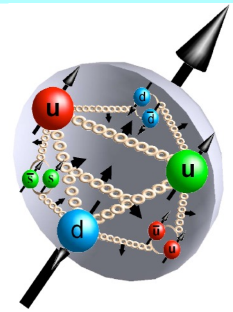
HADRONIC PHYSICS
WITH LEPTON AND HADRON BEAMS
Jefferson Lab, September 5-8, 2017

Lattice QCD effort to calculate parton distributions

Jianwei Qiu
Theory Center, Jefferson Lab

Based on work done with
T. Ishikawa, Y.-Q. Ma, K. Orginos, S. Yoshida, …
and work by many others, …

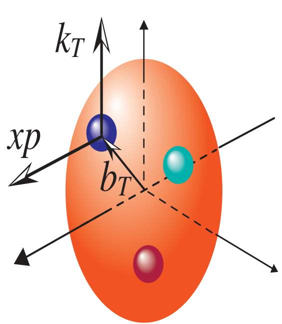
Hadron structure in QCD



- What do we need to know for the structure?
  - ✧ In theory: $\langle P,S|\mathcal{O}(\overline{\psi},\psi,A^\mu)|P,S\rangle$ – Hadronic matrix elements

    with all possible operators: $\mathcal{O}(\overline{\psi},\psi,A^\mu)$
  - ✧ In fact: *None of these matrix elements is a direct physical observable in QCD – color confinement!*
  - ✧ In practice: Accessible hadron structure
    = hadron matrix elements of quarks and gluons, which
    1) can be related to physical cross sections of hadrons and leptons with controllable approximation; and/or
    2) can be calculated in lattice QCD

- Single-parton structure "seen" by a short-distance probe:
  - ✧ 5D structure:

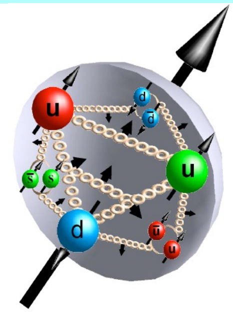


1) $\int d^2b_T \Rightarrow f(x,k_T,\mu)$ – TMDs: *2D confined motion!*

2) $\int d^2k_T \Rightarrow F(x,b_T,\mu)$ – GPDs: *2D spatial imaging!*

3) $\int d^2k_T d^2b_T \Rightarrow f(x,\mu)$ – PDFs: *Number density!*

Hadron structure in QCD



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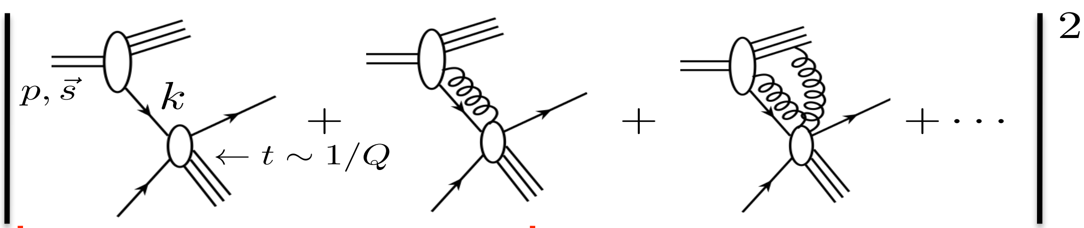
❑ Multi-parton correlations:

$\sigma(Q, \vec{s}) \propto$

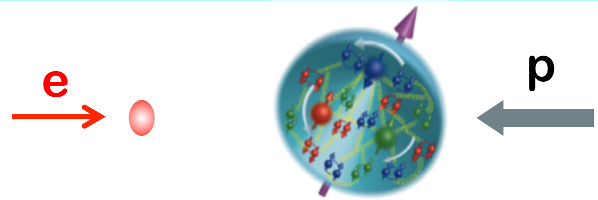


$\left(\frac{\langle k_\perp \rangle}{Q}\right)^n$ – Expansion

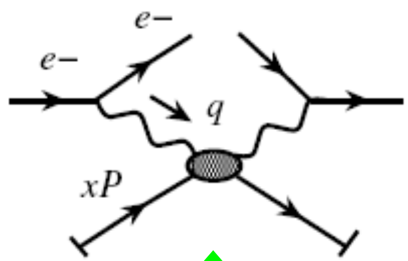
Quantum interference ⟹ 3-parton matrix element – not a probability!

Hard probe and QCD factorization

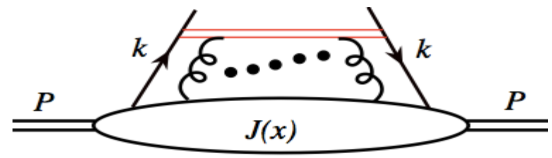
- One hadron:



$\sigma_{\text{tot}}^{\text{DIS}}$:



$\otimes$



$+\, O\left(\dfrac{1}{QR}\right)$

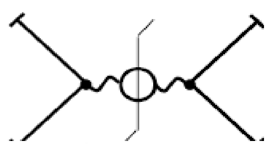
Hard-part
Probe

Parton-distribution
Structure

Power corrections
Approximation

- Two hadrons:

$\sigma_{\text{tot}}^{\text{DY}}$:



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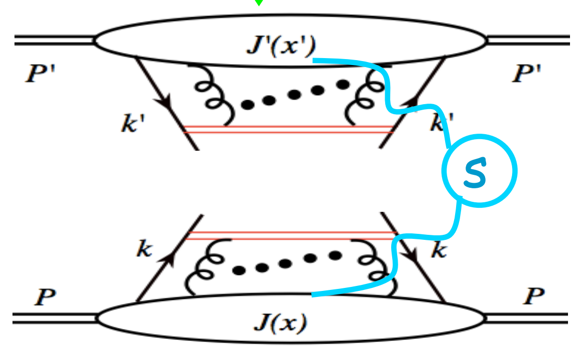

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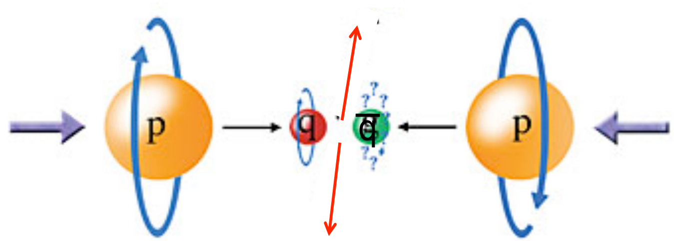


Predictive power:
Universal Parton Distributions

Global QCD analyses – a successful story

- World data with "Q" > 2 GeV

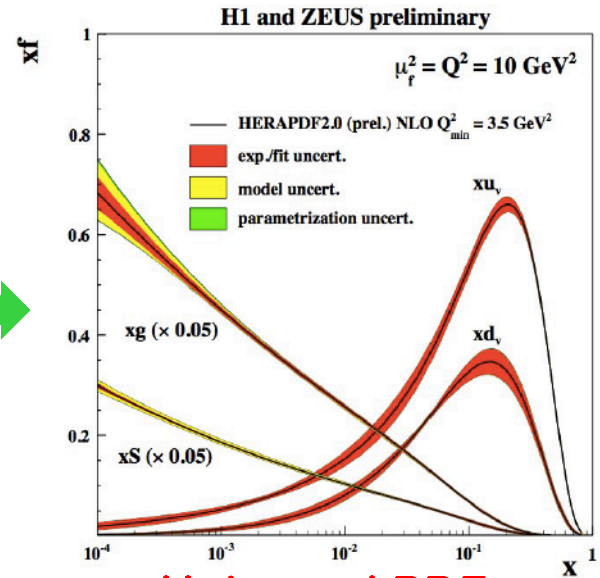
  + Factorization:

DIS: $$F_2(x_B, Q^2) = \Sigma_f C_f(x_B/x, \mu^2/Q^2) \otimes f(x, \mu^2)$$

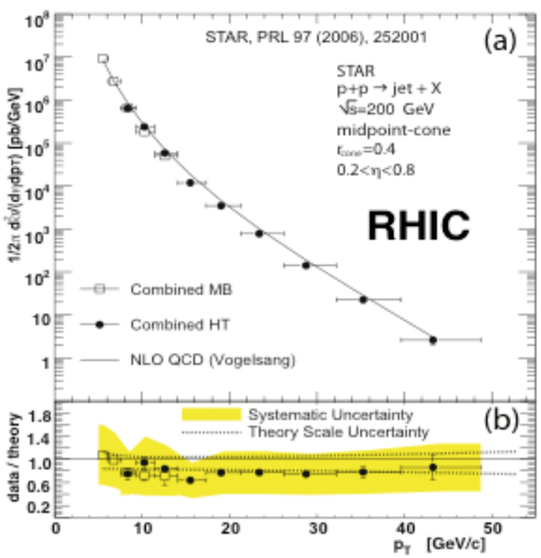
H-H: $$\frac{d\sigma}{dy dp_T^2} = \Sigma_{ff'} f(x) \otimes \frac{d\hat{\sigma}_{ff'}}{dy dp_T^2} \otimes f'(x')$$

