Lattice QCD & the anomalous magnetic moment of the muon

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Hadronic Physics with Lepton and Hadron Beams, Jefferson Lab,
7 September 2017

Cluster of Excellence

$(g-2)_\mu$: a reminder

$$\boldsymbol{\mu} = g\,\mu_B \boldsymbol{s}, \qquad \mu_B = \frac{e}{2m_\mu}$$

- $g = 2$ in Dirac's theory
- $a_\mu \equiv (g-2)/2 = F_2(0) = \frac{\alpha}{2\pi}$ (Schwinger 1948)
- direct measurement (BNL): $a_\mu = (11659208.9 \pm 6.3)\cdot 10^{-10}$
- Standard Model prediction $a_\mu = (11659182.8 \pm 4.9)\cdot 10^{-10}$.
- $a_\mu^{\rm exp} - a_\mu^{\rm th} = (26.1 \pm 8.0)\cdot 10^{-10}$.

Numbers from 1105.3149 Hagiwara et al.

$(g-2)_\mu$: history and near future

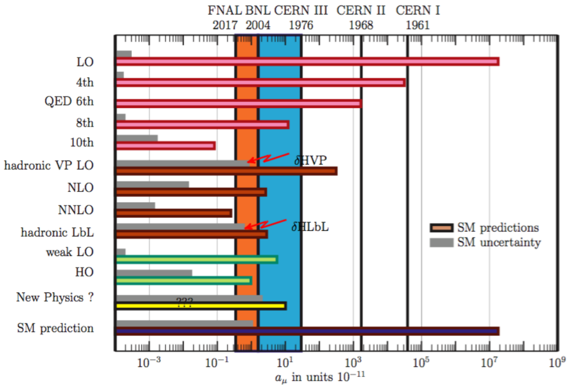
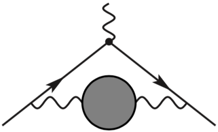
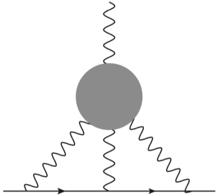


Fig. from Jegerlehner 1705.00263



Hadronic vacuum polarisation (HVP)



Hadronic light-by-light scattering (HLbL)

New experiments: $\times 4$ improvement in accuracy $\Longrightarrow$ theory effort needed:

- HVP target accuracy: $\lesssim 0.5\%$
- HLbL target accuracy: 15%.

HVP: definitions (Euclidean space)

- primary object on the lattice: $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0) \rangle$.

- polarization tensor: $\Pi_{\mu\nu}(Q) \equiv \int d^4x\, e^{iQ\cdot x} G_{\mu\nu}(x)$.

- O(4) invariance and current conservation $\partial_\mu j_\mu = 0$:

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2).$$

- Spectral representation: $\rho(s) = \frac{R(s)}{12\pi^2}$, $R(s) \equiv \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{4\pi\alpha(s)^2/(3s)}$,

$$\Pi(Q^2) - \Pi(0) = Q^2 \int_{4m_\pi^2}^{\infty} ds \frac{\rho(s)}{s(s+Q^2)}.$$

-

$$a_\mu^{\mathrm{hvp}} = 4\alpha^2 \int_0^\infty dQ^2\ K(Q^2; m_\mu^2)\ [\Pi(Q^2) - \Pi(0)]$$

- In the limit $m_\mu \to 0$: $\lim_{m_\mu \to 0} \frac{a_\mu^{\mathrm{hvp}}}{m_\mu^2} = \frac{4}{3}\alpha^2 \Pi'(Q^2 = 0)$.

