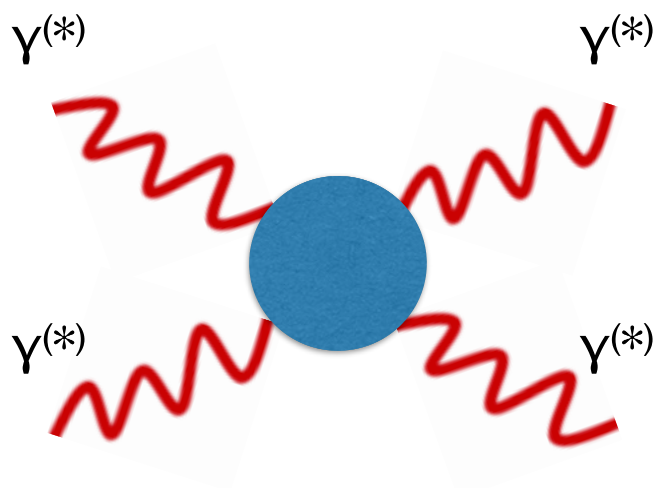



# *Dispersive approaches to hadronic light-by-light scattering and the muon (g-2)*

**Igor Danilkin**

in coll. with Marc Vanderhaeghen

September 7, 2017

SFB 1044
THE LOW-ENERGY FRONTIER
OF THE STANDARD MODEL

JOHANNES GUTENBERG
UNIVERSITÄT MAINZ

JG|U

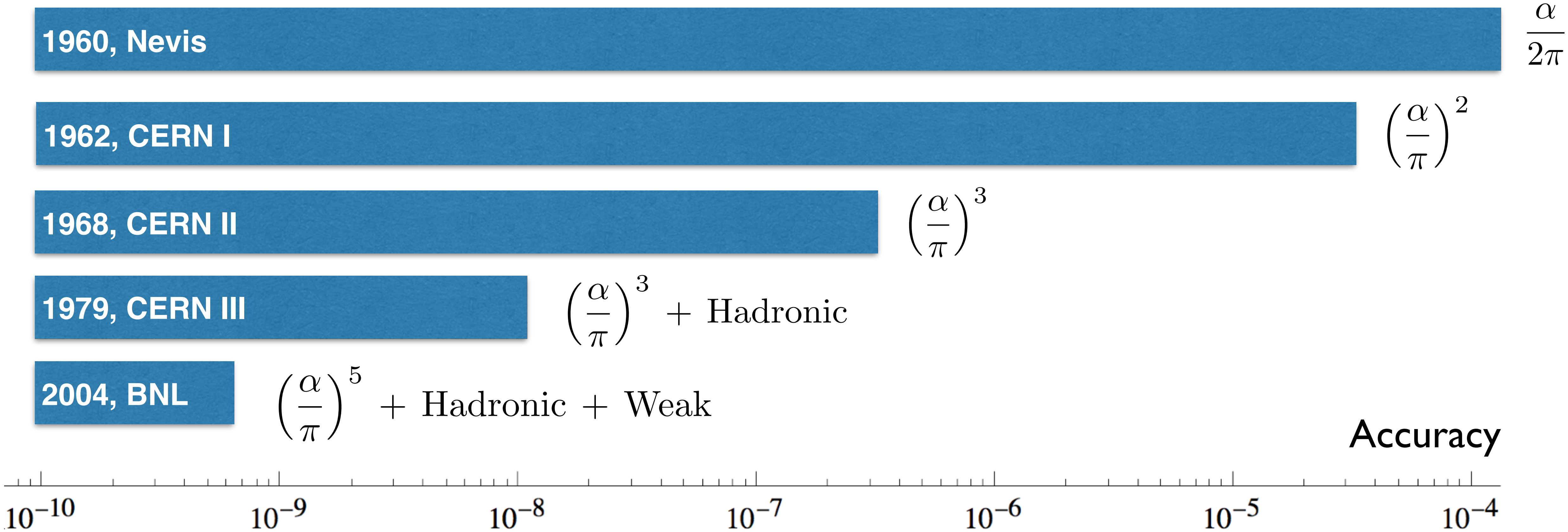
# (g-2) of the muon

- Magnetic moment of the muon

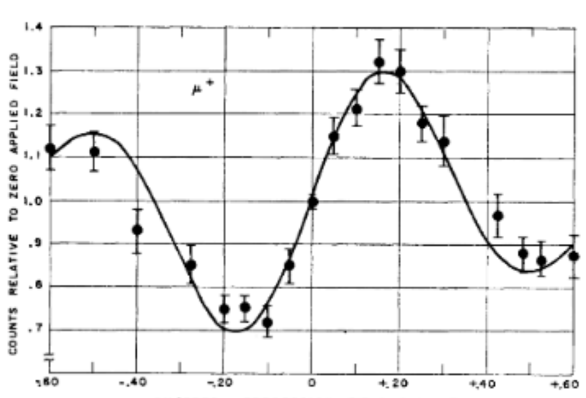
$$\vec{\mu} = \frac{Q}{2m}\, g\, \vec{S}$$

- anomalous part

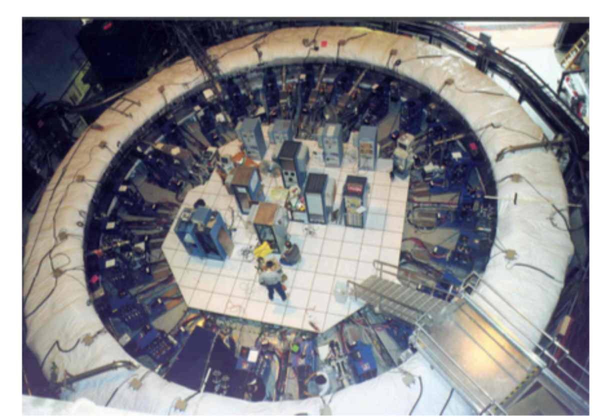
$$a_\mu = \frac{(g-2)_\mu}{2}$$




**Nevis**




**CERN I**



**Brookhaven**

# (g-2) theory vs exp

Experiment:

$$a_\mu^{exp} = (11\,659\,208.9 \pm 6.3) \times 10^{-10}$$

BNL, (2006)
PRD 73 072003

Theory:

$$a_\mu^{SM} = (11\,659\,182.8 \pm 4.9) \times 10^{-10}$$

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J. Ph. G 38, 085003

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$$a_\mu^{exp} - a_\mu^{SM} =$$
$$(26.1 \pm 4.9_{th} \pm 6.3_{exp}) \times 10^{-10}$$

3 - 4 σ
deviation !

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$$1.6_{exp}$$

**3 - 4 σ deviation !**

FNAL, J-PARC experiments

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? $\quad 1.6_{exp}$

3 - 4 σ
deviation !

FNAL, J-PARC
experiments

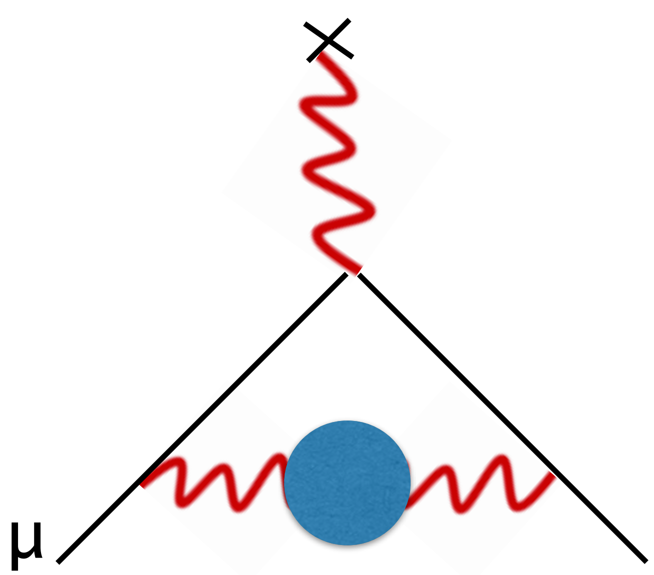
# QCD contribution to (g-2)

$$a_\mu^{QCD} = (695.6 \pm 4.9) \times 10^{-10}$$

Hagiwara (2011)
Jegerlehnner (2015)

## Hadronic vacuum polarization

$$a_\mu^{QCD,VP[LO]} = (694.9 \pm 4.3) \times 10^{-10}$$
$$a_\mu^{QCD,VP[HO]} = (-9.8 \pm 0.1) \times 10^{-10}$$

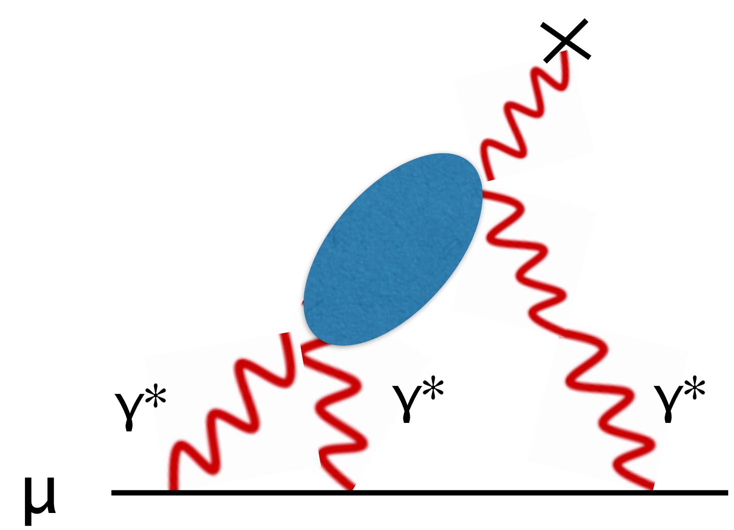



relies on experiment $e^+e^- \rightarrow$ hadrons through unitarity

$$\sigma(s)_{e^+e^- \rightarrow hadrons}$$

## Hadronic light-by-light scattering

$$a_\mu^{QCD,LbL} = (10.5 \pm 2.6) \times 10^{-10}$$
$$= (10.2 \pm 3.9) \times 10^{-10}$$




relies on measurements of **TFF** to reduce model dependence

$$\pi^0\gamma^*\gamma^{(*)}, \eta\gamma^*\gamma^{(*)}, \ldots$$
$$f_1\gamma^*\gamma^{(*)}, f_2\gamma^*\gamma^{(*)}, \ldots$$

# *Light by light scattering contribution*

Hadronic light-by-light scattering

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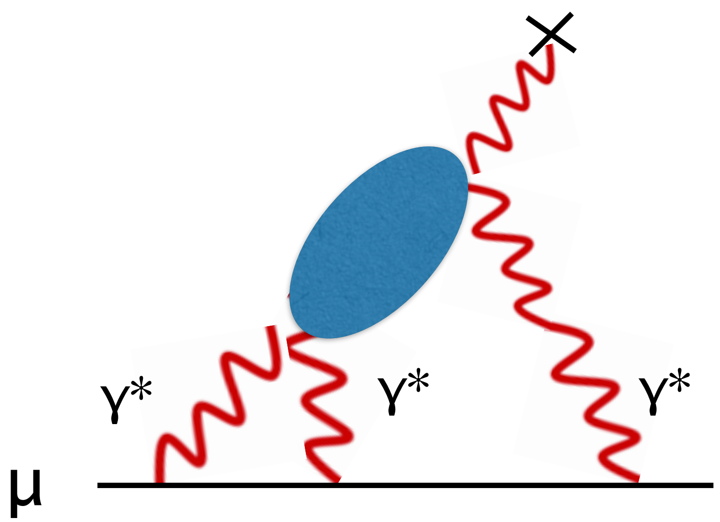


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# *Light by light scattering contribution*

Timelike: KLOE, MAMI/A2, NA62

Hadronic light-by-light scattering

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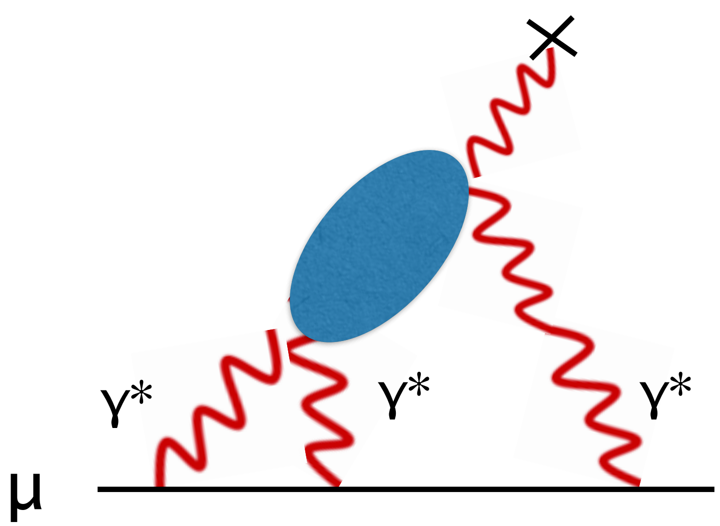
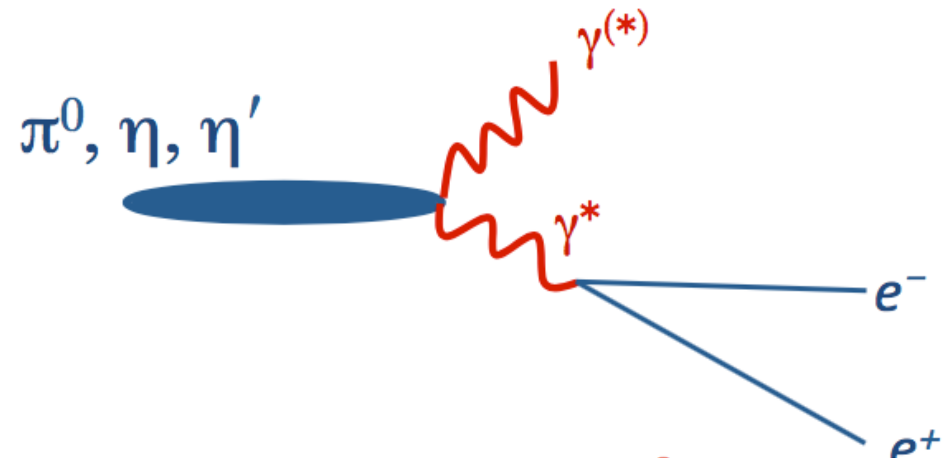
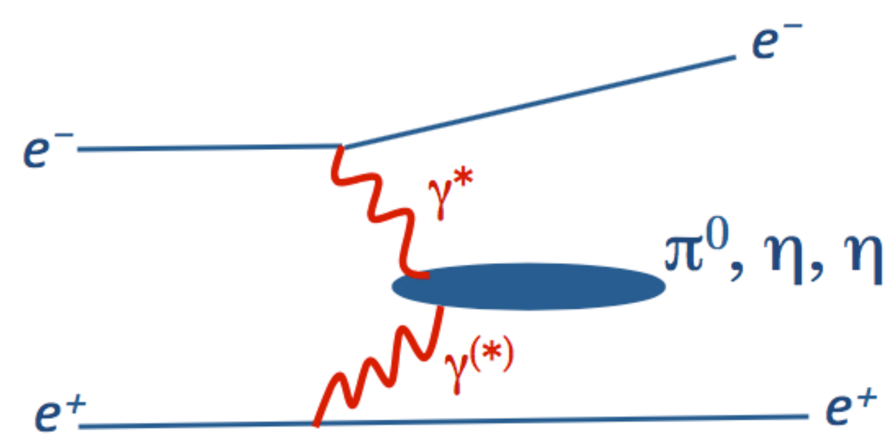



Spacelike: CLEO, BaBar, Belle, BESIII

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$$\pi^0\gamma^*\gamma^{(*)}, \eta\gamma^*\gamma^{(*)}, \ldots$$
$$f_1\gamma^*\gamma^{(*)}, f_2\gamma^*\gamma^{(*)}, \ldots$$

# *Light by light scattering contribution*

HLbL contributions to (g-2) in units $10^{-10}$

| Authors | $\pi^0, \eta, \eta'$ | $\pi\pi$, KK | scalars | axial vectors | quark loops | Total |
|---|---|---|---|---|---|---|
| BPaP(96) | 8.5(1.3) | -1.9(1.3) | -0.68(0.20) | 0.25(0.10) | 2.1(03) | 8.3(3.2) |
| HKS(96) | 8.3(0.6) | -0.5(0.8) | — | 0.17(0.17) | 1.0(1.1) | 9.0(1.5) |
| KnN(02) | 8.3(1.2) | — | — | — | — | 8.0(4.0) |
| MV(04) | 11.4(1.0) | — | — | 2.2(0.5) | — | 13.6(2.5) |
| PdRV(09) | 11.4(1.3) | -1.9(1.9) | -0.7(0.7) | 1.5(1.0) | 0.23 | **10.5(2.6)** |
| N/JN(09) | 9.9(1.6) | -1.9(1.3) | -0.7(0.2) | 2.2(0.5) | 2.1(0.3) | 11.6(3.9) |
| J(15) | 9.9(1.6) | -1.9(1.3) | -0.7(0.2) | **0.75(0.27)** | 2.1(0.3) | **10.2(3.9)** |

B=Bjnens, Pa=Pallante, P=Prades, H=Hayakawa, K=Kinoshita, S=Sanda, Kn=Knecht, N=Nyffeler, M=Melnikov, V=Vainshtein, dR=de Rafael, J=Jegerlehner

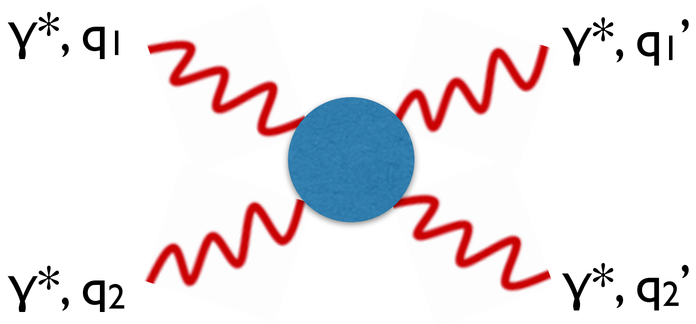
How to improve on the present calculations?

1. Space like doubly-virtual measurement of $\pi^0$ TFF at BESIII ($Q_1^2, Q_2^2$~0.5-1 GeV$^2$)
2. Dispersive analysis for ππ, KK, … loops contribution to (g-2)

Colangelo, Hoferichter, Procura, Stoffer, (2017)

Pauk, Vanderhaeghen, (2014)

# *Light by light scattering*

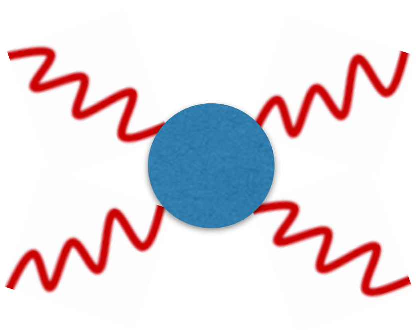



$$\lambda_i = \pm 1, 0$$

$$q_i^2 = -Q_i^2$$

# *Light by light scattering*

Helicity amplitudes



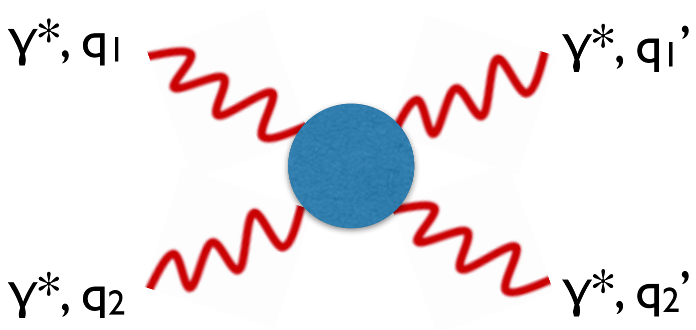

$$\lambda_i = \pm 1, 0$$

$$q_i^2 = -Q_i^2$$

$$M_{\lambda_1' \lambda_2' \lambda_1 \lambda_2} = M^{\mu\nu\alpha\beta} \epsilon^*_\mu(\lambda_1')\, \epsilon^*_\nu(\lambda_2')\, \epsilon_\alpha(\lambda_1)\, \epsilon_\beta(\lambda_2)$$

# Light by light scattering

Helicity amplitudes



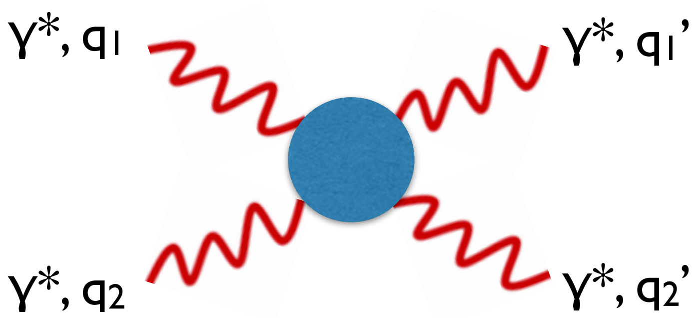

$$M_{\lambda'_1\lambda'_2\lambda_1\lambda_2} = M^{\mu\nu\alpha\beta}\epsilon^*_\mu(\lambda'_1)\,\epsilon^*_\nu(\lambda'_2)\,\epsilon_\alpha(\lambda_1)\,\epsilon_\beta(\lambda_2)$$

Forward scattering $q_1 = q'_1,\, q_2 = q'_2$

$$\lambda_i = \pm 1, 0$$
$$q_i^2 = -Q_i^2$$

$$s = (q_1 + q_2)^2$$
$$t = (q_1 - q'_1)^2 = 0$$

# Light by light scattering

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Forward scattering $q_1 = q_1',\ q_2 = q_2'$

$$s = (q_1 + q_2)^2$$

$$t = (q_1 - q_1')^2 = 0$$

P and T symmetry: **81** → **8** independent amplitudes

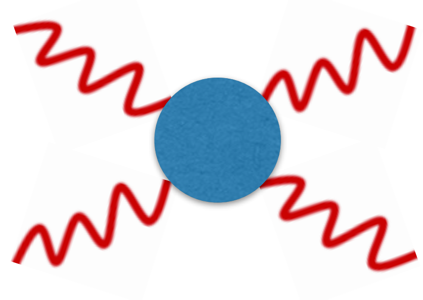
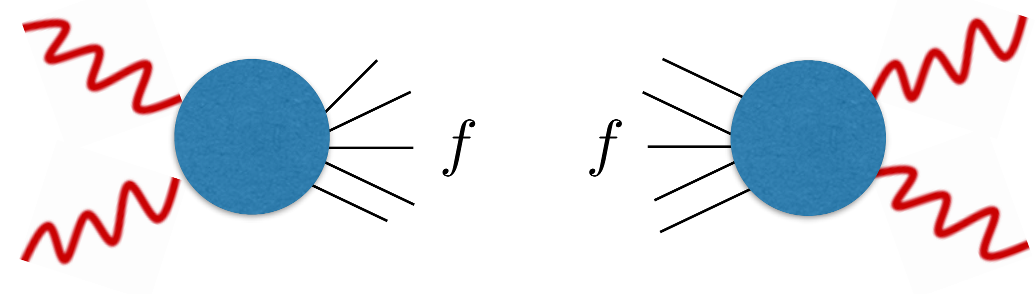
$$M_{++,++},\ M_{+-,+-},\ M_{++,--}$$

$$M_{00,00},\ M_{+0,+0},\ M_{0+,0+}$$

$$M_{++,00},\ M_{0+,-0}$$

# *Light by light scattering*

Unitarity

2 Im  $= \sum_f \int d\Pi_f$ 


For the forward scattering (optical theorem):

$$\mathrm{Im}\, M_{++,++} = 2\sqrt{X}\,\sigma_0$$
$$\mathrm{Im}\, M_{+-,+-} = 2\sqrt{X}\,\sigma_2$$
$$\mathrm{Im}\, M_{++,--} = 2\sqrt{X}\,(\sigma_{\|} - \sigma_{\perp})$$
$$\ldots$$

*X - flux factor*

Observables in: $e^+e^- \to e^-e^+f$

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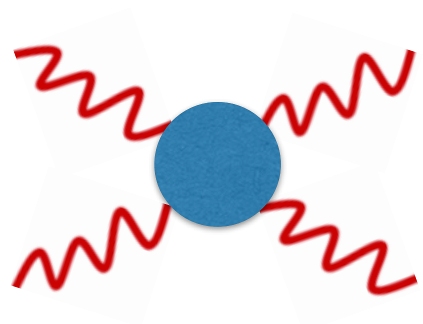
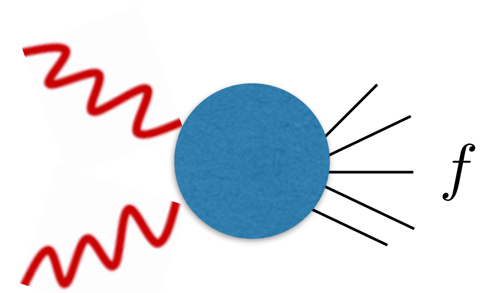
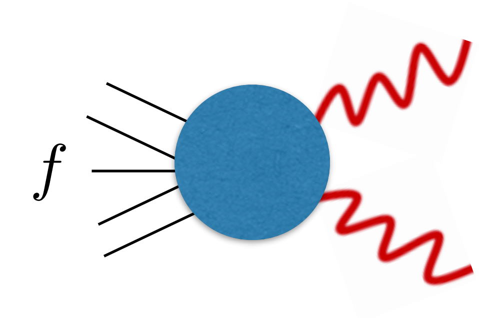
2 Im  $= \sum_f \int d\Pi_f$ 

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Analyticity (fixed *t* dispersion relation)

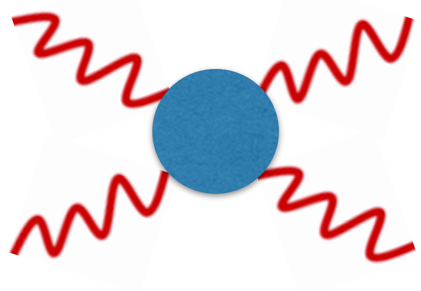
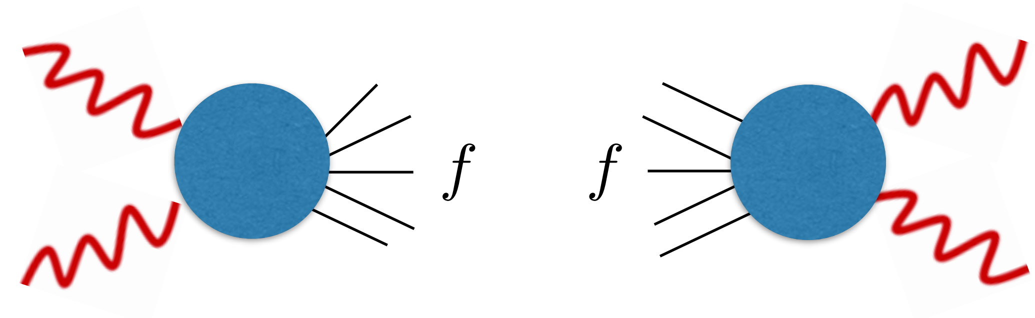
$$M_{++,--}(\nu) = \int_{\nu_0}^{\infty} \frac{d\nu'}{\pi} \frac{2\,\nu'\, \mathrm{Im} M_{++,--}(\nu')}{\nu'^2 - \nu^2 + i\,0}\,, \qquad \nu = \frac{s-u}{4}$$

...

