GLUON DISTRIBUTIONS IN POLARIZED TARGETS OF SPIN ≤ 1

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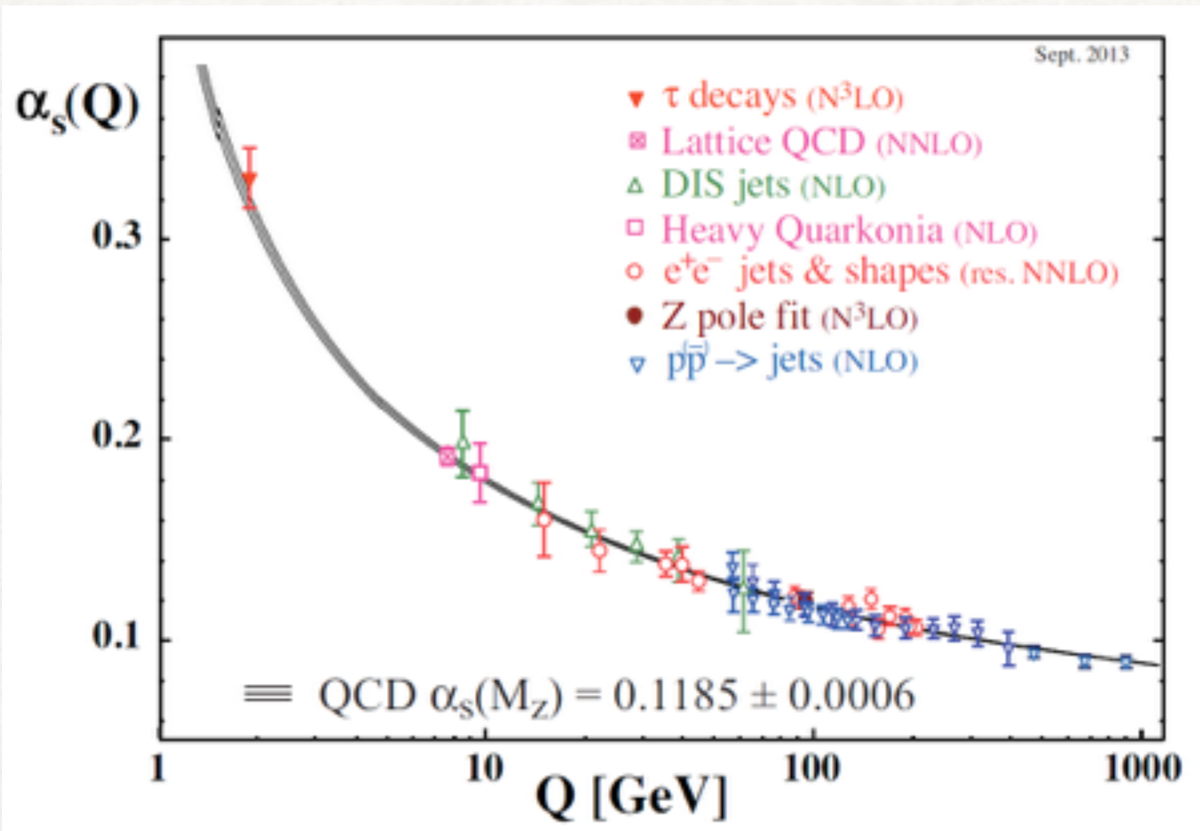
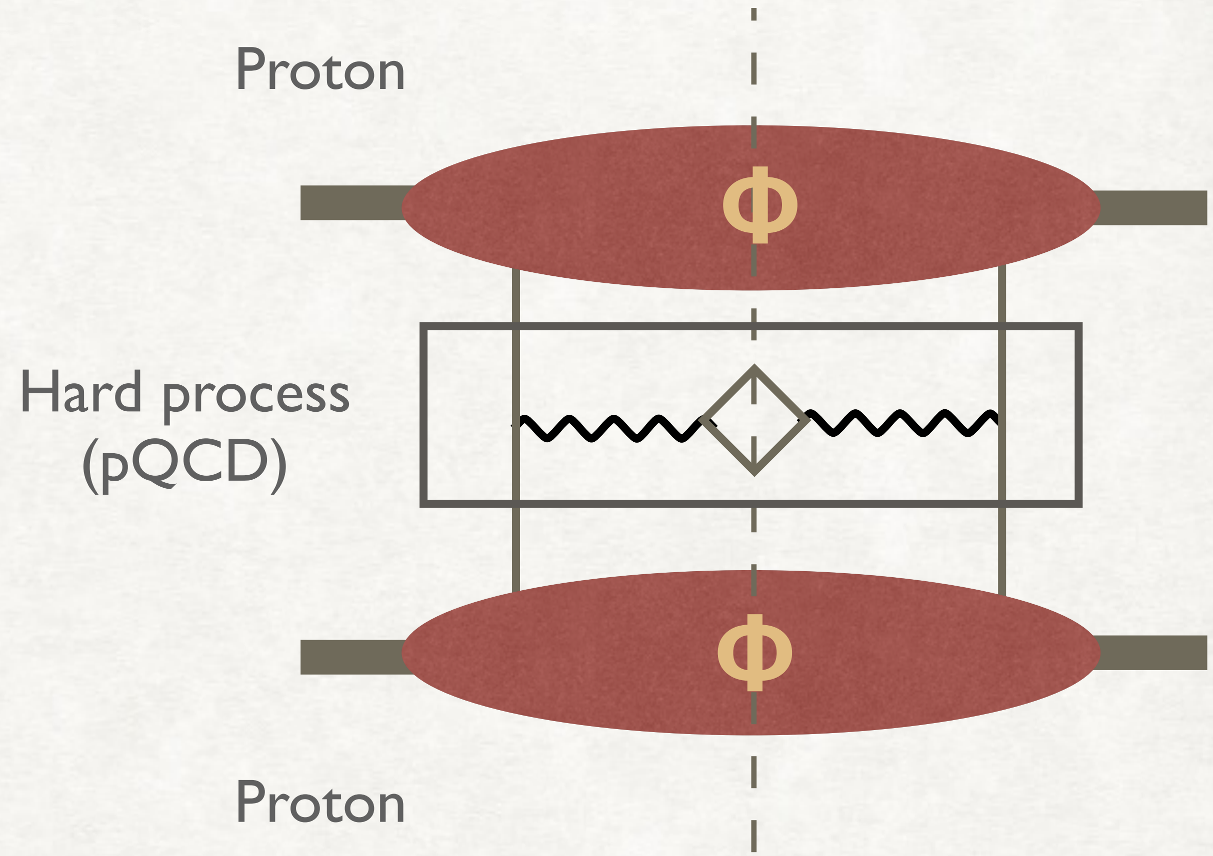
POLARIZED LIGHT-ION PHYSICS WITH EIC

