3D Nucleon Tomography from LQCD
INTRODUCTION

- Goal: Compute properties of hadrons from first principles
  - Parton distribution functions (PDFs) and Generalized Parton distributions (GPDs)
  - Transverse Momentum Dependent densities (TMDs)
  - Form Factos ...

- Lattice QCD is a first principles method
  - For many years calculations focused on Mellin moments
  - Can be obtained from local matrix elements of the proton in Euclidean space
    - Breaking of rotational symmetry \( \rightarrow \) power divergences
    - only first few moments can be computed

- Recently direct calculations of PDFs in Lattice QCD are proposed
  - Y.-Q. Ma J.-W. Qiu (2014) 1404.6860

- First lattice Calculations already available
PDFS: DEFINITION

Light-cone PDFs:

\[ f^{(0)}(\xi) = \int_{-\infty}^{\infty} \frac{d\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P \left| T \overline{\psi}(0, \omega^-, 0_T) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_C. \]

\[ W(\omega^-, 0) = \mathcal{P} \exp \left[ -ig_0 \int_0^{\omega^-} dy^- A^+_{\alpha}(0, y^-, 0_T) T_{\alpha} \right] \]

Moments:

\[ a_{0}^{(n)} = \int_{0}^{1} d\xi \xi^{n-1} \left[ f^{(0)}(\xi) + (-1)^n \overline{f^{(0)}(\xi)} \right] = \int_{-1}^{1} d\xi \xi^{n-1} f(\xi) \]

Local matrix elements:

\[ \left\langle P | O^{\{\mu_1 \cdots \mu_n\}}_0 | P \right\rangle = 2a_{0}^{(n)} (P^{\mu_1} \cdots P^{\mu_n} - \text{traces}) \]

\[ O^{\{\mu_1 \cdots \mu_n\}}_0 = i^{n-1} \overline{\psi}(0) \gamma^{\mu_1} D^{\mu_2} \cdots D^{\mu_n} \frac{\lambda^a}{2} \psi(0) - \text{traces} \]
GPDS: DEFINITION

GPDs:

\[ \bar{u}(P') \left( \gamma^+ H(x, \xi, t) + i \frac{\sigma^{+k} \Delta_k}{2m} E(x, \xi, t) \right) = \int_{-\infty}^{\infty} \frac{d\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P' \left| T \bar{\psi}(0, \omega^-, 0_T) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \right| P \right\rangle_C \]

\[ W(\omega^-, 0) = \mathcal{P} \exp \left[ -ig_0 \int_0^{\omega^-} dy^- A_\alpha^+(0, y^-, 0_T) T_\alpha \right] \]

Moments:

\[ \int_{-1}^{1} dxx^{n-1} \left[ \frac{H(x, \xi, t)}{E(x, \xi, t)} \right] = \sum_{k=0}^{(n-1)/2} (2\xi)^{2k} \left[ A_{n,2k}(t) \begin{array}{c} \mathcal{B}_{n,2k}(t) \\ \mathcal{C}_{n,2k}(t) \end{array} \right] \pm \delta_{n,\text{even}} (2\xi)^n \mathcal{C}_n(t). \]

Matrix elements of twist-2 operators

\[ \mathcal{O}_0^{\mu_1 \ldots \mu_n} = i^{n-1} \bar{\psi}(0) \gamma^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} \frac{\lambda^a}{2} \psi(0) - \text{traces} \]
X. Ji, D. Muller, A. Radyushkin (1994-1997)

Form Factors

Parton Distribution functions

Generalized Parton Distribution functions
Mellin moments are local matrix elements

Can be evaluated in Euclidean space

Lattice QCD calculations are possible

Challenges:

Renormalization and power divergent mixing

Lattice breaks O(4) symmetry

Only few moments can be computed
**LATTICE QCD**

In continuous Euclidian space: \( Z = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}A_\mu \ e^{-S[\bar{q}, q, A_\mu]} \)

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}A_\mu \ \mathcal{O}(\bar{q}, q, A_\mu) \ e^{-S[\bar{q}, q, A_\mu]} \]

Lattice regulator:

Gauge sector:

\[ U_\mu(x) = e^{-iA_\mu(x + \frac{\hat{\mu}}{2})} \]

Fermion sector:

\[ S_f = \bar{\Psi} D \Psi \]

\( \Psi \) is now a vector whose components leave on the sites of the lattice

D is the Dirac matrix which is large and sparse
MONTE CARLO INTEGRATION

\[
\langle \mathcal{O} \rangle = \frac{1}{Z} \int \prod_{\mu,x} dU_{\mu}(x) \ \mathcal{O}[U, D(U)^{-1}] \ \text{det}(D(U)^\dagger D(U))^{n_f/2} \ e^{-S_g(U)}
\]

Monte Carlo Evaluation

\[
\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(U_i)
\]

