|P^{\mu}|p;m;M\rangle = p^{\mu}(2\pi)^4\delta^4(p'-p)\delta_M^{(+)}(p')\delta_{m'm}}$$

→ *Lorentz transformations:* $e^{-\boldsymbol{\beta}\cdot\mathbf{J}}|p;k;M\rangle = \sum_l \mathcal{D}^s_{lk}(\boldsymbol{\beta})|\Lambda(\boldsymbol{\beta})p;l;M\rangle$

$$\boxed{\langle p';m';M|J^{i}|p;m;M\rangle = (2\pi)^4\delta_M^{(+)}(p')\left[S^i_{m'm} - i\delta_{m'm}\epsilon^{ijk}p^j\frac{\partial}{\partial p_k}\right]\delta^4(p'-p)}$$

*(ii)* Compare these expressions with those obtained from the form factor decomposition of the matrix elements

# 3. A distributional matching approach

- In order to define the Poincaré charges in a consistent manner one must smear the currents with appropriate test functions [Kastler et al.]

$$\boxed{P^{\mu} = \lim_{\substack{d \to 0 \\ R \to \infty}} \int d^4x \ f_{d,R}(x) T^{0\mu}(x)}$$

$$\boxed{J^{i} = \frac{1}{2}\epsilon^{ijk} \lim_{\substack{d \to 0 \\ R \to \infty}} \int d^4x \ f_{d,R}(x) \left[ x^j T^{0k}(x) - x^k T^{0j}(x) \right]}$$

$$f_{d,R}(x) := \alpha_d(x_0) F_R(\mathbf{x})$$

$$\int dx_0 \, \alpha_d(x_0) = 1 \qquad \alpha_d(x_0) \xrightarrow{d \to 0} \delta(x_0)$$

$$F_R(0) = 1 \qquad F_R(\mathbf{x}) \xrightarrow{R \to \infty} 1.$$

- One can then relate the matrix elements of the charges to those of the energy-momentum tensor:

$$\langle p'; m'; M | P^{\mu} | p; m; M \rangle = \lim_{\substack{d \to 0 \\ R \to \infty}} \widehat{f}_{d,R}(q) \langle p'; m'; M | T^{0\mu}(0) | p; m; M \rangle$$

$$\langle p'; m'; M | J^{i} | p; m; M \rangle = -i\epsilon^{ijk} \lim_{\substack{d \to 0 \\ R \to \infty}} \frac{\partial \widehat{f}_{d,R}(q)}{\partial q_j} \langle p'; m'; M | T^{0k}(0) | p; m; M \rangle$$

- Identical definitions can also be used to define light-like charges [Jegerlehner]

[F. Jegerlehner, *Helv. Phys. Acta* **46**, 824 (1974)]

[D. Kastler, D. W. Robinson and J. A. Swieca, *Commun. Math. Phys.* **2**, 108 (1966)]

- ***P*$^{\mu}$ matrix element**: matching the coefficients of the distributions with the same momentum dependence implies

$$\boxed{\begin{aligned} &A(q^2)\delta^4(q) = \delta^4(q) \\ &q^j B(q^2)\delta^4(q) = 0 \\ &q^j q^l C(q^2)\delta^4(q) = 0 \end{aligned}}$$

$$\lim_{n\to\infty}\int d^4q\,\delta_n^{\{0\}}(q)A(q^2) = 1$$

$$\boxed{A(0) = 1}$$

- ***J*$^{i}$ matrix element:** performing the same procedure one obtains

$$\boxed{\begin{aligned} &A(q^2)\partial^k\delta^4(q) = \partial^k\delta^4(q) \\ &A(q^2)\delta^4(q) = \delta^4(q) \\ &B(q^2)\delta^4(q) = 0 \\ &q^l\left[A(q^2)+B(q^2)\right]\partial^k\delta^4(q) = 0 \\ &q^j q^l C(q^2)\partial^k\delta^4(q) = 0, \qquad (l\neq k) \\ &q^j C(q^2)\delta^4(q) = 0 \end{aligned}}$$

$$\boxed{A(0) = 1}$$

$$\boxed{B(0) = 0}$$

The first set of constraints are a <u>subset</u> of these

# 3. A distributional matching approach

- This matching approach recovers the standard results for the form factors in the $q \to 0$ limit, but no choice of frame, wavepacket, operator component, or spin component $m$ is required