Lautrup, Peterman & de Rafael Phys.Rept 3 (1972) 193; Blum hep-lat/0212018

The time-momentum representation

- mixed-representation Euclidean correlator:

$$G(x_0) = -\frac{1}{3}\sum_{k=1}^{3}\int d^3\boldsymbol{x}\; G_{kk}(x),$$

- the spectral representation:

$$G(x_0) = \int_0^\infty d\omega\; \omega^2 \rho(\omega^2)\; e^{-\omega|x_0|}, \qquad x_0 \neq 0.$$

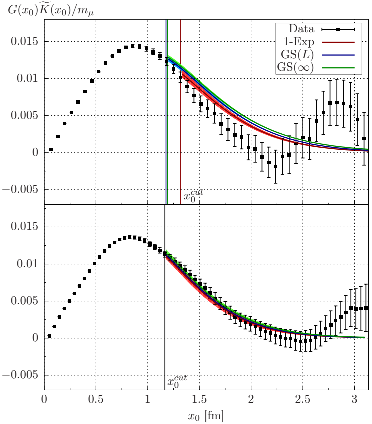
- Finally, the quantity $a_\mu^{\rm hvp}$ is given by

$$\begin{aligned} a_\mu^{\rm hvp} &= \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\; G(x_0)\; \tilde f(x_0), \\ \tilde f(x_0) &= \frac{2\pi^2}{m_\mu^2}\Big[-2 + 8\gamma_{\rm E} + \frac{4}{\hat x_0^2} + \hat x_0^2 - \frac{8}{\hat x_0}K_1(2\hat x_0) \\ &\qquad + 8\log(\hat x_0) + G^{2,1}_{1,3}\left(\hat x_0^2 \Big| \begin{matrix} \frac{3}{2} \\ 0, 1, \frac{1}{2} \end{matrix}\right)\Big] \end{aligned}$$

where $\hat x_0 = m_\mu x_0$, $\gamma_{\rm E} = 0.577216..$ and $G^{m,n}_{p,q}$ is Meijer's function.

Bernecker & Meyer 1107.4388; Mainz-CLS 1705.01775.

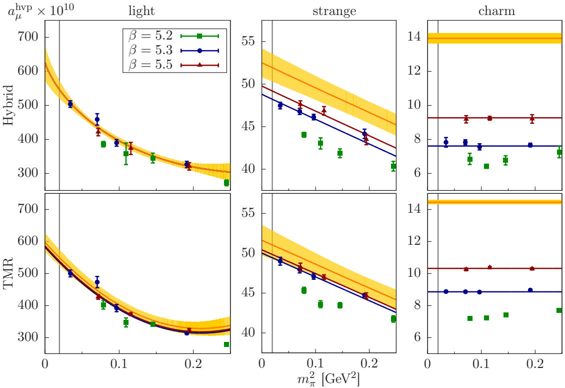
TMR: a look at the integrand (light-quark conn. contribution)



- $m_\pi = 190\,$MeV and $m_\pi = 270\,$MeV
- signal-to-noise deteriorates at long distances
- used various ways to extend the correlator to $x_0 = \infty$:
  - 1-exponential fit: $G(x_0) = Ae^{-m_V x_0}$, extracting m_V from a smeared-smeared correlator
  - Exponential + $\pi\pi$ state: $G(x_0) = Ae^{-m_V x_0} + Be^{-E_{\pi\pi} x_0}$
  - assumed Gounaris-Sakurai timelike pion form factor: correct at the same time for the dominant finite-volume effect.

$N_f = 2$ O(a) improved Wilson quarks: Mainz-CLS 1705.01775.

Chiral & continuum extrapolation

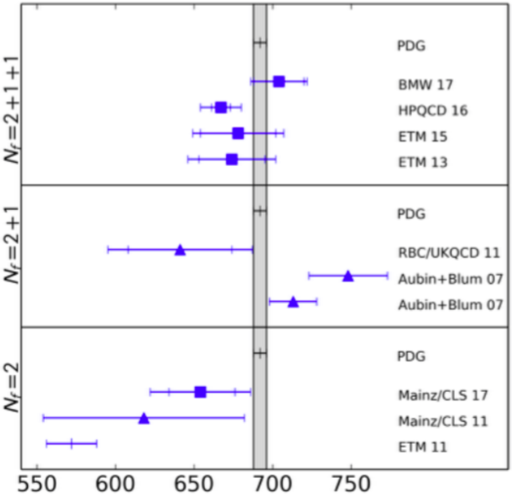


$$a_\mu^{\rm hvp} = (654 \pm 32_{\,\rm stat} \pm 17_{\,\rm syst} \pm 10_{\,\rm scale} \pm 7_{\,\rm FV}\,{}^{+\;0}_{-10\,\rm disc}) \cdot 10^{-10}.$$

- effect of applying finite-volume correction on final result: $+37 \cdot 10^{-10}$.