(modulo subtractions)

# *Light by light scattering*

Unitarity

2 Im  $= \sum_f \int d\Pi_f$ 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...

(modulo subtractions)

Matching around $\nu = 0$ to the LbL Lagrangian

$$\mathcal{L} = c_1 (F_{\mu\nu}F^{\mu\nu})^2 + c_2 (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \ldots$$

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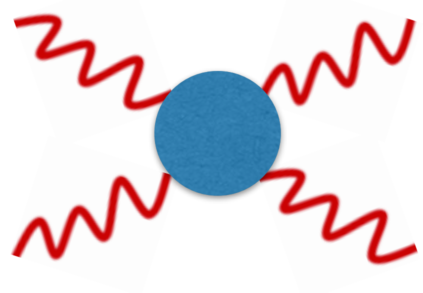
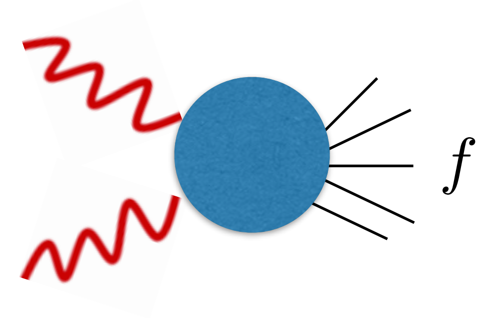
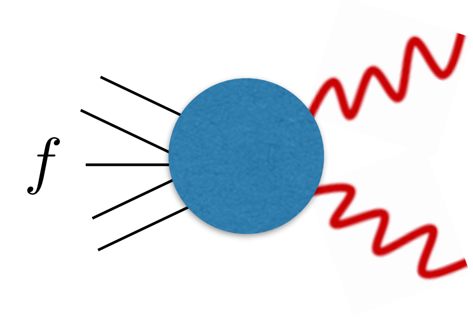
2 Im  $= \sum_f \int d\Pi_f$ 

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yield a number of **constraints on cross sections**

# Light by light sum rules

Three super convergence relations

Gerasimov, Moulin (1975), Brodsky, Schmidt (1995)

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q_1^2)} \left[\sigma_2 - \sigma_0\right]_{Q_2^2=0}$$

Pascalutsa, Vanderhaeghen (2010)

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q_1^2)^2} \left[\sigma_{\parallel} + \sigma_{LT} + \frac{(s+Q_1^2)}{Q_1 Q_2} \tau_{TL}^a\right]_{Q_2^2=0}$$

$$0 = \int_{s_0}^{\infty} ds \left[\frac{\tau_{TL}(s, Q_1^2, Q_2^2)}{Q_1 Q_2}\right]_{Q_2^2=0}$$

Pascalutsa, Pauk Vanderhaeghen (2012), (2014)

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These sum rules have been tested in perturbative QFT both at tree-level and one loop level:

| | |
|---|---|
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| spinor QED | $\phi^4$ theory + resum. |

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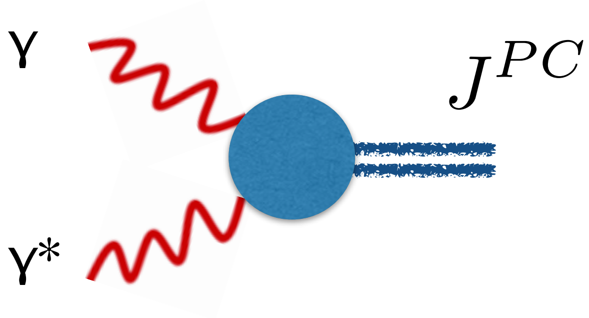
# Light by light sum rules: Meson production




C=+1: $J^{PC}=0^{-+}, 0^{++}, 1^{++}, 2^{++}, \ldots$

$Q^2 \neq 0$ Landau-Yang theorem

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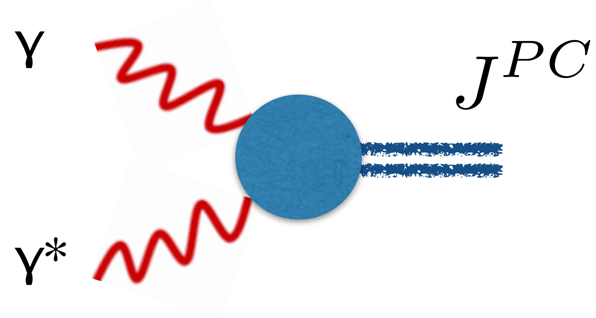


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Landau-Yang theorem

Narrow width approximation

$$\sigma\left(\gamma^*\gamma \to J^P(\Lambda)\right) = \delta(s-m^2)\, 8\pi^2 \frac{(2J+1)\,\Gamma_{\gamma\gamma}\left(J^P\right)}{m} \left(1+\frac{Q^2}{m^2}\right) \left[\mathrm{T}^{(\Lambda)}\left(Q^2\right)\right]^2$$

Sum rules will relate 2γ width or TFFs:

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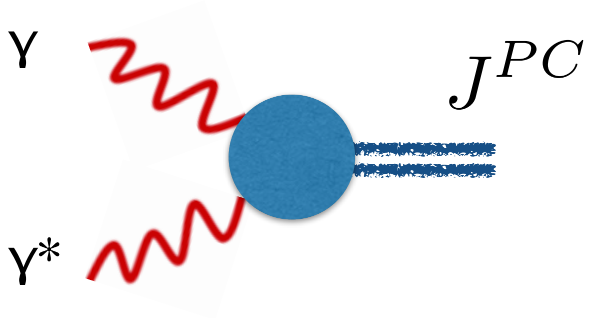


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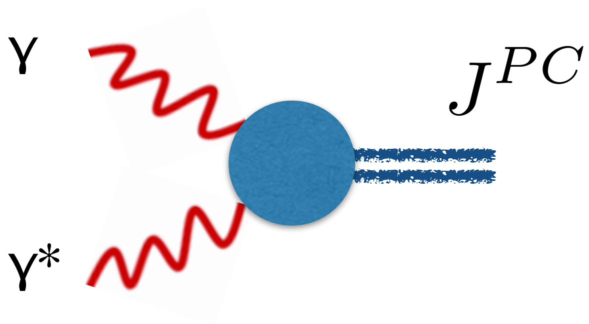
$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q^2)} \left[\sigma_2 - \sigma_0\right]$$

$Q^2 = 0$ →

$$0 = -\sum_{\mathcal{P}} 16\pi^2 \frac{\Gamma_{\gamma\gamma}(\mathcal{P})}{m_{\mathcal{P}}^3} - \sum_{\mathcal{S}} \ldots$$

# *Light by light sum rules: Meson production*

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$Q^2 \neq 0$ Landau-Yang theorem

Narrow width approximation

$$\sigma\left(\gamma^*\gamma \to J^P(\Lambda)\right) = \delta(s-m^2)\, 8\pi^2 \frac{(2J+1)\,\Gamma_{\gamma\gamma}\left(J^P\right)}{m}\left(1+\frac{Q^2}{m^2}\right)\left[\mathrm{T}^{(\Lambda)}\left(Q^2\right)\right]^2$$

Sum rules will relate 2γ width or TFFs:

$$0 = \int_{s_0}^{\infty} ds \frac{1}{(s+Q^2)}\left[\sigma_2 - \sigma_0\right]$$

$Q^2 = 0$ →

$$0 = -\sum_{\mathcal{P}} 16\pi^2 \frac{\Gamma_{\gamma\gamma}(\mathcal{P})}{m_{\mathcal{P}}^3} - \sum_{\mathcal{S}} \ldots$$

$Q^2 \neq 0$ →

$$0 = -\sum_{\mathcal{P}} 16\pi^2 \frac{\Gamma_{\gamma\gamma}(\mathcal{P})}{m_{\mathcal{P}}^3}\left[T(Q^2)\right]^2 - \sum_{\mathcal{S}} \ldots$$

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Pascalutsa, Pauk
Vanderhaeghen
(2012)

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Dominant contributions

| State | m (MeV) | $\Gamma_{\gamma\gamma}$(keV) | | $SR_1$ $(Q^2 = 0)$ (nb) |
|---|---|---|---|---|
| $\eta$ | 547.862±0.017 | 0.516±0.020 | | -193±7 |
| $\eta$’ | 957±0.06 | 4.35±0.25 | | -304±17 |
| $f_2$(1270) | 1275.5±0.8 | 2.93±0.40 | ($\Lambda$=2) | 434±60 |
| | | | ($\Lambda$=0) | ≈0 |
| $f_2$(1565) | 1562±13 | 0.70±0.14 | | 56±11 |
| ..... | | | | |
| sum | | | | -7±64 |

Pascalutsa, Pauk
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Pascalutsa, Pauk Vanderhaeghen (2012)

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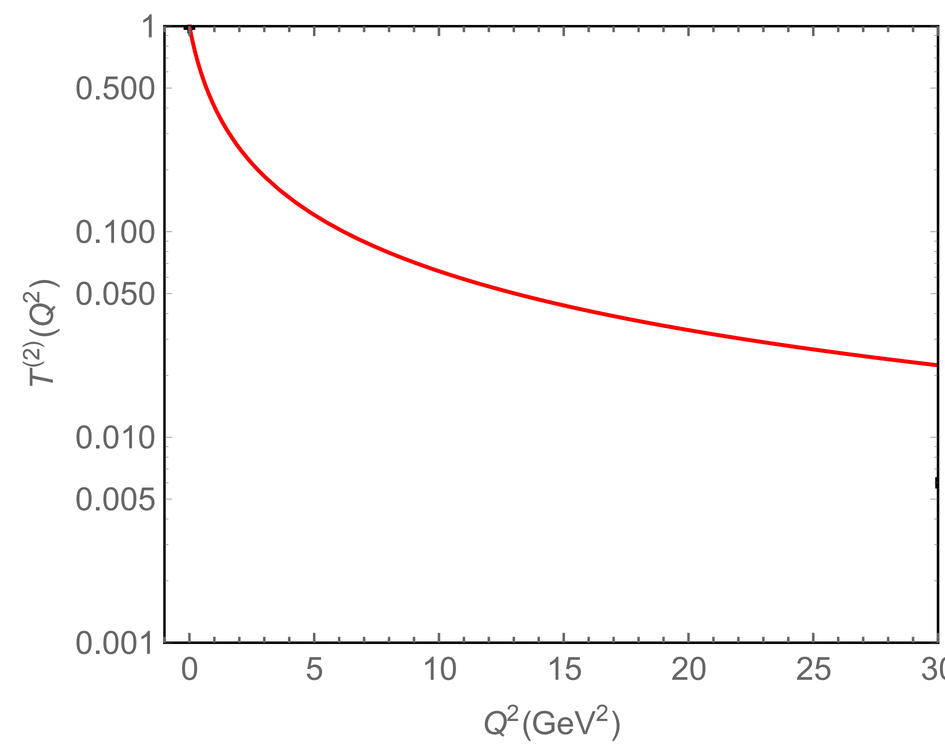


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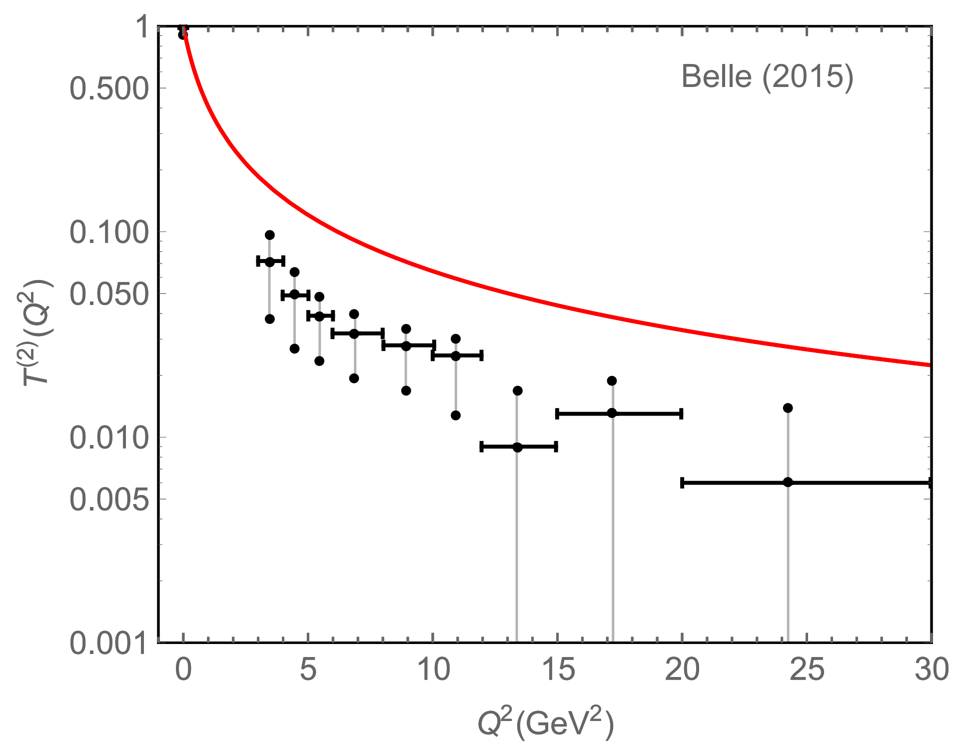


Pascalutsa, Pauk
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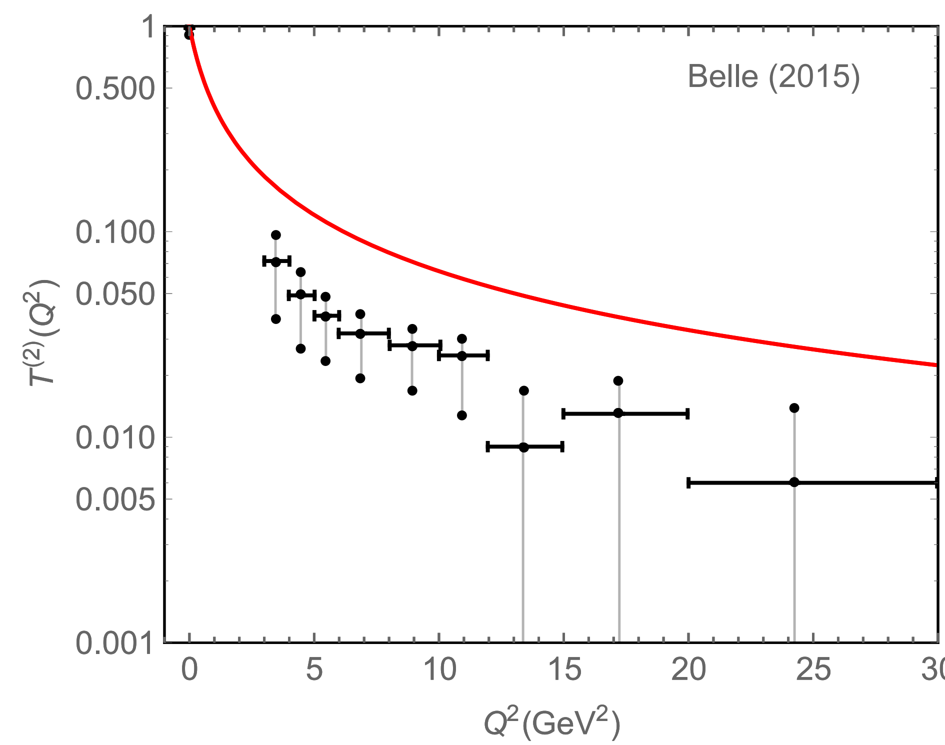
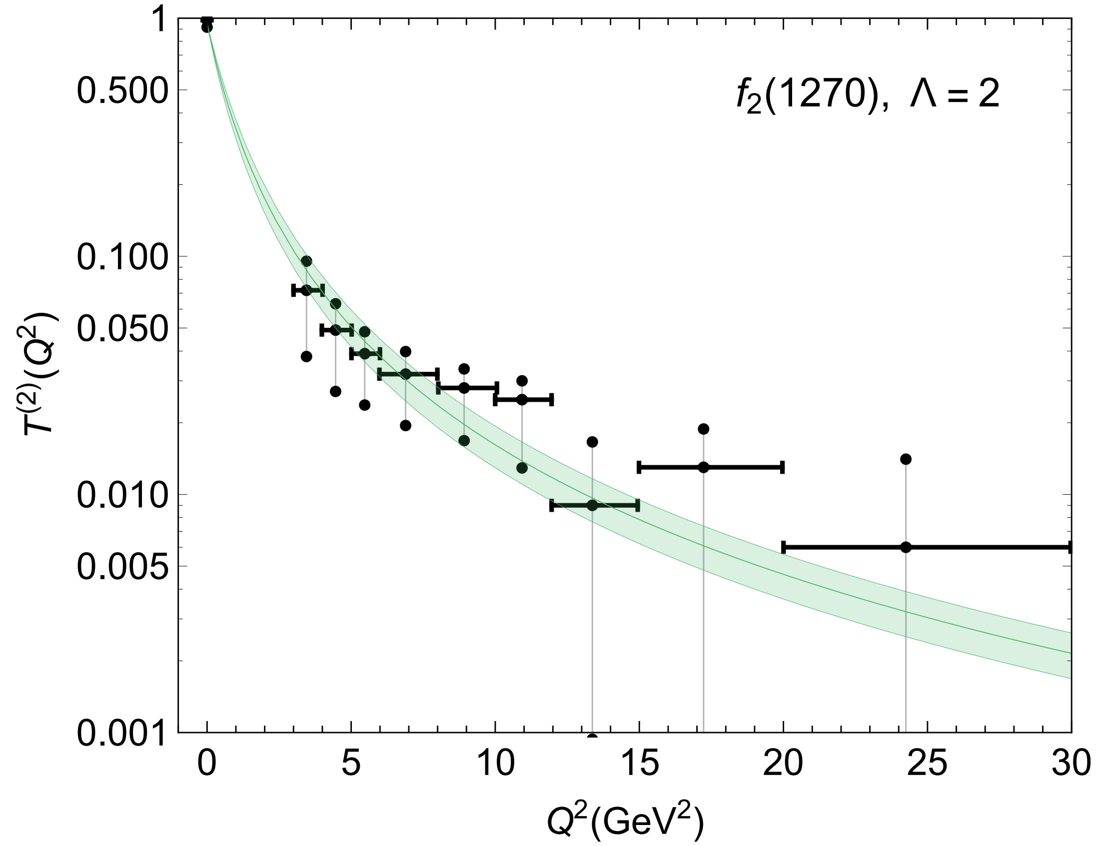
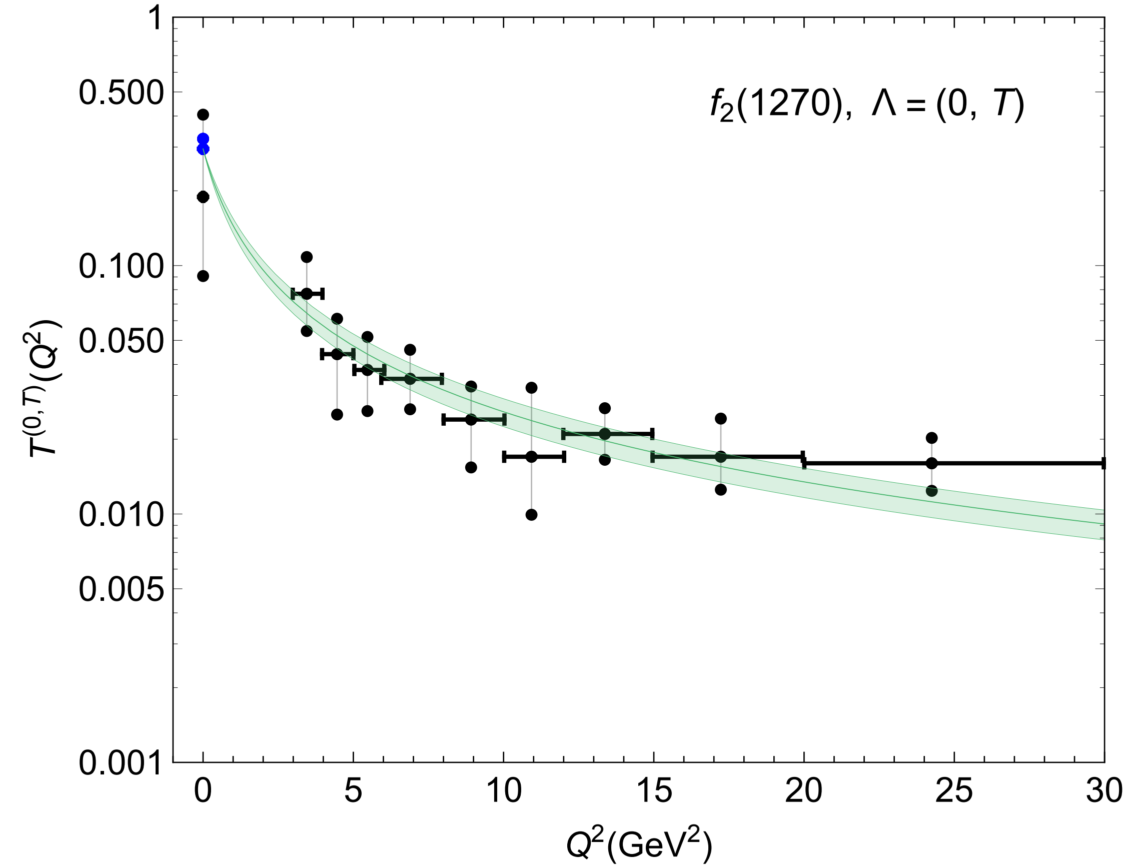



Pascalutsa, Pauk Vanderhaeghen (2012)

# Belle (2015)

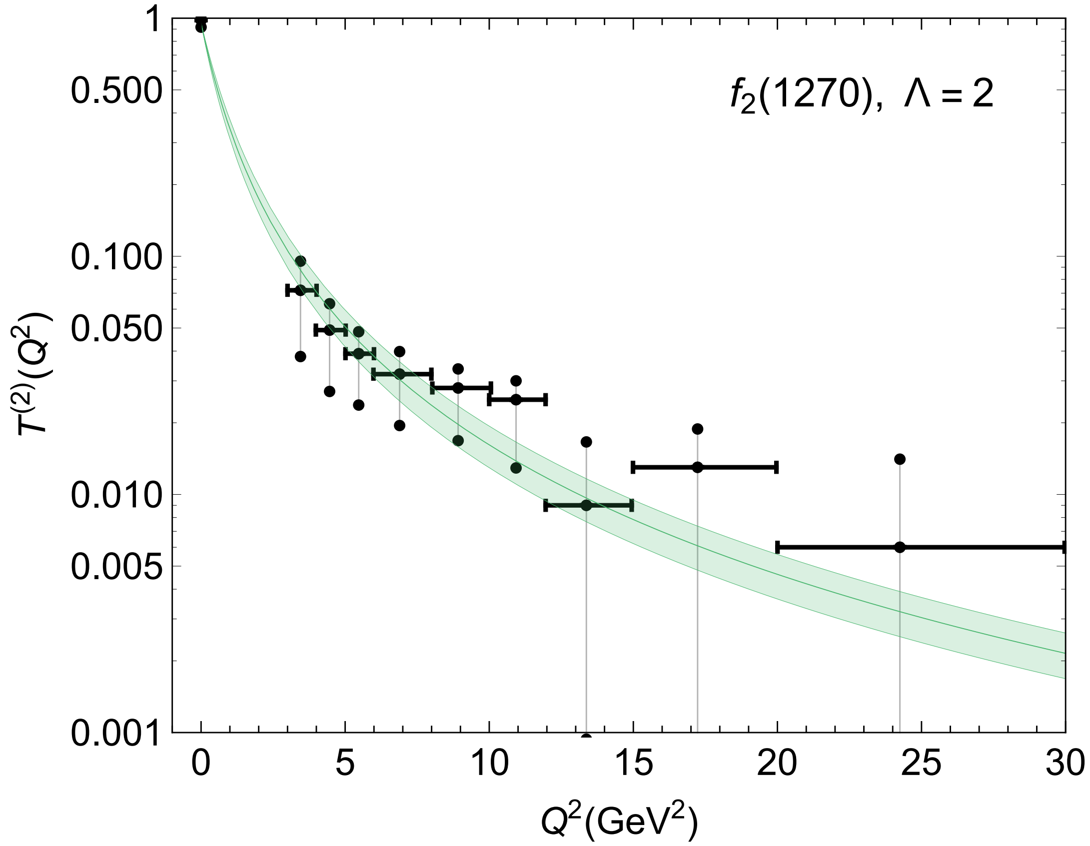


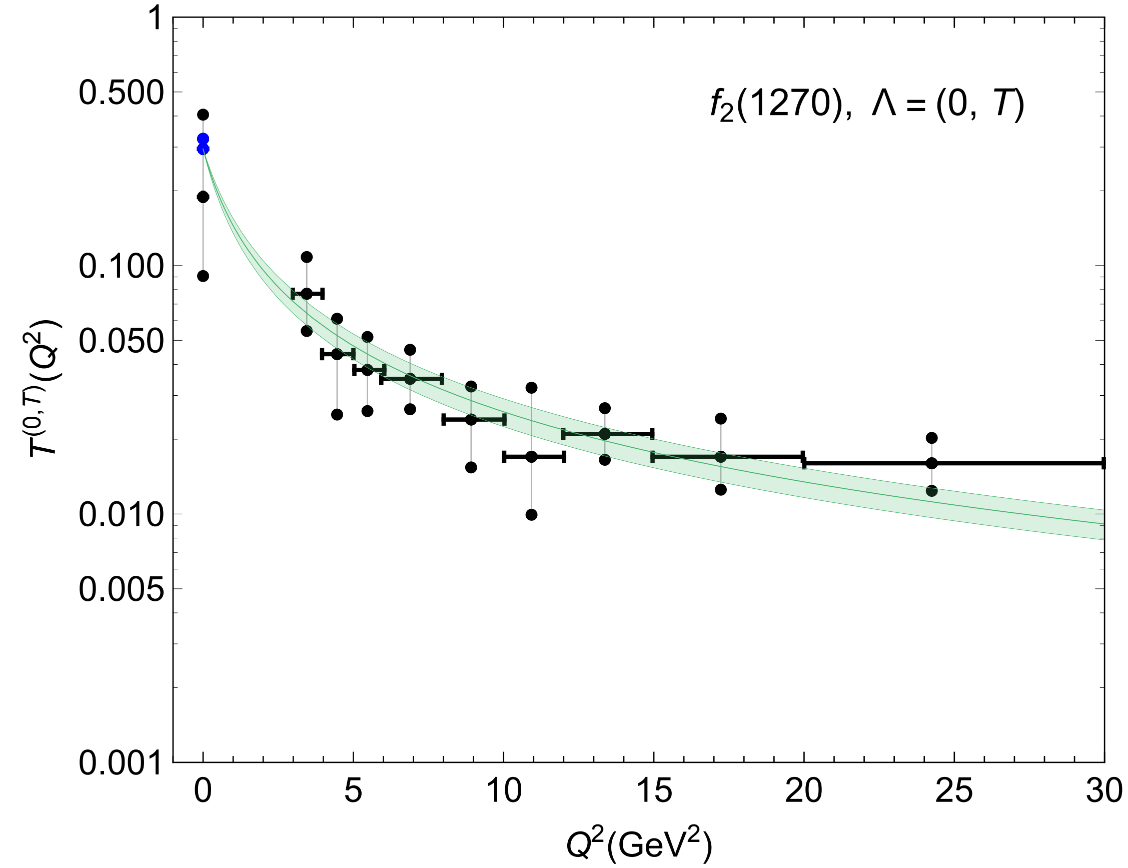
$f_2(1270)$, $\Lambda = 2$
$T^{(2)}(Q^2)$
$Q^2(\text{GeV}^2)$
1
0.500
0.100
0.050
0.010
0.005
0.001
0 5 10 15 20 25 30

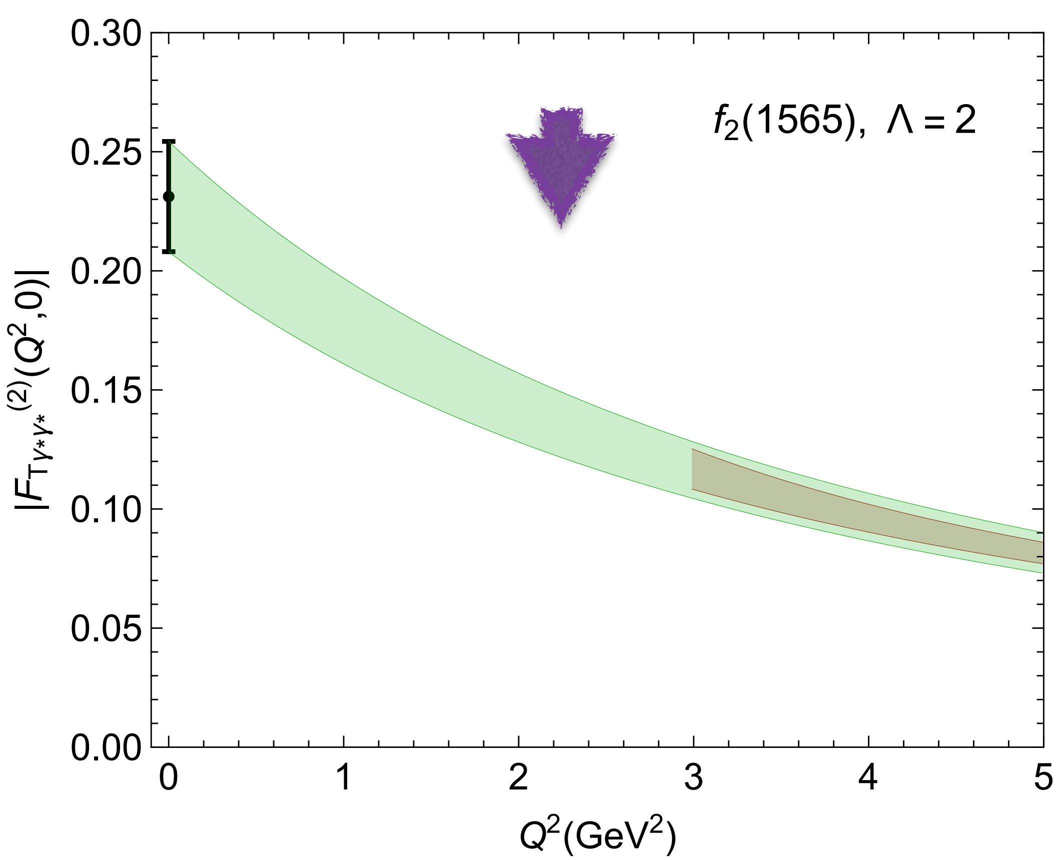



$f_2(1270)$, $\Lambda = (0, T)$
$T^{(0,T)}(Q^2)$
$Q^2(\text{GeV}^2)$
1
0.500
0.100
0.050
0.010
0.005
0.001
0 5 10 15 20 25 30