GHENT UNIVERSITY, 5-9 FEBRUARY 2018

Perturbative VS Non perturbative QCD

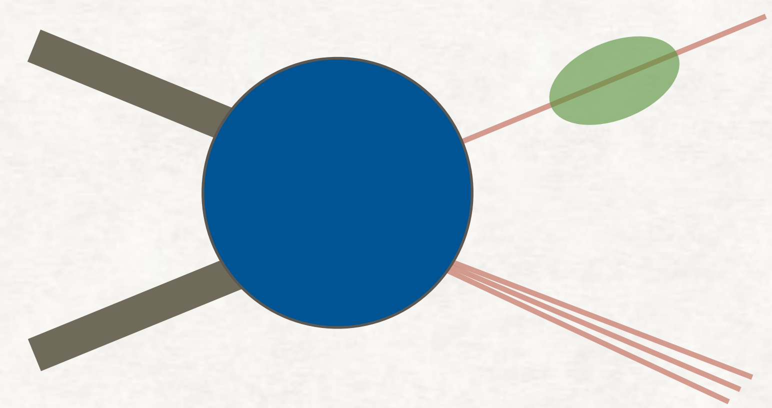




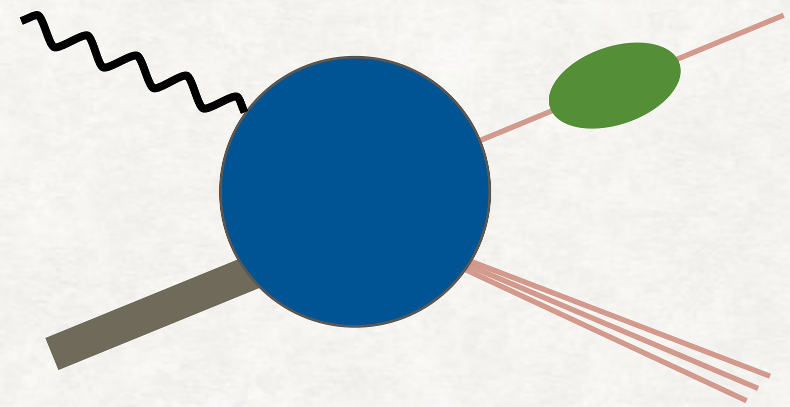
- Two distinct energy scales: the hadronic scale (order Λ_{QCD}) and the hard scale Q of the process.
- For several processes, factorization has been proven.

The hadron structure in high energy processes: examples

Proton-proton pp collision
nucleus



Lepton-proton scattering
nucleus



NONPERTURBATIVE PHYSICS: CORRELATION FUNCTIONS

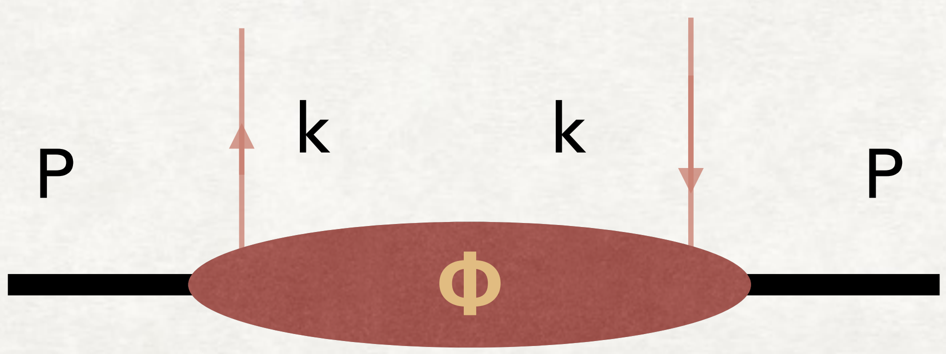
Quantum field theory language

QCD paradigm: quarks and gluons are confined inside hadrons (no free-fields)

$$\sum_s u_i^s(k)\bar{u}_j^s(k) \Rightarrow \Phi_{ij}(k;P)$$

$$= \sum_X \int \frac{d^3 P_X}{(2\pi)^3 2E_X} \langle P| \overline{\psi}_j(0) |X\rangle \langle X| \psi_i(0) |P\rangle \, \delta^4(k+P-P_X)$$

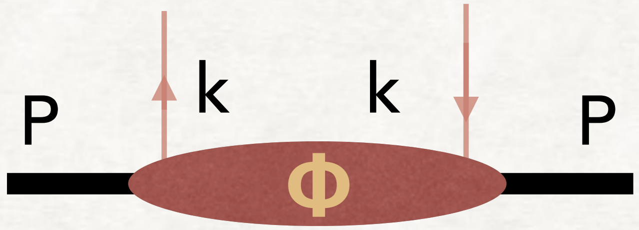
$$= \int \frac{d^4\xi}{(2\pi)^4} e^{ik\cdot\xi} \langle P| \overline{\psi}_j(0)\psi_i(\xi) |P\rangle$$



Transverse momentum dependent (light-front) correlation function for **quarks**

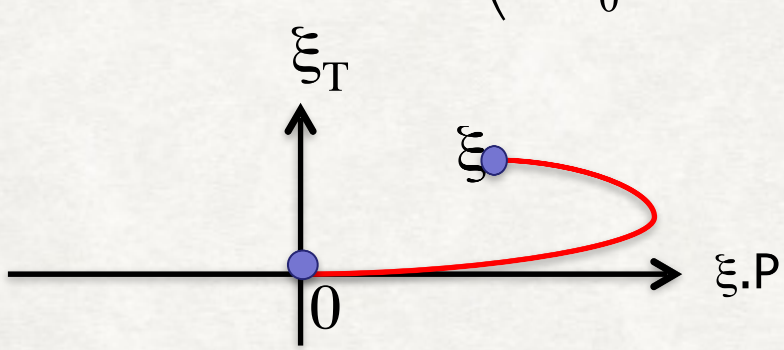
$$\Phi_{ij}^{[U]}(x,\boldsymbol{k}_T;n,P,S) = \int \frac{d\xi\cdot P\, d^2\boldsymbol{\xi}_T}{(2\pi)^3} e^{ik\cdot\xi} \langle P,S|\,\overline{\psi}_j(0) U_{[0,\xi]} \psi_i(\xi)\,|P,S\rangle\Big|_{\xi\cdot n=0}$$

Nonlocal operator: color-gauge invariance?