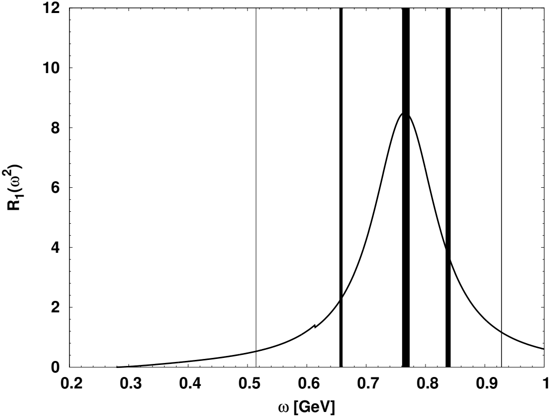
$N_f = 2$ O(a) improved Wilson quarks: Mainz-CLS 1705.01775.

Overview of recent lattice results for a_μ^{hvp}



From H. Wittig, talk at PhiPsi conference, Mainz, 26-29 June 2017

From discrete states on the torus to the timelike pion form factor

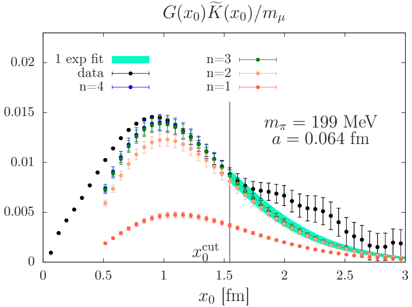


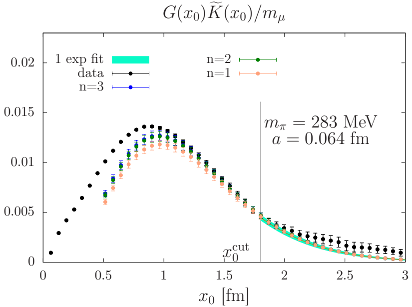
$$\rho(s) = \frac{1}{48\pi^2}\left(1-4m_\pi^2/s\right)^{3/2}|F_\pi(s)|^2, \qquad\qquad s<(4M_\pi)^2$$

$$|F_\pi(s)|^2 = \left(q\phi'(q)+k\frac{\partial\delta_1(k)}{\partial k}\right)\frac{3\pi s}{2k^5L^3}\left|{}_L\left\langle \pi\pi\left|\int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x})\right|0\right\rangle\right|^2.$$

HM 1105.1892

New: TMR on $N_f = 2+1$ CLS ensembles





- 'genuine' spectroscopy including $\pi\pi$ interpolating operators
- the first three states saturate the correlator beyond $x_0 = 1.2$ fm
- benefit 1: if m has been computed with error δm, the relative error on e^{-mx_0} is $\delta m \cdot t \Rightarrow$ only linear growth of the error.
- benefit 2: correct for finite-size effects from having discrete instead of continuum $\pi\pi$ states.

Bernecker & HM 1107.4388.

Preliminary; A. Gérardin et al. + B. Hörz, J. Bulava et al. (1511.02351).

New: a Lorentz-covariant coordinate-space approach

- primary object: $G_{\mu\nu}(x) = \langle j_\mu(x) j_\nu(0)\rangle$.
- $a_\mu^{\rm hvp} = \int d^4x\; G_{\mu\nu}(x)\; H_{\mu\nu}(x) = 2\pi^2 \int_0^\infty d|x|\, |x|^3\, [G_{\mu\nu}(x)\; H_{\mu\nu}(x)]$,