# *Belle (2015)*



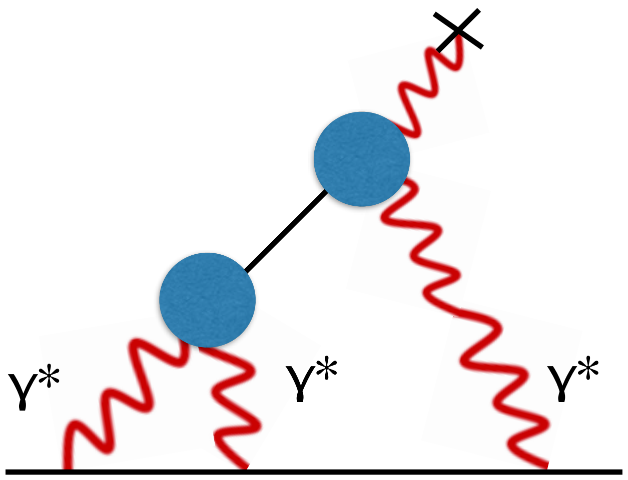

**Prediction:**

$$f_2(1565)$$
$$\lambda_{\Lambda=2} = 2719 \pm 53\,\mathrm{MeV}$$

I.D., Vanderhaeghen (2016)

future Belle data

# *Meson contributions to (g-2)*




$$a_\mu^{LbL} = \lim_{k\to 0} ie^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} T^{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)\, \Pi_{\mu\nu\lambda\sigma}(q_1, k-q_1-q_2, q_2)$$
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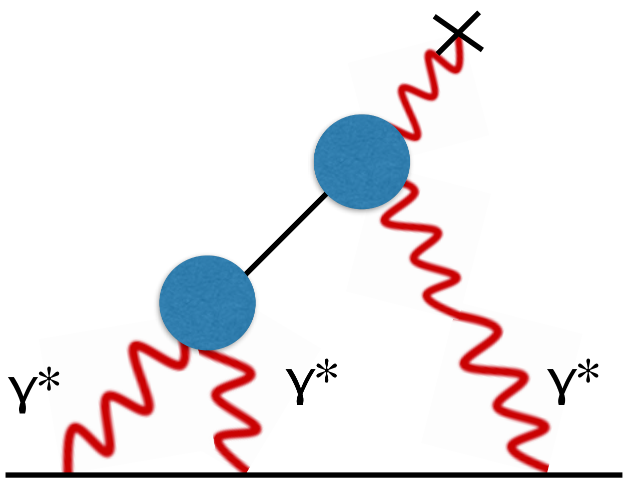


Lepton tensor: well known

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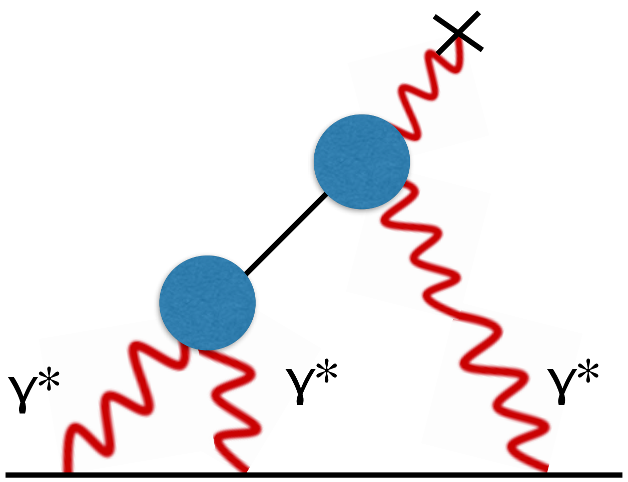


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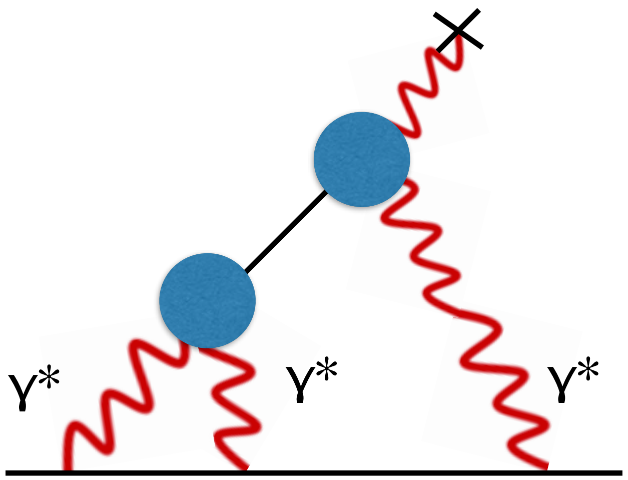


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Results (excluding low energy region):

$$a_\mu[f_2(1270), f_2(1565)] = (0.1 \pm 0.01) \times 10^{-10}$$

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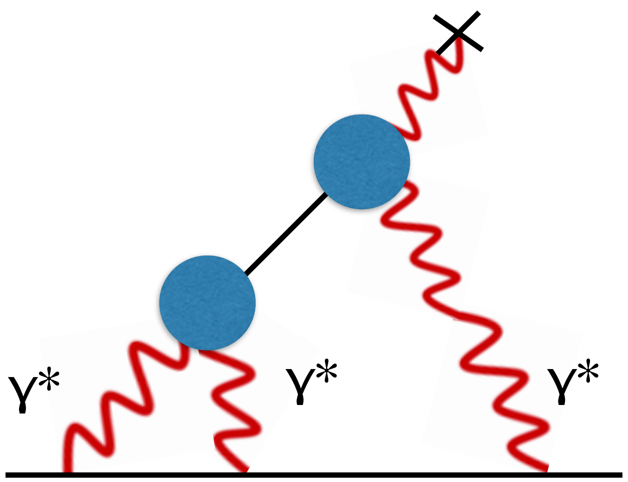


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Pauk, Vdh (2013)
Jegerlehner (2015)

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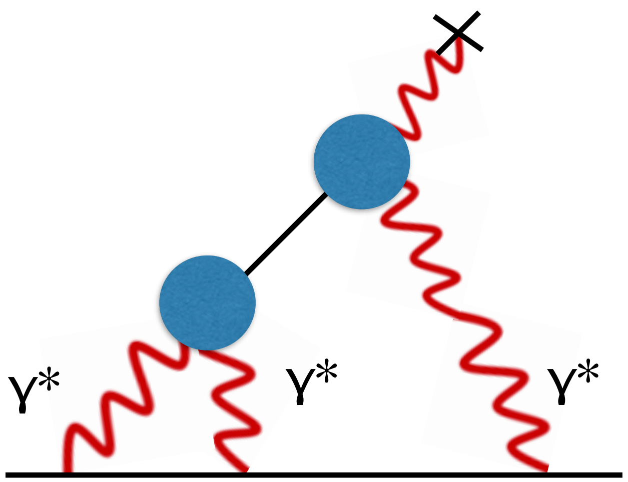


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(which enters the Glasgow consensus)

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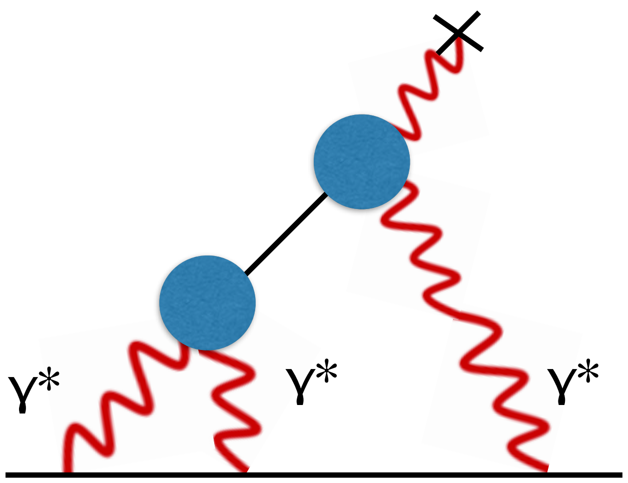


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Jegerlehner (2015)

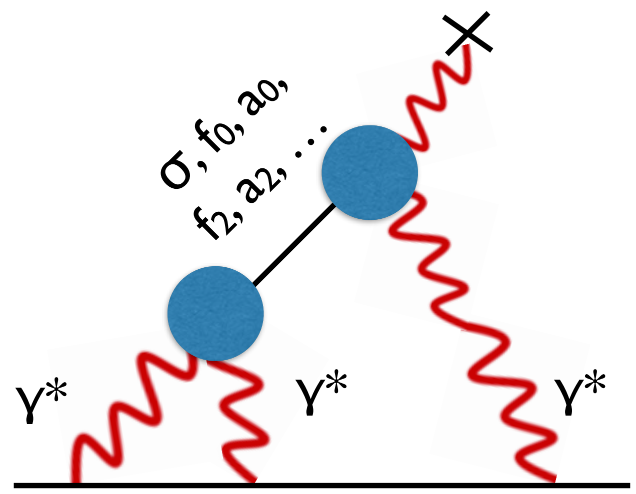
Compared to (1.5 ± 1.0) $10^{-10}$
(which enters the Glasgow consensus)

$$\delta a_\mu^{exp} = 1.6 \times 10^{-10}$$

FNAL, J-PARC experiments

# *Improvements: Multi-meson production*

Important contributions beyond **pseudo-scalar** poles



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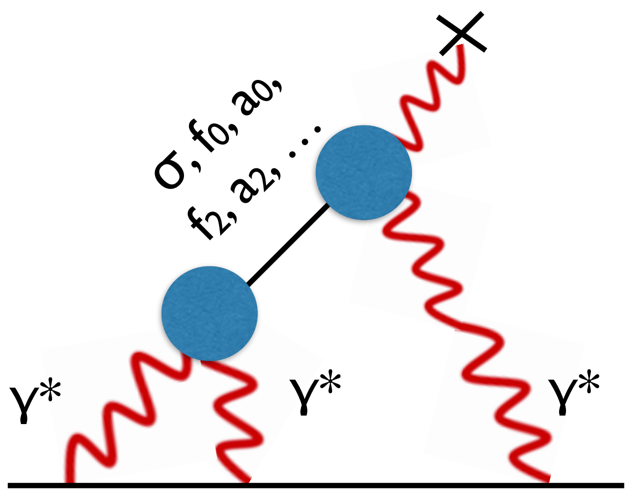
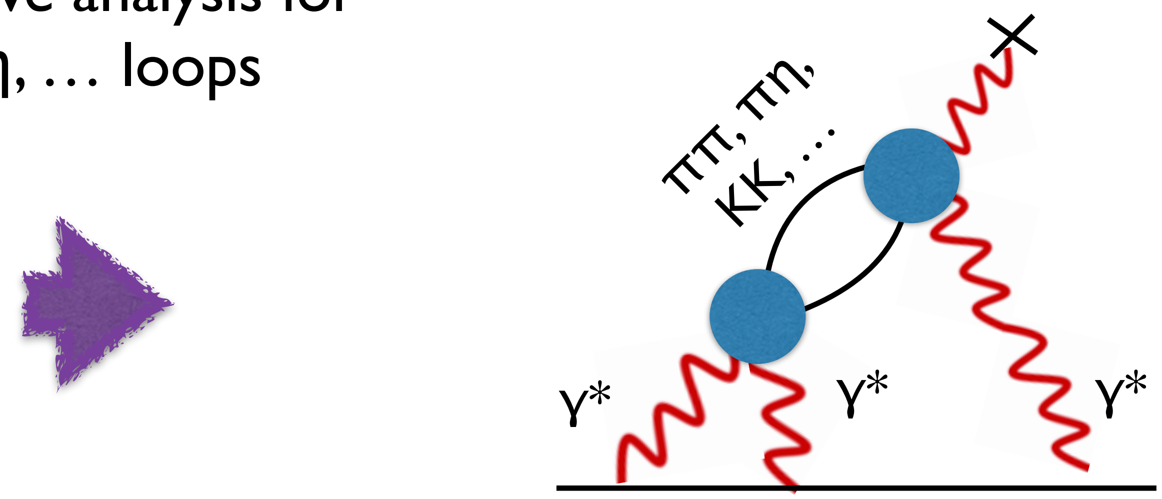



dispersive analysis for
ππ, πη, … loops



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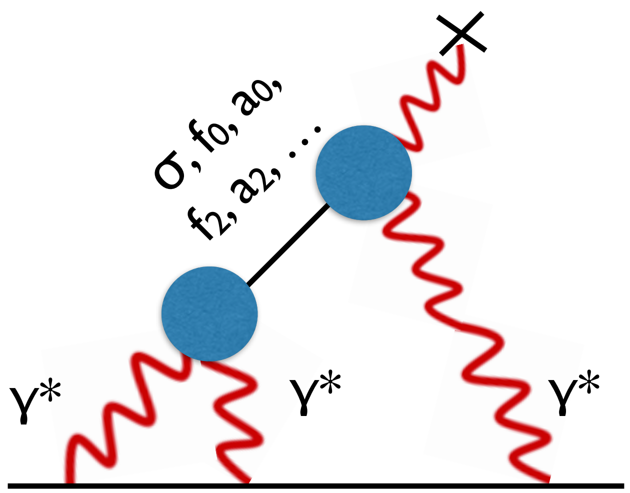


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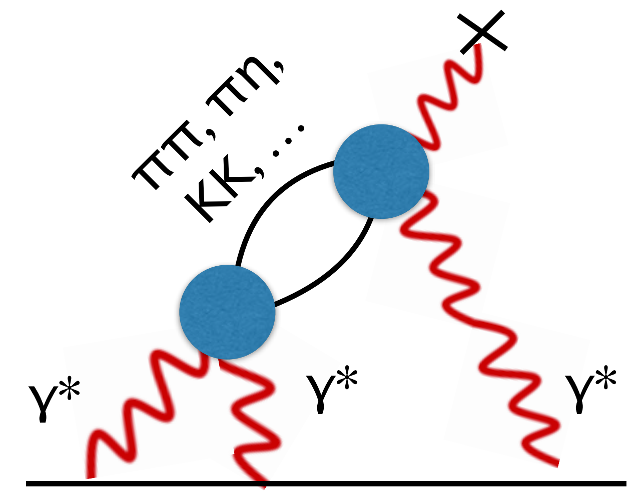


Pauk, Vanderhaeghen, (2014)

Colangelo, Hoferichter, Procura, Stoffer, (2017)

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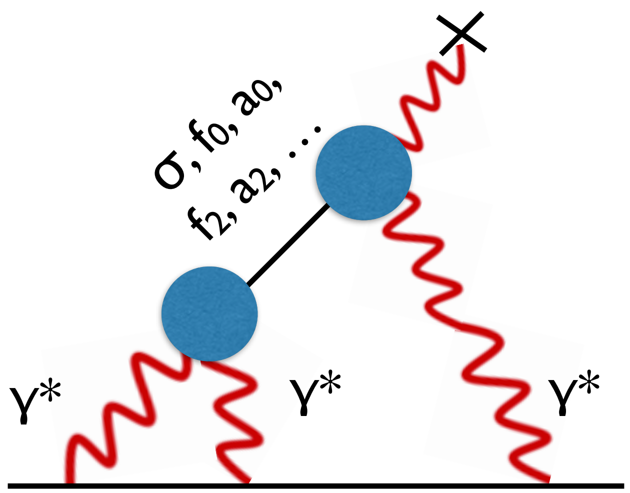


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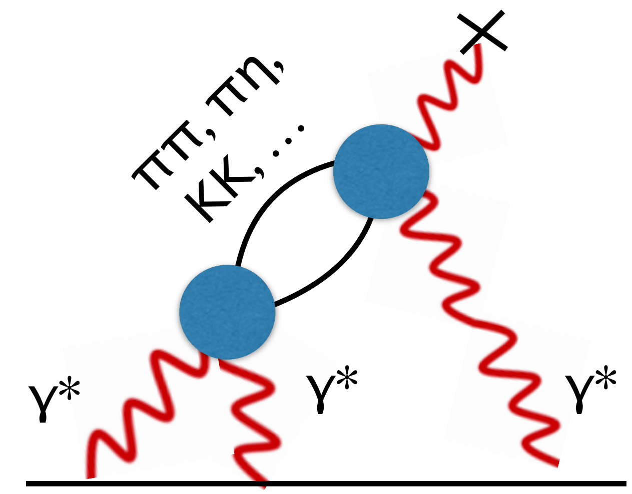
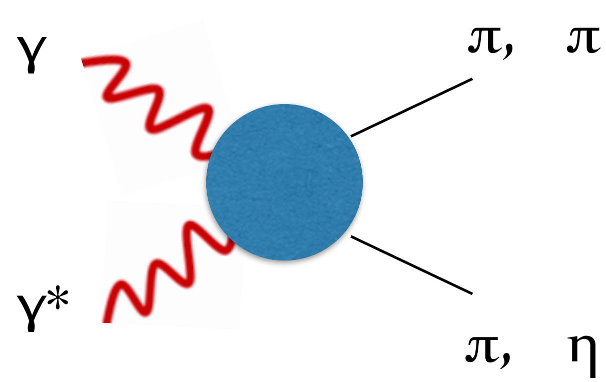



Important ingredient: $\gamma\gamma^* \rightarrow \pi\pi, \pi\eta, \ldots$

Pauk, Vanderhaeghen, (2014)

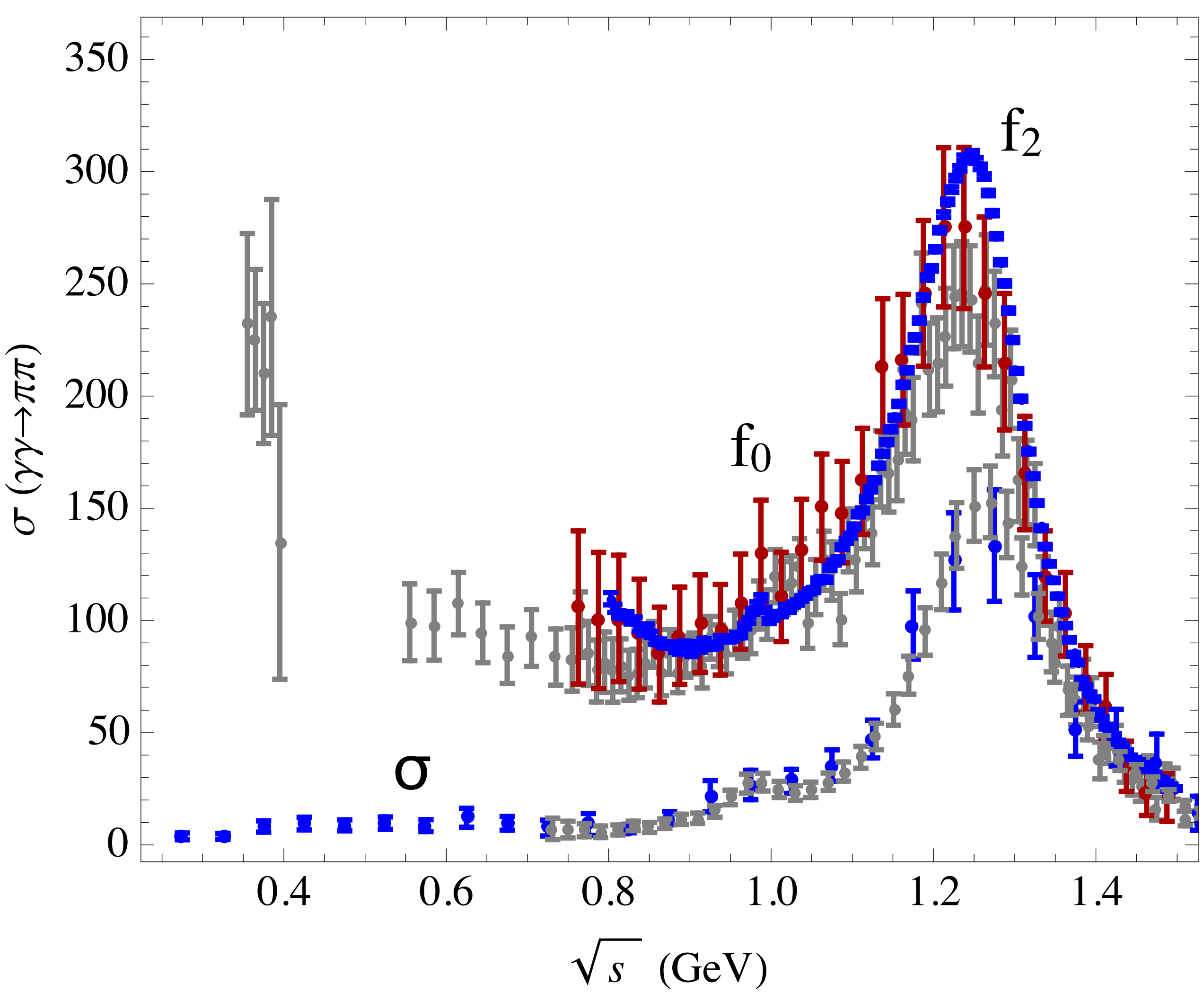
Colangelo, Hoferichter, Procura, Stoffer, (2017)



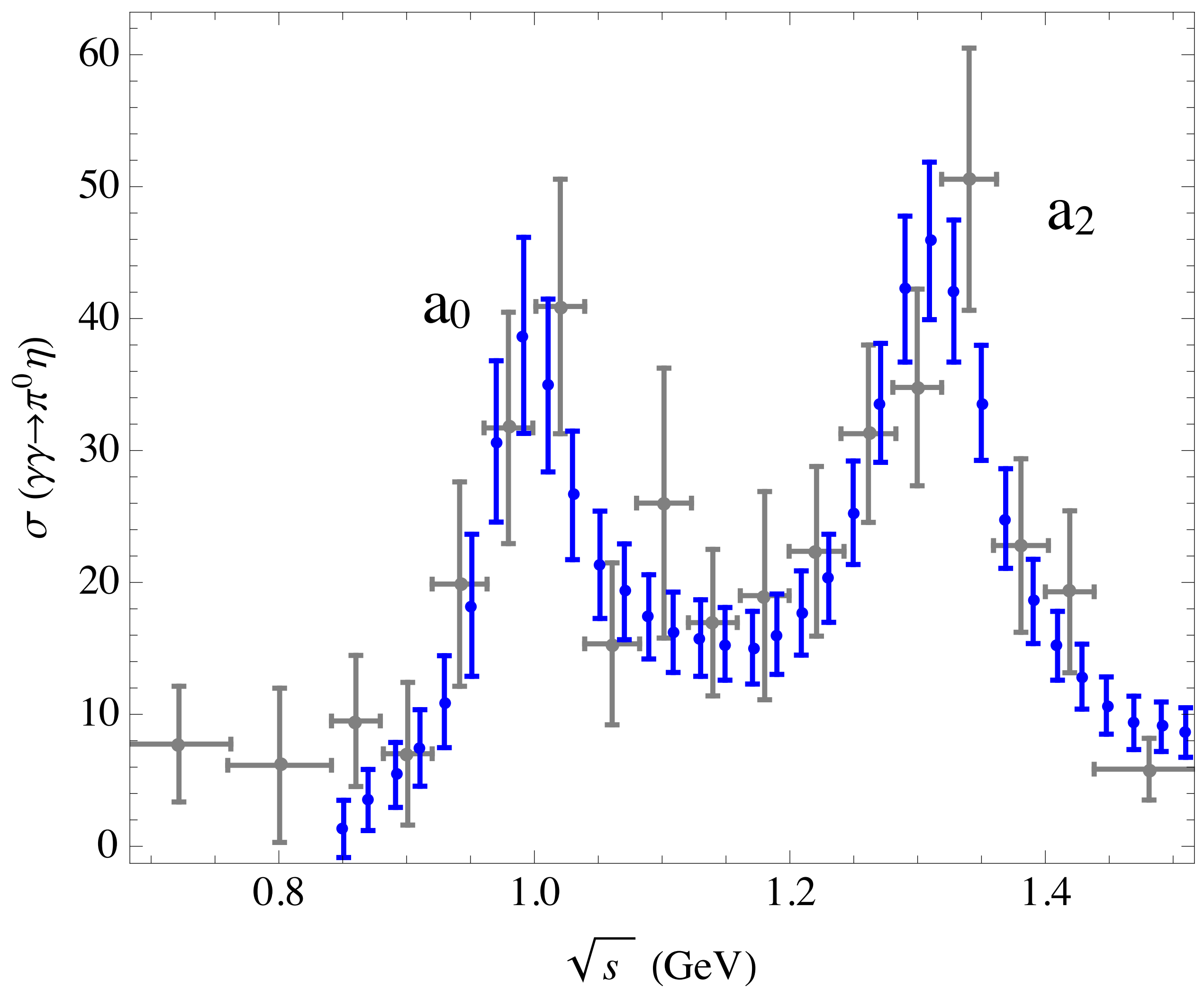

$\gamma\gamma \rightarrow \pi\pi$, KK, $\eta\eta$, $\pi\eta$ (Belle: 07,08, 09,10, ..)
$\gamma\gamma^* \rightarrow \pi\pi$, $\pi\eta$ (BESIII in progress)