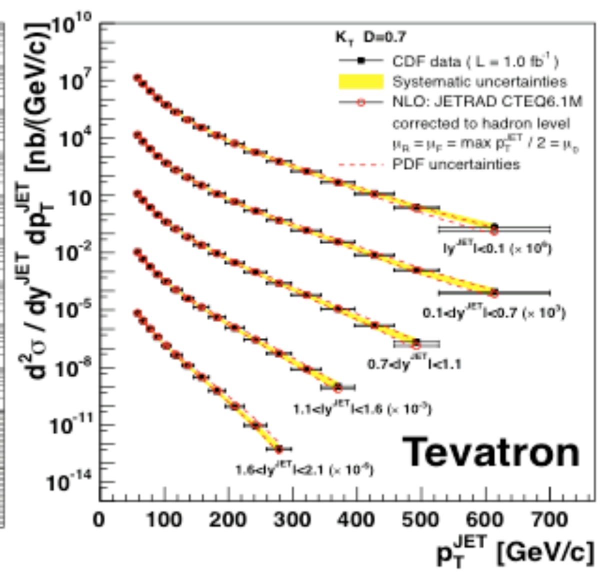
+ DGLAP Evolution:

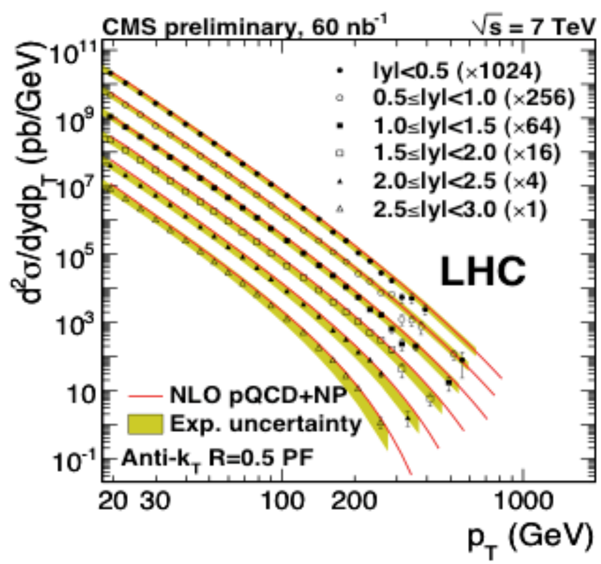
$$\frac{\partial f(x, \mu^2)}{\partial \ln \mu^2} = \Sigma_{f'} P_{ff'}(x/x') \otimes f'(x', \mu^2)$$



Universal PDFs







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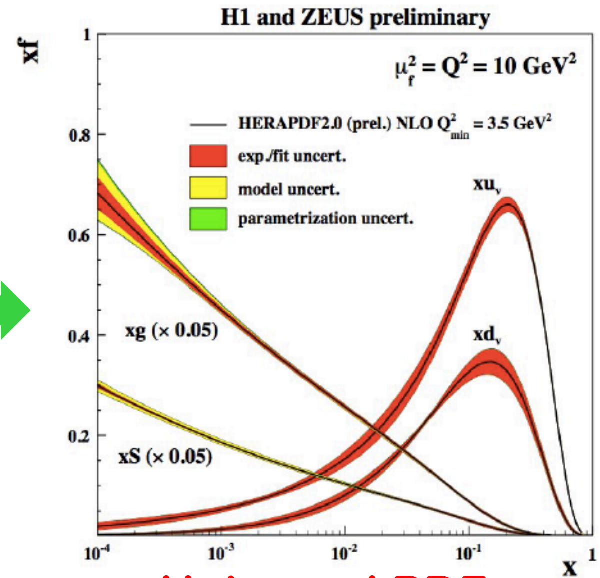


Universal PDFs

- **The "BIG" question(s)**

  Why these PDFs behave as what have been extracted from the fits?

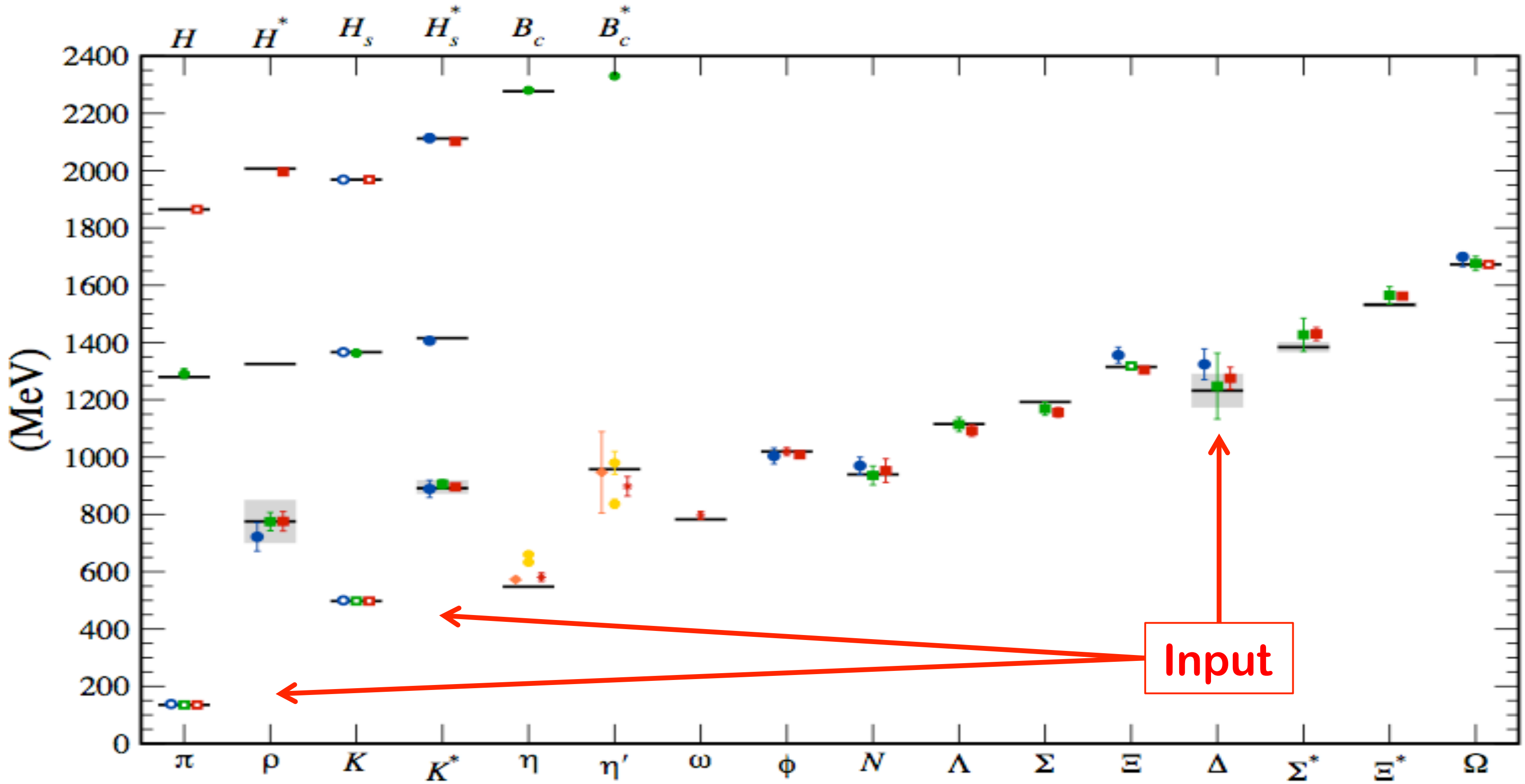
  What have been tested is the evolution from μ_1 to μ_2
  But, does not explain why they have the shape to start with!

  Can QCD calculate and predict the shape of PDFs at the input scale, and other parton correlation functions?

Lattice QCD

- Hadron masses:

Predictions with limited inputs

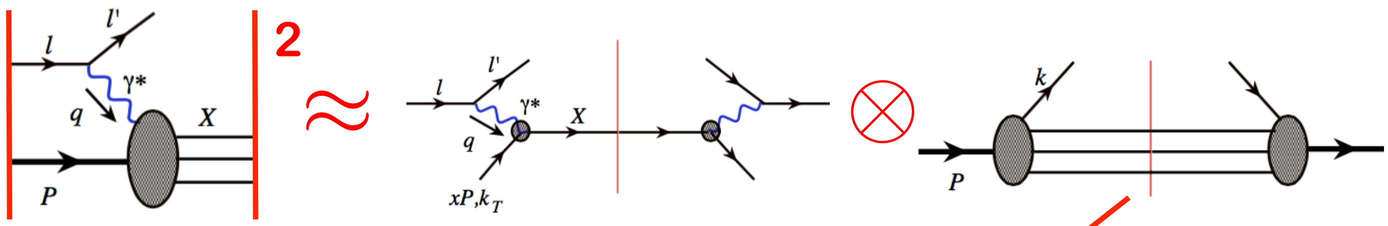


- Lattice "time" is Euclidean: $\tau = i\,t$

Cannot calculate PDFs, TMDs, …, directly, whose operators are time-dependent

Operator definition of PDFs

❑ Definition – from QCD factorization:

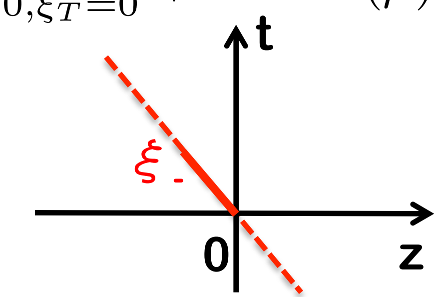


$$+\,\mathcal{O}\left(\frac{\langle k^2\rangle}{Q^2}\right)$$

$$\Phi^{[U]}(x;P,\mu) = \int \frac{d\xi^-}{(2\pi)}\, e^{i\,k\cdot\xi}\, \langle P|\overline{\psi}(0)U(0,\xi)\psi(\xi)|P\rangle_{\xi^+=0,\vec{\xi}_T=0} + \mathrm{UVCT}(\mu)$$