Wilson lines: connect the two space-time points and restore gauge invariance

$$U(0,\xi) = \mathcal{P}\,\exp\left(-ig\int_0^{\xi} ds^{\mu} A_{\mu}\right)$$

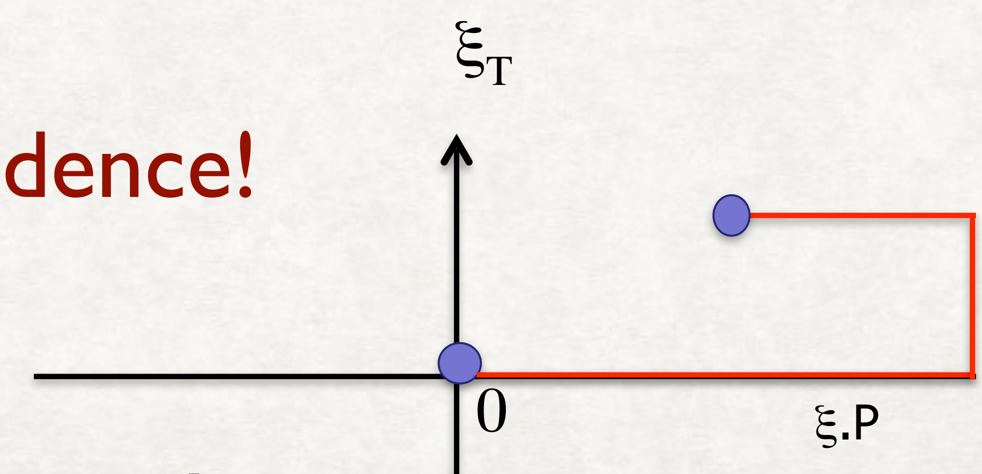


Type of path in high energy processes:

staple-like

(future- or past-pointing)

Process dependence!



[Brodsky, Hwang,Schmidt, 2002; Belitsky, Ji, Yuan, 2003; Boer, Mulders,Pijlman, 2003]

Process dependence of the TMDs

The path connecting the end points depends on the color flow

$\longrightarrow$ process dependence

Famous sign-change of the Sivers and the Boer-Mulders functions

$$f_{1T}^{\perp[+]}(x, \boldsymbol{k}_T^2) = -f_{1T}^{\perp[-]}(x, \boldsymbol{k}_T^2) \qquad h_1^{\perp[+]}(x, \boldsymbol{k}_T^2) = -h_1^{\perp[-]}(x, \boldsymbol{k}_T^2)$$

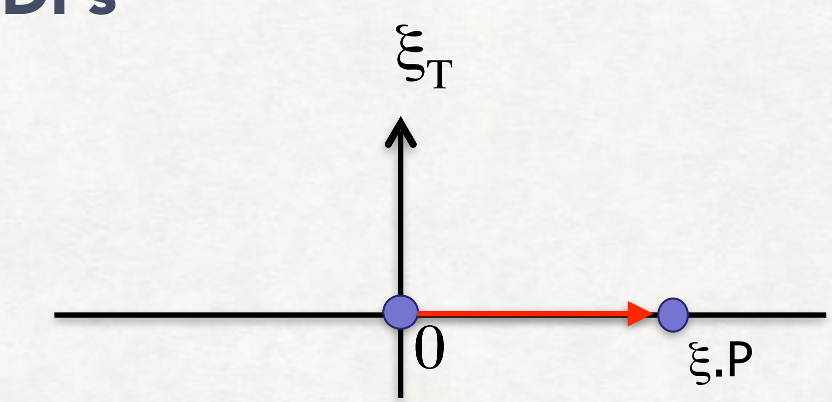
SIDIS Drell-Yan SIDIS Drell-Yan

[Collins, 2002]

Universality of PDFs

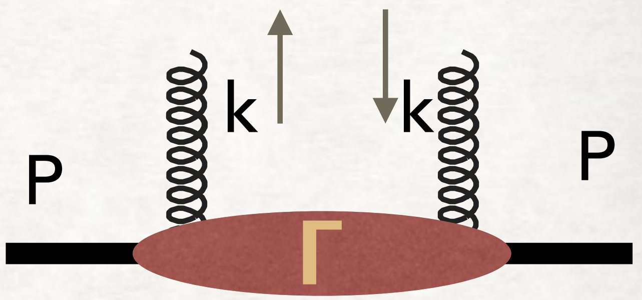
Gauge link reduces to a line:

$$\int dk_T \longrightarrow$$



PDFs are universal!

Transverse momentum dependent (light-front) correlation function for gluons



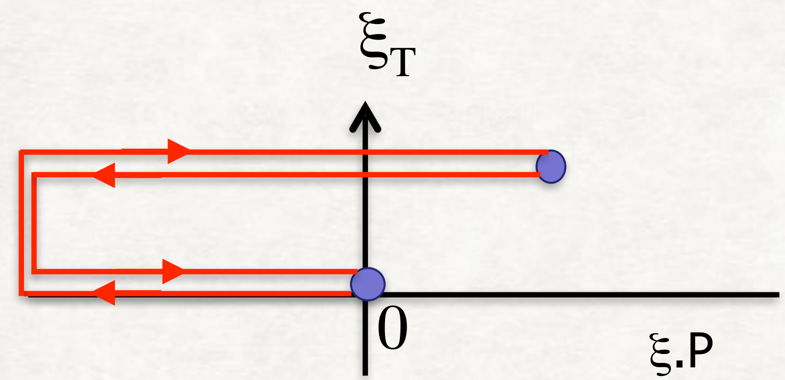
$$\sum_\lambda \epsilon_\lambda^{*\mu}(k)\epsilon_\lambda^{\nu}(k) \Rightarrow \Gamma^{\mu\nu}(k;n,P)$$
$$= \int \frac{d^4\xi}{(2\pi)^4} e^{ik\cdot\xi} \langle P| F^{n\mu}(0)F^{n\nu}(\xi)|P\rangle$$