- *What about the boost matrix elements?* Interestingly, by using the matrix elements for the boost operator [Bakker, Leader, Trueman]:

$$\langle p'; m'; M | K^i | p; m; M \rangle = (2\pi)^4 \delta_M^{(+)}(p') \left[ \frac{(\mathbf{p} \times \boldsymbol{\sigma}_{m'm})^i}{2(p^0 + M)} + i\delta_{m'm} p^0 \frac{\partial}{\partial p_i} \right] \delta^4(p' - p)$$

one obtains *precisely the same* constraints as those derived using the angular momentum operator $J^i$

- These findings demonstrate that the $q \to 0$ constraints imposed on the form factors $A(q^2)$, $B(q^2)$ and $C(q^2)$ are actually a consequence of the physical on-shell requirement of the states, and the manner in which they transform under Poincaré transformations

[Bakker, Leader, Trueman; hep-ph/0406139]

# 4. Summary and outlook

- Using a distributional matching approach one can derive low-energy ($q \to 0$) constraints on the form factors associated with the matrix elements of $T^{\mu\nu}$ [PL, Chiu, Brodsky, 1707.06313]

- This approach involves expanding the matrix elements of Poincaré charges in two different ways, and then comparing the coefficients

- The same set of constraints: $A(0)=1$, $B(0)=0$ are derived from the angular momentum and boost matrix elements

  → these constraints are a consequence of the on-shellness and Poincaré transformation properties of the states

- This procedure has potentially interesting generalisations

  → form factors associated with different currents

  → matrix elements involving states with higher spin

# Backup

<u>Different spin states</u>:

**Canonical:** $|p, \boldsymbol{s}\rangle = B(\boldsymbol{v})|0, \boldsymbol{s}\rangle = B(\boldsymbol{v})\mathcal{D}^{1/2}_{m1/2}(R(\boldsymbol{s}))|0, m\rangle$

**Jacob-Wick helicity:** $|p, \lambda\rangle_{JW} = R_z(\phi)\, R_y(\theta)\, R_z(-\phi)\, B_z(v)|0, m = \lambda\rangle$

**Wick helicity:** $|p, \lambda\rangle = R_z(\phi)\, R_y(\theta)\, B_z(v)|0, m = \lambda\rangle$

- Wick helicity states have more complicated matrix elements:

$$_W\langle p'; m'; M|J^i|p; m; M\rangle_W = (2\pi)^4\delta^{(+)}_M(p')\left[m\,\delta_{m'm}\frac{(\delta^{i1}p^1+\delta^{i2}p^2)|\mathbf{p}|}{(p^1)^2+(p^2)^2} - i\delta_{m'm}\epsilon^{ijk}p^j\frac{\partial}{\partial p_k}\right]\delta^4(p'-p)$$

- In the spin-½ case: $S^i_{m'm} = \frac{1}{2}\sigma^i_{m'm}$ but for higher spin states this matrix is more complicated [Bakker, Leader, Trueman]

[Bakker, Leader, Trueman; hep-ph/0406139]

## Definitions used:

$$\boxed{h(q)\,\partial^k\delta^4(q) = h(0)\,\partial^k\delta^4(q) - (\partial^k h)(0)\,\delta^4(q)}$$

$$\boxed{q = p' - p, \quad \bar{p} = \tfrac{1}{2}(p' + p)}$$

$$\frac{\partial}{\partial q_k}\left\{\left[\bar{u}_{m'}\left(\bar{p}+\tfrac{1}{2}q\right)u_m\left(\bar{p}-\tfrac{1}{2}q\right)\right]_{q^0=\frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{\bar{p}^0}}\right\}_{q=0} = \frac{i}{(\bar{p}^0+M)}\epsilon^{kln}\,\bar{p}^l\,\sigma^n_{m'm},$$

$$\bar{u}_{m'}(\bar{p})\sigma^{jk}u_m(\bar{p}) = 2\epsilon^{jkl}\left[\bar{p}^0\sigma^l_{m'm} - \frac{\bar{p}^l(\bar{\mathbf{p}}\cdot\boldsymbol{\sigma}_{m'm})}{\bar{p}^0+M}\right],$$