✧ **Depends on the choice of the gauge link:**

$$U(0,\xi) = e^{-ig\int_0^\xi ds^\mu A_\mu}$$

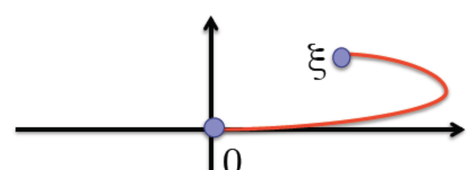


PDFs are not direct physical observables, but, well defined in QCD

❑ Transverse momentum dependent PDFs (TMDs):

$$\Phi^{[U]}(x,k_T;P,\mu) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3}\, e^{i\,k\cdot\xi}\, \langle P|\overline{\psi}(0)U(0,\xi)\psi(\xi)|P\rangle_{\xi^+=0} + \mathrm{UVCT}(\mu)$$

✧ **General gauge link:**

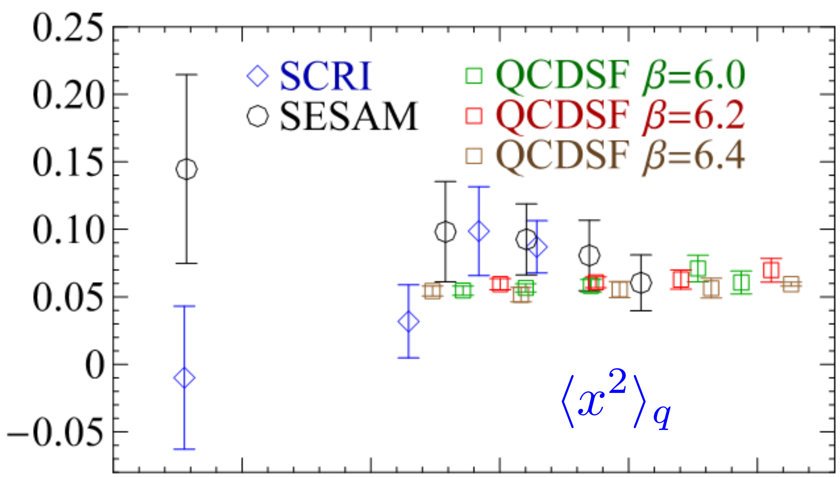


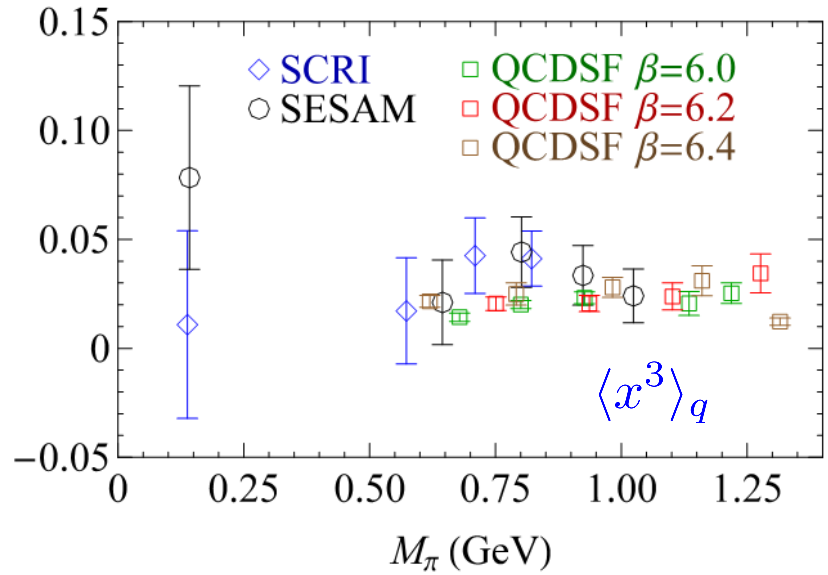
PDFs from lattice QCD

- Moments of PDFs – matrix elements of local operators

$$\langle x^n(\mu^2)\rangle_q \equiv \int_0^1 dx\, x^n\, q(x,\mu^2)$$

- Works, but, hard and limited moments:





Dolgov et al., hep-lat/0201021

Gockeler et al., hep-ph/0410187

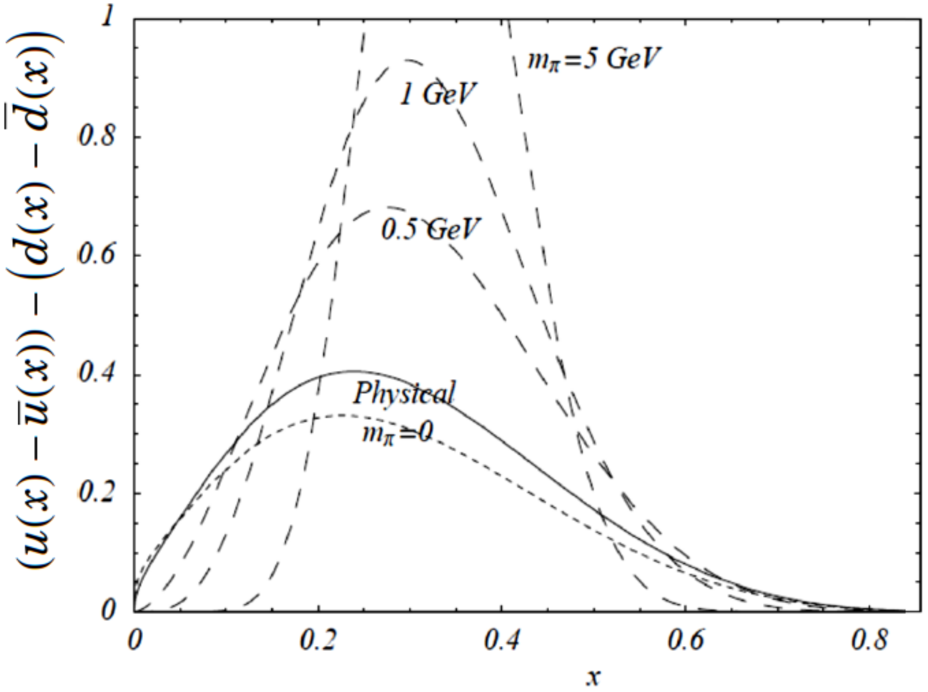
Limited moments – hard to get the full x-dependent distributions!

PDFs from lattice QCD

❑ How to get x-dependent PDFs with a limited moments?

- ✧ Assume a smooth functional form with some parameters
- ✧ Fix the parameters with the lattice calculated moments

$$xq(x) = a\,x^b(1-x)^c(1+\epsilon\sqrt{x}+\gamma\,x)$$



W. Dermold et al., Eur.Phys.J.direct C3 (2001) 1-15

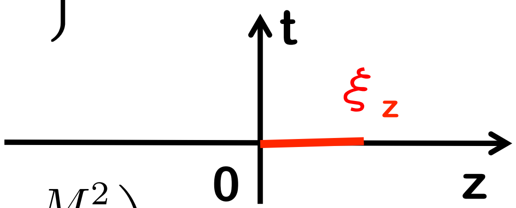
Cannot distinguish valence quark contribution from sea quarks

From quasi-PDFs to PDFs (Ji's idea)

Ji, arXiv:1305.1539

❑ "Quasi" quark distribution (spin-averaged):

$$\tilde{q}(x,\mu^2,P_z) \equiv \int \frac{d\xi_z}{4\pi} e^{-ixP_z\xi_z} \langle P|\overline{\psi}(\xi_z)\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle + \mathrm{UVCT}(\mu^2)$$



❑ Proposed matching:

$$\tilde{q}(x,\mu^2,P_z) = \int_x^1 \frac{dy}{y}\, Z\left(\frac{x}{y},\frac{\mu}{P_z}\right) q(y,\mu^2) + \mathcal{O}\left(\frac{\Lambda^2}{P_z^2},\frac{M^2}{P_z^2}\right)$$

- Size of $O(1/P_z^2)$ terms, non-perturbative subtraction of power divergence
- Mixing with lower dimension operators cannot be treated perturbatively, …

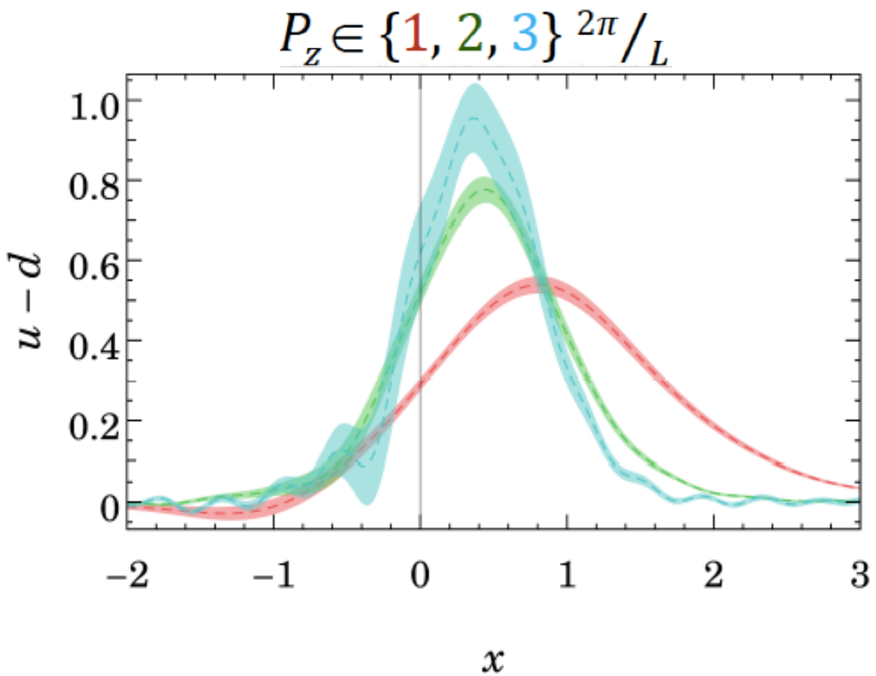
❑ Features:

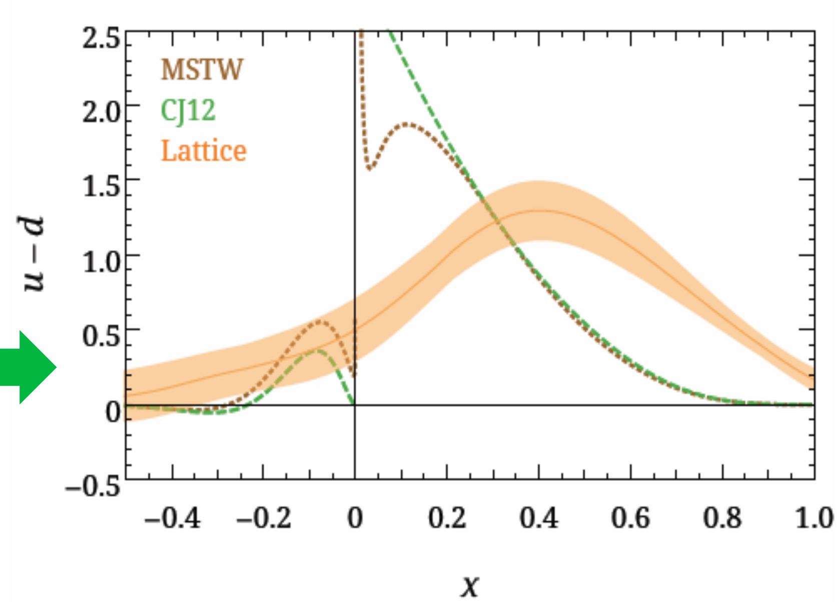
- Quark fields separated along the z-direction – not boost invariant!
- Perturbatively UV power divergent: $\propto (\mu/P_z)^n$ with $n > 0$ - renormalizable?
- Quasi-PDFs could be calculated using standard lattice method
- Quasi-PDFs ➔ *Normal PDFs when P_z ➔ ∞ ?*

Lattice calculation of quasi-PDFs

Lin *et al.*, arXiv:1402.1462

- Exploratory study:





Quasi-Quark Distribution with different P_z → **Predicted quark distribution along with global fitted one**

Matching – taking into account:

Target mass: $(M_N/P_z)^2$
High twist: $a+b/P_z^2$

Our observation

- Quasi-PDFs are NOT defined by "twist-2" operators:

$$\tilde{q}(x,\mu^2,P_z) \equiv \int \frac{d\xi_z}{4\pi} e^{-ixP_z\xi_z} \langle P|\overline{\psi}(\xi_z)\gamma_z \exp\left\{-ig\int_0^{\xi_z} d\eta_z A_z(\eta_z)\right\}\psi(0)|P\rangle + \mathrm{UVCT}(\mu^2)$$

They have power UV divergence! Twist = Dimension - Spin

- Renormalization scale dependence does not obey DGLAP:

$$\mu^2 \frac{d}{d\mu^2}\widetilde{q}(x,\mu^2,P_z) \neq \mathrm{DGLAP}$$

- Questions to ask:
  - ✧ Is the operators defining quasi-PDFs renormalizable – continuum limit?
  - ✧ Does the renormalization mix with other operators? within a close set?
  - ✧ Does renormalized quasi-PDFs and PDFs share the same CO properties?
  - ✧ Reliability to extract PDFs from the renormalized quasi-PDFs?
  - ✧ Lattice calculation: nonperturbative renormalization?
  - ✧ …?
- Extract hadron structure beyond quasi-DPFs?

Our observation

- **Facts:**
  - ✧ PDFs are time-independent, so as the factorized cross sections!
  - ✧ The operators, defining PDFs, located on the light-cone is a consequence of the approximation defining the twist-2 factorization

    More precisely, the collinear approximation

- **Our idea:**
  - ✧ NOT try to calculate PDFs directly from lattice QCD calculations
  - ✧ Identify a set of time-independent (fixed or integrated over time) and *good* single hadron matrix elements:
    - Calculable in lattice QCD
    - Factorizable to PDFs with calculable coefficients with controllable approximations

    Call these matrix elements as "lattice cross sections (LCS)"
  - ✧ Derive PDFs from Global Analysis of "data" on lattice cross sections

    Just like what we do now to extract PDFs from experimental data

Our proposal

Ma and Qiu, 2014, 2017

❑ Lattice cross sections – definition:

$$\sigma_n(\xi^2,\omega,P^2) = \langle P|T\{\mathcal{O}_n(\xi)\}|P\rangle \qquad \omega = P\cdot\xi$$

where the operator is defined as

$$\mathcal{O}_{j_1 j_2}(\xi) \equiv \xi^{d_{j_1}+d_{j_2}-2}\, Z_{j_1}^{-1}\, Z_{j_2}^{-1} j_1(\xi)\, j_2(0)$$

with

d_j : Dimension of the current

Z_j : Renormalization constant of the current

❑ Lattice cross sections – requirements:

- ✧ is calculable in lattice QCD with an Euclidean time
- ✧ has a well-defined continuum limit as the lattice spacing, $a \to 0$ and
- ✧ has the same and factorizable logarithmic CO divergences as PDFs

❑ Lattice cross sections – two-current correlations:

$$j_S(\xi) = \xi^2 Z_S^{-1}[\overline{\psi}_q\psi_q](\xi), \qquad j_V(\xi) = \xi Z_V^{-1}[\overline{\psi}_q\gamma\cdot\xi\psi_q](\xi),$$

$$j_{V'}(\xi) = \xi Z_{V'}^{-1}[\overline{\psi}_q\gamma\cdot\xi\psi_{q'}](\xi), \qquad j_G(\xi) = \xi^3 Z_G^{-1}[-\frac{1}{4}F_{\mu\nu}^c F_{\mu\nu}^c](\xi)\ , \ldots$$

❑ Lattice cross sections – quasi-PDFs:

$$\mathcal{O}_q(\xi) = Z_q^{-1}(\xi^2)\overline{\psi}_q(\xi)\,\gamma\cdot\xi\Phi(\xi,0)\,\psi_q(0) \qquad \Phi(\xi,0) = \mathcal{P}e^{-ig\int_0^1 \xi\cdot A(\lambda\xi)\,d\lambda}$$

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❑ Identify good lattice cross sections:

$$\overline{\sigma}_{\mathrm{E}}^{\mathrm{Lat}}(\xi_z, 1/a, P_z) \xleftrightarrow{\mathcal{Z}} \sigma_{\mathrm{E}}(\xi_z, \tilde{\mu}^2, P_z) \qquad \text{– Renormalization}$$

$$\Updownarrow$$

$$\sigma_{\mathrm{M}}(\xi_z, \tilde{\mu}^2, P_z) \xleftrightarrow{\mathcal{C}} f_i(x, \mu^2) \, , \qquad \text{– Factorization}$$

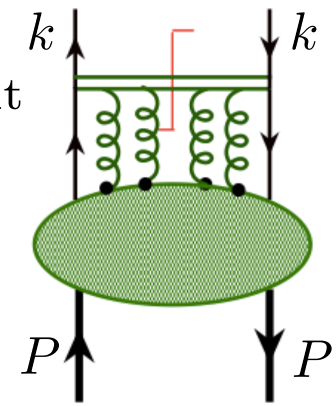
Rest of my talk: using quasi-PDFs as a case study

The case study

❑ **Quasi-quark distribution could be a good LCS:**

$$\tilde{q}(\tilde{x},\tilde{\mu}^2,P_z) = \int \frac{dy_z}{4\pi} e^{i\tilde{x}P_z y_z} \langle P|\overline{\psi}(y_z)\gamma_z \exp\left\{-ig\int_0^{y_z} dy'_z A_z(y'_z)\right\}\psi(0)|P\rangle$$

✧ **Feynman diagram representation:** $\Phi_{n_z}^{(f,a)}(\{\xi_z,0\}) = \Phi_{n_z}^{\dagger(f,a)}(\{\infty,\xi_z\})\,\Phi_{n_z}^{(f,a)}(\{\infty,0\})$

$\Sigma_{\rm cut}$  $\longleftarrow \frac{\gamma_z}{2P_z}\delta\left(\tilde{x}-\frac{k_z}{P_z}\right)$

$P_z \sim \text{“}\sqrt{s}\text{”} \sim \tilde{\mu}/\tilde{x}$ ***Sufficiently large***

$P^2 \sim 0$

✧ **Like PDFs, it is IR finite**

✧ **Like PDFs, it is UV divergent, but, worse (linear UV divergence)**

Potential trouble! - need to show that it is multiplicative renormalizable?

✧ **Like PDFs, it is CO divergent – factorizes CO divergence into PDFs**

Show to all orders in perturbation theory?