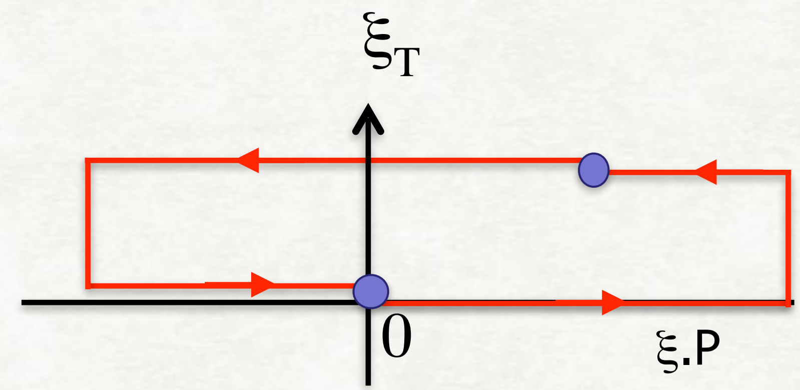
$$\Gamma^{\mu\nu\,[U,U']}(x,\boldsymbol{k}_T;n,P,S) = \int \frac{d\xi\cdot P\, d^2\boldsymbol{\xi}_T}{(2\pi)^3} e^{ik\cdot\xi} \langle P,S| F^{n\mu}(0)U_{[0,\xi]}F^{n\nu}(\xi)U'_{[\xi,0]}|P,S\rangle\Big|_{\xi\cdot n=0}$$

Gauge invariant thanks to the Wilson lines (not unique)

Parametrized in terms of gluon TMDs

Transverse momentum dependent (light-front) correlation function for gluons



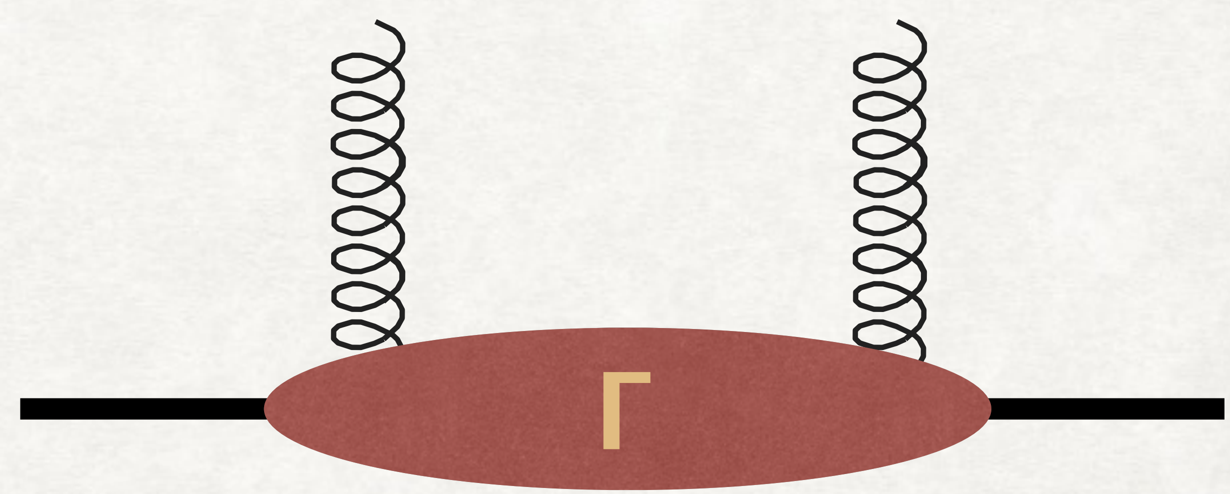


Leading twist contribution: μ and ν transverse
Gauge link structures: staple-like

[Dominguez,Marquet,Xiao,Yuan,2011]

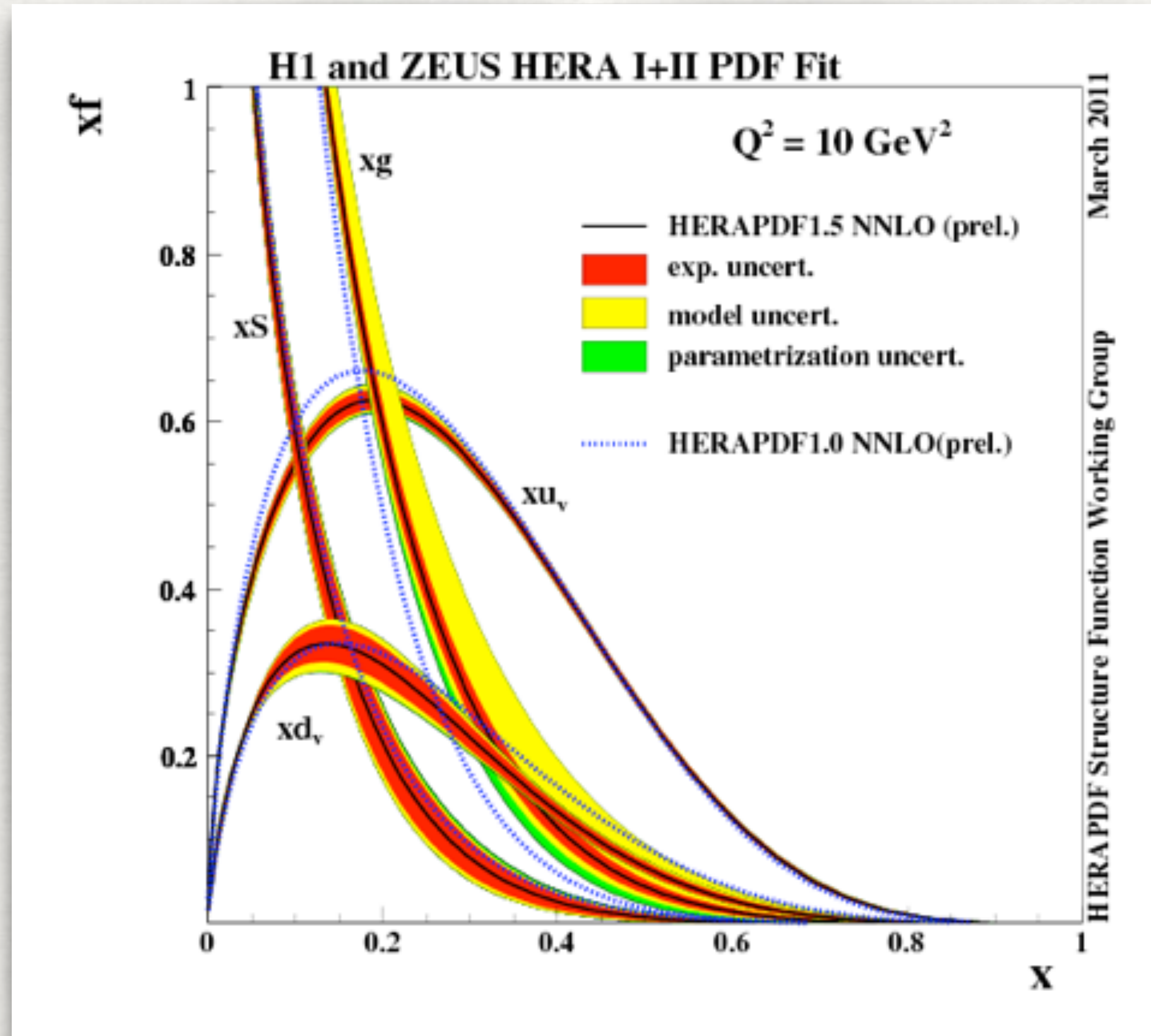
$$\Gamma^{[U,U']ij}(x,\boldsymbol{k}_T;n,P,S) = \int \frac{d\xi\cdot P\, d^2\boldsymbol{\xi}_T}{(2\pi)^3} \langle P,S|\, F^{ni}(0) U_{[0,\xi]} F^{nj}(\xi) U'_{[\xi,0]}\, |P,S\rangle\Big|_{\xi\cdot n=0}$$