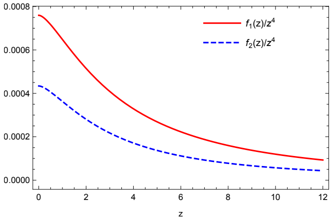
$$H_{\mu\nu}(x) = -\delta_{\mu\nu}\mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2}\,\mathcal{H}_2(|x|)$$

a transverse tensor with $\mathcal{H}_i(|x|) = \frac{8\alpha^2}{3m_\mu^2} f_i(m_\mu|x|)$ and

$$f_2(z) = \frac{G^{2,2}_{2,4}\left(z^2\Big|\begin{array}{c}\frac{7}{2},4\\ 4,5,1,1\end{array}\right) - G^{2,2}_{2,4}\left(z^2\Big|\begin{array}{c}\frac{7}{2},4\\ 4,5,0,2\end{array}\right)}{8\sqrt{\pi}z^4},$$

$$f_1(z) = f_2(z) - \frac{3}{16\sqrt{\pi}}\cdot\left[G^{2,3}_{3,5}\left(z^2\Big|\begin{array}{c}1,\frac{3}{2},2\\ 2,3,-2,0,0\end{array}\right) - G^{2,3}_{3,5}\left(z^2\Big|\begin{array}{c}1,\frac{3}{2},2\\ 2,3,-1,-1,0\end{array}\right)\right].$$

$f_i(z)/z^4$



HM, 1706.01139.

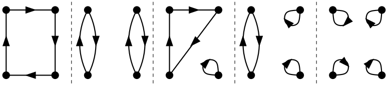
Hadronic light-by-light contribution to $(g-2)_\mu$

Columbia-RIKEN/BNL-Connecticut group: [Hayakawa et al. hep-lat/0509016]

- QCD+QED on the lattice method: fully connected [1407.2923,1510.07100]
- fully connected and leading disconnected sampling vertices [1610.04603]
- mixed analytic/numerical calculation of positions-space QED kernel [1705.01067]

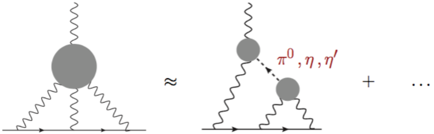
Mainz group:

- the $\pi^0 \to \gamma^*\gamma^*$ transition form factor [A. Gérardin, HM, A. Nyffeler 1607.08174]
- analysis of the forward $\gamma^*\gamma^* \to \gamma^*\gamma^*$ scattering amplitudes [A. Gérardin, J. Green, V. Pascalutsa et al.; 1507.01577 and in prep.]
- (semi-)analytic calculation of position-space QED kernel with averaging over muon momentum [N. Asmussen, J. Green, HM, Nyffeler 1510.08384, 1609.08454]



Quark Wick contraction topologies

Models for a_μ^{HLbL}



Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99 ± 16
axial vectors	2.5±1.0	1.7±1.7	–	22±5	–	15 ± 10	22 ± 5
scalars	−6.8±2.0	–	–	–	–	−7±7	−7±2
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π,K loops +subl. N_C	–	–	–	0±10	–	–	–
quark loops	21±3	9.7±11.1	–	–	–	2.3 (c-quark)	21±3
Total	83±32	89.6±15.4	80±40	136±25	110±40	105 ± 26	116 ± 39

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

Recent estimates: (mostly, much smaller axial-vector contribution)

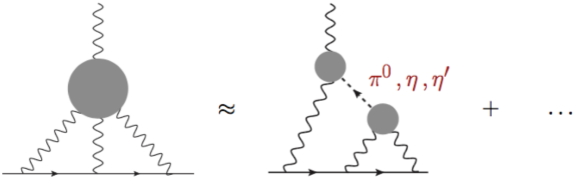
$$a_\mu^{\mathrm{HLbL}} = (98 \pm 26) \times 10^{-11} \qquad \text{Prades, de Rafael, Vainshtein}$$

$$a_\mu^{\mathrm{HLbL}} = (102 \pm 39) \times 10^{-11} \qquad \text{Jegerlehner, Nyffeler}$$

In development: dispersive approaches [Colangelo, Hoferichter, Procura, Stoffer '14-'17; Pauk, Vanderhaeghen '14].