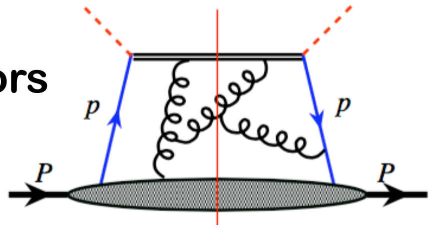
Renormalization

Ishikawa, et al. arXiv:1707.03107

❑ Different from PDFs:

✧ **PDFs – moments – twist-2 operators:**

Twist-2 operators



$$\overline{\psi}(\xi^-)\gamma^+\Phi_n(\xi^-,0)\psi(0) = \sum_m \frac{(i\xi^-)^m}{m!}\mathcal{O}^{\mu_1\ldots\mu_m}(0)n_{\mu_1}\ldots n_{\mu_m}$$

Moments of PDFs ⟷ Matrix-elements of twist-2 operators

Renormalization of PDFs ⟷ Renormalization of twist-2 operators

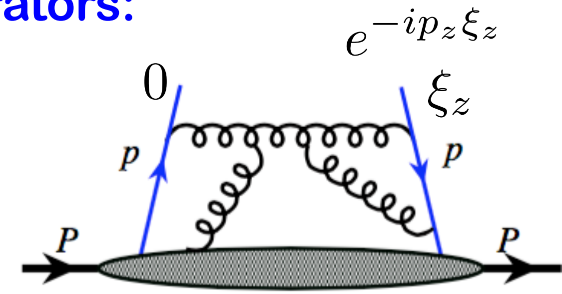
Mixing of all twist-2 operators

✧ **Quasi-PDFs – NO moments – NOT by twist-2 operators:**

In $A\cdot n_z = 0$, NO gauge link!

Renormalization of QCD in $A\cdot n_z = 0$ gauge

NO guarantee for quasi-PDFs renormalization



✧ **Most challenge part of quasi-PDFs renormalization:**

Renormalization of the bi-local/composite operators!

❑ Conclusion from arXiv:1707.03107:

$$\tilde{F}^R_{i/p}(\xi_z,\tilde{\mu}^2,p_z) = e^{-C_i|\xi_z|}Z^{-1}_{wi}Z^{-1}_{vi}\tilde{F}^b_{i/p}(\xi_z,\tilde{\mu}^2,p_z).$$

No mix with other flavors or gluon!

Renormalization

❑ Coordinate-space definition:

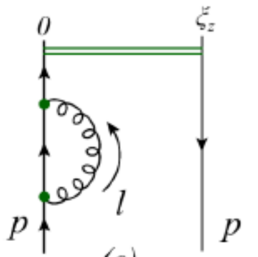
$$\tilde{F}_{q/p}(\xi_z, \tilde{\mu}^2, p_z) = \frac{e^{ip_z\xi_z}}{p_z} \langle h(p)|\overline{\psi}_q(\xi_z)\, \frac{\gamma_z}{2} \Phi_{n_z}^{(f)}(\{\xi_z, 0\})\, \psi_q(0)|h(p)\rangle$$

❑ Why the proof is hard:

- Because of z-direction dependence, Lorentz symmetry is broken, hard to exhaust all possible UV divergences
- Renormalization of composite operator is needed

❑ Broken Lorentz symmetry:

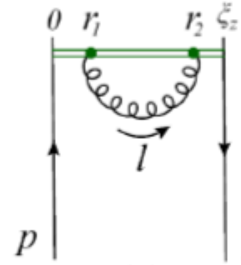
Both 3D and 4D loop-integration can generate UV divergences



(c)

UV: 4-D integration

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 (p-l)^2}$$



(a)

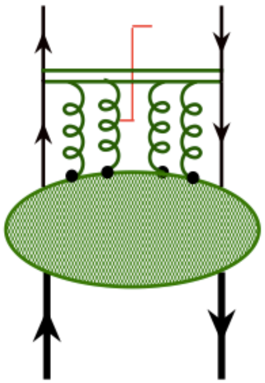
$$l^\mu = \bar{l}^\mu + l_z n_z^\mu$$

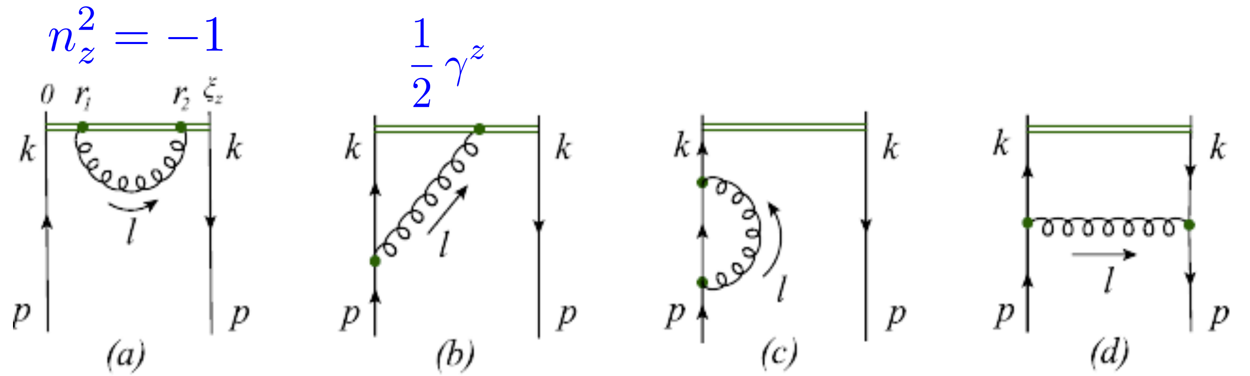
UV: 3-D integration

$$\int \frac{d^3\bar{l}}{l^2} = \int \frac{d^3\bar{l}}{\bar{l}^2 - l_z^2}$$

Renormalization

- Quasi-quark at one-loop:





- Fig. 1(a):

$$M_{1a} = \frac{e^{ip_z\xi_z}}{p_z}\frac{1}{N_c}\mathrm{Tr}_c[T^aT^a]\int_a^{\xi_z-2a} dr_1 \int_{r_1+a}^{\xi_z-a} dr_2$$
$$\times \int \frac{d^4l}{(2\pi)^4} e^{-ip_z\xi_z}\, e^{il_z(r_2-r_1)}\left(\frac{-ig_{\mu\nu}}{l^2}\right)$$
$$\times (-ig_s n_z^\mu)(-ig_s n_z^\nu)\mathrm{Tr}\left[\frac{1}{2}\not{p}\,\frac{1}{2}\gamma_z\right]$$
$$= \frac{\alpha_s C_F}{4i\pi^3}\int_a^{\xi_z-2a} dr_1 \int_{r_1+a}^{\xi_z-a} dr_2 \int d^4l \frac{e^{il_z(r_2-r_1)}}{l^2}$$

$$M_{1a} \overset{\mathrm{div}}{=} -\frac{\alpha_s C_F}{\pi}\frac{|\xi_z|}{a} + \frac{\alpha_s C_F}{\pi}\ln\frac{|\xi_z|}{a}$$

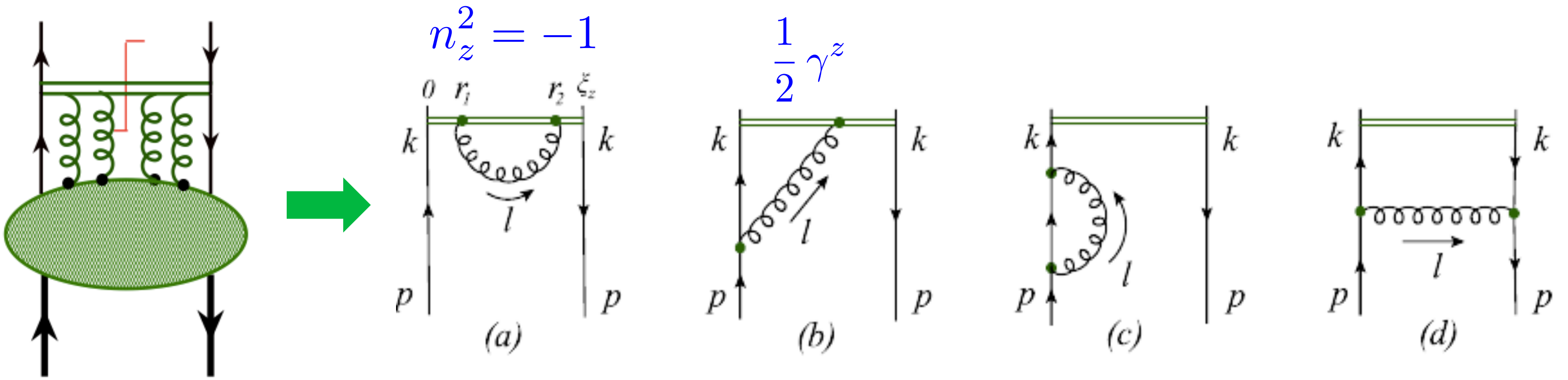
- ✧ Cutoff "a" between fields
- ✧ Conclusion independent of regulator
- ✧ 3D-integration: $d^4l = d^3\bar{l}\,dl_z$

$$\int \frac{d^3\bar{l}}{l^2} = \int \frac{d^3\bar{l}}{\bar{l}^2 - l_z^2}$$
$$= \int d^3\bar{l}\left(\frac{1}{\bar{l}^2} + \frac{l_z^2}{(\bar{l}^2 - l_z^2)\bar{l}^2}\right)$$
$$\int dl_z e^{il_z(r_2-r_1)} = 2\pi\delta(r_2 - r_1)$$

1st term vanishes for $r_1 \neq r_2$

Renormalization

- Quasi-quark at one-loop:



- Complete one-loop contribution:

$$M^{(1)} \overset{\text{div}}{=} M_{1a} + 2\times M_{1b} + 2\times\frac{1}{2}M_{1c} + M_{1d}$$
$$= \frac{\alpha_s C_F}{\pi}\left(-\frac{|\xi_z|}{a} + 2\ln\frac{|\xi_z|}{a} - \frac{1}{4\epsilon}\right).$$

✧ At one-loop, all 3D integrations are finite
✧ Divergence only come from the region when all momentum components go to infinity

Localized UV divergence in all directions!