## Form factor calculation – angular momentum case:

$$(2\pi)^4\left[\frac{1}{2}\sigma^i_{m'm} + i\delta_{m'm}\epsilon^{ijk}\bar{p}^j\frac{\partial}{\partial q_k}\right]\delta^4(q) =$$

$$(2\pi)\,\delta\left(q^0 - \frac{\bar{\mathbf{p}}\cdot\mathbf{q}}{\bar{p}^0}\right)\lim_{\substack{d\to 0\\ R\to\infty}}\left\{\frac{1}{2}\sigma^i_{m'm}\,\widehat{f}_{d,R}(q)A(q^2) + i\delta_{m'm}\epsilon^{ijk}\bar{p}^j\frac{\partial\widehat{f}_{d,R}}{\partial q_k}A(q^2)\right.$$

$$+\left[\frac{\bar{p}^0}{2M}\sigma^i_{m'm} - \frac{\bar{p}^i(\bar{\mathbf{p}}\cdot\boldsymbol{\sigma}_{m'm})}{2M(\bar{p}^0+M)}\right]\widehat{f}_{d,R}(q)B(q^2) - \epsilon^{ijk}\,\frac{\bar{p}^{\{0}\bar{u}_{m'}(\bar{p})\,\sigma^{j\}\rho}u_m(\bar{p})\,q_\rho}{8M\bar{p}^0}\,\frac{\partial\widehat{f}_{d,R}}{\partial q_k}\left[A(q^2)+B(q^2)\right]$$

$$\left.+\,i\delta_{m'm}\,\epsilon^{ijk}\,\frac{q^0q^j}{\bar{p}^0}\,\frac{\partial\widehat{f}_{d,R}}{\partial q_k}C(q^2) + \epsilon^{ijk}\left[i\,\frac{q^j\bar{p}^k}{(\bar{p}^0)^2}\delta_{m'm} - \frac{\bar{p}_l q^l q^j(\bar{\mathbf{p}}\times\boldsymbol{\sigma}_{m'm})^k}{2M(\bar{p}^0)^2(\bar{p}^0+M)}\right]\widehat{f}_{d,R}(q)C(q^2)\right\}$$

# Backup

Local QFT approaches are defined by a core set of axioms:

**Axiom 1 (Hilbert space structure).** *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}}^{\uparrow}_{+}$.*

**Axiom 2 (Spectral condition).** *The spectrum of the energy-momentum operator $P^{\mu}$ is confined to the closed forward light cone $\overline{V}^{+} = \{p^{\mu} \mid p^2 \geq 0, \, p^0 \geq 0\}$, where $U(a,1) = e^{iP^{\mu}a_{\mu}}$.*

**Axiom 3 (Uniqueness of the vacuum).** *There exists a unit state vector $|0\rangle$ (the vacuum state) which is a unique translationally invariant state in $\mathcal{H}$.*

**Axiom 4 (Field operators).** *The theory consists of fields $\varphi^{(\kappa)}(x)$ (of type $(\kappa)$) which have components $\varphi^{(\kappa)}_{l}(x)$ that are operator-valued tempered distributions in $\mathcal{H}$, and the vacuum state $|0\rangle$ is a cyclic vector for the fields.*

**Axiom 5 (Relativistic covariance).** *The fields $\varphi^{(\kappa)}_{l}(x)$ transform covariantly under the action of $\overline{\mathscr{P}}^{\uparrow}_{+}$:*