A. Nyffeler, PhiPsi, Mainz, June 2017

The $\pi^0 \to \gamma^*\gamma^*$ transition form factor from the lattice



$$M_{\mu\nu}(p, q_1) \equiv i \int \mathrm{d}^4x \, e^{iq_1 x} \, \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p) \rangle = \epsilon_{\mu\nu\alpha\beta} \, q_1^\alpha \, q_2^\beta \, F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2) \, ,$$

Lattice: $M_{\mu\nu}(p, q_1) = (i^{n_0}) M^{\mathrm{E}}_{\mu\nu}(p, q_1)$, where

$$M^{\mathrm{E}}_{\mu\nu}(p, (\omega_1, \boldsymbol{q}_1)) = \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \, e^{\omega_1 \tau} \, \widetilde{A}_{\mu\nu}(\tau),$$

$$\widetilde{A}_{\mu\nu}(\tau) = \lim_{t_\pi \to +\infty} e^{E_\pi(t_f - t_0)} C^{(3)}_{\mu\nu}(\tau, t_\pi),$$

$$C^{(3)}_{\mu\nu}(\tau, t_\pi) = a^6 \sum_{\boldsymbol{x},\boldsymbol{z}} \left\langle T \left\{ J_\mu(\boldsymbol{z}, t_i) J_\nu(\boldsymbol{0}, t_f) P^\dagger(\boldsymbol{x}, t_0) \right\} \right\rangle e^{i\boldsymbol{p}\,\boldsymbol{x}} \, e^{-i\boldsymbol{q}_1 \boldsymbol{z}} \, .$$

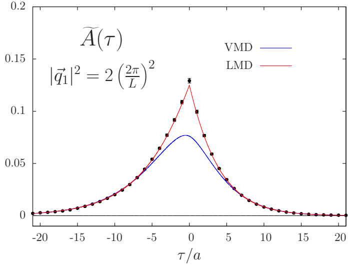
$\tau = t_i - t_f$, $t_\pi = \min(t_f - t_0, t_i - t_0)$.

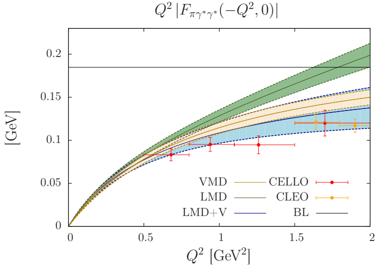
Ji, Jung hep-lat/0101014; Dudek, Edwards, hep-ph/0607140; Feng et al. 1206.1375; A. Gérardin, HM, A. Nyffeler 1607.08174.

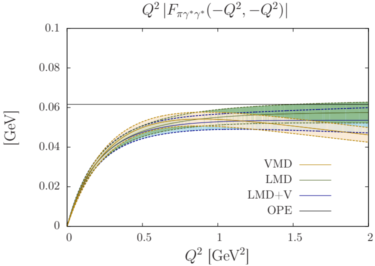
The $\pi^0 \to \gamma^*\gamma^*$ transition form factor from the lattice (II)

$$\boldsymbol{p} = 0: \quad \widetilde{A}_{kl}(\tau) = -i m_\pi \epsilon_{kli} q_1^i \widetilde{A}(\tau)$$

OPE: VMD model
$F^{\rm VMD}_{\pi^0\to\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$
too soft in the UV $\Rightarrow$ does not account for the cusp at $\tau = 0$.







Double-virtual case is a prediction: no experimental data at this point.
Open question: a systematically improvable parametrization of $F_{\pi\gamma^*\gamma^*}(q_1^2, q_2^2)$.
A. Gérardin, HM, A. Nyffeler 1607.08174.

Light-by-light amplitudes from the lattice

Forward helicity amplitudes: functions of the virtualities q_1^2 and q_2^2 and $\nu = q_1 \cdot q_2$,

$$\mathcal{M}_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(q_1,q_2) = \mathcal{M}_{\mu\nu\rho\sigma}(q_1,q_2)\,\epsilon^{*\mu}(\lambda_1',q_1)\,\epsilon^{*\nu}(\lambda_2',q_2)\,\epsilon^{\rho}(\lambda_1,q_1)\,\epsilon^{\sigma}(\lambda_2,q_2)\,.$$

e.g. $M_{TT} = \frac{1}{2}(M_{++,++} + M_{+-,+-})$. Dispersive sum rule:

$$\overline{M}_{TT} = M_{TT}(\nu) - M_{TT}(0) = \frac{2\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu'\,\frac{\sqrt{\nu'^2 - q_1^2 q_2^2}\;(\sigma_0+\sigma_2)(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