Process dependence!



process dependence, non universality and more:
[Boer, Mulders,Pijlman, 2003;Bomhof, Mulders, Pijlman, 2006]
[Bomhof, Mulders, 2006,Buffing, Mukherjee, Mulders,2012]

Relevant regions for gluon distributions



GLUON PDFs

- Large-x: suppressed
- Small-x: dominant

TARGET SPIN

Polarized Hadrons

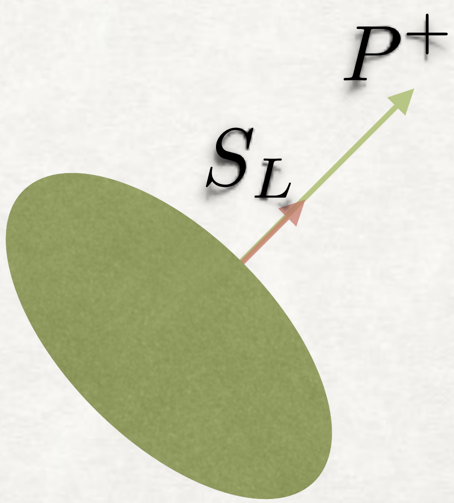
- Parent hadron momentum P ;
- Parton momentum $k^\mu = xP^\mu + k_T^\mu + (k \cdot P - xM^2)n^\mu$;
- n, light-like vector satisfying $P \cdot n = 1$;

Vector polarization:

- Construction of the space-like spin vector satisfying $P \cdot S = 0$;

$$S^\mu = S_L \frac{P^\mu}{M} + S_T^\mu - M S_L n^\mu$$

"Light-cone helicity"
(IMF helicity of the particle)



- Simple interpretation in light-front formalism.

Tensor polarization (relevant for spin-1 and higher)

- Construction of the symmetric traceless spin tensor satisfying $P_\mu T^{\mu\nu} = 0$;

$$T^{\mu\nu} = \frac{1}{2}\left[\frac{2}{3}S_{LL}\, g_T^{\mu\nu} + \frac{4}{3}S_{LL}\frac{P^\mu P^\nu}{M^2} + \frac{S_{LT}^{\{\mu}P^{\nu\}}}{M} + S_{TT}^{\mu\nu} \right.$$
$$\left. - \frac{4}{3}S_{LL}P^{\{\mu}n^{\nu\}} - M S_{LT}^{\{\mu}n^{\nu\}} + \frac{4}{3}M^2 S_{LL}\, n^\mu n^\nu \frac{P^\mu P^\nu}{M^2}\right]$$

- Five more spin components (representing combinations of probability of finding the system in a certain spin state → less simple to visualize)

[Bacchetta,PhD Thesis,2002]
[Leader, "Spin in Particle Physics", 2001]

Gluon TMDs in spin-1/2 target

TARGET SPIN	GLUON POLARIZATION: Unpolarized	Circular	Linear
U	$\boldsymbol{f_1}$		$h_1^{\perp}$
L		$\boldsymbol{g_1}$	$h_{1L}^{\perp}$
T	$f_{1T}^{\perp}$	g_{1T}	$h_1, h_{1T}^{\perp}$

Spin 0 and 1/2
[Mulders,Rodrigues,2001]
[Meissner, Metz and Goeke, 2007]

- The functions have dependence:

$$f^{[U,U']}(x, \boldsymbol{k}_T^2)$$

Scales dependence on the TMDs is frozen (more comments later..)

- **Bold face**: collinear PDFs
- Red Boxes: T-odd
- The gluon Boer-Mulders is T-even
- **There is no gluon analogue of the transversity PDF**

etc…

Gluon TMDs in spin-1/2 target

TARGET SPIN	GLUON POLARIZATION: Unpolarized	Circular	Linear
U	$\boldsymbol{f_1}$		$h_1^\perp$
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T	$f_{1T}^\perp$	g_{1T}	$h_1, h_{1T}^\perp$

Spin 0 and 1/2
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- The functions have dependence:

$$f^{[U,U']}(x, \boldsymbol{k}_T^2)$$

- **Bold face**: collinear PDFs
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- The gluon Boer-Mulders is T-even
- **There is no gluon analogue of the transversity PDF**

$$UU: \quad g_T^{ij} f_1$$

$$LL: \quad \frac{i\epsilon_{T\alpha}^{\{i} k_T^{j\}\alpha}}{M^2} S_L h_{1L}^\perp$$

$$TU: \quad \frac{g_T^{ij}\epsilon^{k\cdot S_T}}{M} f_{1T}^\perp$$

$$TC: \quad \frac{i\epsilon_T^{ij} k_T\cdot S_T}{M} g_{1T}$$

etc…

Gluon TMDs in spin-1 target

TARGET SPIN	GLUON POLARIZATION: Unpolarized	Circular	Linear
U	$\boldsymbol{f_1}$		$h_1^\perp$
L		$\boldsymbol{g_1}$	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	$h_1, h_{1T}^\perp$
LL	$\boldsymbol{f_{1LL}}$		$h_{1LL}^\perp$
LT	f_{1LT}	g_{1LT}	$h_{1LT}, h_{1LT}^\perp$
TT	f_{1TT}	g_{1TT}	$\boldsymbol{h_{1TT}}, h_{1TT}^\perp, h_{1TT}^{\perp\perp}$

Two additional collinear functions

$$LLU: \quad g_T^{ij} S_{LL} f_{1LL}$$

$$LTL: \quad \frac{S_{LT}^{\{i} k_T^{j\}}}{M} h_{1LT} + \frac{k_T^{ij\alpha}}{M^3} S_{LT\alpha} h_{1LT}^\perp$$

$$TTC: \quad i\epsilon^{ij} \frac{\epsilon_{T\gamma}^{\beta} k_T^{\gamma\alpha} S_{TT\alpha\beta}}{M^2} g_{1TT}$$

etc…

Spin 1
[Jaffe & Manohar, 1989]
[Boer, C, vanDaal, Mulders, Signori, Zhou, 2016]