# Experimental data

$\gamma\gamma \to \pi^+\pi^-$, $\pi^0\pi^0$

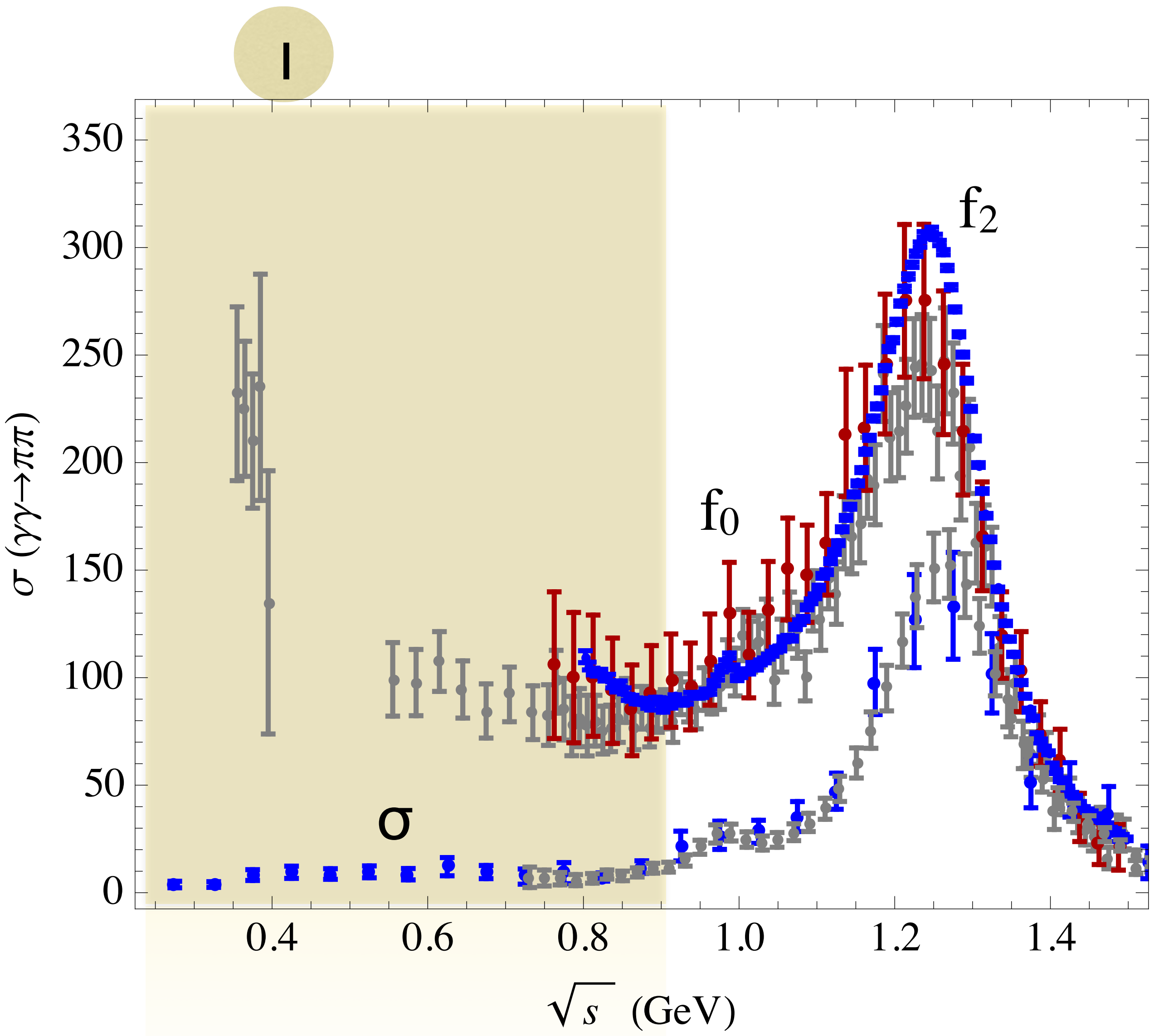



$\gamma\gamma \to \pi^0\eta$

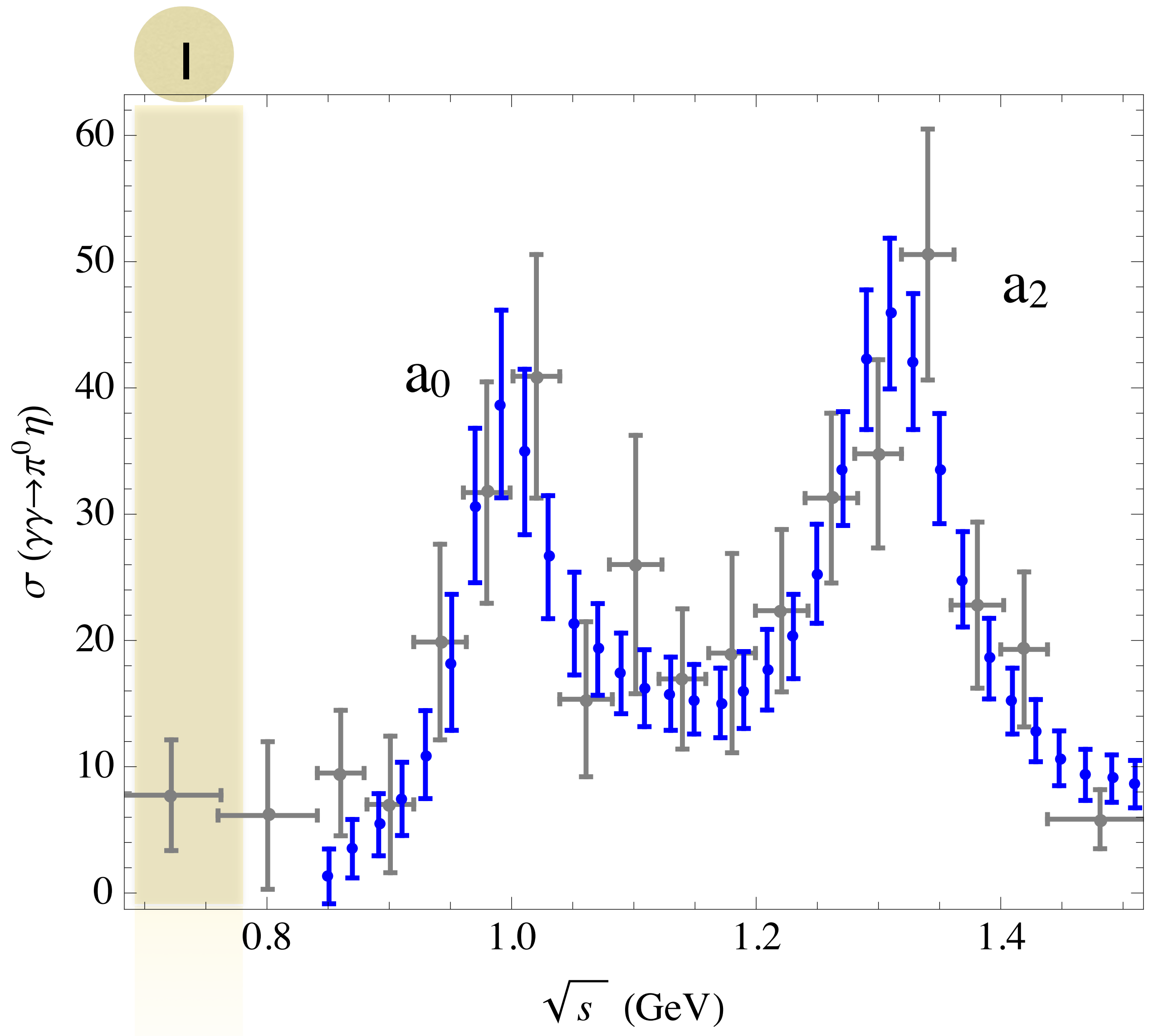


# Experimental data

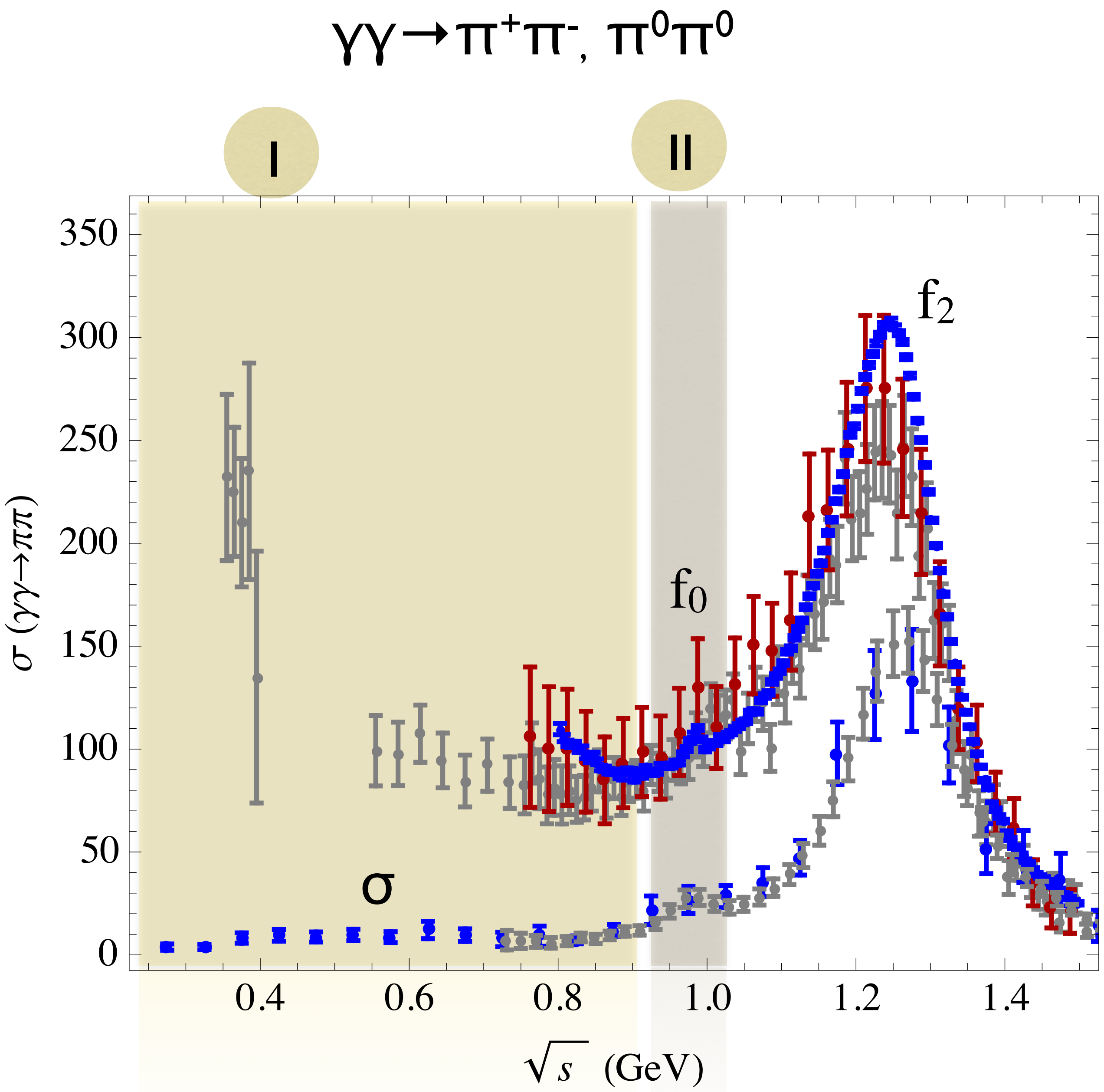
$\gamma\gamma \to \pi^+\pi^-$, $\pi^0\pi^0$

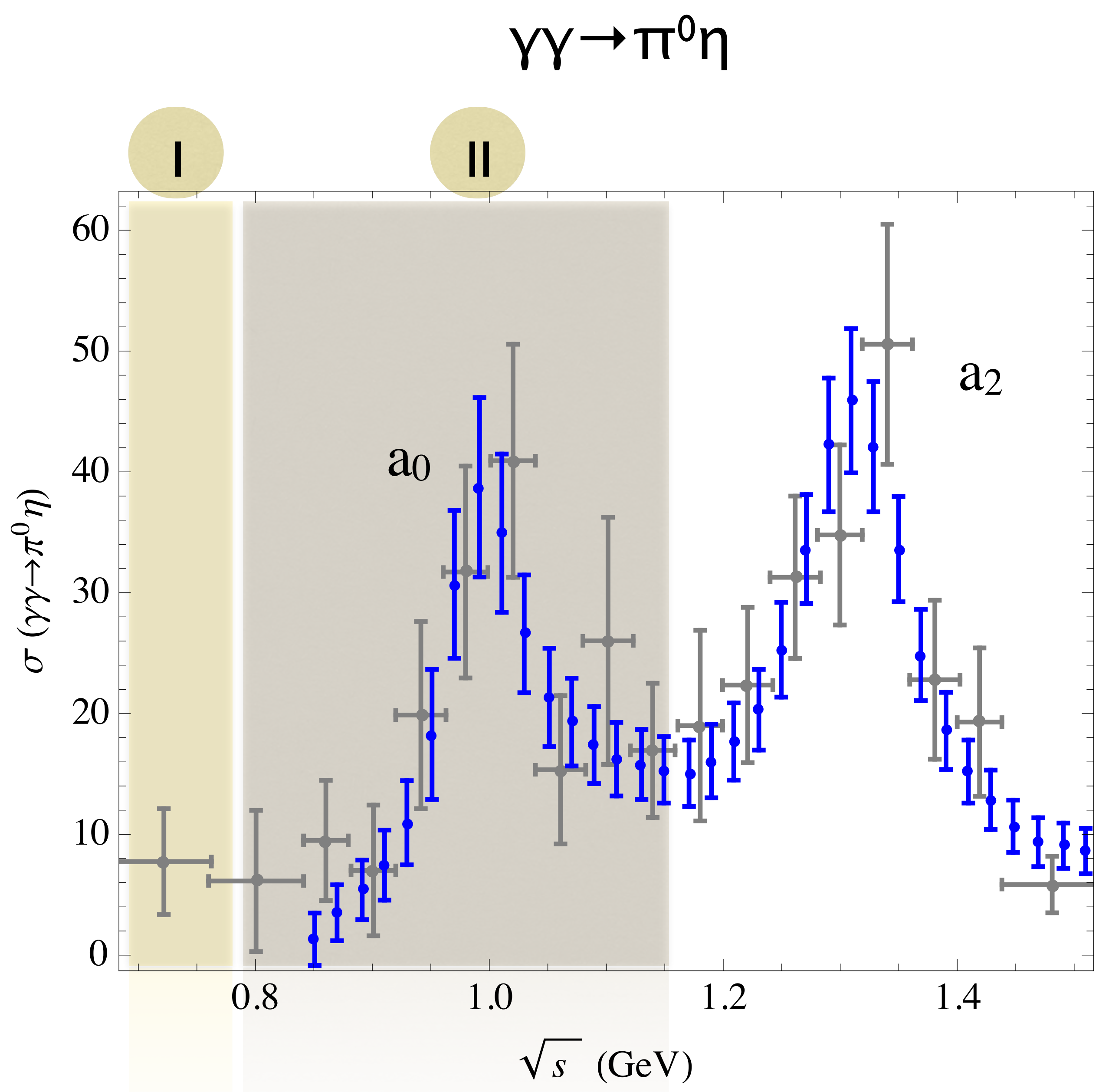



$\gamma\gamma \to \pi^0\eta$



# *Experimental data*



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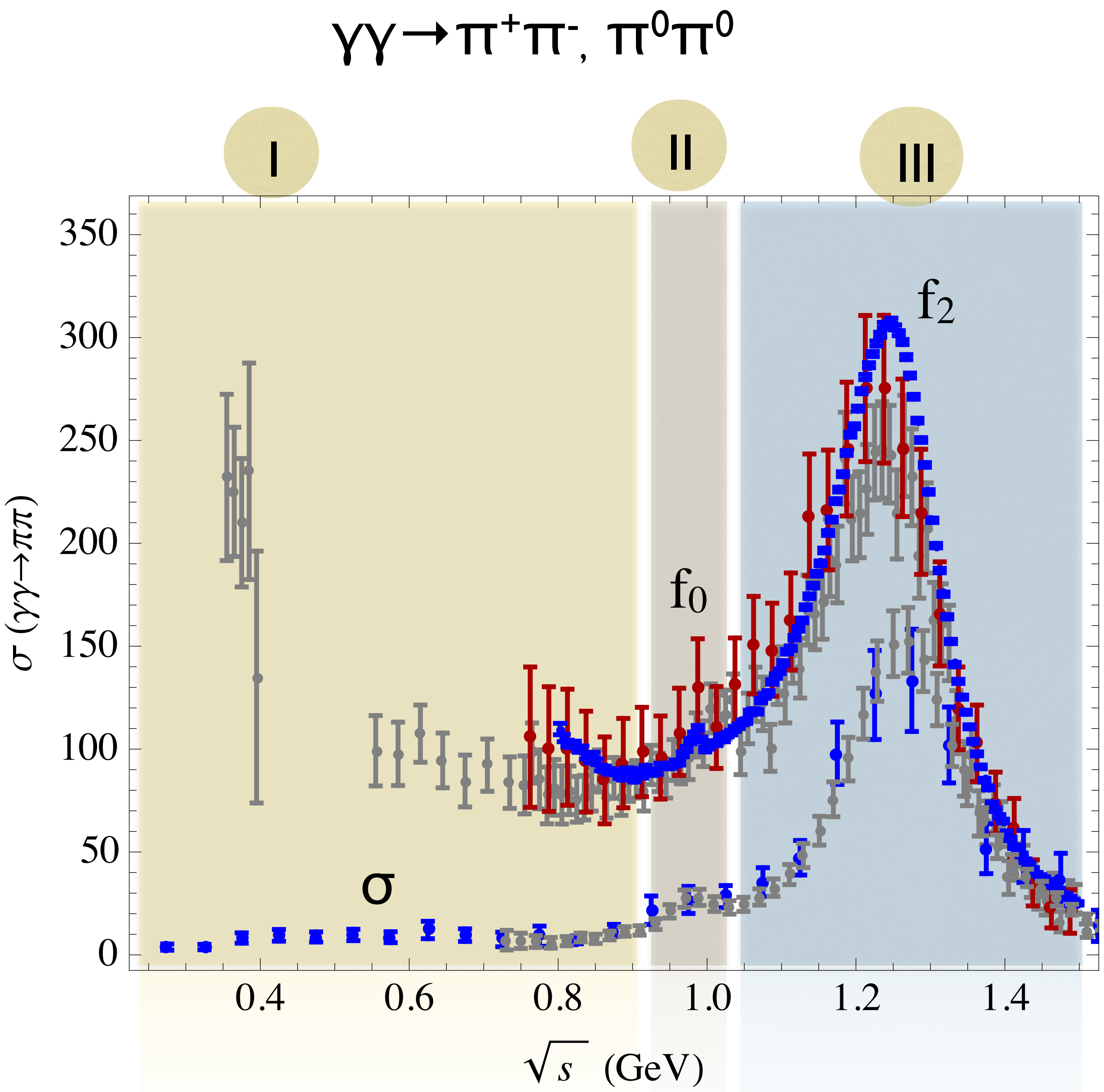
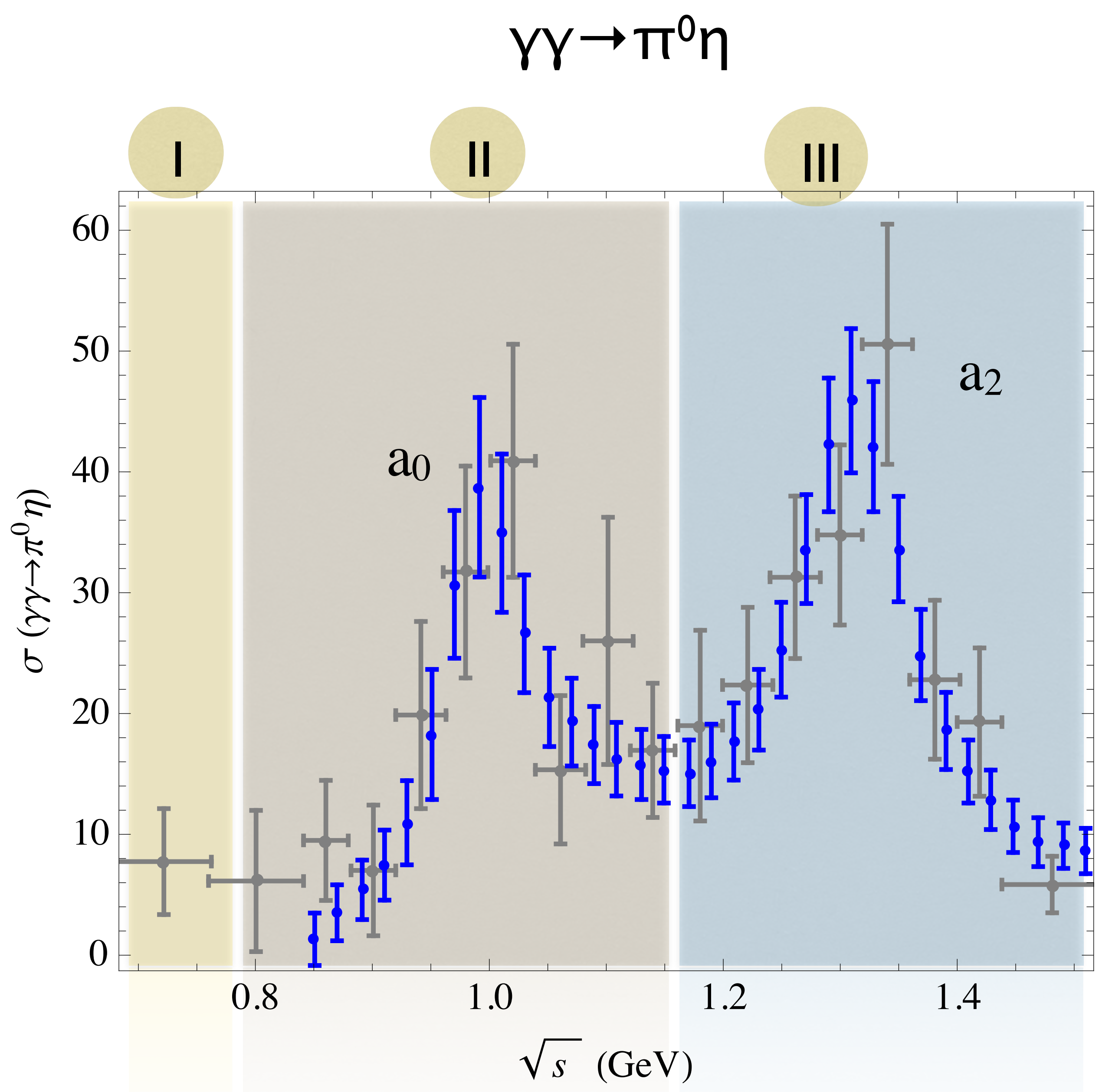
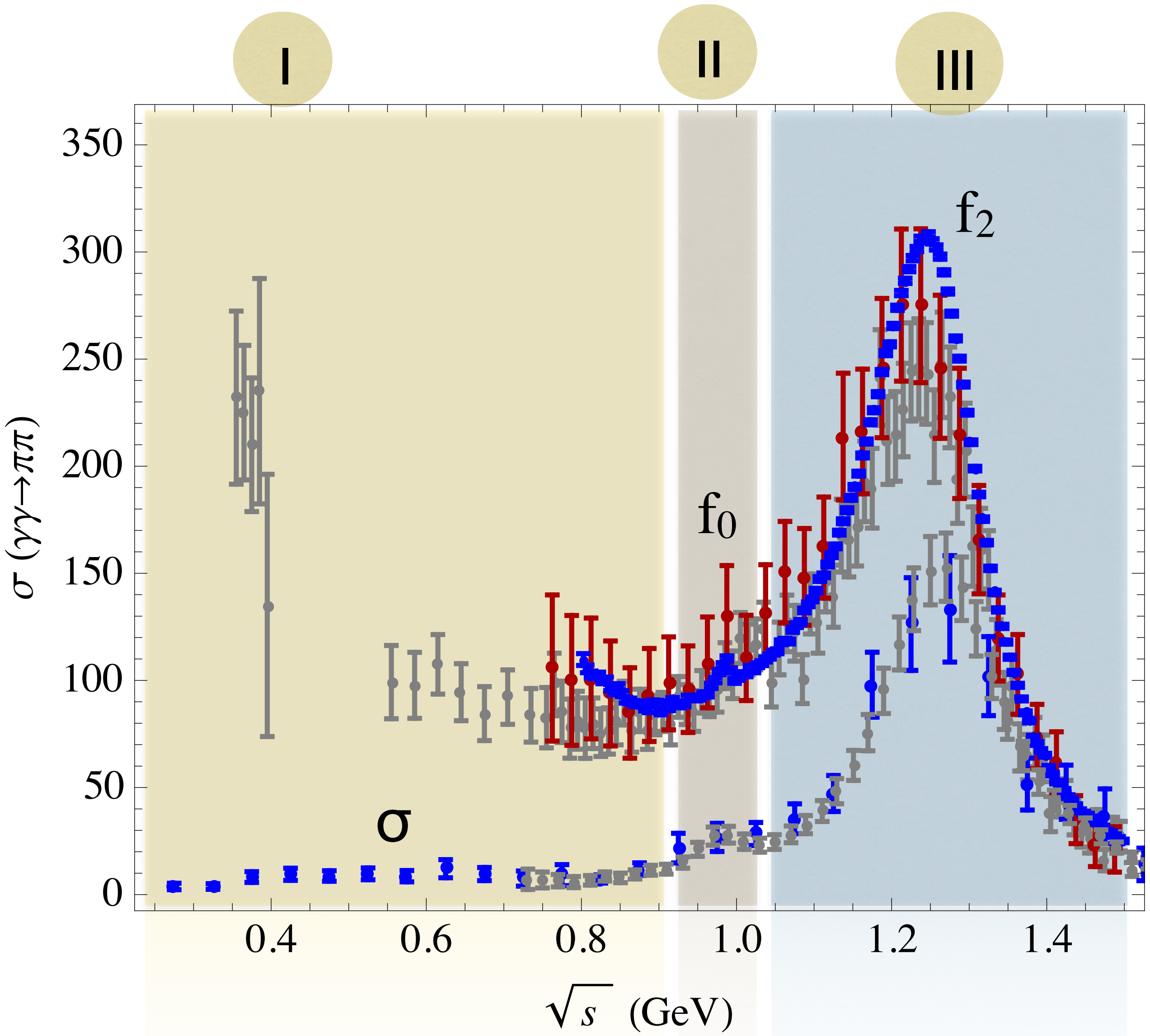


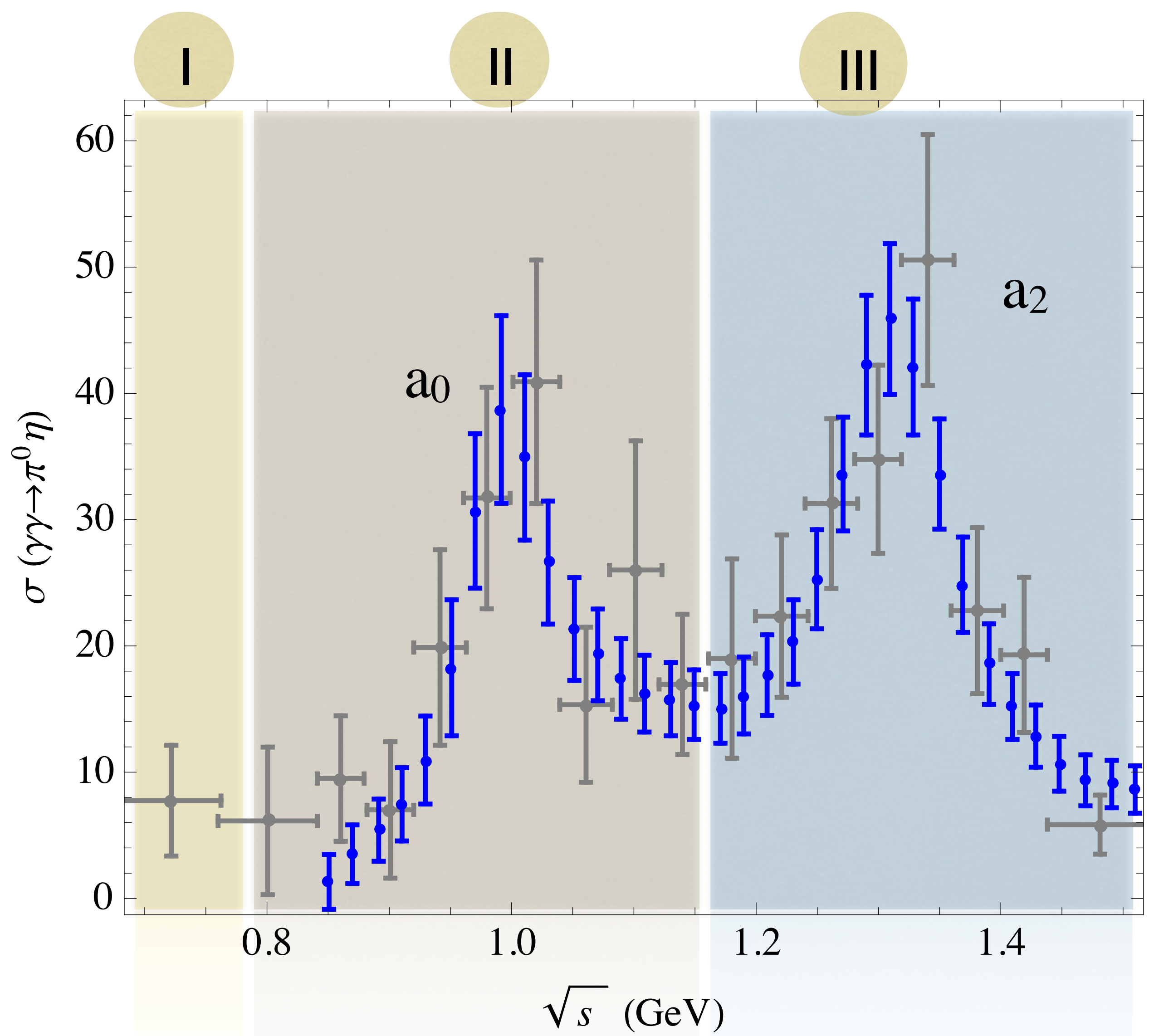



# Experimental data

γγ→π⁺π⁻, π⁰π⁰




γγ→π⁰η




$\gamma\gamma \to \pi^+\pi^-$: Mark II ('90), CELLO ('92), Belle ('07)
$\gamma\gamma \to \pi^0\pi^0$: Crystal Ball ('90), Belle ('09)
$\gamma\gamma \to \pi^0\eta$: Crystal Ball ('86), Belle ('09)

$\gamma\gamma \to \eta\eta$: Belle ('10)
$\gamma\gamma \to KK$: ARGUS ('90), TASSO ('85), CELLO ('89), Belle ('13)

Ongoing experiment
$\gamma\gamma^* \to \pi\pi$, $\pi\eta$: BES III

# What has been done so far?

| $Q^2 = 0$ | Approach | Inelasticity | Number of fitted parameters to $\sigma_{\gamma\gamma\to MM}$ | Range of applicability |
|---|---|---|---|---|
| [Hoferichter et. al. 2011] | Roy-Steiner | $\pi\pi$ | 0 | $\sqrt{s} < 0.98$ GeV |
| [Morgan et. al. 1998] | Disp, Omnes | $\pi\pi$ | 0 | $\sqrt{s} \lesssim 0.6$ GeV |
| [Dai et. al. 2014] | Amplitude anal. | $\pi\pi$, KK | >20 | $\sqrt{s} < 1.5$ GeV |
| [Garcia-Martin et.al. 2010] | Disp, Omnes | $\pi\pi$, KK | 6 | $\sqrt{s} < 1.3$ GeV |
| [Current work] | Disp, Omnes | $\pi\pi$, KK<br>$\pi\eta$, KK | 0 | $\sqrt{s} < 1.4$ GeV |
| $Q^2 \neq 0$ | | | | |
| [Moussallam 2013] | Disp, Omnes | $\pi\pi$, J=0 | 0 | $\sqrt{s} \lesssim 0.8$ GeV |
| [Colangelo et.al. 2017] | Roy-Steiner | $\pi\pi$, J=0 | 0 | $\sqrt{s} \lesssim 0.8$ GeV |
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Only dispersive analyses are shown