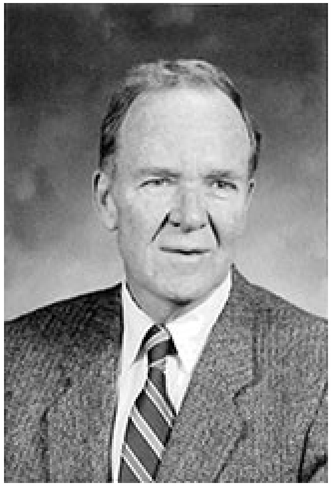
$$U(a,\alpha)\varphi^{(\kappa)}_{i}(x)U(a,\alpha)^{-1} = S^{(\kappa)}_{ij}(\alpha^{-1})\varphi^{(\kappa)}_{j}(\Lambda(\alpha)x + a)$$

*where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}}^{\uparrow}_{+}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}}^{\uparrow}_{+}$.*

**Axiom 6 (Local (anti-)commutativity).** *If the support of the test functions $f,g$ of the fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$ are space-like separated, then:*

$$[\varphi^{(\kappa)}_{l}(f), \varphi^{(\kappa')}_{m}(g)]_{\pm} = \varphi^{(\kappa)}_{l}(f)\varphi^{(\kappa')}_{m}(g) \pm \varphi^{(\kappa')}_{m}(g)\varphi^{(\kappa)}_{l}(f) = 0$$

*when applied to any state in $\mathcal{H}$, for any fields $\varphi^{(\kappa)}_{l}, \varphi^{(\kappa')}_{m}$.*



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]

R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

# Backup

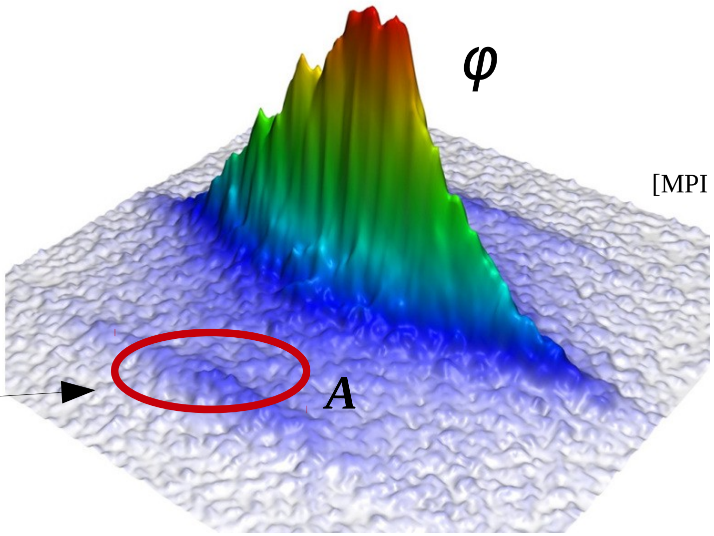
- Quantum fields $\varphi(x)$ are distributions – *what difference does this make?*

  → This means that they cannot be evaluated at a single point (e.g. think of the Dirac delta $\delta(x)$ at $x=0$)

  → Need to 'average them out' over some spacetime region ***A***

$$\boxed{\mathcal{M}_\varphi := \int_A d^4x\, \varphi(x) f(x)}$$

Can think of this as the performance of a measurement $M_\varphi$ in the region ***A*** where $f(x)$ is non-zero



[MPI Munich (2004)]

- But why? – Heisenberg's uncertainty principle! $\Delta x \Delta p \sim \frac{\hbar}{2}$

# Backup

- The distributional nature of form factors implies that these objects are not in general continuous. Nevertheless, form factors $F(q^2)$ are seemingly measured at specific values of $q^2$. In order to reconcile these points of view one must recognise that one cannot ever physically measure a form factor at a specific value of $q^2$, since this would require an experiment with infinite precision.

- In practice, a measurement of $F(q^2)$ at $q^2 = Q^2$ is really a measurement of an averaged-out quantity $\boldsymbol{F}(Q^2; \Delta)$ in some small but non-vanishing region $[Q^2 - \Delta, Q^2 + \Delta]$.

- $\boldsymbol{F}(Q^2; \Delta)$ is the convolution of $F(q^2)$ with a test function $f_\Delta(q^2)$, which characterises the resolution $\Delta$ of the experiment

$$\overline{F}(Q^2; \Delta) := (F * f_\Delta)(Q^2)$$