Statistical error

\[
\frac{1}{\sqrt{N}}
\]
Form Factors

PACS: $N_f=2+1$ $m_\pi = 145$ MeV 8.1 fm box

ETMC: $N_f=2+1$ $m_\pi = 131$ MeV 4.5 fm box

PNDME: mixed action $m_\pi = 138$ MeV 5.6 fm box
Strange quark contribution to nucleon form factors

![Graphs showing form factors vs. Q^2](image)

Dynamical 2 + 1 flavors of Clover fermions

32^3 x 96 lattice of dimensions (3.6 fm)^3 x (10.9 fm)
a = 0.115 fm, pion mass 317 MeV

**z-expansion fit:**

\[ G(Q^2) = \sum_{k}^{k_{\text{max}}} a_k z^k, \quad z = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}}}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}}}}, \]

Comparison with experiments

Experiment: forward-angle parity-violating elastic e-p scattering

\[ G_E^s + \eta G_M^s \quad \eta = A Q^2, \quad A = 0.94 \]

Prediction: very hard for such experiments to measure a non-zero result
One of the fundamental nucleon observables is the axial charge to understand its origin, so that we have confidence in its reliability and precision. It is essential for lattice QCD to be measured precisely. It is a benchmark quantity to understand its origin, so that we have confidence in its reliability and precision.

Over the last years, simulations at or near the physical point have become available, which eliminate the uncontrolled systematic uncertainties.

The data close to the physical pion mass have found that to the current statistics, volume and lattice spacing dependence before reaching final conclusions. Slight upward tendencies towards the experimental value: 

The plotted results obtained from the plateau method without continuum corrected systematic uncertainties are not well under control yet and it is a benchmark quantity to understand its origin, so that we have confidence in its reliability and precision.

In Fig. 2 we plot $g_{A}$ vs $m_{\pi}^2$ dependence before reaching to final conclusions.

Axial Charge

From M. Constantinou: arXiv:1701.02855
FIG. 8: Unpolarized (vector) generalized $n = 2$ for factor $s$ for the flavor combination $u^-d$ (left) and $u^+d$ (right). Disconnected contributions are not included.

In the continuum, because of Lorentz invariance, the totally symmetric operator $\bar{q} \left[ \gamma^5 \right] \gamma^5 \{ \mu \nu \} \{ \rho \sigma \} q$ cannot mix with the mixed symmetry operator $\bar{q} \left[ \gamma^5 \right] \gamma^5 \{ \mu \nu \} \{ \rho \sigma \} q$, where the square brackets denote antisymmetrization. In contrast, on the lattice, both operators appear in the same representation, $\tau(8)$, so that they can mix. However, the mixing coefficient $[39, 40], Z_{Oij} = 2.88 \times 10^{-3}$, is very small, so that we have ignored the contribution of the mixed symmetry operator in this present work.

All results below have been transformed to a scale of $\mu^2 = 4 GeV^2$.

Moments of GPDs
LHPC: arXiv:0705.4295
Moments of PDFs (Lattice vs Experiment)

LHPC: 2007
Can Lattice QCD go beyond moments?

Lattice QCD can only compute time local matrix elements

Euclidean space
QPDFS: MAIN IDEA

\[ \lim_{P_z \to \infty} q^{(0)} (x, P_z) = f(x) \]


Euclidean space time local matrix element is equal to the same matrix element in Minkowski space.
A more general point of view:  

\[ \sigma(x, a, P_z) \xrightarrow{a \to 0} \tilde{\sigma}(x, \tilde{\mu}^2, P_z) \]

**Minkowski space factorization:**

\[ \tilde{\sigma}(x, \tilde{\mu}^2, P_z) = \sum_{\alpha = \{q, \bar{q}, g\}} H_\alpha \left( x, \frac{\tilde{\mu}}{P_z}, \frac{\tilde{\mu}}{\mu} \right) \otimes f_\alpha(x, \mu^2) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^2}{\tilde{\mu}^2} \right) \]

\( H_\alpha \) computable in perturbation theory

Related ideas see:

Practical calculations require a regulator (Lattice)

Continuum limit has to be taken

renormalization

Momentum has to be large compared to hadronic scales to suppress higher twist effects

Practical issue with LQCD calculations at large momentum ... signal to noise ratio

\[ q(x, P_z) = \int_{-1}^{1} \frac{d\xi}{\xi} \tilde{Z} \left( \frac{x}{\xi}, \frac{\mu}{P_z} \right) f(\xi, \mu) + \mathcal{O}(\Lambda_{\text{QCD}}/P_z, M_N/P_z) \]

The matching kernel can be computed in perturbation theory

QUASI-PDFS

\[ h^{(s)} \left( \frac{z}{\sqrt{\tau}}, \sqrt{\tau}P_z, \sqrt{\tau}\Lambda_{QCD}, \sqrt{\tau}M_N \right) = \frac{1}{2P_z} \left\langle P_z \left| \overline{\chi}(z;\tau)W(0, z; \tau)\gamma_z \frac{\lambda^a}{2}\chi(0; \tau) \right| P_z \right\rangle \]