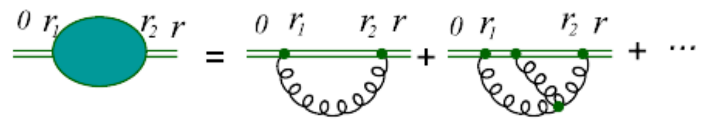
Very different from the UV behavior of normal PDFs: $(1, \lambda^2, \lambda)$, $\lambda \rightarrow \infty$

Renormalization

Ishikawa, Ma, Qiu, Yoshida (2017)

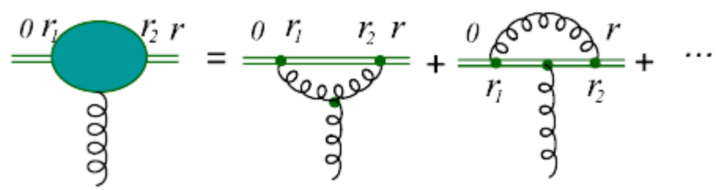
❑ Power counting and divergent sub-diagrams:

(a) - 1/a, ln(1/a):



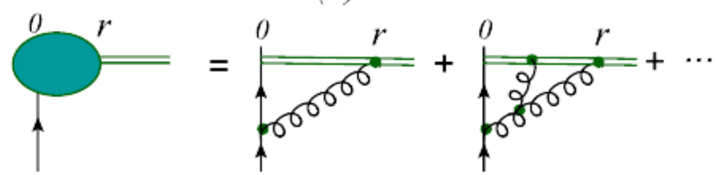
(a)

(b) - ln(1/a):



(b)

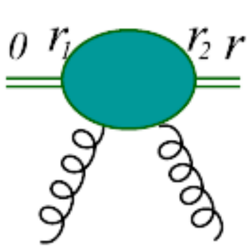
(c) - ln(1/a):

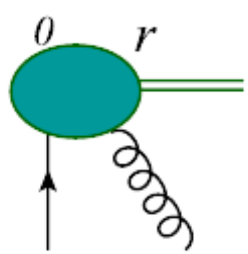


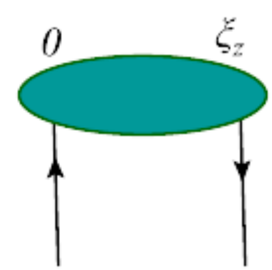
(c)

Happen only when all loop momenta go to infinity – localized!

❑ Example of convergent sub-diagrams:



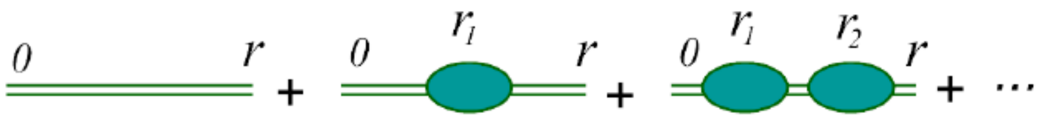




Renormalized

Ishikawa, Ma, Qiu, Yoshida (2017)

❑ Power divergence:



$$1 + c\int_0^r dr_1 + c^2 \int_0^r dr_1 \int_{r_1}^r dr_2 + \cdots$$
$$= \mathcal{P} e^{\,c\int_0^r dr'} = e^{\,c\,r}\,,$$

- It is allowed to introduce an overall factor $e^{-c|\xi_z|}$ to remove all power UV divergences

❑ Interpretation:

- Mass renormalization of test particle

Dotsenko, Vergeles, NPB (1980)

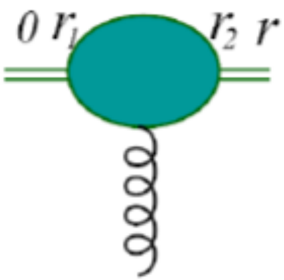
❑ Log divergence in from gauge link:

- Besides power divergence, there are also logarithmic UV divergences
- It is known that these divergences can be removed by a "wave function" renormalization of the test particle, Z_{wq}^{-1}.

Renormalized

Ishikawa, Ma, Qiu, Yoshida (2017)

❑ Log divergence from gluon-gauge link vertex:



- Logarithmic UV: can be absorbed by the coupling constant renormalization of QCD.

❑ UV from vertex correction:

- The most dangerous UV diagram, may mix with other operators
- Locality of UV divergence: no dependence on $r_2 - r_1$ or p
- UV divergence is proportional to quark-gaugelink vertex at lowest order, with a constant coefficient
- A constant counter term is able to remove this UV divergence.

❑ Renormalization to all orders:

- Using bookkeeping forests subtraction method, the net effect is to introduce a constant multiplicative renormalizaton factor $Z_{vq}^{-1}$for the quark-gaugelink vertex.

Renormalized

Ishikawa, Ma, Qiu, Yoshida (2017)

- With renormalized QCD Lagrangian:
  - All UV divergences (too all orders) can be removed by the following renormalization

$$\tilde{F}^{R}_{i/p}(\xi_z, \tilde{\mu}^2, p_z) = e^{-C_i|\xi_z|} Z_{wi}^{-1} Z_{vi}^{-1} \tilde{F}^{b}_{i/p}(\xi_z, \tilde{\mu}^2, p_z)$$

- Renormalization:

  Multiplicative factor – not mix with other operators

  - Significantly different from normal PDFs

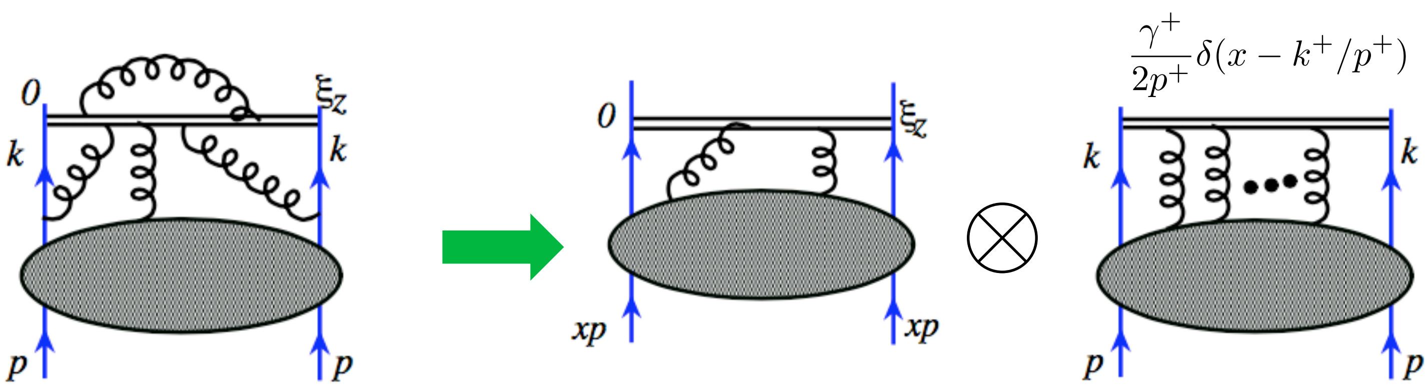
- Quasi-quark PDF could be a good "lattice cross sections"

  If it can be factorized into PDFs

Factorization

- ❑ Does the renormalized quasi-PDFs and PDFs share the same CO properties?

- ❑ Can we extract PDFs from renormalized quasi-PDFs reliably?