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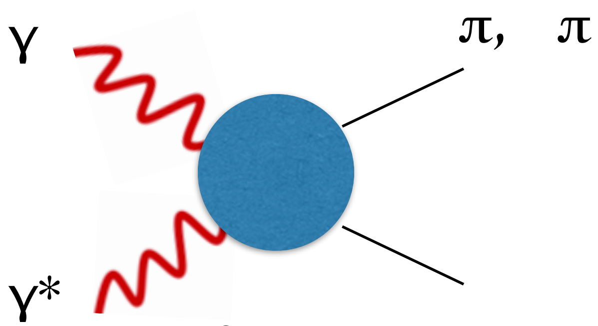
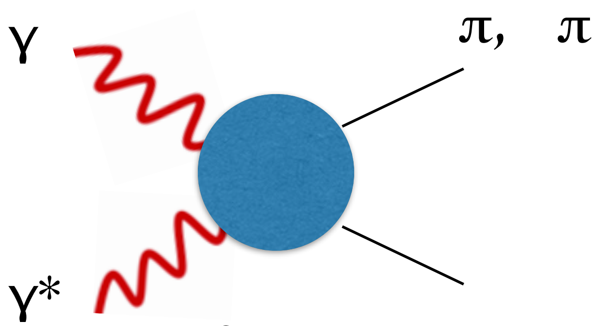



$$q_1^2 = 0$$
$$q_2^2 = -Q^2$$

C=+1: $J^{PC}=0^{++}, 2^{++}, 1^{-+}, \ldots$

$Q^2 \neq 0$ Landau-Yang theorem

# Cross section

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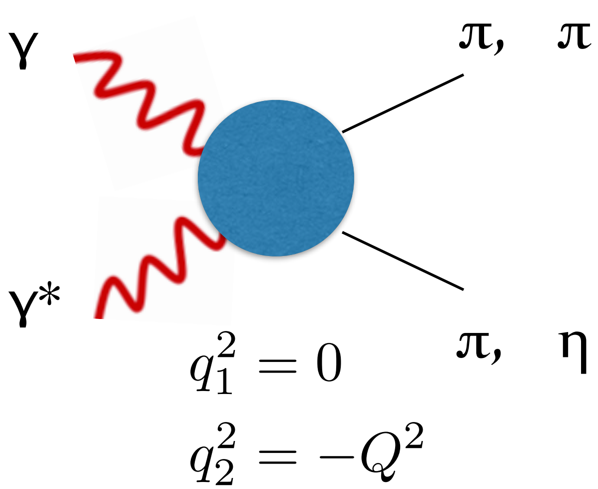
Landau-Yang theorem

Helicity amplitudes

$$H_{\lambda_1 \lambda_2} = H^{\mu\nu} \epsilon_\mu(\lambda_1)\, \epsilon_\nu(\lambda_2), \quad \lambda_1 = \pm 1,\ \lambda_2 = \pm 1, 0$$

# Cross section

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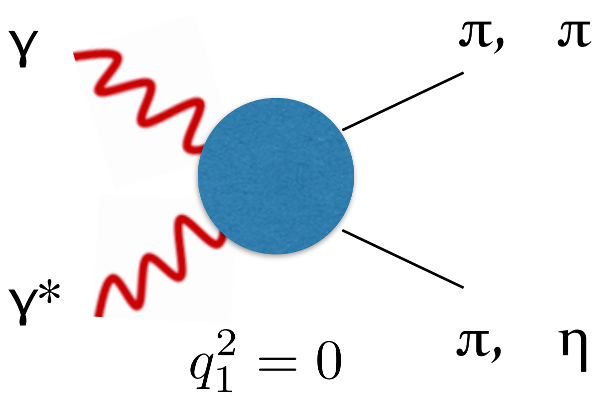
P symmetry: **6** ➜ **3** independent amplitudes

$$H_{++}, H_{+-}, H_{+0}$$

# Cross section

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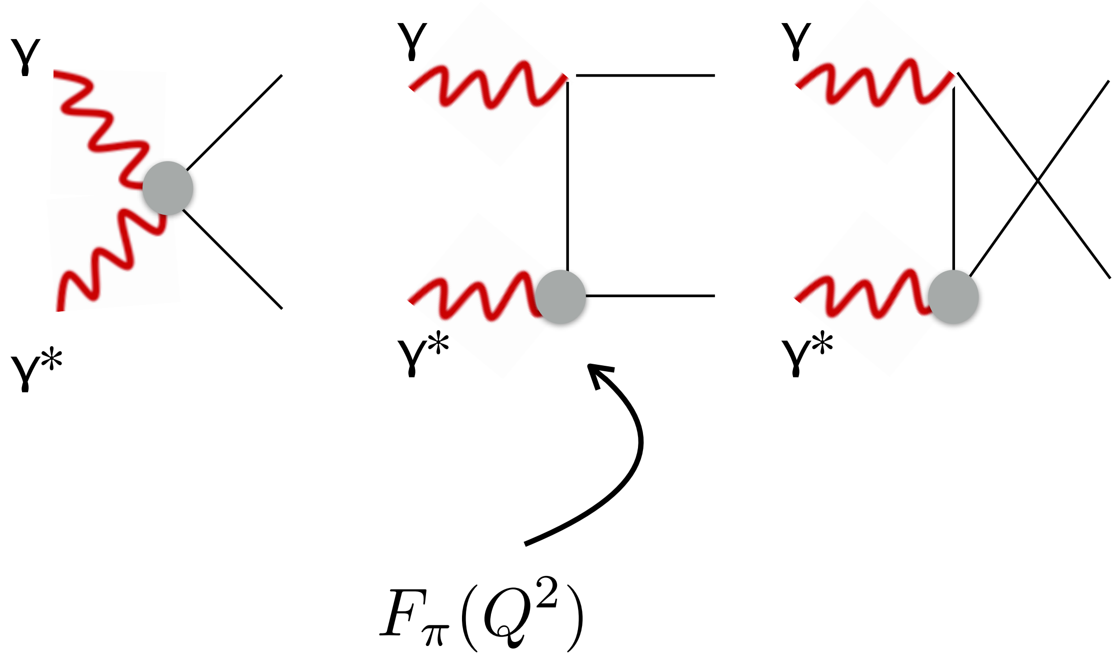
P symmetry: **6** → **3** independent amplitudes

$$H_{++}, H_{+-}, H_{+0}$$

Differential cross section

$$\frac{d\sigma}{d\ \cos\theta} = \pi\alpha^2 \frac{\rho(s)}{4\,(s+Q^2)} \left(|H_{++}|^2 + |H_{+-}|^2 + |H_{+0}|^2\right)$$

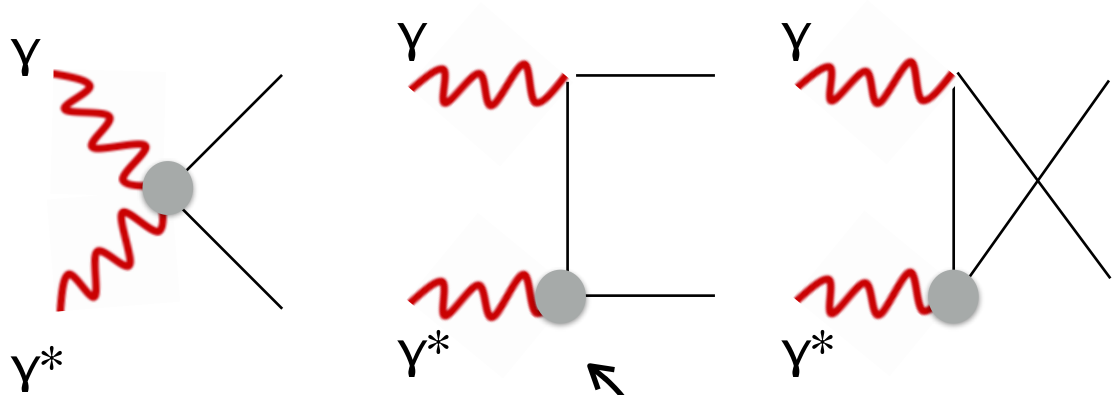
# *Born amplitudes ($Q^2 \neq 0$)*




Vertex ππγ*

$$\langle \pi^+ | j_\mu(0) | \pi^+(p') \rangle = (p + p')_\mu \, F_\pi(Q^2)$$

# Born amplitudes ($Q^2 \neq 0$)




$F_\pi(Q^2)$

Vertex ππγ*

$$\langle \pi^+ | j_\mu(0) | \pi^+(p') \rangle = (p+p')_\mu \, F_\pi(Q^2)$$

Space-like region

$$F_\pi(Q^2) = \frac{1}{1 + Q^2/M_\rho^2}$$

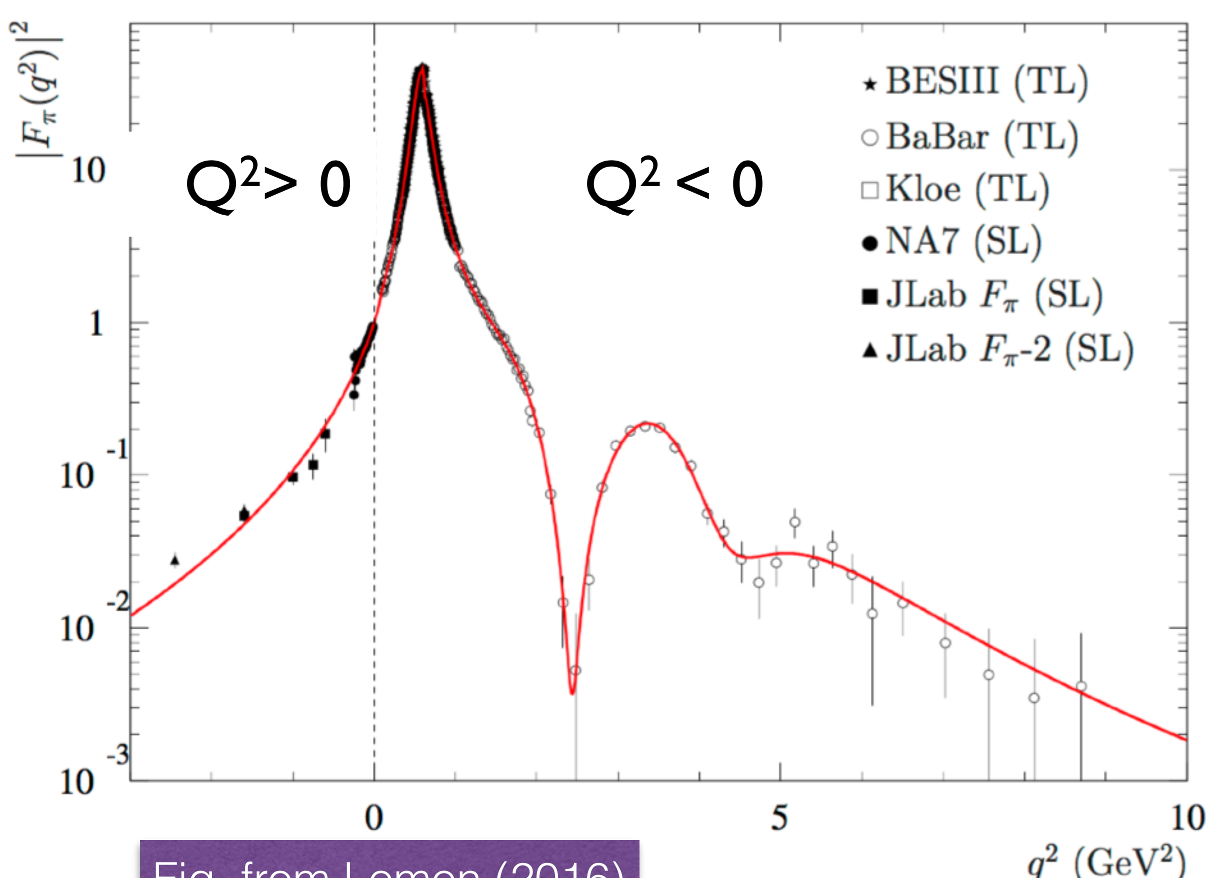
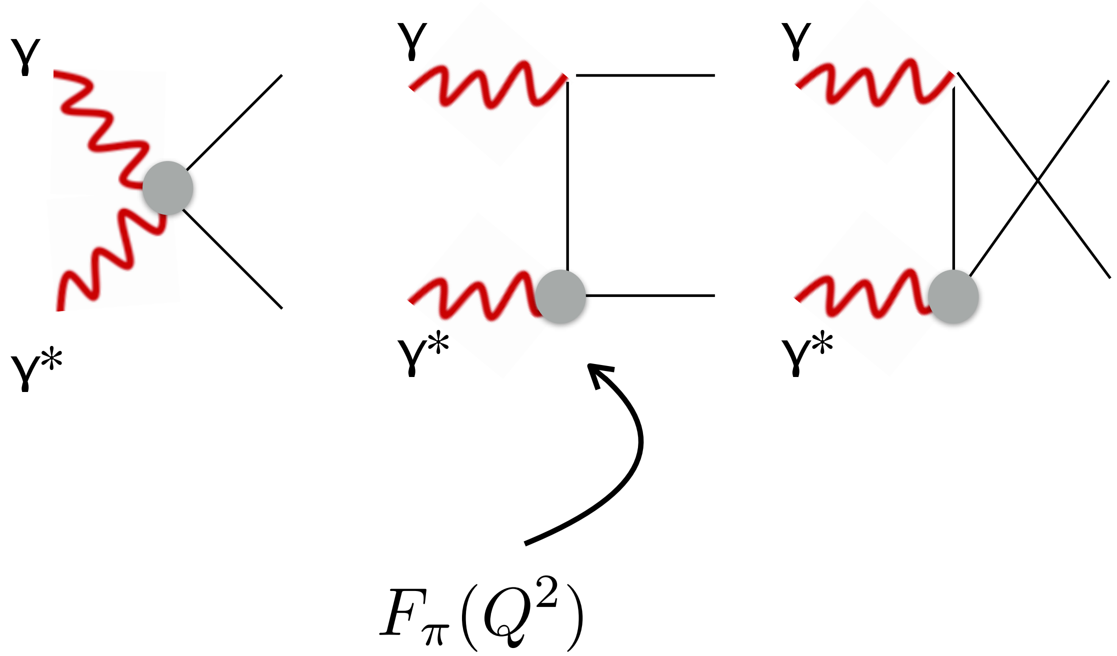



Fig. from Lomon (2016)

# *Born amplitudes ($Q^2 \neq 0$)*

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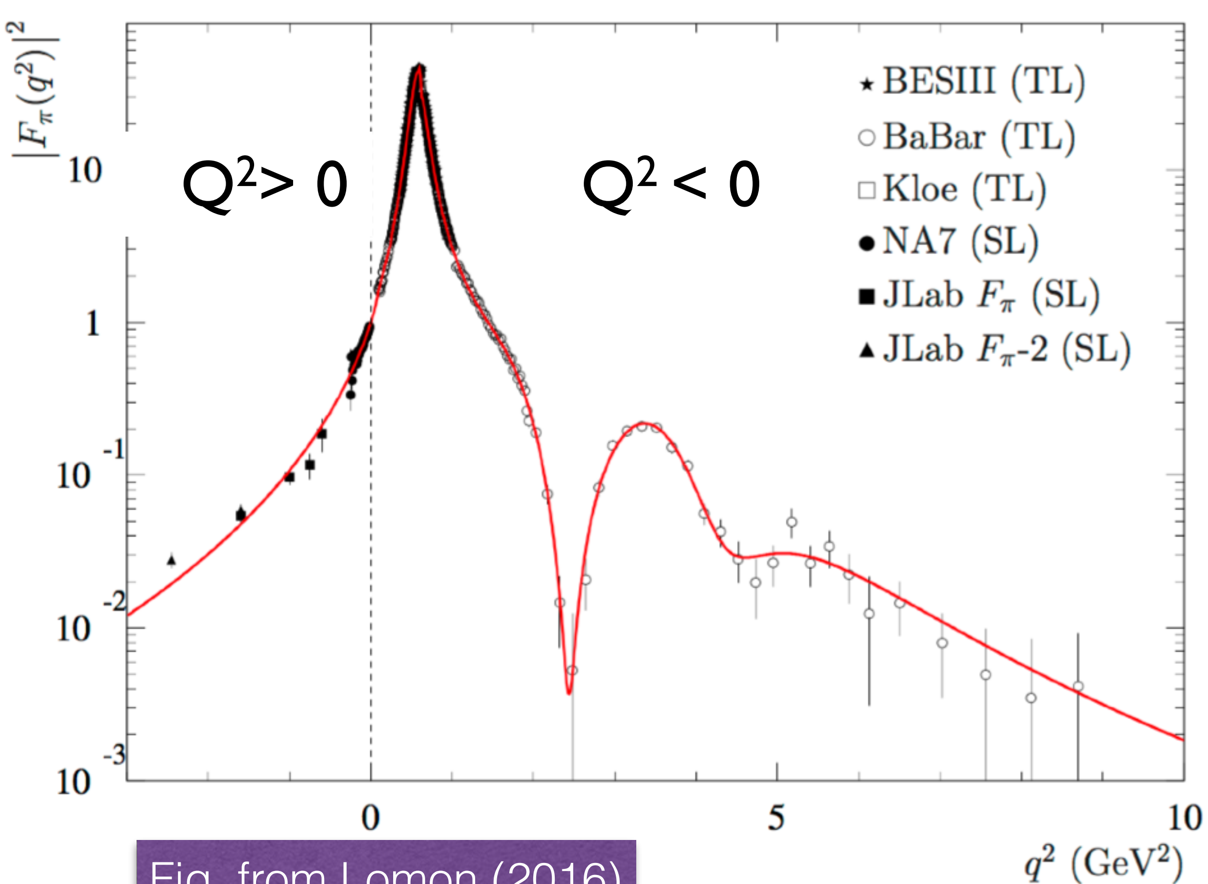


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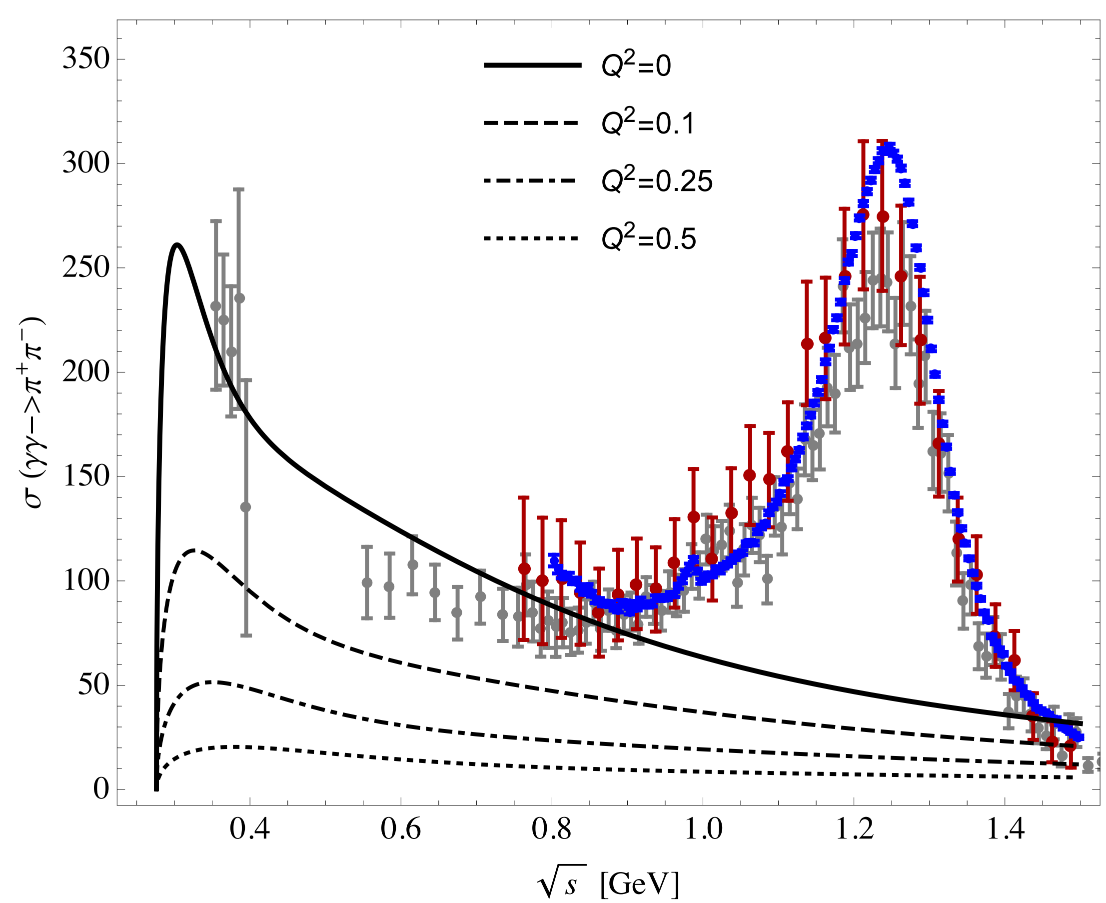
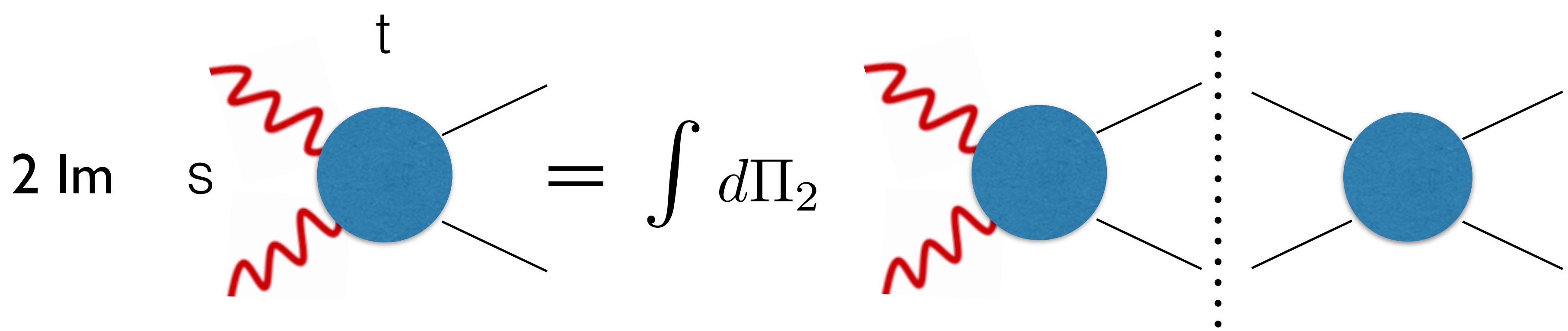


# *Unitarity*



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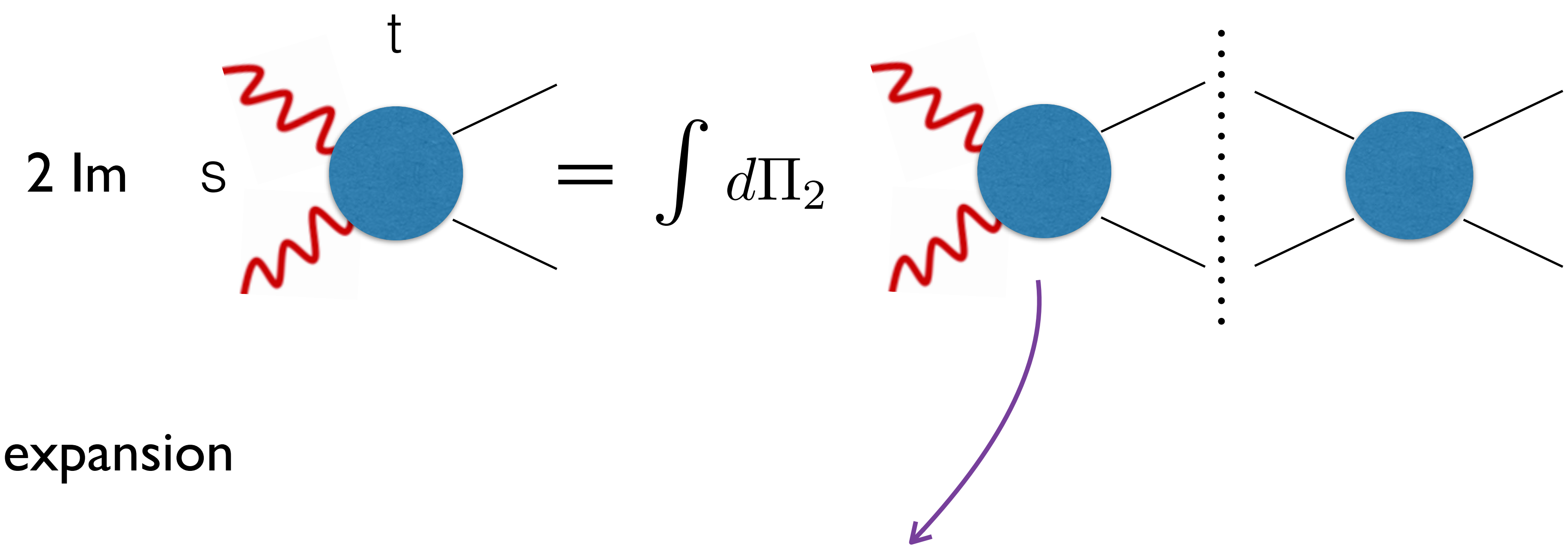
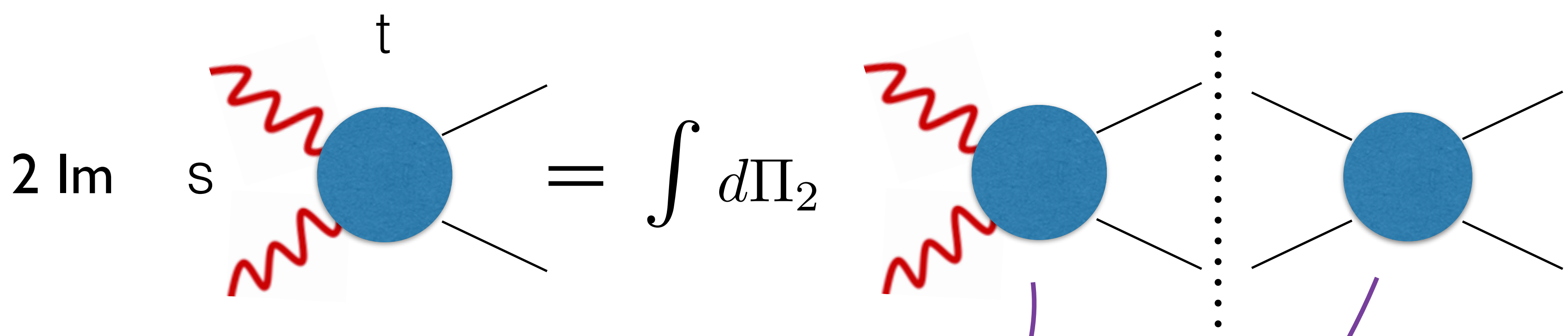



Partial wave expansion

$$H_{\lambda_1 \lambda_2}(s,t) = \sum_{J=0}^{\infty} (2J+1)\, h_{J,\lambda_1\lambda_2}(s)\, d^J_{\lambda_1-\lambda_2,0}(\theta)$$

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$$T(s,t) = \sum_{J=0}^{\infty}(2J+1)\, t_J(s)\, P_J(\theta)$$

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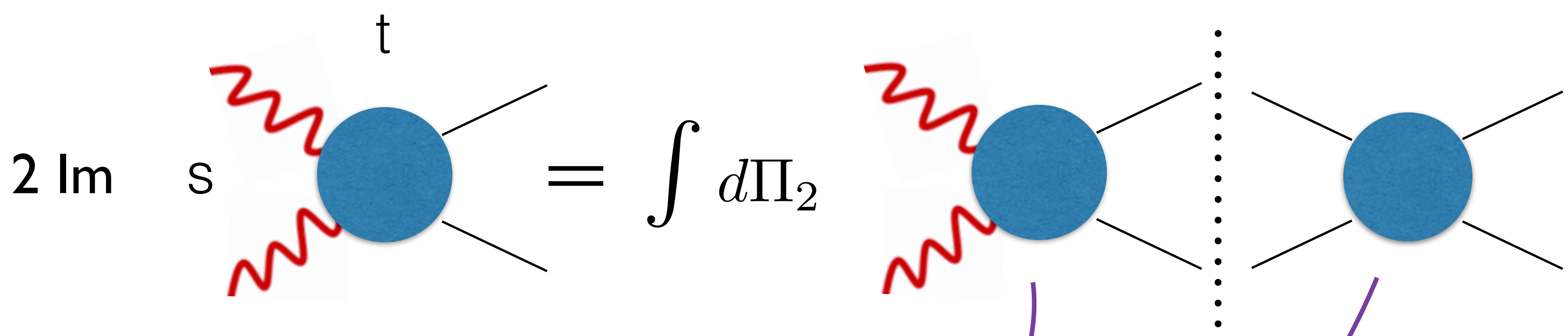


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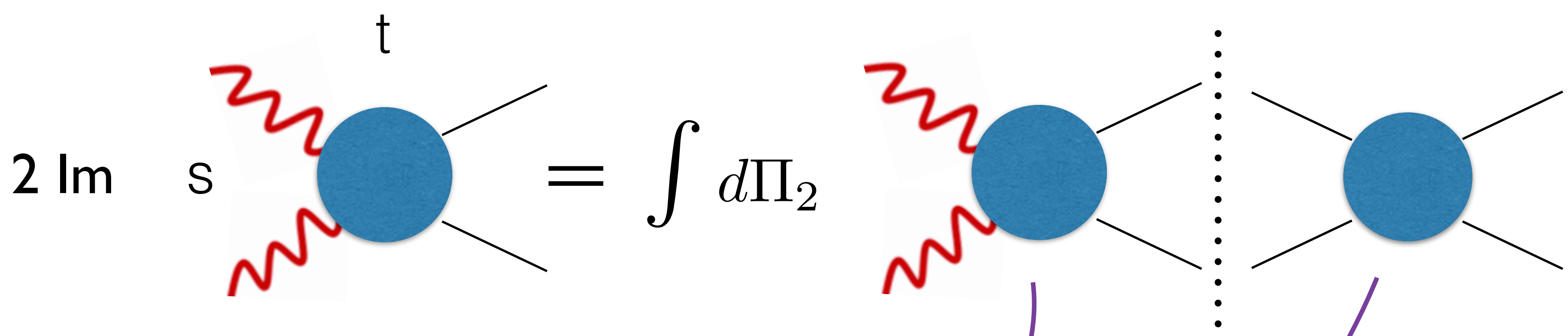
$$T(s,t) = \sum_{J=0}^{\infty}(2J+1)\, t_J(s)\, P_J(\theta)$$