$\sigma_{0,2}$ are the $\gamma^*\gamma^* \to$ hadrons cross-sections for total helicity 0 resp. 2.

$$\Pi^E_{\mu\nu\rho\sigma}(Q_1,Q_2) = \sum_{X_1,X_2,X_3} \langle J_\mu(X_1) J_\nu(X_2) J_\rho(X_3) J_\sigma(0)\rangle_E\, e^{iQ_1(X_1-X_3)}\, e^{iQ_2 X_2}\,,$$

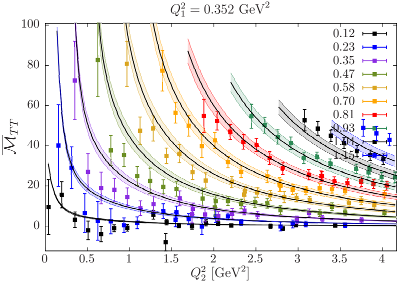
Analytic continuation:

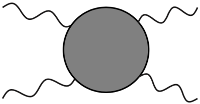
$$\mathcal{M}_{\mu\nu\rho\sigma}(q_1,q_2) = e^4\, i^{n_0}\,\Pi^E_{\mu\nu\rho\sigma}(Q_1,Q_2), \qquad q_i^0 = -iQ_i^0,\ \boldsymbol{q} = \boldsymbol{Q}.$$

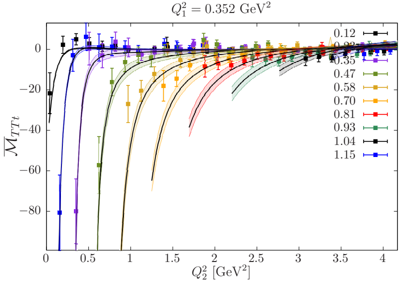
$n_0 =$ number of temporal indices among (μ,ν,ρ,σ).

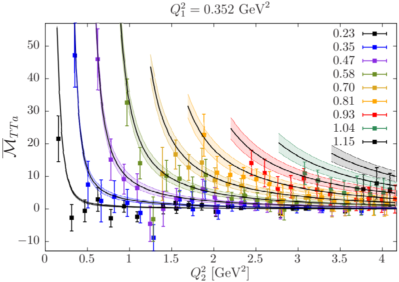
1507.01577; sum rules: Pascalutsa, Pauk, Vanderhaeghen 1204.0740.

The magnificent eight: forward light-by-light amplitudes









Dispersive sum rule:

$$\overline{M}_{TT} = \frac{4\nu^2}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{X'}\sigma_{TT}(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

σ_{TT} = total cross-section $\gamma^*\gamma^* \to$ hadrons.

A. Gérardin, J. Green et al. (Mainz-CLS), $N_f = 2$ ensemble F7, $48^3 \times 96$, $m_\pi = 270$ MeV

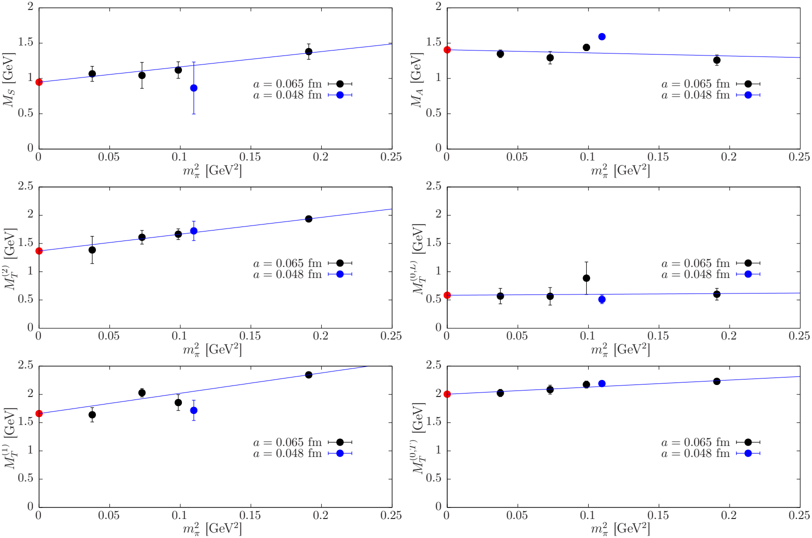
Model for photon-photon fusion cross-section