Gluon PDFs in spin-1 target

TARGET SPIN	GLUON POLARIZATION		
	Unpolarized	Circular	Linear
U	f_1		
L		g_1	
T			
LL	f_{1LL}		
LT			
TT			$h_{1TT}(x)$

[Jaffee,Manohar,1989]
[Artru,Mekhfi,1990]

Volume 223, number 2 PHYSICS LETTERS B 8 June 1989

NUCLEAR GLUONOMETRY ☆

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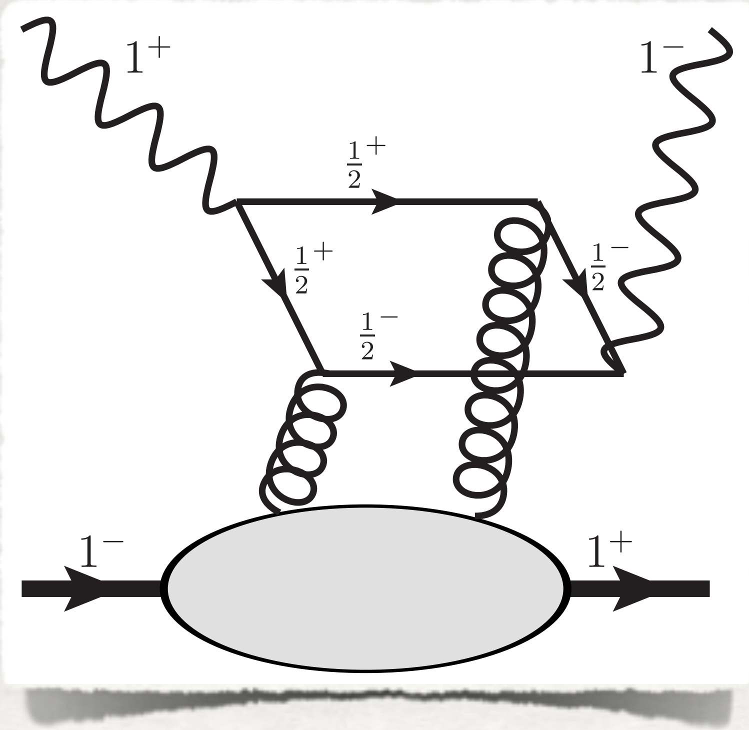
Received 24 March 1989

We identify a new leading twist structure function in QCD which can be measured in deep elastic scattering from polarized targets (such as nuclei) with spin ⩾ 1. The structure function measures a gluon distribution in the target and vanishes for a bound state of protons and neutrons, thereby providing a clear signature for exotic gluonic components in the target.



[Detmold,Shanahan,2016]

$\Delta(x)$ Double helicity flip structure function

$h_{1TT}(x)$ Probability to find linearly polarized gluons inside a transversely tensor polarized target (misleadingly called "gluon transversity")

Lattice calculations

[NPLQCD collaboration,2017]

$h_{1TT}(x)$ Clear signal of a non-vanishing first Mellin moment

$f_{1LL}(x)$ With the current precision not resolvable from zero

POSITIVITY BOUNDS

Matrix representation of the correlator

Single out hadron spin:

$$\rho = \frac{1}{3}\left(I + \frac{3}{2} S^i \Sigma^i + 3 T^{ij} \Sigma^{ij} \right).$$

$$\Gamma^{ij} = \sum_{ss'} \rho_{ss'} G^{ij}_{ss'}$$

Gamma: matrix in the gluon polarization space

G: matrix in gluon $\otimes$ target spin space

Circular polarization bases $|\pm\rangle = \mp\frac{1}{\sqrt{2}}(|x\rangle \pm i|y\rangle);$

$$G^{ij} = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix}$$

Quarks:
[Bacchetta,Boglione,Henneman, Mulders,2000]
[Bacchetta, Mulders2001]
Gluons:
[Mulders,Rodrigues,2001]
[Meissner, Metz and Goeke,2007]

SEMI-POSITIVE DEFINITE MATRIX.
ALLOWS FOR THE INTERPRETATION OF SOME FUNCTIONS AS DENSITIES

Positivity bounds on gluon TMDs

6x6 matrix in gluon $\otimes$ target spin space

$$G^{ij}_{ss'}(\boldsymbol{k}_T) = \left(\begin{array}{ccc|ccc} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

Explicit form in arXiv: 1709.07827

$$\frac{\boldsymbol{k}_T^4}{2M^4}\left|h_{1TT}^{\perp\perp}\right| \leq f_1 + \frac{f_{1LL}}{2} - g_1$$

$$\frac{\boldsymbol{k}_T^2}{2M^2}\left|h_1^{\perp} - h_{1LL}^{\perp}\right| \leq f_1 - f_{1LL}$$

Bounds can be sharpened!

$$\frac{\boldsymbol{k}_T^2}{2M^2}\left(h_1^2 + 4h_{1LT}^2\right) \leq \left(f_1 - f_{1LL}\right)\left(f_1 + \frac{f_{1LL}}{2} + g_1\right)$$

... 6 more inequalities from the remaining 2x2 principal minors

Bounds on the gluon PDFs in spin-1 targets

Only few gluon functions survive upon integration over transverse momenta:

$$G^{ij}_{ss'}(x) =$$

$$\left(\begin{array}{ccc|ccc}
f_1 + \frac{f_{1LL}}{2} - g_1 & 0 & 0 & 0 & 0 & 0 \\
0 & f_1 - f_{1LL} & 0 & 0 & 0 & 0 \\
0 & 0 & f_1 + \frac{f_{1LL}}{2} + g_1 & -2h_{1TT} & 0 & 0 \\
\hline
0 & 0 & -2h_{1TT} & f_1 + \frac{f_{1LL}}{2} + g_1 & 0 & 0 \\
0 & 0 & 0 & 0 & f_1 - f_{1LL} & 0 \\
0 & 0 & 0 & 0 & 0 & f_1 + \frac{f_{1LL}}{2} - g_1
\end{array}\right)$$