These "diagonalise unitarity" and contain resonance information

Definite: $J$, $\lambda_1$, $\lambda_2$

$$\mathrm{Im}\, h_{\gamma\gamma^*\to\pi\pi}(s) = h_{\gamma\gamma^*\to\pi\pi}(s)\, \rho_{\pi\pi}(s)\, t^*_{\pi\pi\to\pi\pi}(s)$$

# *Unitarity*




Partial wave expansion

$J_{max} = 2$

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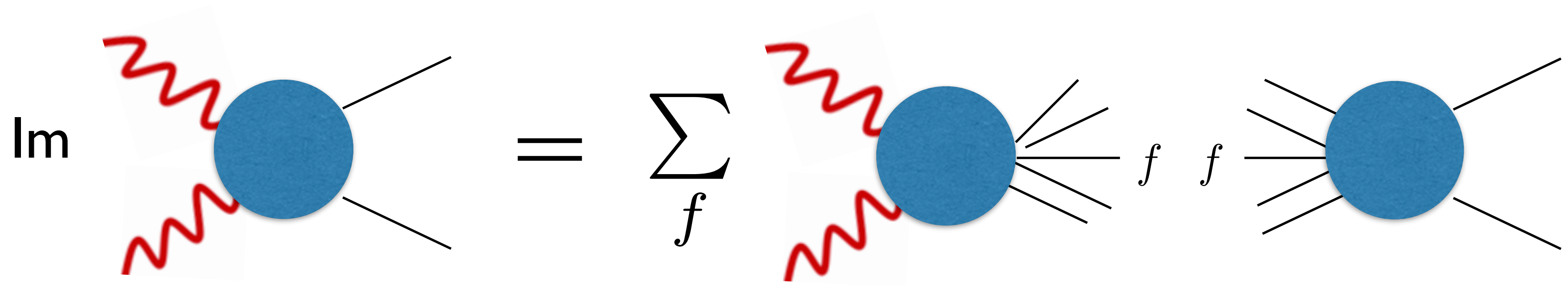
# Coupled channel Unitarity

Coupled-channel unitarity

Definite: $J$, $\lambda_1$, $\lambda_2$




$$\mathrm{Im}\, h_{\gamma\gamma^*,b}(s) = \sum_f h_{\gamma\gamma^*,f}(s)\, \rho_f(s)\, t^*_{fb}(s)$$

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Coupled-channel unitarity

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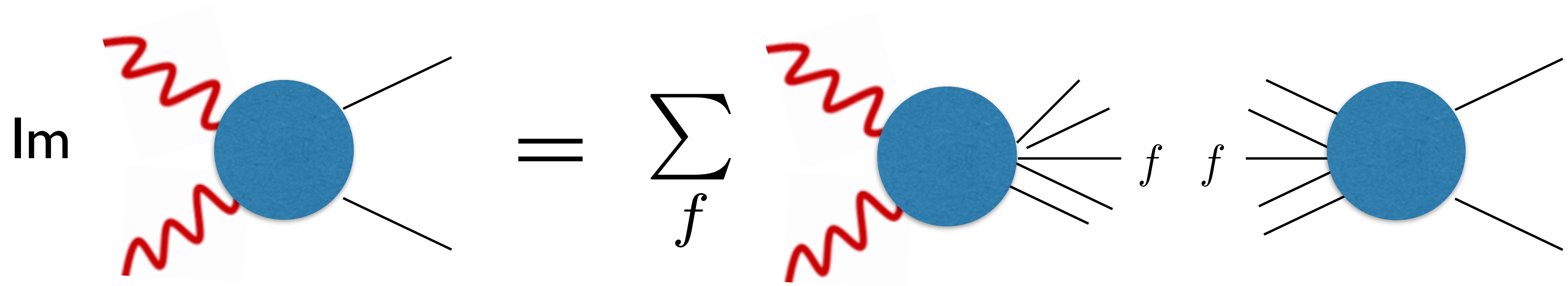


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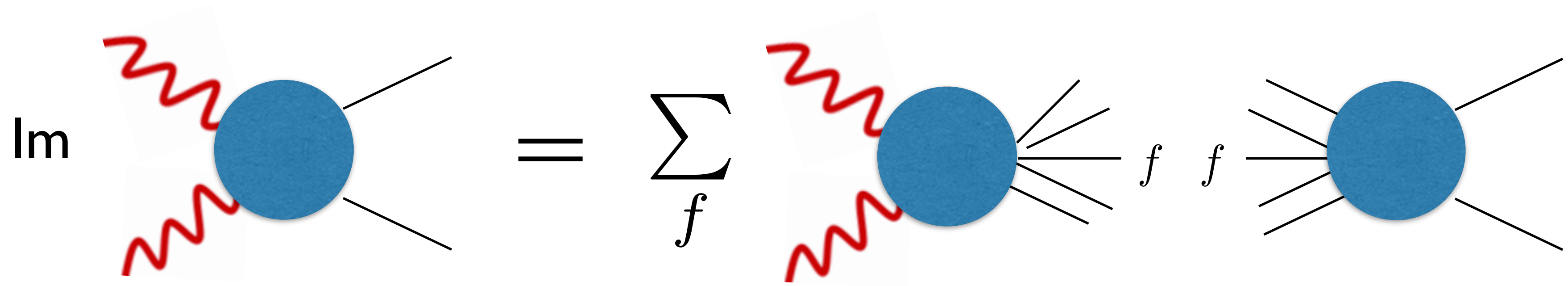
$$\begin{aligned}
\mathrm{Im}\, h_{\gamma\gamma^*,1}(s) &= \rho_1\, h_{\gamma\gamma^*,1}\, t^*_{11} + \rho_2\, h_{\gamma\gamma^*,2}\, t^*_{21} \\
\mathrm{Im}\, h_{\gamma\gamma^*,2}(s) &= \rho_1\, h_{\gamma\gamma^*,1}\, t^*_{12} + \rho_2\, h_{\gamma\gamma^*,2}\, t^*_{22}
\end{aligned}$$

1 = $\pi\pi$
2 = KK

# Coupled channel Unitarity

Coupled-channel unitarity

Definite: $J, \lambda_1, \lambda_2$

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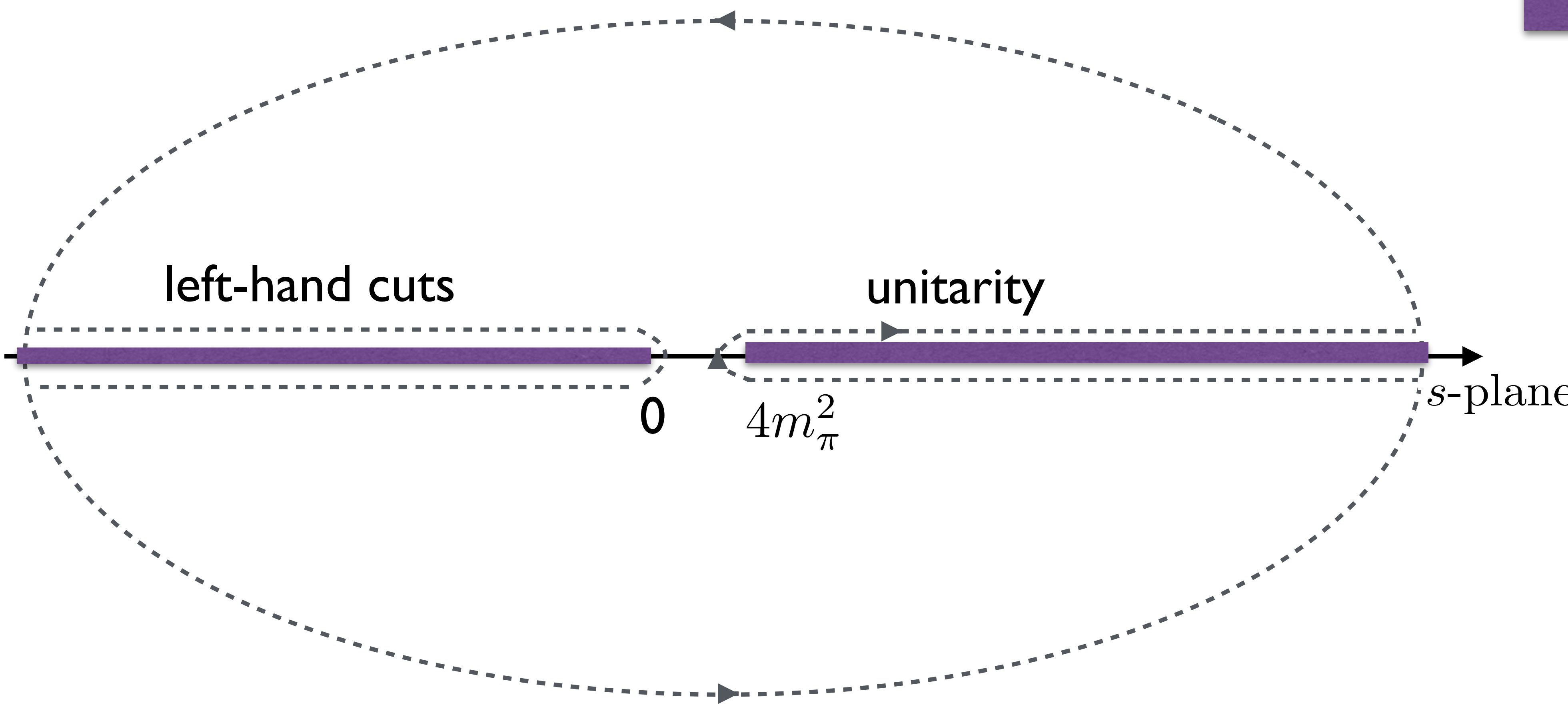
1 = $\pi\pi$
2 = KK

Entire dynamical information that does not depend on the underlying theory (e.g. QCD) comes from **unitarity**

# Dispersion relation

Definite: $J$, $\lambda_1$, $\lambda_2$




$$h(s) = \frac{1}{2\pi i}\int_C ds' \frac{h(s')}{s'-s} = \int_{-\infty}^{0} \frac{ds'}{\pi}\frac{\mathrm{Im}\, h(s')}{s'-s} + \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi}\frac{\mathrm{Im}\, h(s')}{s'-s}$$

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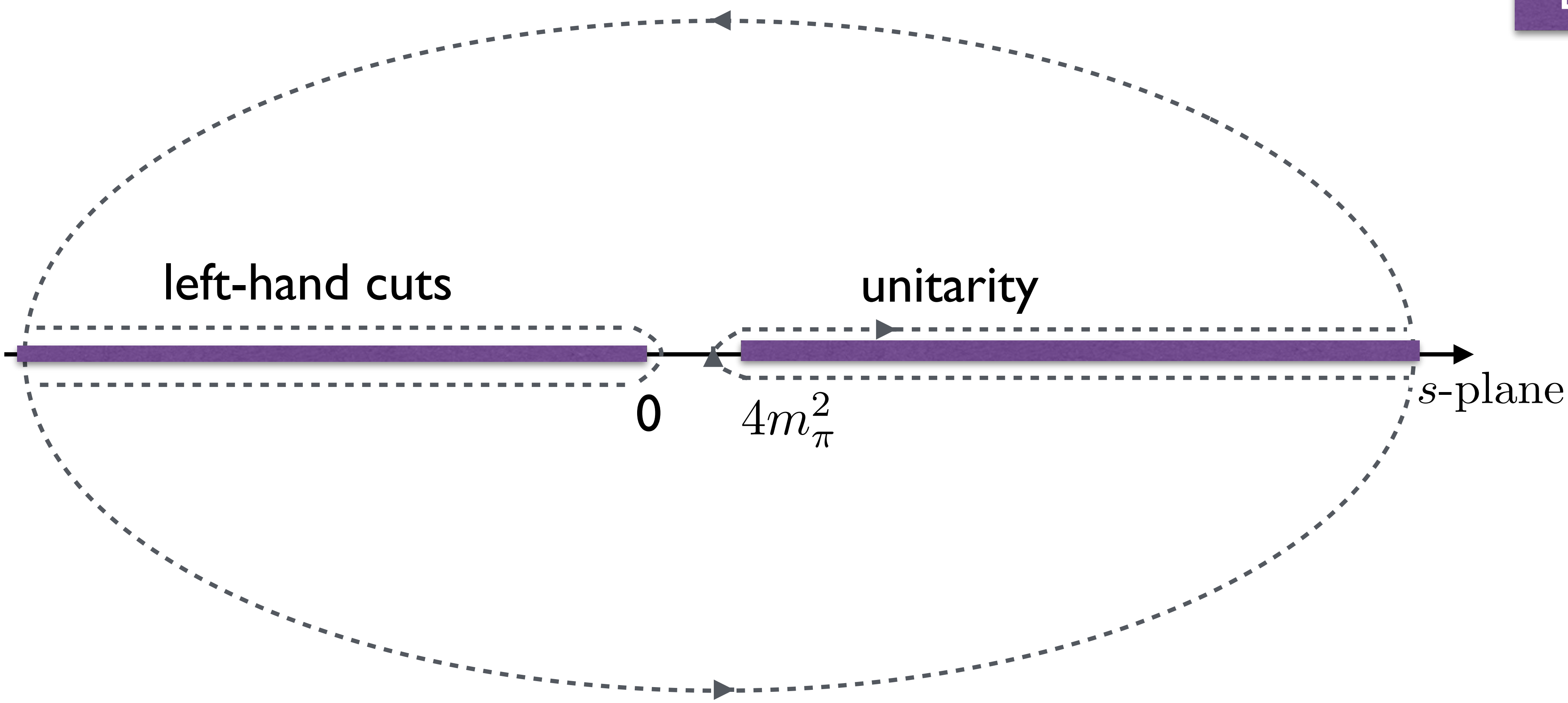


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**analyticity** relates scattering amplitude at different energies

# *Dispersion relation*

Left and right-hand cuts

Definite: $J$, $\lambda_1$, $\lambda_2$

$$h(s) = \int_{-\infty}^{0} \frac{ds'}{\pi} \frac{\operatorname{Im} h(s')}{s'-s} + \int_{4m_\pi^2}^{\infty} \frac{ds'}{\pi} \frac{\operatorname{Im} h(s')}{s'-s}$$

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Looking for a solution in the form

$$h(s) = h^{Born}(s) + \Omega(s)\, N(s)$$

$$s > 4m_\pi^2$$
$$\mathrm{Im}\,\Omega(s) = \Omega(s)\,\rho(s)\,t^*(s)$$
$$\mathrm{Im}\, h(s) = h(s)\,\rho(s)\,t^*(s)$$

Omnes (1958)
Morgan et. al. (1998)

Dispersive integral (twice subtracted) for J=0

$$h(s) = h^{Born}(s) + \Omega(s)\Big(a + b\, s + \frac{s^2}{\pi} \int_{-\infty}^{-s_L} \frac{ds'}{s'^2} \frac{\mathrm{Im}(h(s'))\Omega^{-1}(s)}{s'-s} - \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{h^{Born}(s')\,\mathrm{Im}(\Omega^{-1}(s'))}{s'-s}\Big)$$

similar eq. for coupled-channel (ππ,KK)

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Definite: $J, \lambda_1, \lambda_2$

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$Q^2$ - dependent

see also Moussallam (2013)

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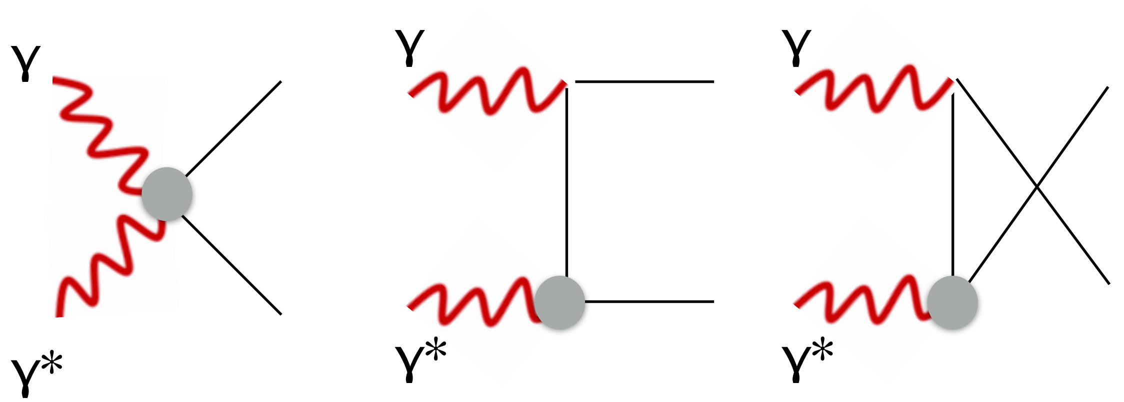
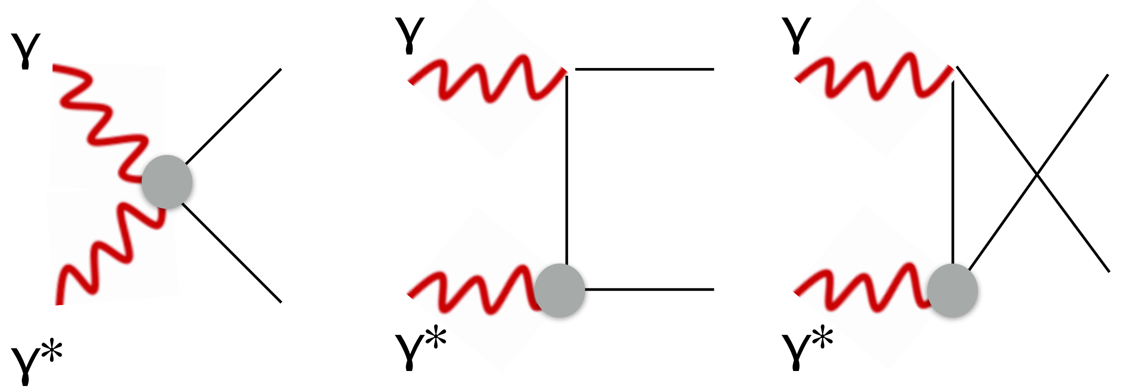
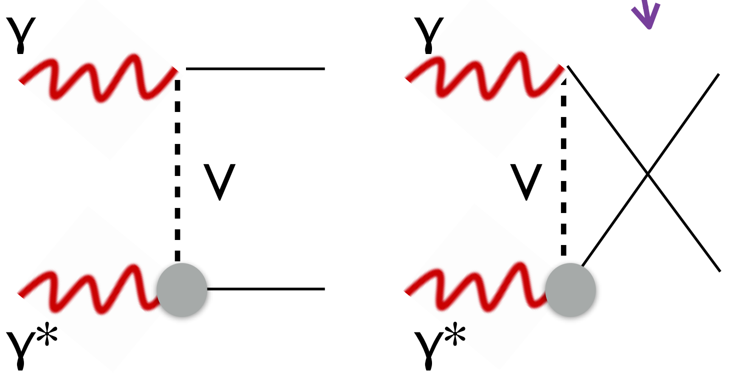


Dispersive integral for J=0

$$h(s) = h^{Born}(s) + \Omega(s)\Big(a + b\,s + \frac{s^2}{\pi}\int_{-\infty}^{-s_L}\frac{ds'}{s'^2}\frac{\mathrm{Im}(h(s'))\Omega^{-1}(s)}{s'-s} - \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{h^{Born}(s')\,\mathrm{Im}(\Omega^{-1}(s'))}{s'-s}\Big)$$

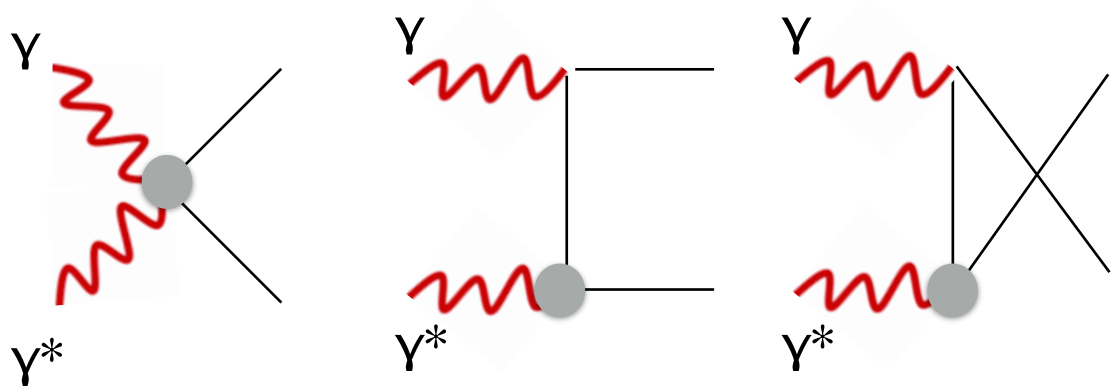


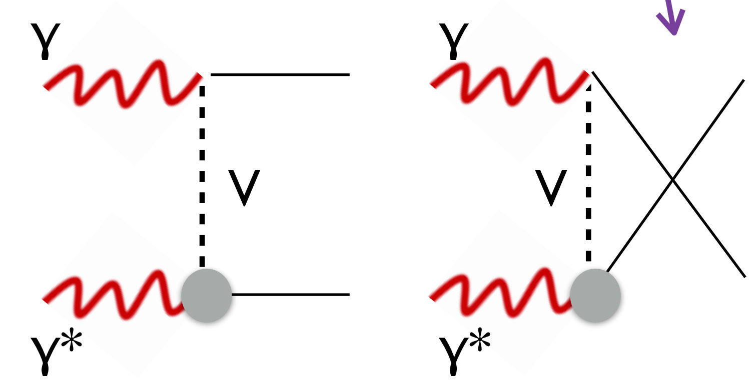



# Dispersion relation

Dispersive integral for J=0

$$h(s) = h^{Born}(s) + \Omega(s)\Big(a + b\,s + \frac{s^2}{\pi}\int_{-\infty}^{-s_L}\frac{ds'}{s'^2}\frac{\mathrm{Im}(h(s'))\Omega^{-1}(s)}{s'-s} - \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{h^{Born}(s')\,\mathrm{Im}(\Omega^{-1}(s'))}{s'-s}\Big)$$

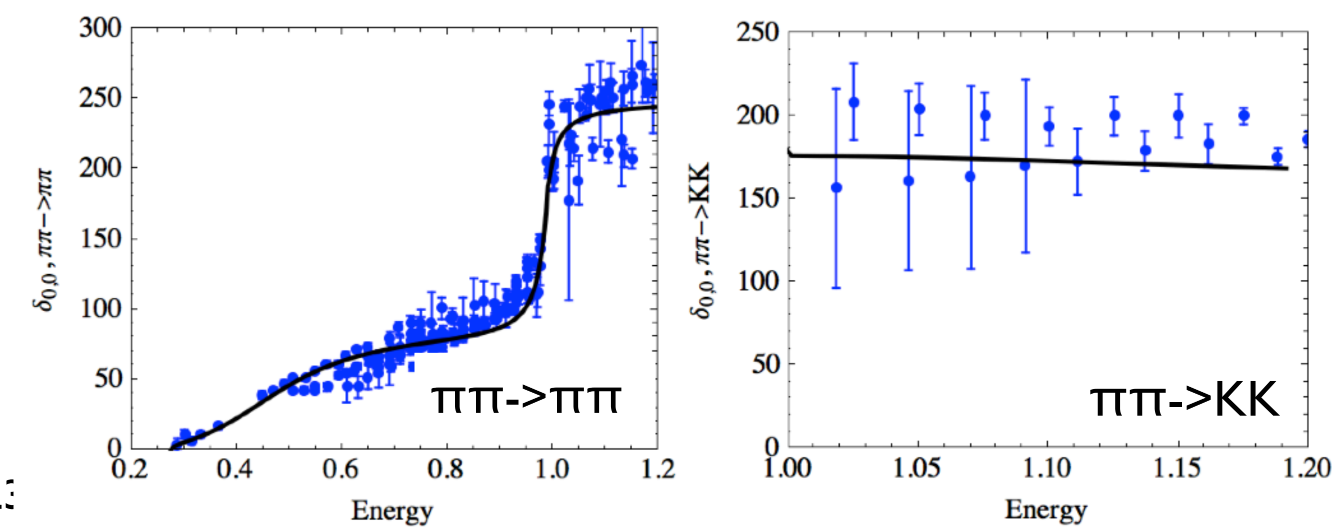



Coupled channel Omnes

$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\pi\to\pi\pi} & \Omega_{\pi\pi\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\pi} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

p.w. disp. relation: left-hand cuts (conformal map)



# Subtraction constants

Dispersive integral for J=0

$$h(s) = h^{Born}(s) + \Omega(s)\Big(a + b\,s + \frac{s^2}{\pi}\int_{-\infty}^{-s_L} \frac{ds'}{s'^2}\frac{\mathrm{Im}(h(s'))\Omega^{-1}(s)}{s'-s}$$
$$- \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\frac{h^{Born}(s')\,\mathrm{Im}(\Omega^{-1}(s'))}{s'-s}\Big)$$

# Subtraction constants

Dispersive integral for J=0

$$h(s) = h^{Born}(s) + \Omega(s)\Big(a + b\,s + \frac{s^2}{\pi}\int_{-\infty}^{-s_L}\frac{ds'}{s'^2}\frac{\mathrm{Im}(h(s'))\Omega^{-1}(s)}{s'-s} - \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{h^{Born}(s')\,\mathrm{Im}(\Omega^{-1}(s'))}{s'-s}\Big)$$

Soft photon limit ($q_1$=0)

$$H_{\lambda_1\lambda_2} \to H^{Born}_{\lambda_1\lambda_2}$$
$$s = -Q^2,\, t = u = m_\pi^2$$

# Subtraction constants

Dispersive integral for J=0

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Soft photon limit ($q_1$=0)

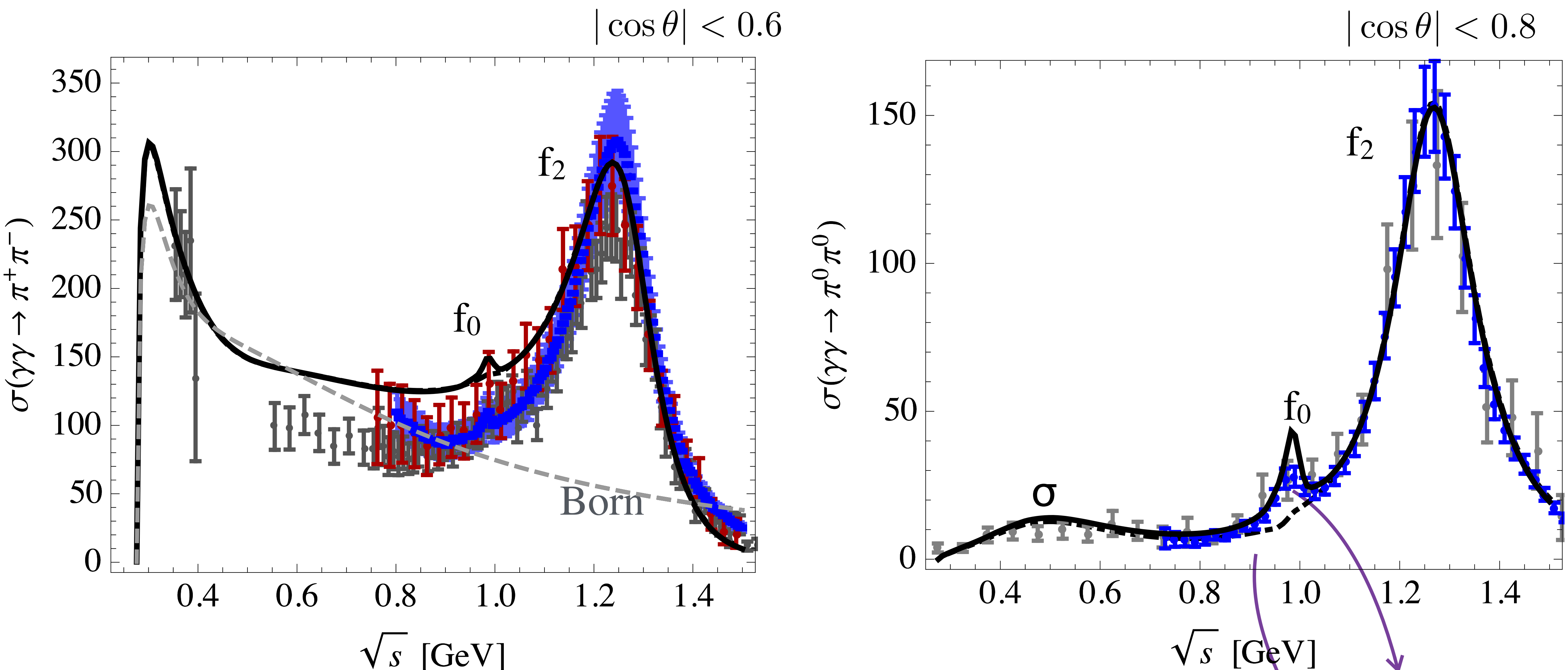
$$H_{\lambda_1\lambda_2} \to H^{Born}_{\lambda_1\lambda_2}$$
$$s = -Q^2,\ t = u = m_\pi^2$$