$+$ Corrections

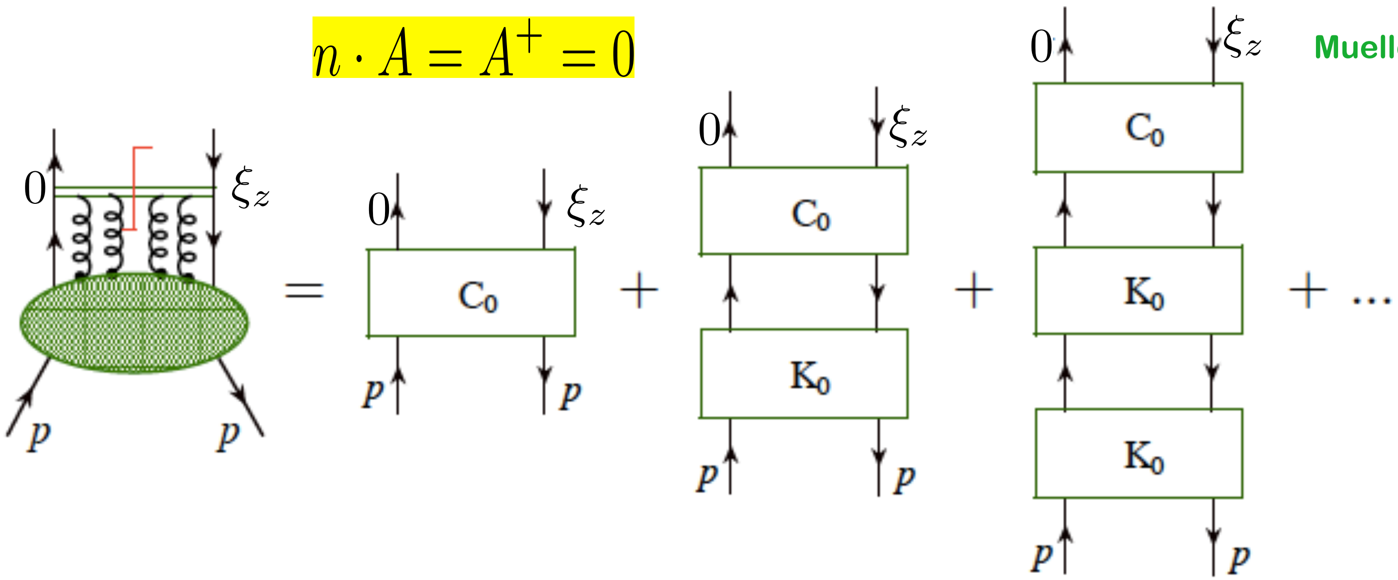
Factorization of CO divergence

Ma and Qiu, arXiv:1404.6860

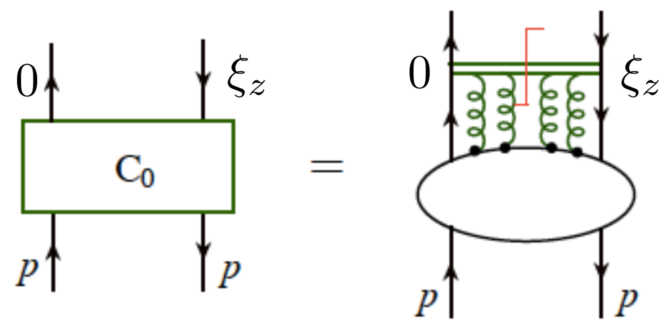
- Generalized ladder decomposition in a physical gauge

$n \cdot A = A^{+} = 0$

Mueller, PRD 1974



- $C_0, \;\; K_0$:2PI kernels
  - ✧ Only process dependence:

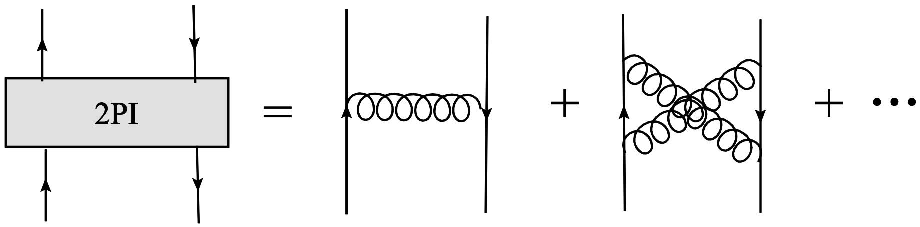


  - ✧ 2PI are finite in a physical gauge for fixed *k* and *p*:

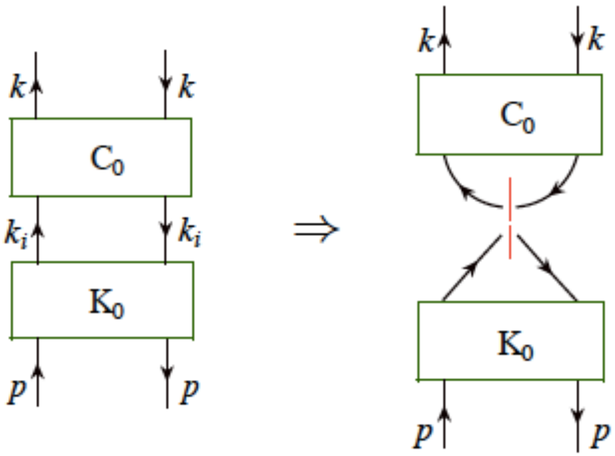
Ellis, Georgi, Machacek, Politzer, Ross, 1978, 1979

Factorization of CO divergence

❑ 2PI kernels – Diagrams:



❑ Ordering in virtuality: $P^2 \ll k^2 \lesssim \tilde{\mu}^2$ – Leading power in $\dfrac{1}{\tilde{\mu}}$



$\leftarrow \frac{1}{2}\gamma\cdot p$

$\leftarrow \frac{\gamma\cdot n}{2p\cdot n}\,\delta\left(x_i - \frac{k_i\cdot n}{p\cdot n}\right)$ **+ power suppressed**

Cut-vertex for normal quark distribution
Logarithmic UV and CO divergence

❑ Renormalized kernel - UV & IR safe - parton PDF:

$$K \equiv \int d^4k_i\,\delta\left(x_i - \frac{k^+}{p^+}\right)\mathrm{Tr}\left[\frac{\gamma\cdot n}{2p\cdot n}\,K_0\,\frac{\gamma\cdot p}{2}\right] + \mathrm{UVCT}_{\mathrm{Logarithmic}}$$

Factorization of CO divergence

- Projection operator for CO divergence:

$\widehat{\mathcal{P}}\, K$ Pick up the logarithmic CO divergence of *K*

- Factorization of CO divergence:

$$\tilde{f}_{q/p} = \lim_{m\to\infty} C_0 \sum_{i=0}^{m} K^i + \mathrm{UVCTs}$$

If multiplicative

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=0}^{m-1} K^i (1-\widehat{\mathcal{P}})K\right]_{\mathrm{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K$$

$$= \lim_{m\to\infty} C_0 \left[1 + \sum_{i=1}^{m} \left[(1-\widehat{\mathcal{P}})K\right]^i\right]_{\mathrm{ren}} + \tilde{f}_{q/p}\,\widehat{\mathcal{P}}\,K$$

$$\Rightarrow \quad \tilde{f}_{q/P} = \left[C_0 \frac{1}{1-(1-\widehat{\mathcal{P}})K}\right]_{\mathrm{ren}} \left[\frac{1}{1-\widehat{\mathcal{P}}K}\right]$$

Normal Quark distribution

CO divergence free

All CO divergence of quasi-quark distribution

$$\Rightarrow \quad \tilde{f}_{i/h}(\tilde{x},\tilde{\mu}^2,P_z) \approx \sum_j \int_0^1 \frac{dx}{x}\, \mathcal{C}_{ij}\!\left(\frac{\tilde{x}}{x},\tilde{\mu}^2,P_z\right) f_{j/h}(x,\mu^2) + \text{Power corrections}$$

One-loop example: quark →quark

Ma and Qiu, arXiv:1404.6860

- **Expand the factorization formula:**

$$\tilde{f}_{i/h}(\tilde{x},\tilde{\mu}^2,P_z) \approx \sum_j \int_0^1 \frac{dx}{x}\, \mathcal{C}_{ij}\!\left(\frac{\tilde{x}}{x},\tilde{\mu}^2,P_z\right) f_{j/h}(x,\mu^2)$$

To order α_s :

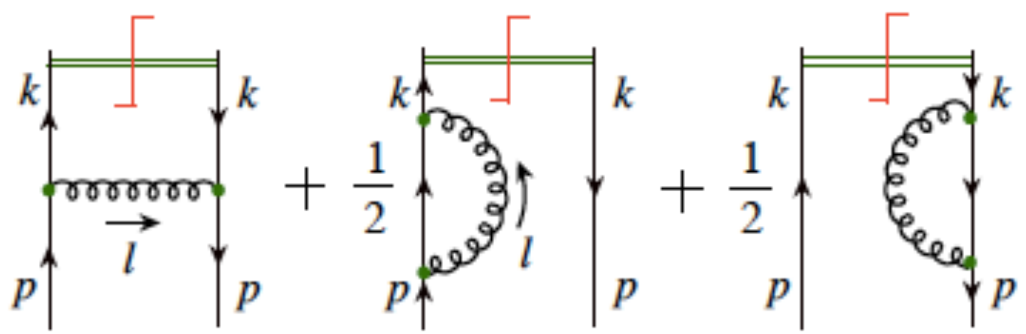
$$\tilde{f}^{(1)}_{q/q}(\tilde{x}) = f^{(0)}_{q/q}(x) \otimes \mathcal{C}^{(1)}_{q/q}(\tilde{x}/x) + f^{(1)}_{q/q}(x) \otimes \mathcal{C}^{(0)}_{q/q}(\tilde{x}/x)$$

$$\Rightarrow \quad \mathcal{C}^{(1)}_{q/q}(t,\tilde{\mu}^2,\mu^2,P_z) = \tilde{f}^{(1)}_{q/q}(t,\tilde{\mu}^2,P_z) - f^{(1)}_{q/q}(t,\mu^2)$$

- **Feynman diagrams:**

Same diagrams for both

$\tilde{f}_{q/q}$ and $f_{q/q}$



But, in different gauge: $n_z \cdot A = 0$ for $\tilde{f}_{q/q}$ $\qquad n \cdot A = 0$ for $f_{q/q}$