$$f_{1LL} \le f_1,$$

Bounds come from the diagonalization of the full matrix

$$|g_1| \le f_1 + \frac{f_{1LL}}{2},$$

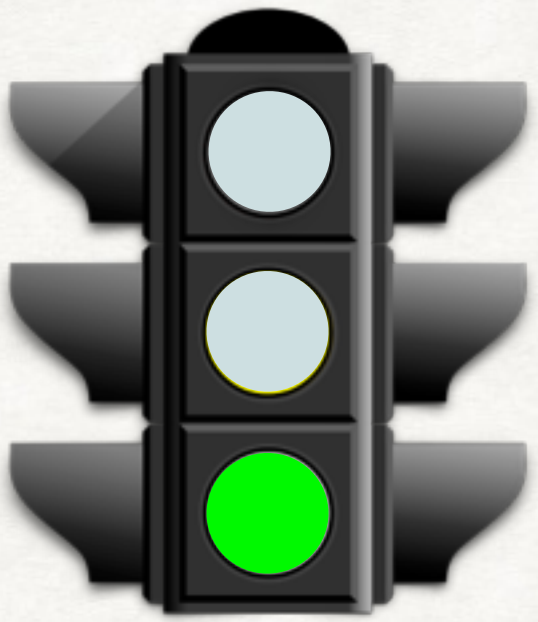
$$\boxed{|h_{1TT}| \le \frac{1}{2}\left(f_1 + \frac{f_{1LL}}{2} + g_1\right)}$$

Lattice calculations on the first moment of this bound

[Detmold,Shanahan,2016]

REMARKS ON POSITIVITY BOUNDS

- Model-independent inequalities;
- Rigorous test of QCD;
- Allow for an estimate of the magnitude of the functions.



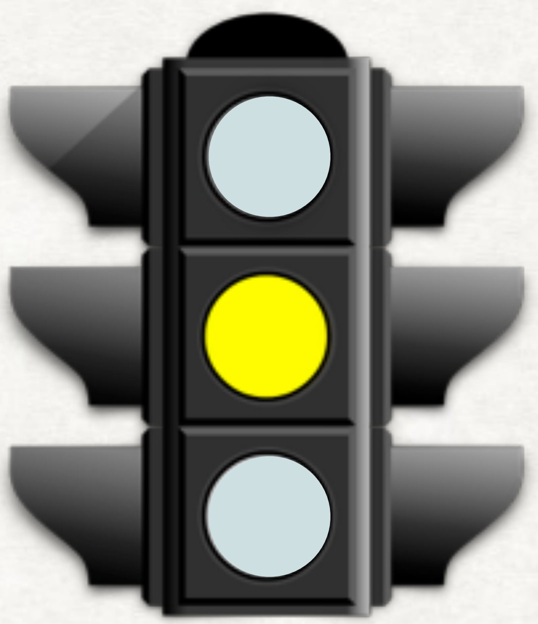
- Process dependence:

  The matrix G is positive semidefinite only for operators in the form $O^\dagger(0)O(\xi)$. The simplest gauge link structures for which this holds are [+,+], [−,−], [+,−], and [−,+] ;

- QCD evolution effects:

  Multi-scale evolution: no studies have been performed on the (in)stability of the TMD bounds after evolution.

  Collinear bounds are stable under DGLAP equations



EXPERIMENTAL RELEVANCE

Experimental relevance at present facilities:

Spin-1 program at JLab:

- Proposals on tensor polarized experiments using nitrogen targets: extraction of the gluon structure function Δ (encouraged for full submission at JLab by PAC 44)

In the future…

Possibilities at LHC:

- COMPASS and AFTER@LHC (it allows, in principle, for polarized targets)

Electron Ion Collider EIC

- Would allow to thoroughly study many gluon observables.

CONCLUSIONS

Conclusions

- Gluon TMDs are fundamental tools to understand hadron internal structure (3D distribution of momentum);
- Their knowledge would reveal a lot about the internal dynamics of gluons in hadrons, which is at present almost unknown;
- TMDs allows one to study fundamental problems such as gauge invariance and process dependence.
- Gluon distributions (PDFs and TMDs) are dominant is in the small-x limit: it is important that future facilities access this kinematical region.
- When hadron polarization is included, the additional degrees of freedom could open up a wide range of new phenomena (signs of different types of parton-hadron correlations);
- Positivity bounds can be used as model-independent tools to estimate magnitude of mainly unknown functions.
- "Exotic" or non-nucleonic gluonic effects within nuclei would also allow to study more thoroughly the binding between the constituents;

Thank you!

BACK UP SLIDES

	Unpolarized	Longitudinal	Transverse
U	$\boldsymbol{f_1}$		$h_1^\perp$
L		$\boldsymbol{g_1}$	$h_{1L}^\perp$
T	$f_{1T}^\perp$	g_{1T}	$\boldsymbol{h_1}, h_{1T}^\perp$
LL	$\boldsymbol{f_{1LL}}$		$h_{1LL}^\perp$
LT	f_{1LT}	g_{1LT}	$h_{1LT}, h_{1LT}^\perp$
TT	f_{1TT}	g_{1TT}	$h_{1TT}, h_{1TT}^\perp$

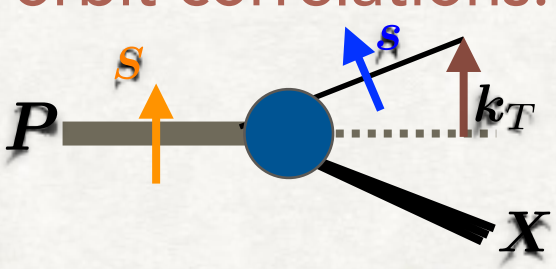
Functions depend on:

$$f^{q[U]}(x, \boldsymbol{k}_T^2)$$

Bold-face: Collinear

Red Box: T-odd

- Spin-orbit correlations:



Sivers function

$$\boldsymbol{S} \cdot (\boldsymbol{P} \times \boldsymbol{k}_T)$$

Boer-Mulders function

$$\boldsymbol{s} \cdot (\boldsymbol{P} \times \boldsymbol{k}_T)$$

- h_{1LT} is T-odd and survives k_T-integration
- f_{1LL} enters in the spin-1 structure function b_1