For space like photons: generalized polarizabilities

$$\pm\frac{2\alpha}{m_\pi}\frac{H^n_{+\pm}}{s+Q^2} = (\alpha_1 \mp \beta_1)_{\pi^0} + \ldots$$
$$\pm\frac{2\alpha}{m_\pi}\frac{(H^c_{+\pm} - H^{Born}_{+\pm})}{s+Q^2} = (\alpha_1 \mp \beta_1)_{\pi^+} + \ldots$$

- COMPASS data on $(\alpha_1-\beta_1)_{\pi^+}$
- future Hall D (JLab) experiment

# $\gamma\gamma \to \pi\pi \ (Q^2\!=\!0)$




<u>Experiment:</u>

$\gamma\gamma \to \pi^+\pi^-$: Mark II ('90), CELLO ('92), Belle ('07)

$\gamma\gamma \to \pi^0\pi^0$: Crystal Ball ('90), Belle ('09)

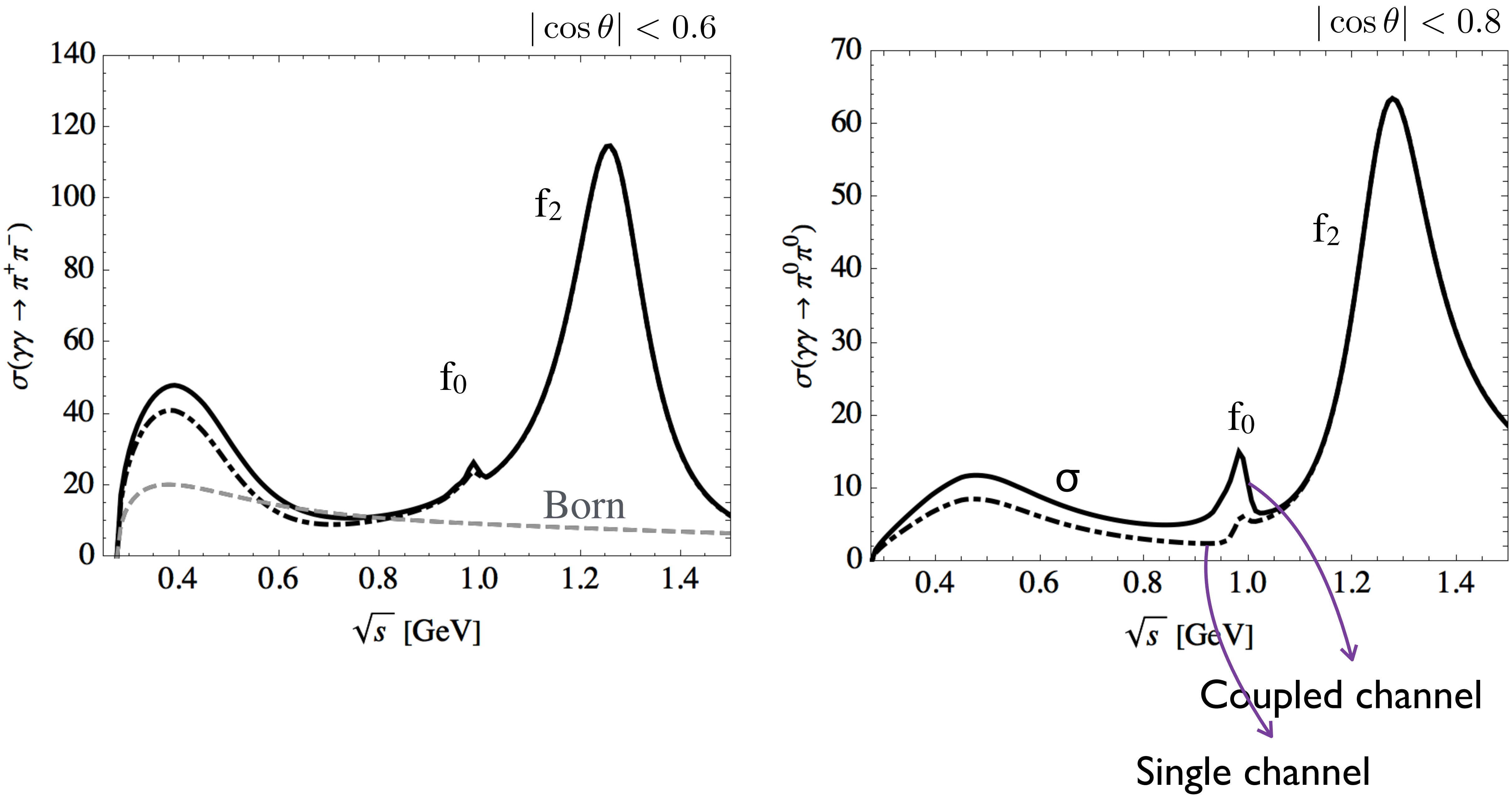
Coupled channel

Single channel

I.D., Vanderhaeghen (work in progress)

see also Dai ('14), Hoferichter ('11), Garcia-Martin et. al ('10)

# $\gamma\gamma \to \pi\pi$ ($Q^2$=0.5)

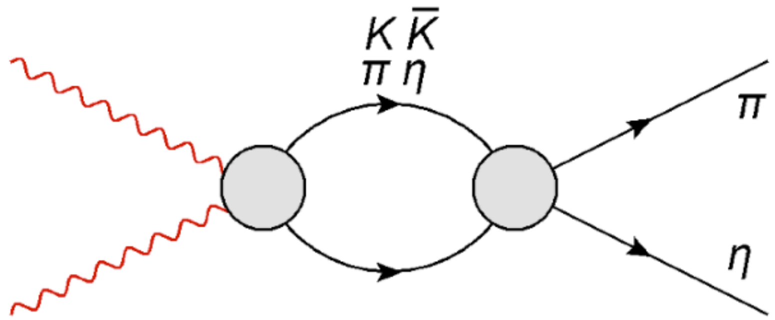



Currently results for $Q^2$=0.5 without VM in the left-hand cut…

Ongoing experiment: BES III

I.D., Vanderhaeghen (work in progress)

# $\gamma\gamma \to \pi\eta \ (Q^2=0)$

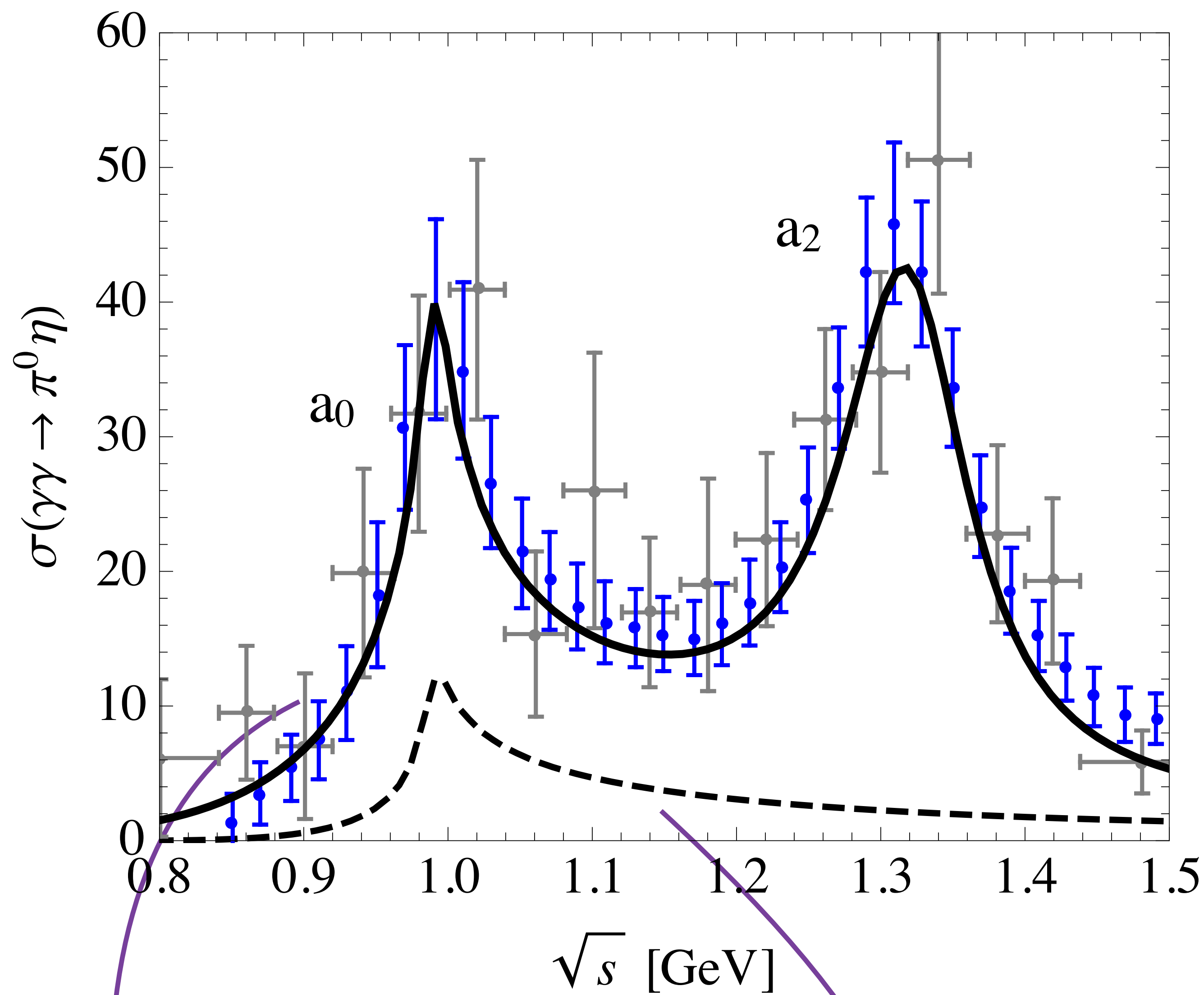



$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\eta\to\pi\eta} & \Omega_{\pi\eta\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\eta} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

I.D., Gil, Lutz
(2011), (2013)

**Coupled-channel** dispersive treatment for J=0 is **crucial**

$a_2$(1230) described as a Breit Wigner resonance




Coupled channel:
**with VM**

Coupled channel:
**no VM**

I.D., Deineka,
Vanderhaeghen
(work in progress)

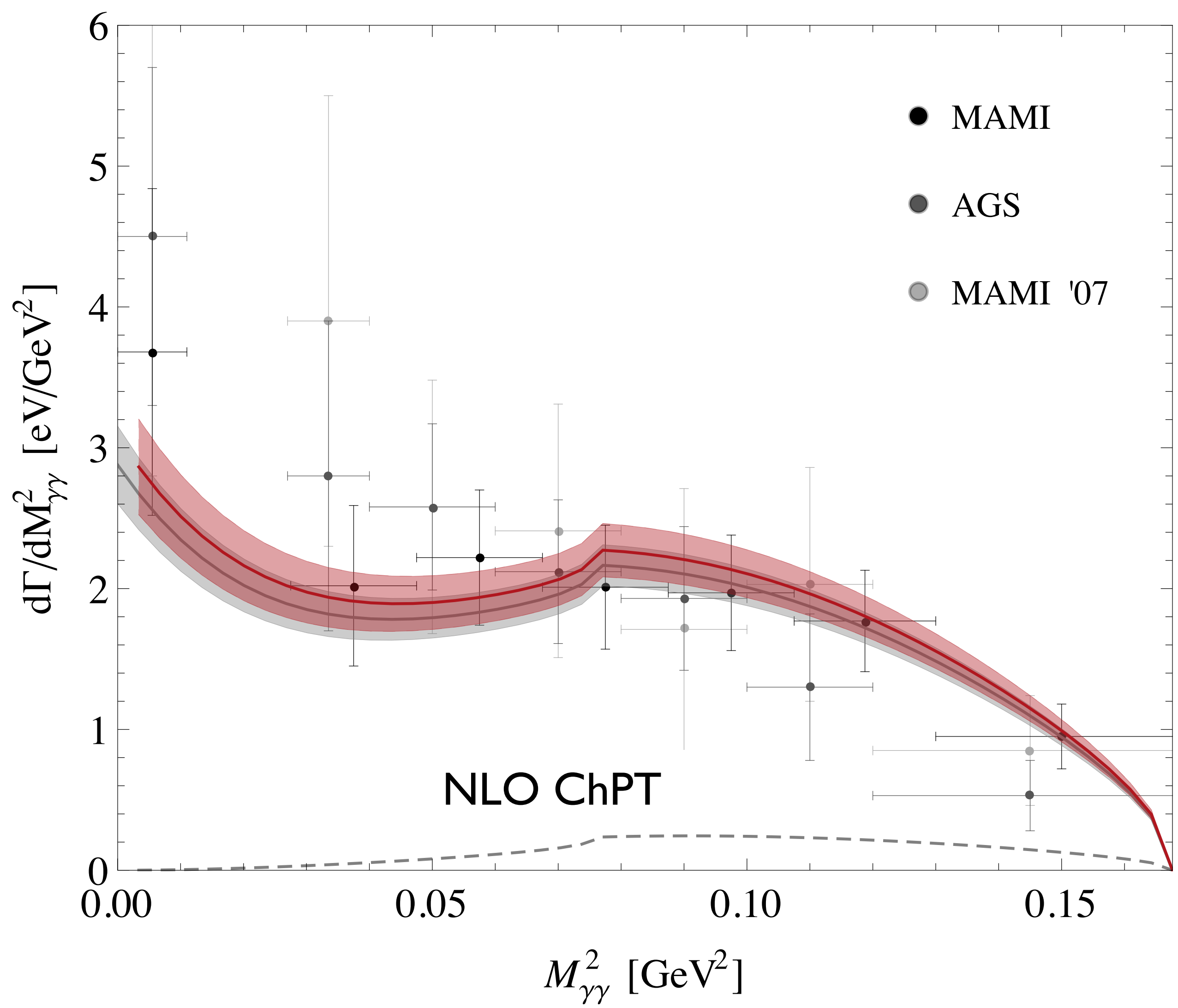
# $\eta \rightarrow \pi^0\gamma\gamma$

$\eta \rightarrow \pi^0\gamma\gamma$ is linked to $\gamma\gamma \rightarrow \pi^0\eta$ by crossing symmetry

$$\frac{d^2\Gamma}{ds\,dt} = \frac{1}{(2\pi)^3}\frac{1}{32\,m_\eta^3}\sum_{\lambda_1,\lambda_2}|H_{\lambda_1\lambda_2}|^2$$

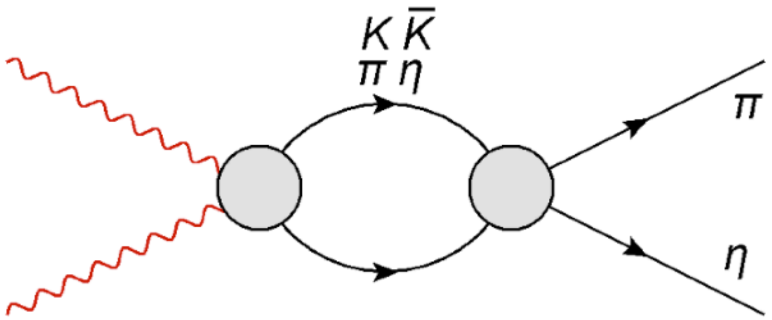
We find

$$\Gamma_{\eta\to\pi^0\gamma\gamma} = 0.291 \pm 0.022 \text{ eV}$$
$$\Gamma^{\text{PDG}}_{\eta\to\pi^0\gamma\gamma} = 0.335 \pm 0.035 \text{ eV}$$




Study η rare decays in **Hall D** (JLab)

# $\gamma\gamma \to \pi\eta$ $(Q^2=0.5)$




$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\eta\to\pi\eta} & \Omega_{\pi\eta\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\eta} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$

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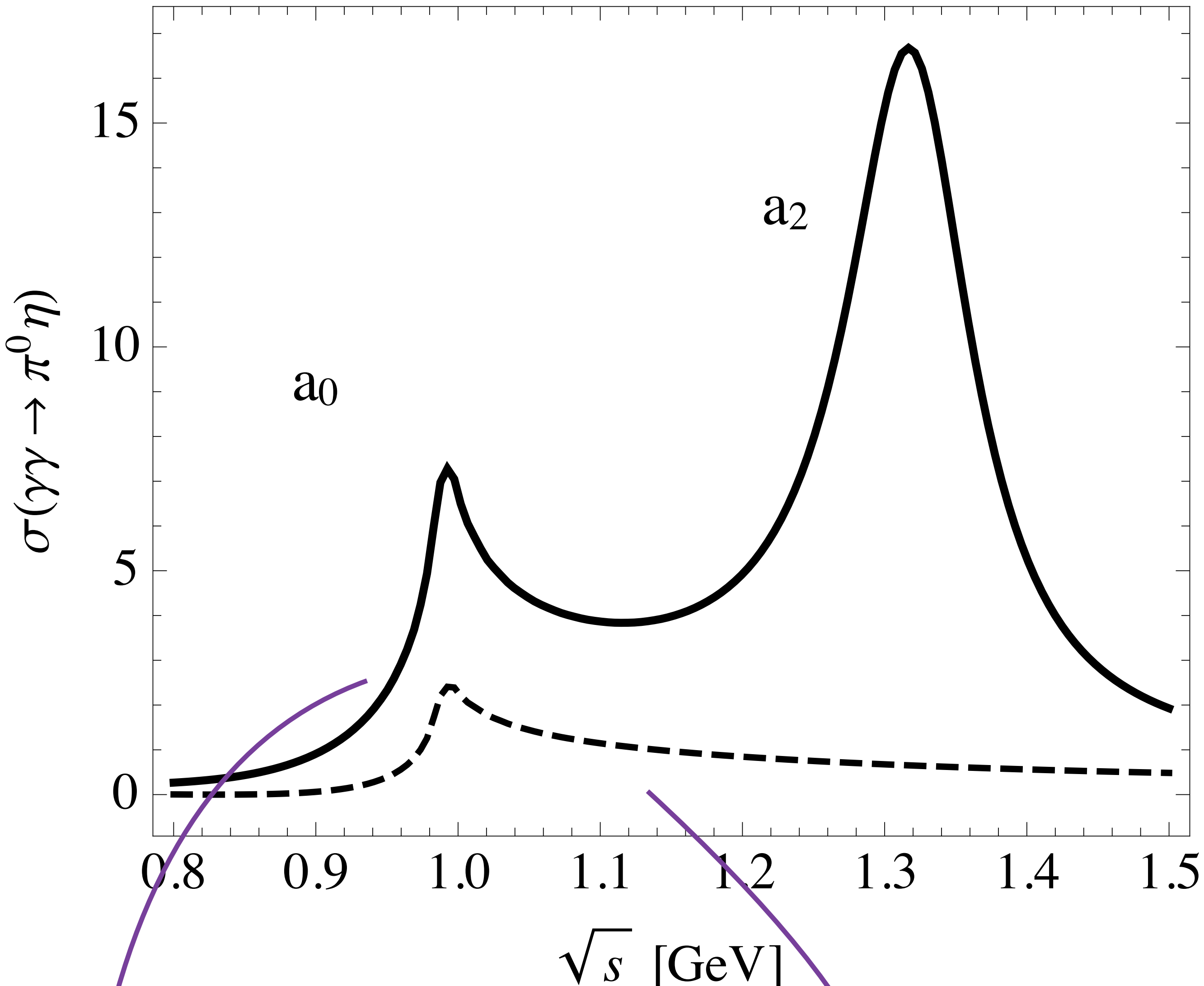



Coupled channel: **with VM**

Coupled channel: **no VM**

I.D., Deineka, Vanderhaeghen (work in progress)

# *Summary and Outlook*

- In light of the **new Belle data (2015)** for $f_2(1270)$ TFFs and using LbL sum rules we **predicted** ($\Lambda$=2) TFF for $f_2(1565)$
- **Update for meson contributions to (g-2) LbL**
  Tensor mesons contributions found to be small compared to anticipated exp. uncertainty $1.6*10^{-10}$

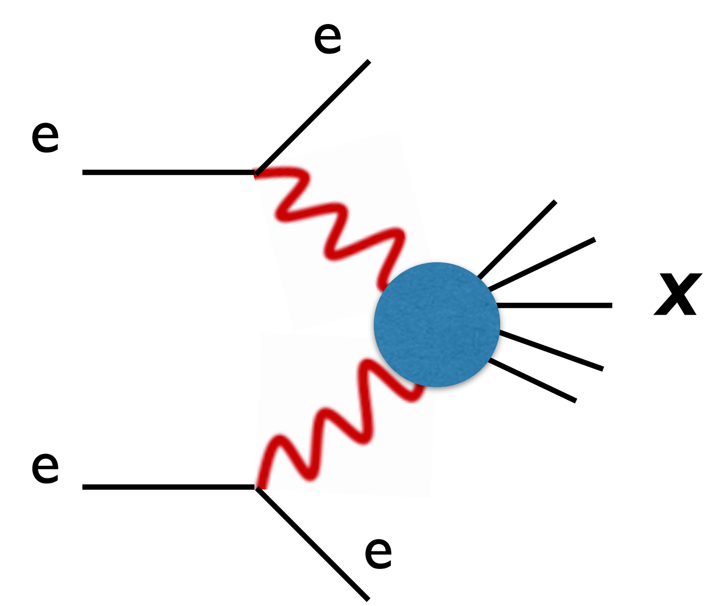
  Axial vector mesons contributions (satisfying Landau-Yang theorem constraint) evaluated by 2 groups and found to be between $(0.64 - 0.75 \pm 0.27)10^{-10}$
- **Next steps?**
  Need to take into account $f_0(500)$ and non resonant contributions in a dispersive approach
- Main ingredients: $\boldsymbol{\gamma\gamma^* \to \pi\pi, \pi\eta}$,… (work in progress). Can be used in **different** (g-2) dispersive approaches.

  It is important to **validate** dispersive treatment of $\boldsymbol{\gamma\gamma^* \to \pi\pi, \pi\eta}$,… with upcoming BES III data

# Extra slides

# Light by light scattering

Observables in experiment $e^+e^- \to e^-e^+X$




$$d\sigma = \frac{\alpha^2}{16\pi^4 Q_1^2 Q_2^2} \frac{2\sqrt{X}}{s(1-4m^2/s)} \cdot \frac{d^3\vec{p}_1'}{E_1'} \cdot \frac{d^3\vec{p}_2'}{E_2'}$$
$$\times \left\{4\rho_1^{++}\rho_2^{++}\sigma_{TT} + \rho_1^{00}\rho_2^{00}\sigma_{LL} + 2\rho_1^{++}\rho_2^{00}\sigma_{TL} + 2\rho_1^{00}\rho_2^{++}\sigma_{LT}\right.$$
$$+2\left(\rho_1^{++}-1\right)\left(\rho_2^{++}-1\right)\left(\cos 2\tilde{\phi}\right)\tau_{TT} + 8\left[\frac{\left(\rho_1^{00}+1\right)\left(\rho_2^{00}+1\right)}{\left(\rho_1^{++}-1\right)\left(\rho_2^{++}-1\right)}\right]^{1/2}\left(\cos\tilde{\phi}\right)\tau_{TL}$$
$$\left.+h_1h_2\,4\left[\left(\rho_1^{00}+1\right)\left(\rho_2^{00}+1\right)\right]^{1/2}\tau_{TT}^a + h_1h_2\,8\left[\left(\rho_1^{++}-1\right)\left(\rho_2^{++}-1\right)\right]^{1/2}\left(\cos\tilde{\phi}\right)\tau_{TL}^a\right\},$$

# Born amplitudes ($Q^2 \neq 0$)

Differential cross section

$$\frac{d\sigma}{d\,\cos\theta} = \pi\alpha^2 \frac{\rho(s)}{4\,(s+Q^2)}\,\left(|H_{++}|^2 + |H_{+-}|^2 + |H_{+0}|^2\right)$$

# *Born amplitudes ($Q^2 \neq 0$)*

Differential cross section

$$\frac{d\sigma}{d\cos\theta} = \pi\alpha^2 \frac{\rho(s)}{4\,(s+Q^2)} \left(|H_{++}|^2 + |H_{+-}|^2 + |H_{+0}|^2\right)$$

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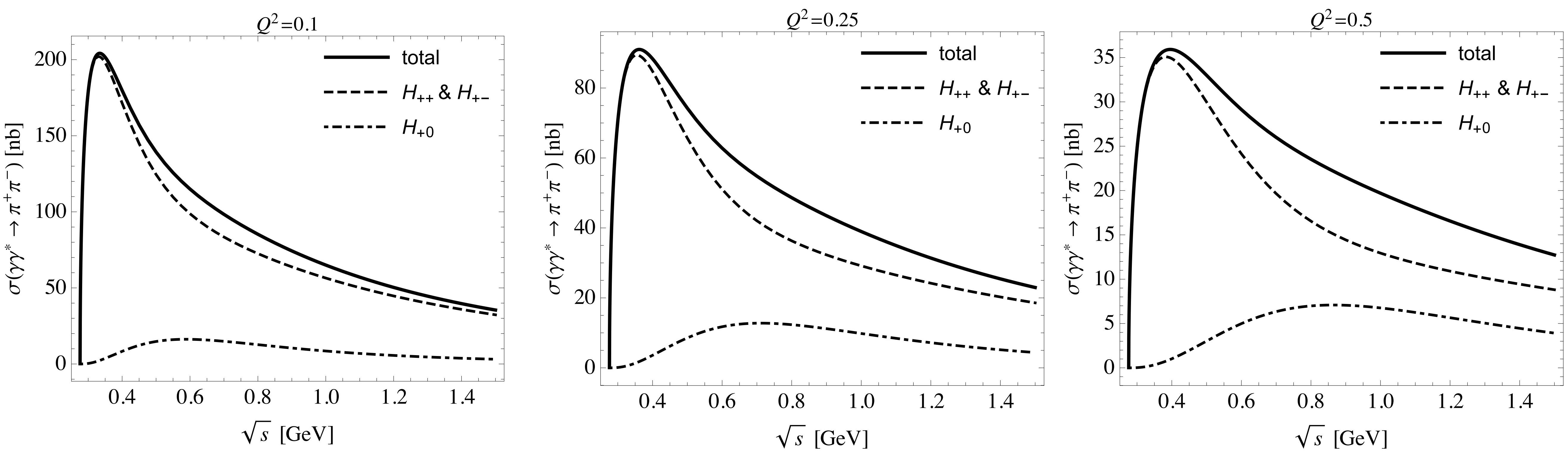

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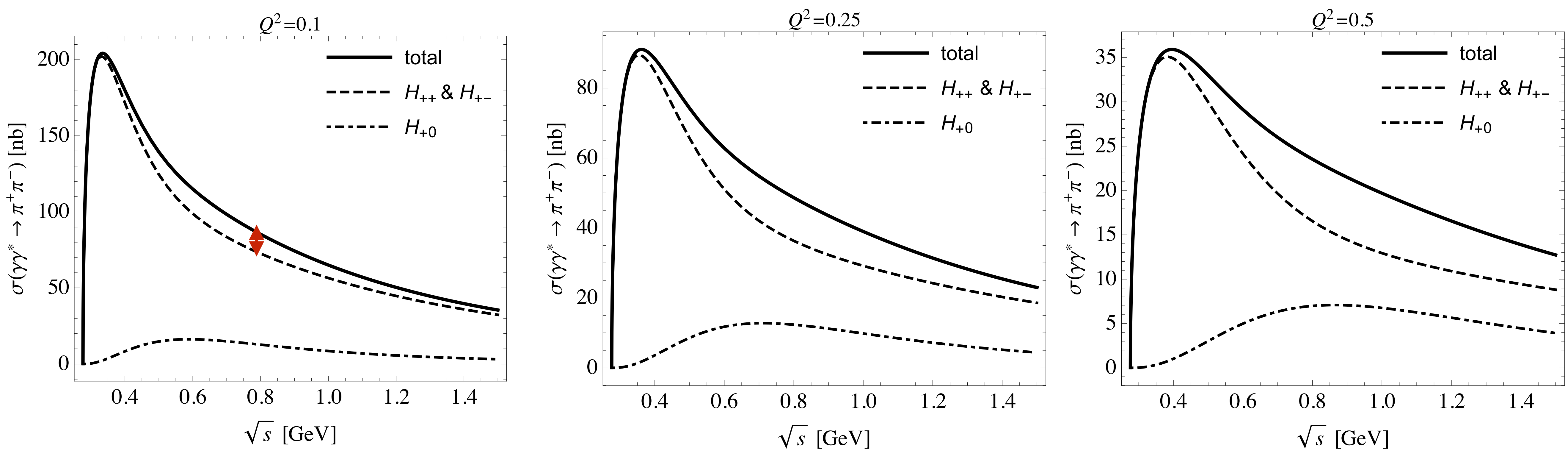