- **Gluon propagator in n_z . A = 0:**

$$\tilde{d}^{\alpha\beta}(l) = -g^{\alpha\beta} + \frac{l^\alpha n_z^\beta + n_z^\alpha l^\beta}{l_z} - \frac{n_z^2\, l^\alpha l^\beta}{l_z^2}$$

with $n_z^2 = -1$

One-loop "quasi-quark" distribution in a quark

Ma and Qiu, arXiv:1404.6860

❑ Real + virtual contribution:

$$\tilde{f}^{(1)}_{q/q}(\tilde{x},\tilde{\mu}^2,P_z) = C_F\frac{\alpha_s}{2\pi}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\int_0^{\tilde{\mu}^2}\frac{dl_\perp^2}{l_\perp^{2+2\epsilon}}\int_{-\infty}^{+\infty}\frac{dl_z}{P_z}\left[\delta\left(1-\tilde{x}-y\right)-\delta\left(1-\tilde{x}\right)\right]\left\{\frac{1}{y}\left(1-y+\frac{1-\epsilon}{2}y^2\right)\right.$$
$$\left.\times\left[\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}}\right]+\frac{(1-y)\lambda^2}{2y^2\sqrt{\lambda^2+y^2}}+\frac{\lambda^2}{2y\sqrt{\lambda^2+(1-y)^2}}+\frac{1-\epsilon}{2}\frac{(1-y)\lambda^2}{[\lambda^2+(1-y)^2]^{3/2}}\right\}$$

where $y = l_z/P_z,\ \lambda^2 = l_\perp^2/P_z^2,\ C_F = (N_c^2-1)/(2N_c)$

❑ Cancelation of CO divergence:

$$\frac{y}{\sqrt{\lambda^2+y^2}}+\frac{1-y}{\sqrt{\lambda^2+(1-y)^2}} = 2\theta(0<y<1)-\left[\mathrm{Sgn}(y)\frac{\sqrt{\lambda^2+y^2}-|y|}{\sqrt{\lambda^2+y^2}}+\mathrm{Sgn}(1-y)\frac{\sqrt{\lambda^2+(1-y)^2}-|1-y|}{\sqrt{\lambda^2+(1-y)^2}}\right]$$

Only the first term is CO divergent for $0 < y < 1$, which is the same as the divergence of the normal quark distribution – necessary!

❑ UV renormalization:

Different treatment for the upper limit of $l_\perp^2$ integration - "scheme"

Here, a UV cutoff is used – other scheme is discussed in the paper

One-loop coefficient functions

Ma and Qiu, arXiv:1404.6860

- MS scheme for $f_{q/q}(x,\mu^2)$:

$$\mathcal{C}^{(1)}_{q/q}(t,\tilde{\mu}^2,\mu^2,P_z) = \tilde{f}^{(1)}_{q/q}(t,\tilde{\mu}^2,P_z) - f^{(1)}_{q/q}(t,\mu^2)$$

CO, UV IR finite!

$$\frac{\mathcal{C}^{(1)}_{q/q}(t)}{C_F\frac{\alpha_s}{2\pi}} = \left[\frac{1+t^2}{1-t}\ln\frac{\tilde{\mu}^2}{\mu^2}+1-t\right]_+ + \left[\frac{t\Lambda_{1-t}}{(1-t)^2}+\frac{\Lambda_t}{1-t}+\frac{\mathrm{Sgn}(t)\Lambda_t}{\Lambda_t+|t|}\right.$$
$$\left.-\frac{1+t^2}{1-t}\left[\mathrm{Sgn}(t)\ln\left(1+\frac{\Lambda_t}{2|t|}\right)+\mathrm{Sgn}(1-t)\ln\left(1+\frac{\Lambda_{1-t}}{2|1-t|}\right)\right]\right]_N$$

where $\Lambda_t = \sqrt{\tilde{\mu}^2/P_z^2+t^2}-|t|$, $\mathrm{Sgn}(t)=1$ if $t\geq 0$, and -1 otherwise.

- Generalized "+" description: $t=\tilde{x}/x$

$$\int_{-\infty}^{+\infty} dt\left[g(t)\right]_N h(t) = \int_{-\infty}^{+\infty} dt\, g(t)\,[h(t)-h(1)]$$

For a testing function $h(t)$

Explicit verification of the CO factorization at one-loop

Note: $\Lambda_t \to \mathcal{O}\left(\frac{\tilde{\mu}}{P_Z}\right)$ as $P_Z\to\infty$ the linear power UV divergence!

Go beyond quasi-PDFs

- Recall: good lattice cross sections – time independent!

$$\sigma_n(\xi^2, \omega, P^2) = \langle P | T\{\mathcal{O}_n(\xi)\} | P \rangle \qquad \omega = P \cdot \xi$$

$$\mathcal{O}_{j_1 j_2}(\xi) \equiv \xi^{d_{j_1}+d_{j_2}-2} \, Z_{j_1}^{-1} \, Z_{j_2}^{-1} j_1(\xi) \, j_2(0)$$

With renormalized and/or conserved currents – No power divergence!

✧ **Lattice calculable:**

Calculable using lattice QCD with an Euclidean time

✧ **UV Renormalizable:**

Ensure a well-defined continuum limit, UV & IR finite!

✧ **CO Factorizable:**

Share the same perturbative collinear divergences with PDFs

Factorizable to PDFs with IR-safe hard coefficients

with controllable power corrections

P and ξ define the "collision" kinematics – 1/$\xi \sim \mu$ defines the hard scale

to ensure the necessary condition for the factorization

Summary and outlook

- “lattice cross sections” = single hadron matrix elements
  calculable in Lattice QCD, renormalizable + factorizable in QCD

Going beyond the quasi-PDFs

- Extract PDFs by global analysis of data on “Lattice x-sections”.
  Same should work for other distributions (TMDs, GPDs)

$$\widetilde{\sigma}_{\rm E}^{\rm Lat}(\tilde{x}, \frac{1}{a}, P_z) \approx \sum_i \int_0^1 \frac{dx}{x} f_{i/h}(x, \mu^2)\, \widetilde{C}_i(\frac{\tilde{x}}{x}, \frac{1}{a}, \mu^2, P_z)$$

- Conservation of difficulties – complementarity:
  High energy scattering experiments
  – less sensitive to large x parton distribution/correlation
  “Lattice factorizable cross sections”
  – more suited for large x PDFs, but limited to large x for now

- Quasi-PDFs are renormalizable & factorizable

- Lattice QCD can be used to study hadron structure, but,
  more works are needed!

Thank you!

BACKUP SLIDES

"Quasi-PDFs" have no parton interpretation

- **Normal PDFs conserve parton momentum:**

$$M = \sum_q \left[\int_0^1 dx\, x f_q(x) + \int_0^1 dx\, x f_{\bar{q}}(x)\right] + \int_0^1 dx\, x f_g(x)$$
$$= \sum_q \int_{-\infty}^{\infty} dx\, x f_q(x) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, x f_g(x)$$
$$= \frac{1}{2(P^+)^2}\langle P|T^{++}(0)|P\rangle = \text{constant}$$

$T^{\mu\nu}$
Energy-momentum tensor

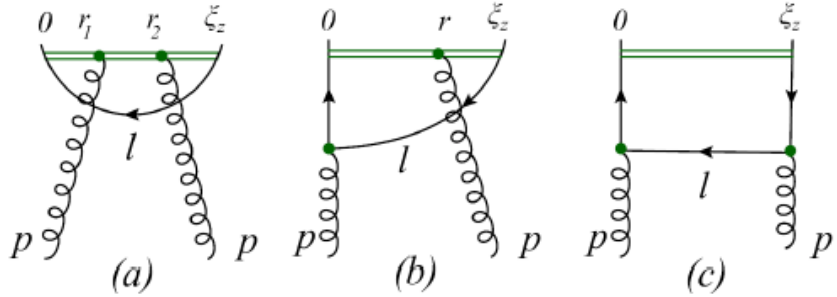
- **"Quasi-PDFs" do not conserve "parton" momentum:**

$$\widetilde{\mathcal{M}} = \sum_q \left[\int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_q(\tilde{x}) + \int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_{\bar{q}}(\tilde{x})\right] + \int_0^\infty d\tilde{x}\, \tilde{x} \tilde{f}_g(\tilde{x})$$
$$= \sum_q \int_{-\infty}^{\infty} d\tilde{x}\, \tilde{x} \tilde{f}_q(\tilde{x}) + \frac{1}{2}\int_{-\infty}^{\infty} d\tilde{x}\, \tilde{x} \tilde{f}_g(\tilde{x})$$
$$= \frac{1}{2(P_z)^2}\langle P|\left[T^{zz}(0) - g^{zz}(...)\right]|P\rangle \neq \text{constant}$$

Note: "Quasi-PDFs" are not boost invariant

Renormalization

❑ Gluon-to-quark at one-loop:



$$M_{2a} \propto \int_0^{\xi_z} dr_1 \int_{r_1}^{\xi_z} dr_2 \int d^4 l\, e^{-il_z\xi_z} \frac{l_z}{l^2}$$
$$= \frac{\xi_z^2}{2} \int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l} \left(\frac{1}{\bar{l}^2} + \frac{l_z^2}{(\bar{l}^2 - l_z^2)\bar{l}^2} \right)$$

- UV divergence from 3-D $\propto \delta'(\xi_z)$, vanishes for finite ξ_z

❑ Caution for momentum-space version:

Finite-term:

$$\frac{\xi_z^2}{2} \int dl_z\, e^{-il_z\xi_z}\, l_z \int d^3\bar{l} \frac{l_z^2}{(\bar{l}^2 - l_z^2)\bar{l}^2}$$
$$\propto \frac{\xi_z^2}{2} \int dl_z\, e^{-il_z\xi_z} \frac{l_z^3}{|l_z|} = \frac{2i}{\xi_z}$$

- Divergent as $\xi_z \to 0$
- Result in bad large $\tilde{x}$ behavior in momentum space