Parametrization of the gluon correlator:

$$\Gamma^{ij[U]}(x,k_T) = \frac{x}{2}\Big\{-g_T^{ij}\, f_1^{[U]}(x,k_T^2) + \frac{k_T^{ij}}{M^2}\, h_1^{\perp\,[U]}(x,k_T^2)\Big\}$$

$$\Gamma_L^{ij[U]}(x,k_T) = \frac{x}{2}\Big\{i\epsilon_T^{ij}\, S_L g_1^{[U]}(x,k_T^2) + \frac{\epsilon_{T\,\alpha}^{\{i}k_T^{j\}\alpha}}{M^2}\, S_L h_{1L}^{\perp\,[U]}(x,k_T^2)\Big\}$$

$$\Gamma_T^{ij[U]}(x,k_T) = \frac{x}{2}\Big\{\frac{g_T^{ij}\epsilon_T^{kS_T}}{M}\, f_{1T}^{\perp[U]}(x,k_T^2) - \frac{i\epsilon_T^{ij}k_T{\cdot}S_T}{M}\, g_{1T}^{[U]}(x,k_T^2)$$
$$- \frac{\epsilon_T^{k\{i}S_T^{j\}} + \epsilon_T^{S_T\{i}k_T^{j\}}}{4M}\, h_1(x,k_T^2) - \frac{\epsilon_{T\,\alpha}^{\{i}k_T^{j\}\alpha S_T}}{2M^3}\, h_{1T}^{\perp}(x,k_T^2)\Big\}$$

$$\Gamma_{LL}^{ij[U]]}(x,k_T) = \frac{x}{2}\Big\{-g_T^{ij}S_{LL}\, f_{1LL}^{[U]}(x,k_T^2) + \frac{k_T^{ij}}{M^2}\, S_{LL}h_{1LL}^{\perp[U]}(x,k_T^2)\Big\}$$

$$\Gamma_{LT}^{ij[U]}(x,k_T) = \frac{x}{2}\Big\{-g_T^{ij}\frac{k_T{\cdot}S_{LT}}{M}\, f_{1LT}^{[U]}(x,k_T^2) + i\epsilon_T^{ij}\frac{\epsilon_T^{S_{LT}k}}{M}\, g_{1LT}^{[U]}(x,k_T^2)$$
$$+ \frac{S_{LT}^{\{i}k_T^{j\}}}{M}\, h_{1LT}^{[U]}(x,k_T^2) + \frac{k_T^{ij\alpha}S_{LT\alpha}}{M^3}\, h_{1LT}^{\perp[U]}(x,k_T^2)\Big\}$$

$$\Gamma_{TT}^{ij[U]}(x,k_T) = \frac{x}{2}\Big\{-g_T^{ij}\frac{k_T^{\alpha\beta}S_{TT\alpha\beta}}{M^2}\, f_{1TT}^{[U]}(x,k_T^2) + i\epsilon_T^{ij}\frac{\epsilon_{T\,\gamma}^{\beta}k_T^{\gamma\alpha}S_{TT\alpha\beta}}{M^2}\, g_{1TT}^{[U]}(x,k_T^2)$$
$$+ S_{TT}^{ij}\, h_{1TT}^{[U]}(x,k_T^2) + \frac{S_{TT\alpha}^{\{i}k_T^{j\}\alpha}}{M^2}\, h_{1TT}^{\perp[U]}(x,k_T^2)$$
$$+ \frac{k_T^{ij\alpha\beta}S_{TT\alpha\beta}}{M^4}\, h_{1TT}^{\perp\perp[U]}(x,k_T^2)\Big\}$$

Density matrix

$$\rho = \frac{1}{3}\left(I + \frac{3}{2}S^i\Sigma^i + 3T^{ij}\Sigma^{ij}\right).$$

$$\rho = \begin{pmatrix} \frac{S_L}{2} + \frac{S_{LL}}{3} + \frac{1}{3} & \frac{S_{LTx}-iS_{LTy}}{2\sqrt{2}} + \frac{S_{Tx}-iS_{Ty}}{2\sqrt{2}} & \frac{1}{2}(S_{TTxx} - iS_{TTxy}) \\ \frac{S_{LTx}+iS_{LTy}}{2\sqrt{2}} + \frac{S_{Tx}+iS_{Ty}}{2\sqrt{2}} & \frac{1}{3} - \frac{2SLL}{3} & \frac{-S_{LTx}+iS_{LTy}}{2\sqrt{2}} + \frac{S_{Tx}-iS_{Ty}}{2\sqrt{2}} \\ \frac{1}{2}(S_{TTxx} + iS_{TTxy}) & \frac{-S_{LTx}-iS_{LTy}}{2\sqrt{2}} + \frac{S_{Tx}+iS_{Ty}}{2\sqrt{2}} & -\frac{S_L}{2} + \frac{S_{LL}}{3} + \frac{1}{3} \end{pmatrix}$$

Positivity bounds: construction of the matrix

$$G = \frac{x}{2}\begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$$

C is the transformed of A under Parity

$$A = \begin{pmatrix} f_1 + \frac{f_{1LL}}{2} - g_1 & \frac{e^{-i\phi}k}{\sqrt{2}M}\left(f_{1LT} + i f_{1T}^\perp - g_{1T} - i g_{1LT} + h_{1LT}\right) & \frac{e^{-2i\phi}k^2}{M^2}\left(f_{1TT} + i g_{1TT} - h_{1TT}^\perp\right) \\ \frac{e^{i\phi}k}{\sqrt{2}M}\left(f_{1LT} - i f_{1T}^\perp - g_{1T} + i g_{1LT} + h_{1LT}\right) & f_1 - f_{1LL} & -\frac{e^{-i\phi}k}{\sqrt{2}M}(f_{1LT} - i f_{1T}^\perp + g_{1T} - i g_{1LT} + h_1 \\ \frac{e^{2i\phi}k^2}{M^2}\left(f_{1TT} - i g_{1TT} - h_{1TT}^\perp\right) & -\frac{e^{i\phi}k}{\sqrt{2}M}\left(f_{1LT} + i f_{1T}^\perp + g_{1T} + i g_{1LT} + h_{1LT}\right) & f_1 + \frac{f_{1LL}}{2} + g_1 \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{e^{-2i\phi}k^2}{4M^2}\left(2h_1^\perp + h_{1LL}^\perp - 2i h_{1L}^\perp\right) & \frac{e^{-3i\phi}k^3}{2\sqrt{2}M^3}\left(h_{1LT}^\perp + i h_{1T}^\perp\right) & -\frac{e^{-4i\phi}k^4}{2M^4} h_{1TT}^{\perp\perp} \\ -\frac{e^{-i\phi}k}{\sqrt{2}M}\left(2h_{1LT} - i h_1\right) & -\frac{e^{-2i\phi}k^2}{2M^2}\left(h_1^\perp - h_{1LL}^\perp\right) & -\frac{e^{-3i\phi}k^3}{2\sqrt{2}M^3}\left(h_{1LT}^\perp - i h_{1T}^\perp\right) \\ -2h_{1TT} & \frac{e^{-i\phi}k}{\sqrt{2}M}\left(2h_{1LT} + i h_1\right) & -\frac{e^{-2i\phi}k^2}{4M^2}\left(2h_1^\perp + h_{1LL}^\perp + 2i h_{1L}^\perp\right) \end{pmatrix}$$