	M_{TT}	M_{TT}^{τ}	M_{TT}^{a}	M_{TL}	M_{LT}	M_{TL}^{τ}	M_{TL}^{a}	M_{LL}
Pseudoscalar	$\sigma_0/2$	$-\sigma_0$	$\sigma_0/2$	$\times$	$\times$	$\times$	$\times$	$\times$
Scalar	$\sigma_0/2$	σ_0	$\sigma_0/2$	$\times$	$\times$	τ_{TL}	τ_{TL}	σ_{LL}
Axial	$\sigma_0/2$	$-\sigma_0$	$\sigma_0/2$	σ_{TL}	σ_{LT}	τ_{TL}	$-\tau_{LT}$	$\times$
Tensor	$\frac{\sigma_0+\sigma_2}{2}$	σ_0	$\frac{\sigma_0-\sigma_2}{2}$	σ_{TL}	σ_{LT}	τ_{TL}	τ_{TL}^{a}	σ_{LL}
Scalar QED	σ_{TT}	τ_{TT}	τ_{TT}^{a}	σ_{TL}	σ_{LT}	τ_{TL}	τ_{TL}^{a}	σ_{LL}

Contribution of meson pole to a $\gamma^*\gamma^* \to$ hadrons cross-section is proportional to $\delta(s-M^2)\Gamma_{\gamma\gamma}\left[\frac{F_{M\gamma^*\gamma^*}(Q_1^2,Q_2^2)}{F_{M\gamma^*\gamma^*}(0,0)}\right]^2$.

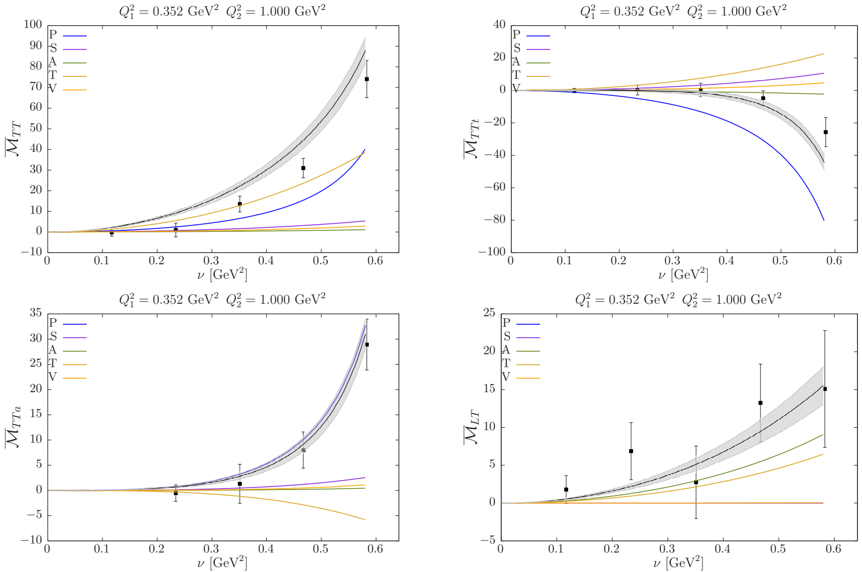
- $\pi^0 \to \gamma^*\gamma^*$ transition form factor was determined in dedicated calculation
- seven other TFFs were parametrized by $1/(1+Q^2/M^2)^k$ $(k=1,2)$ and the parameters M fitted.