\( \tau \) is the a regulator scale
\( \chi \) quark field
\( \mathcal{W} \) is the regulated gauge link

\[ q^{(s)} (\xi, \sqrt{\tau}P_z, \sqrt{\tau}\Lambda_{QCD}, \sqrt{\tau}M_N) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{i\xi z P_z} P_z h^{(s)} (\sqrt{\tau}z, \sqrt{\tau}P_z, \sqrt{\tau}\Lambda_{QCD}, \sqrt{\tau}M_N) \]

At fixed flow time the quasi-PDF is finite in the continuum limit
Using the previous definitions we have

\[
\left( \frac{i}{P_z} \frac{\partial}{\partial z} \right)^{n-1} h^{(s)} \left( \frac{z}{\sqrt{\tau}}, \sqrt{\tau} P_z, \sqrt{\tau} \Lambda_{QCD}, \sqrt{\tau} M_N \right) = \int_{-\infty}^{\infty} d\xi \, \xi^{n-1} e^{-i\xi P_z} q^{(s)} (\xi, \sqrt{\tau} P_z, \sqrt{\tau} \Lambda_{QCD}, \sqrt{\tau} M_N)
\]

By introducing the moments

\[
b_n^{(s)} \left( \sqrt{\tau} P_z, \frac{\Lambda_{QCD}}{P_z}, \frac{M_N}{P_z} \right) = \int_{-\infty}^{\infty} d\xi \, \xi^{n-1} q^{(s)} (\xi, \sqrt{\tau} P_z, \sqrt{\tau} \Lambda_{QCD}, \sqrt{\tau} M_N)
\]
Taking the limit of $z$ going to 0 we obtain:

\[
\begin{align*}
\frac{b_n^{(s)}}{P_z} \left( \sqrt{\tau P_z}, \frac{\Lambda_{\text{QCD}}}{P_z}, \frac{M_N}{P_z} \right) &= \frac{c_n^{(s)}(\sqrt{\tau P_z})}{2P_z^n} \left< P_z \left| \left[ \bar{\chi}(z; \tau) \gamma_z(i\not{D}_z)^{(n-1)} \frac{\lambda^a}{2} \chi(0; \tau) \right] \right| z=0 \right> _{C}.
\end{align*}
\]

i.e. the moments of the quasi-PDF are related to local matrix elements of the smeared fields

These matrix elements are not twist-2. Higher twist effects enter as corrections that scale as powers of

\[
\frac{\Lambda_{\text{QCD}}}{P_z}, \quad \frac{M_N}{P_z}
\]

after removing $M_N/P_z$ effects

\[
b_n^{(s)} \left( \sqrt{\tau P_z}, \sqrt{\tau \Lambda_{\text{QCD}}} \right) = c_n^{(s)}(\sqrt{\tau P_z})b_n^{(s,\text{twist-2})} \left( \sqrt{\tau \Lambda_{\text{QCD}}} \right) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^2}{P_z^2} \right)
\]

Introducing a kernel function such that:

\[ C_n^{(0)}(\sqrt{\tau} \mu, \sqrt{\tau} P_z) = \int_{-\infty}^{\infty} dx \, x^{n-1} \tilde{Z}(x, \sqrt{\tau} \mu, \sqrt{\tau} P_z) \]

We can undo the Melin transform:

\[ q^{(s)}(x, \sqrt{\tau} \Lambda_{QCD}, \sqrt{\tau} P_z) = \int_{-1}^{1} \frac{d\xi}{\xi} \tilde{Z} \left( \frac{x}{\xi}, \sqrt{\tau} \mu, \sqrt{\tau} P_z \right) f(\xi, \mu) + O(\sqrt{\tau} \Lambda_{QCD}) \]

Therefore regulated quasi-PDFs are related to PDFs if

\[ \Lambda_{QCD}, M_N \ll P_z \ll \tau^{-1/2}. \]
PROCEDURE OUTLINE

- Compute equal time matrix elements in Euclidean space using Lattice QCD at sufficiently large momentum in order to suppress higher twist effects

- Take the continuum limit (renormalization)

- Equal time: Minkowski – Euclidean equivalence

- Perform the matching Kernel calculation in the continuum
First Lattice results (Chen et. al)

Convergence with momentum extrapolation

Including the 1-loop matching kernel

Plots taken from: Chen et al. arXiv:1603.06664

Similar results have been achieved by Alexandrou et. al (ETMC)
Along these lines one can compute:

- TMDs (see Engelhardt et. al.)
- GPDs
- Distribution amplitudes
- Gluonic PDFs
- ……
CONCLUSIONS

- Lattice QCD calculations have made a lot of progress and in some cases precision results are being obtained
  - Physical quark masses, large volumes, large scale calculations
- Quasi-PDFs provide a novel way to study hadron structure in Lattice QCD
- Lattice calculations from several groups are on the way
- Several ideas for dealing with the continuum limit are now developing
- Promising new ideas: Stay tuned!
NUCLEON FORM FACTOR

Connected

Disconnected

Strange quark : disconnected only