$\sqrt{s}$ =0.8 GeV: 15%

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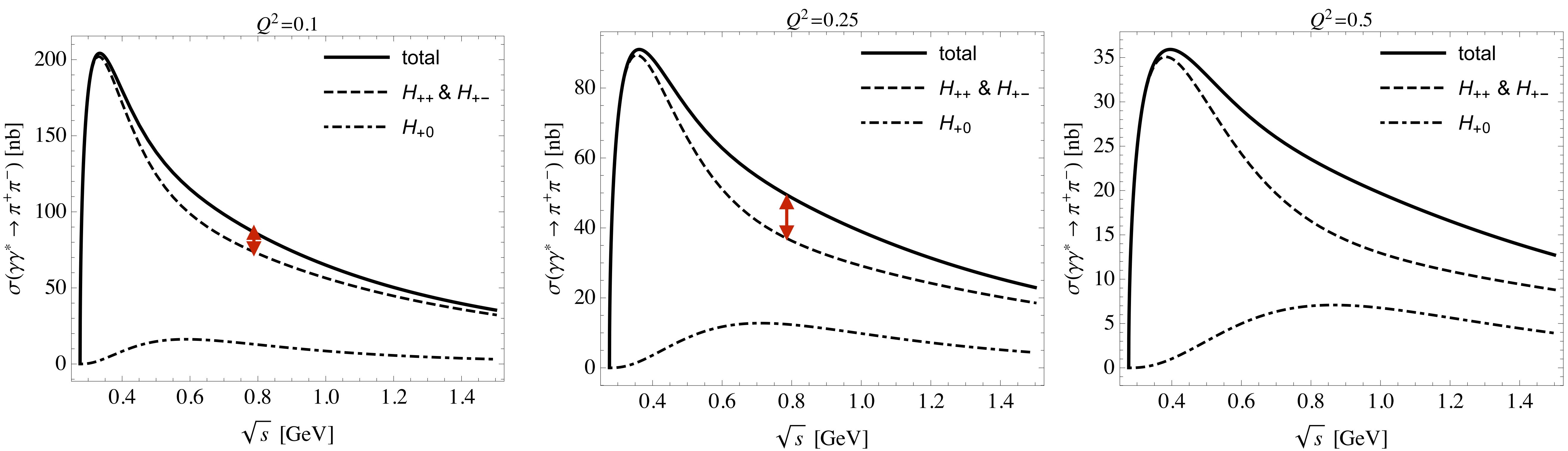


$\sqrt{s}$ =0.8 GeV: 15% 25%

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Differential cross section

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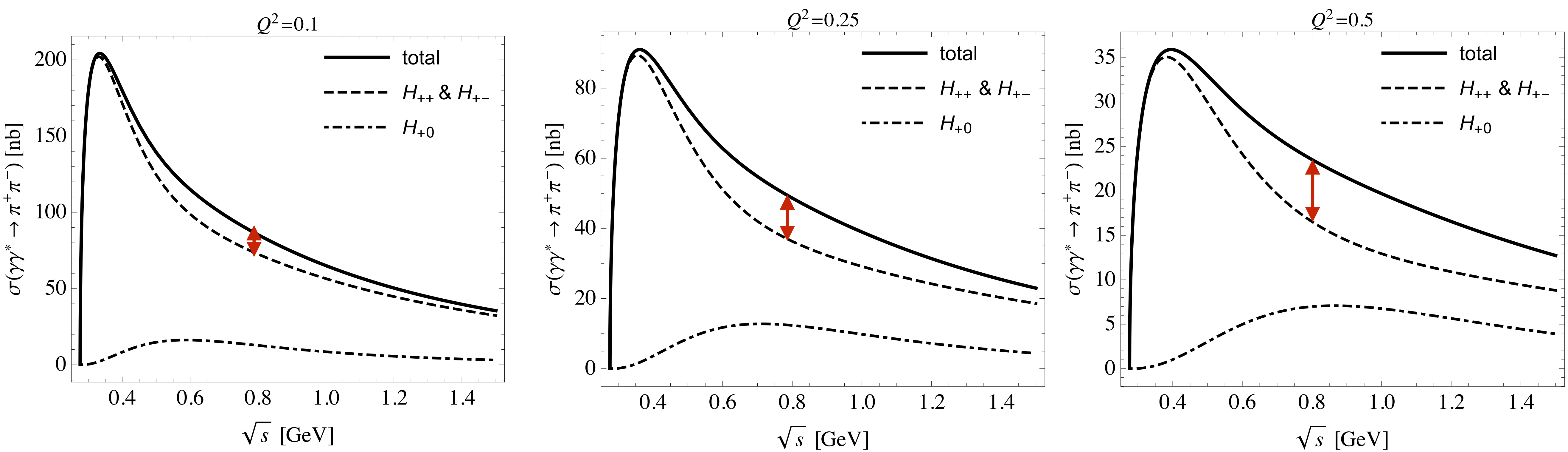



$\sqrt{s}$ =0.8 GeV: 15% 25% 30%

# N/D technique

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\downarrow$$

$$\sum_k C_k\,\xi(s)^k$$

conformal mapping expansion
$C_k$ fitted to exp data and Roy solutions or matched to ChPT

Chew, Mandelstam
Lutz, I.D, Gasparyan

$$T(s) = \frac{N(s)}{D(s)} = \Omega(s)\,N(s)$$

$$N(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')N(s')(U(s)-U(s'))}{s'-s}$$

$$D(s) = 1 - \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')N(s')}{s'-s}$$

# *Conformal mapping*

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

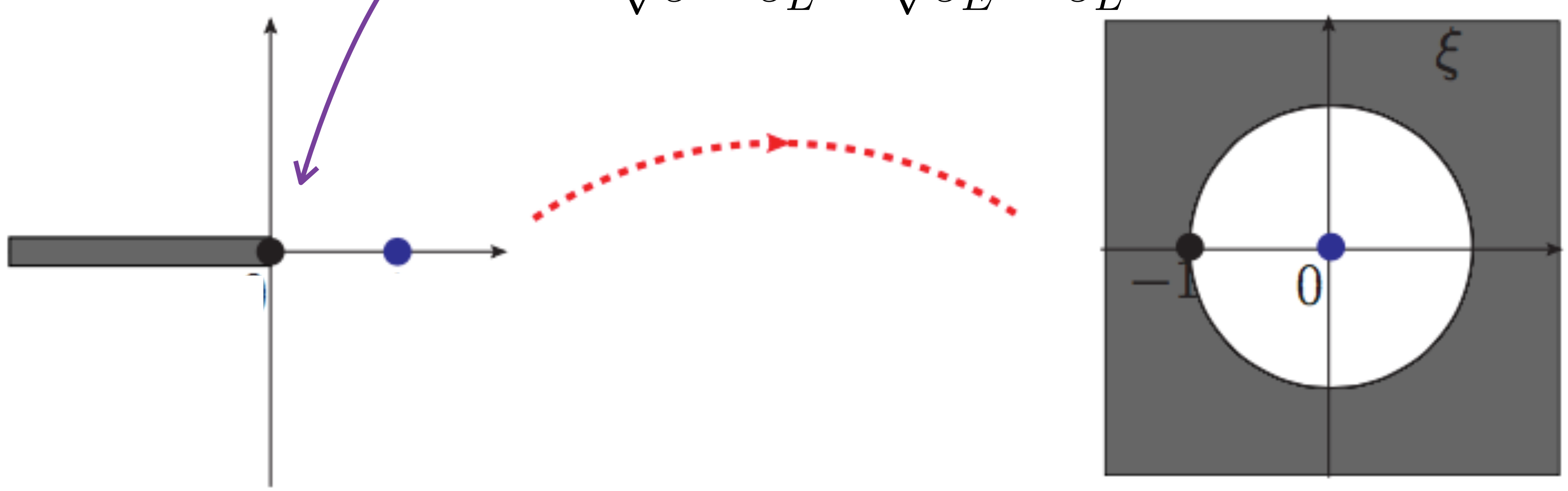
$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k$$

conformal mapping expansion
$C_k$ fitted to exp data and Roy solutions or matched to ChPT

Chew, Mandelstam
Lutz, I.D, Gasparyan

$$\xi(s) = \frac{\sqrt{s-s_L}-\sqrt{s_E-s_L}}{\sqrt{s-s_L}-\sqrt{s_E-s_L}}$$

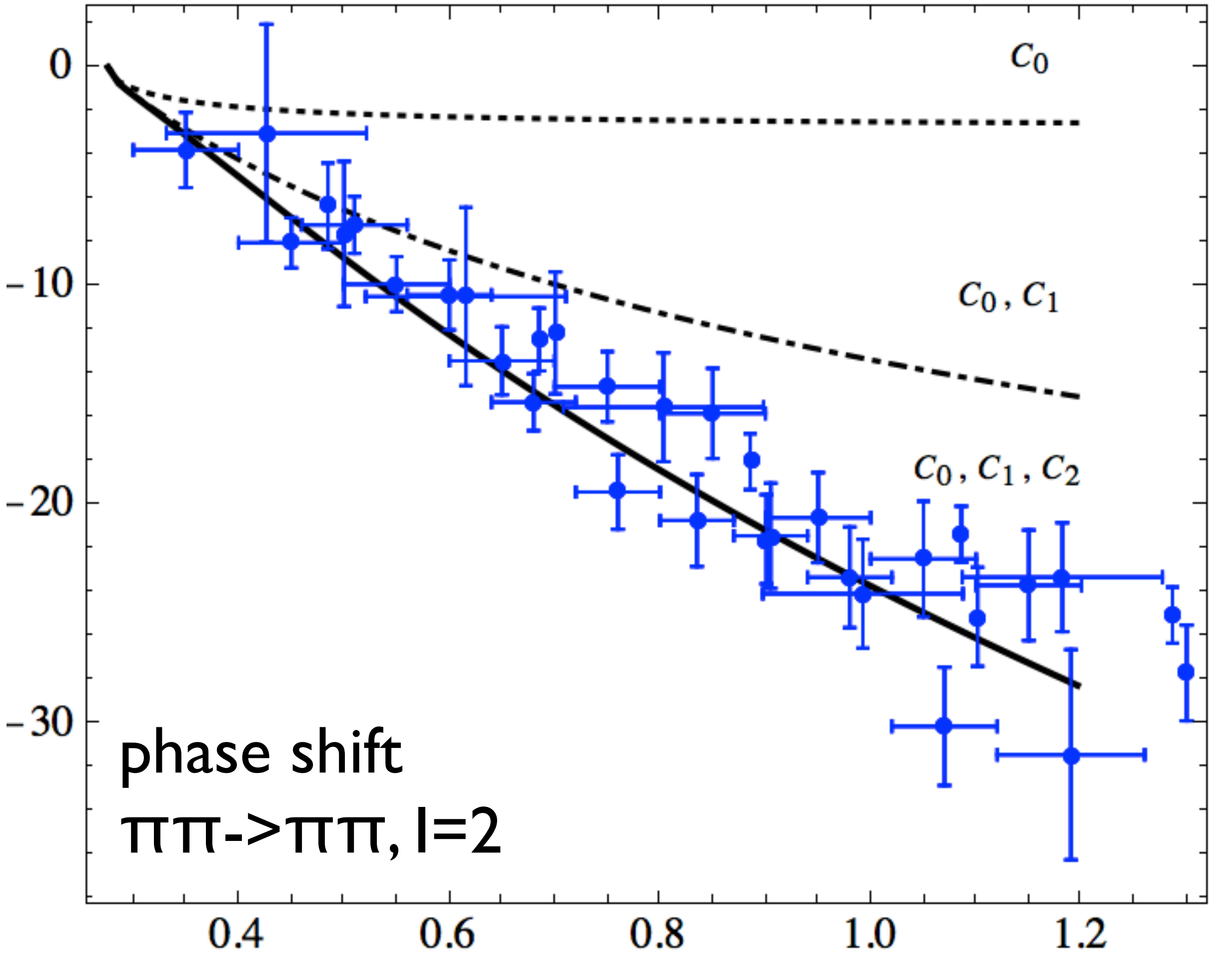


# Example: {ππ}, I=2

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k = C_0 + C_1\xi(s) + C_2\xi(s)^2 + \ldots$$

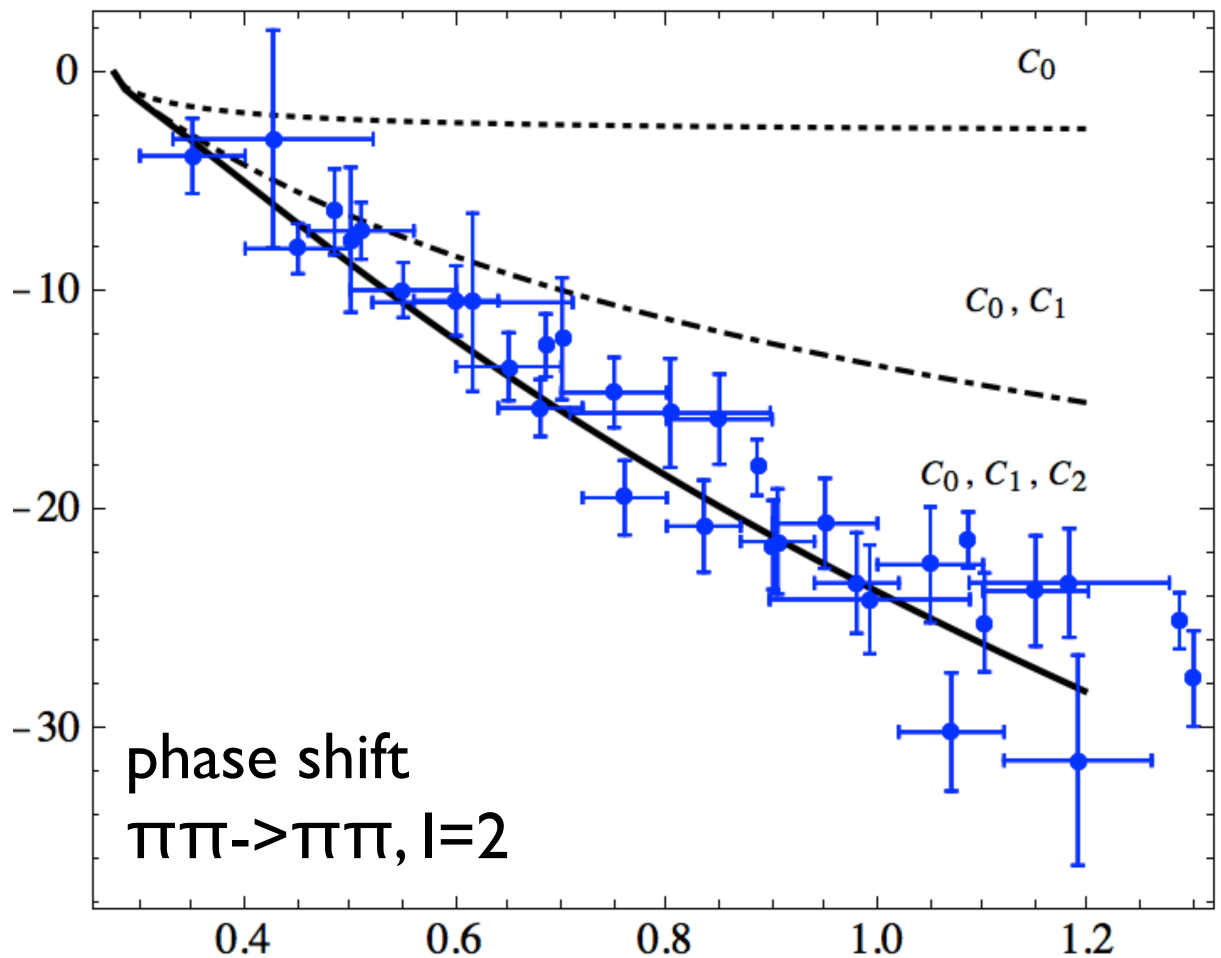


# *Example: {ππ}, I=2*

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k = C_0 + C_1\xi(s) + C_2\xi(s)^2 + \ldots$$



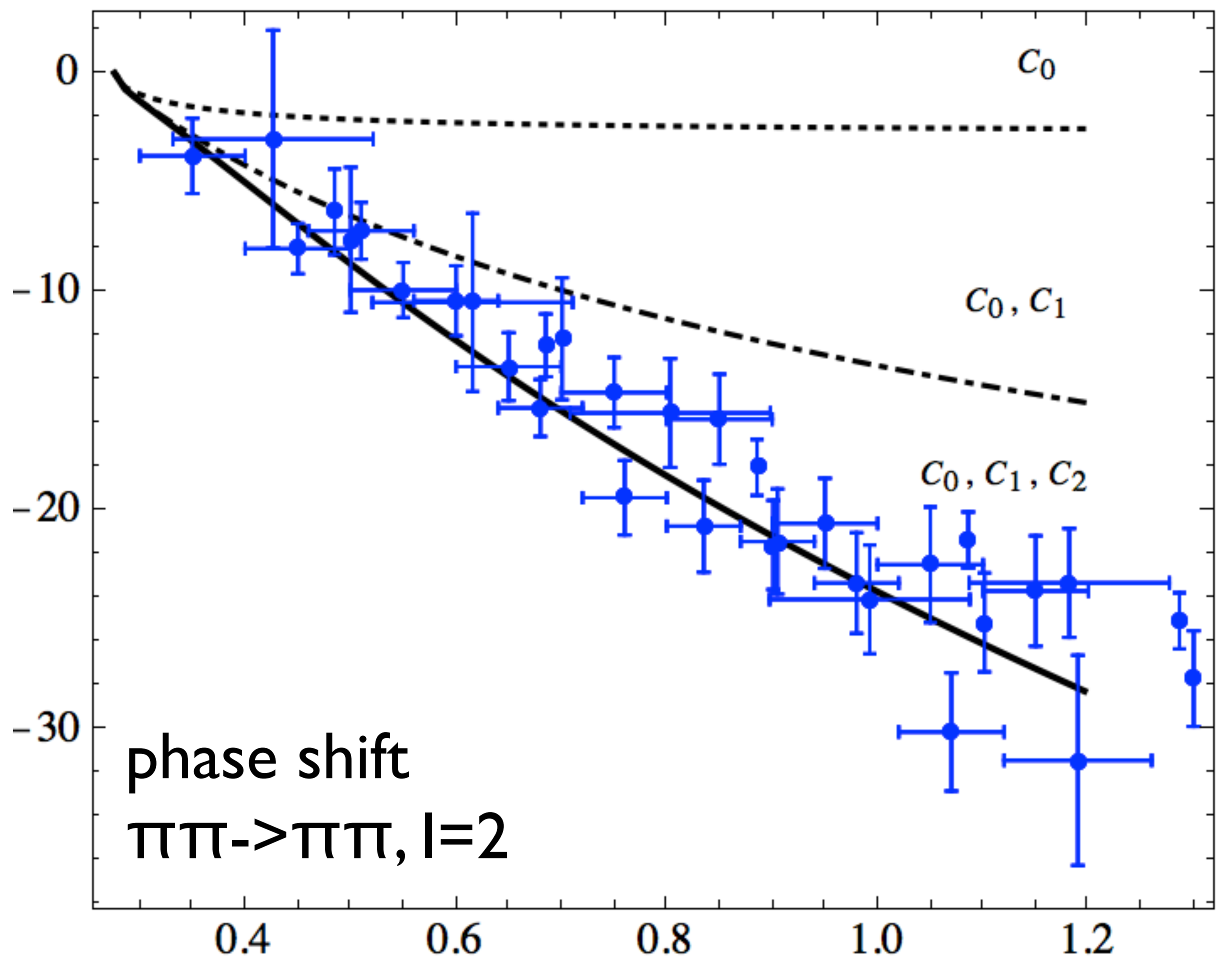

fixed from the threshold parameter **$a_2$** (scattering length)

# *Example: {ππ}, I=2*

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k = C_0 + C_1\xi(s) + C_2\xi(s)^2 + \ldots$$




fixed from the threshold parameter **$a_2$** (scattering length)

fixed from the threshold parameters **$a_2$ , $b_2$**

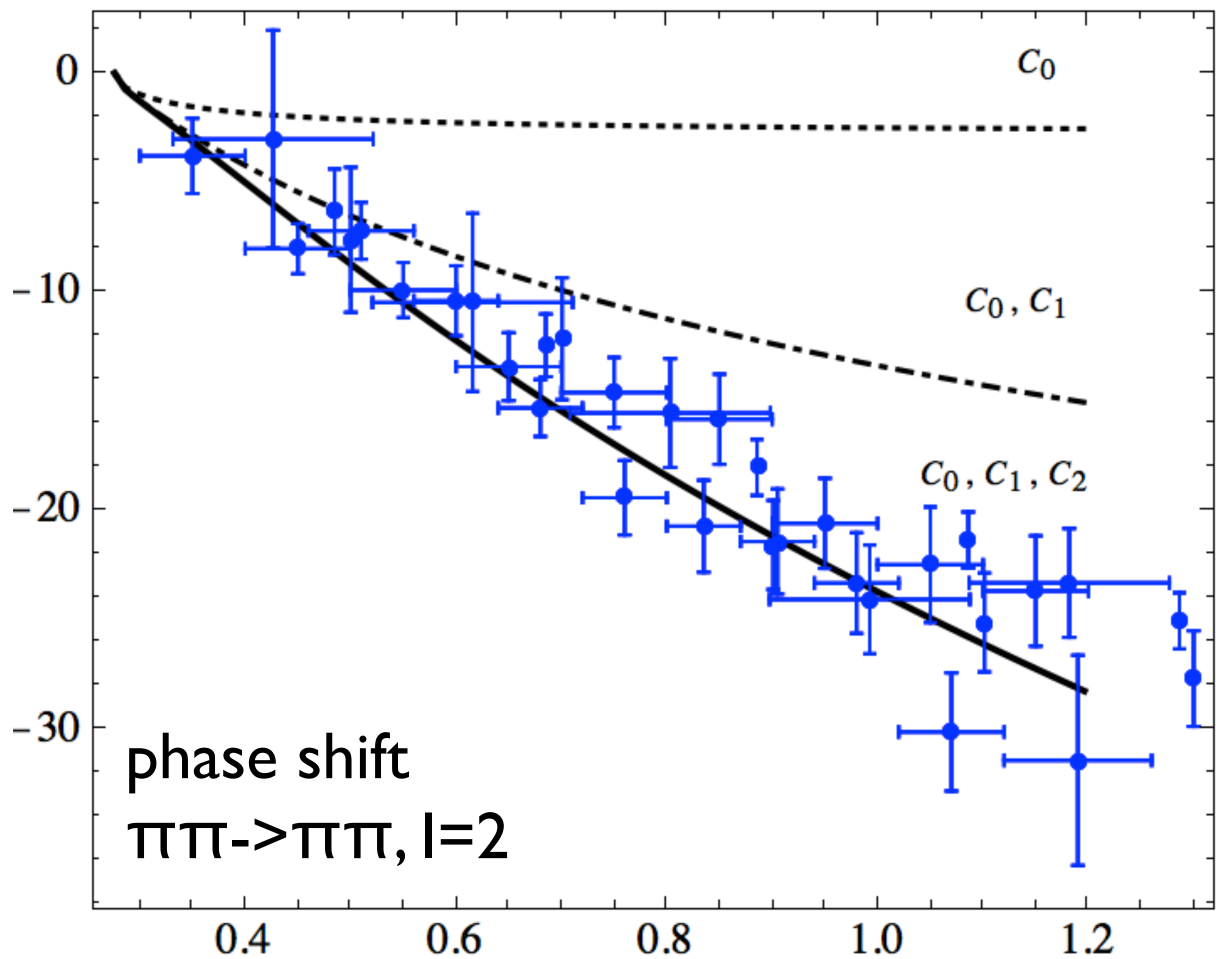
$$\frac{1}{m_\pi}\mathrm{Re}\,(T_{\pi\pi\to\pi\pi}/16\pi) = a + b\,p_{\rm cm}^2 + \ldots$$

# Example: {ππ}, I=2

Solve p.w. dispersion relation using N/D technique using model-independent form for the left-hand cuts

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k = C_0 + C_1\xi(s) + C_2\xi(s)^2 + \ldots$$




fixed from the threshold parameter **a₂** (scattering length)

fixed from the threshold parameters **a₂ , b₂**

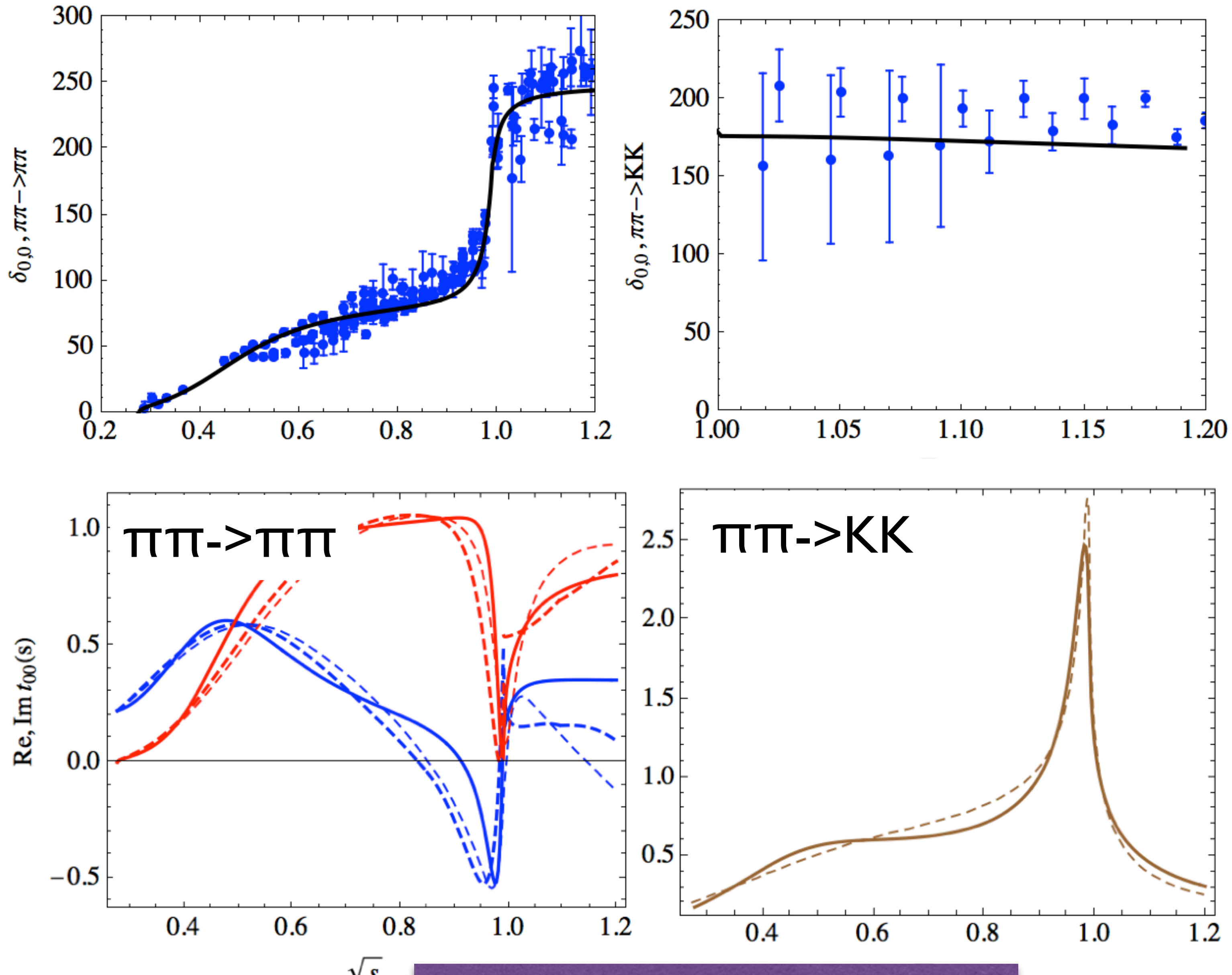
$$\frac{1}{m_\pi}\mathrm{Re}\,(T_{\pi\pi\to\pi\pi}/16\pi) = a + b\,p_{\mathrm{cm}}^2 + \ldots$$

one parameter fit

# $\{\pi\pi, KK\}$ scattering

Coupled channel Omnes

$$\Omega(s) = \begin{pmatrix} \Omega_{\pi\pi\to\pi\pi} & \Omega_{\pi\pi\to K\bar{K}} \\ \Omega_{K\bar{K}\to\pi\pi} & \Omega_{K\bar{K}\to K\bar{K}} \end{pmatrix}$$



Roy eq. solutions (dashed curves)
Coupled channel N/D (solid curves)

**Bounded** p.w. amplitudes and Omnes at large energies

$$T(s) = \frac{N(s)}{D(s)} = \Omega(s)\,N(s)$$

$$N(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\,\frac{\rho(s')N(s')(U(s)-U(s'))}{s'-s}$$

$$D(s) = 1 - \frac{s}{\pi}\int_R \frac{ds'}{s'}\,\frac{\rho(s')N(s')}{s'-s}$$

$$U(s) = \sum_{k=0}^{2} C_k\,\xi(s)^k$$

# {πη, KK} scattering

Solve p.w. dispersion relation using N/D technique: conformal coefficients **matched** to SU(3) **ChPT** at threshold

$$T(s) = U(s) + \frac{s}{\pi}\int_R \frac{ds'}{s'}\frac{\rho(s')|T(s')|^2}{s'-s}$$

$$\sum_k C_k\,\xi(s)^k = C_0 + C_1\xi(s) + C_2\xi(s)^2 + \dots$$

I.D., Gil, Lutz (2011), (2013)