Chiral extrapolation of monopole/dipole masses

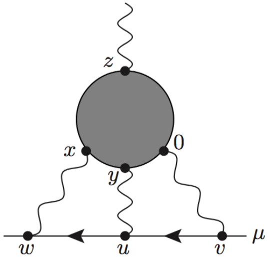


- with these results, new evaluation of a_μ^{HLbL} is possible.
- main limitations: only fully connected contribution, no strange quark.

Forward LbL amplitudes: contributions of individual mesons



Euclidean coordinate-space approach to $a_\mu^{\rm HLbL}$



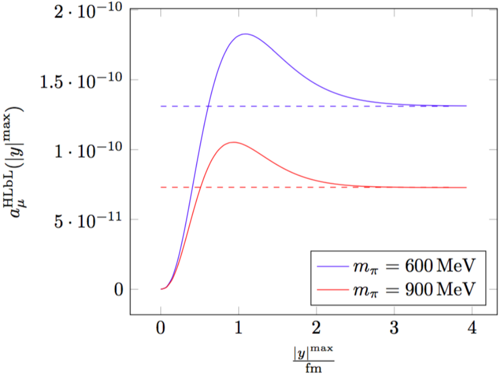
$$a_\mu^{\rm HLbL} = \frac{me^6}{3} \underbrace{\int d^4y}_{=2\pi^2|y|^3 d|y|} \Big[\int d^4x \, \underbrace{\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)}_{\rm QED} \; \underbrace{i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y)}_{\rm QCD} \Big].$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z \; z_\rho \left\langle \, j_\mu(x) \, j_\nu(y) \, j_\sigma(z) \, j_\lambda(0) \right\rangle.$$

- $\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ computed in the continuum & infinite-volume
- no finite-volume effects from the photons & affordable way (1d integral) to sample the integrand for the fully connected contribution.

[N. Asmussen, J. Green, HM, Nyffeler 1510.08384, 1609.08454]

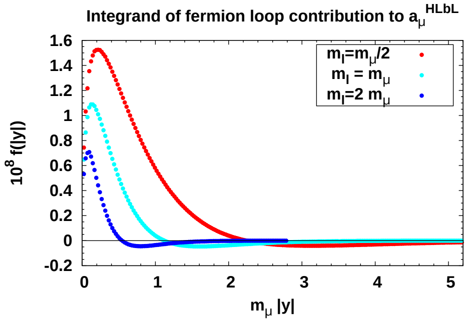
Test I: contribution of the π^0 to $a_\mu^{\rm HLbL}$



From [III]. Dashed line = result from momentum-space integration (Knecht & Nyffeler PRD65, 073034 (2001)).

- Contribution is perhaps surprisingly long-range.
- The observation depends to some extent on the precise QED kernel used.
- The integrand can be roughly described by the form $(c_1|y|^4 + c_2|y|^2)e^{-m_\pi|y|}$.

Test II: contribution of a lepton loop to $a_\mu^{\rm LbL}$



m_ℓ/m_μ	$a_\mu^{\rm LbL}\cdot 10^9$	position-space
0.5	12.291	12.575(62)(24)
1.0	4.6497	4.706(23)(21)
2.0	1.5031	1.504(7)(17)

first uncertainty: estimated error of 3d integrator
second uncertainty: uncertainty from extrapolation to $|y| = 0$.
Remaining discrepancy: probably due to insufficient number of multipoles at large x, y.

'Exact' values: Laporta, Remiddi, PLB 301 (1993) 440; M. Passera, private communication.

Conclusion

HVP:

- main difficulty: many effects enter at the few percent level,
  - strong isospin breaking, 1-photon irreducible QED contributions
  - finite-size effects
  - quark disconnected contributions
  - scale setting: need 0.3% accuracy.
- ... but rapid progress on all fronts!
- auxiliary 'spectroscopy' calculations needed (timelike pion FF, scale setting).

HLbL:

- A high-quality lattice calculation of $\pi^0 \to \gamma^*\gamma^*$ seems possible now. In the near future, $\eta, \eta' \to \gamma^*\gamma^*$?
- Analysis of the eight forward light-by-light scattering amplitudes, constraining the resonance transition form factors.
- The covariant position-space method is a promising approach.