Fluctuations of conserved charges from the lattice

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Wuppertal-Budapest collaboration.
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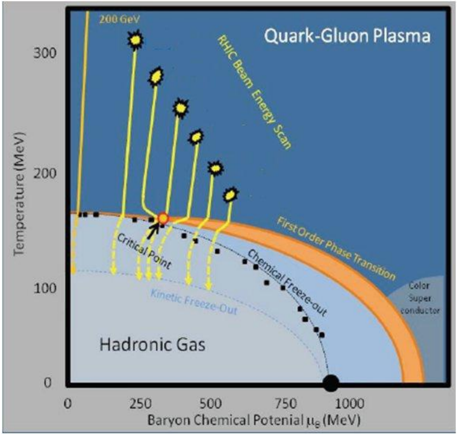
Beam energy scan and freeze-out curve



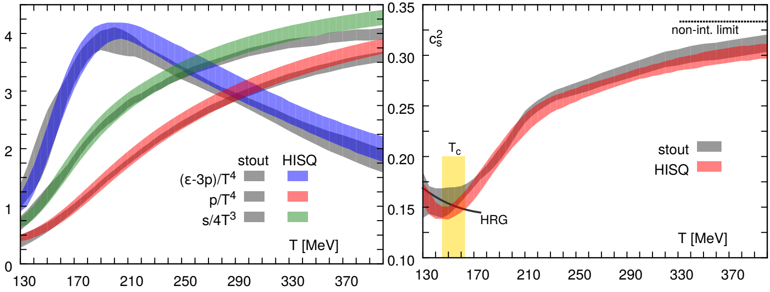
Chiral crossover region from lattice:
$T_c = 147 \ldots 157$
Wuppertal-Budapest:
[hep-lat/0611014,hep-lat/0609068,0903.4155,1005.3508]

HotQCD: [1111.1710]

At RHIC a broad energy range $\sqrt{s_{\mathrm{NN}}} = 7.7 \ldots 200$ has been scanned with heavy ion collisions.
Last inelastic scattering:
chemical freeze-out.
For each energy the chemical freeze-out is described as a grand canonical ensemble with one temperature and chemical potential.
Traditional method:
Hadron Resonance Gas (HRG)-based statistical fit of pion, kaon, proton, etc yields.
Fit result at $\sqrt{s_{\mathrm{NN}}} = 130\mathrm{GeV}$
$\mu_B = 38(12)\ \mathrm{MeV}$ and
$T_{ch} = 165(5)\ \mathrm{MeV}$.
[Andronic et al nucl-th/0511071]

Equation of state with up,down and strange quarks

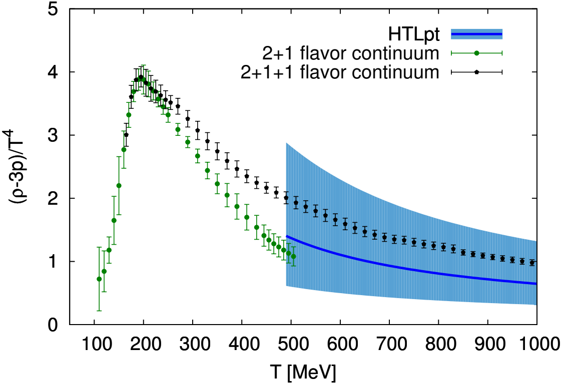
stout result: Wuppertal-Budapest group [1309.5258]
HISQ result: Bielefeld-Brookhaven group [1407.6397]



Around the transition temperature $(\epsilon - 3p)/T^4$ has a steepest point, the speed-of-sound has a minimum

The Wuppertal-Budapest equation of state has recently beeen updated: [1606.07494].

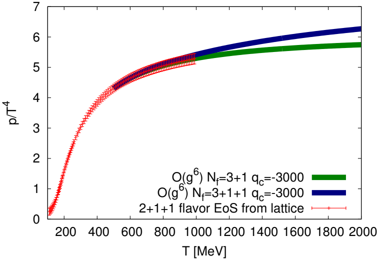
2+1+1 flavor simulations (with the charm quark), the effect of the bottom quark is estimated.

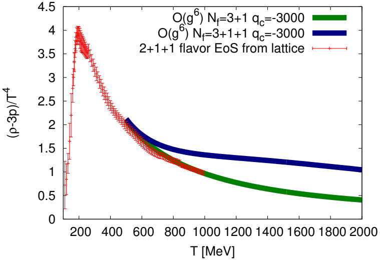


The ratio of the two results can be described by a tree-level threshold function.

Perturbative parametrization at high temperatures

- Applying the tree-level charm threshold to the perturbative pressure
$$\frac{p}{T^4} = \# + \#g^2 + \#g^3 + \#g^4 + \#g^4 \log(g) + \#g^5 + \#g^6 \log(g) + ?g^6$$
[Kajantie 2002]
- The g^6 term is fitted to lattice ($-3200 < q_c < -2700$).
- The fit describes the pressure **and** trace anomaly from 500 MeV.
- Next we can introduce the bottom quark treshold keeping q_c fixed.





Fluctuations in a grand canonical ensemble

The expectation value of a conserved charge is a derivative with respect to the chemical potential.

$$\langle N_q \rangle = T \frac{\partial \log Z(T, V, \{\mu_q\})}{\partial \mu_q}$$

The response of the system to the thermodynamic force μ_q is proportional to the fluctuation of the conserved charge:

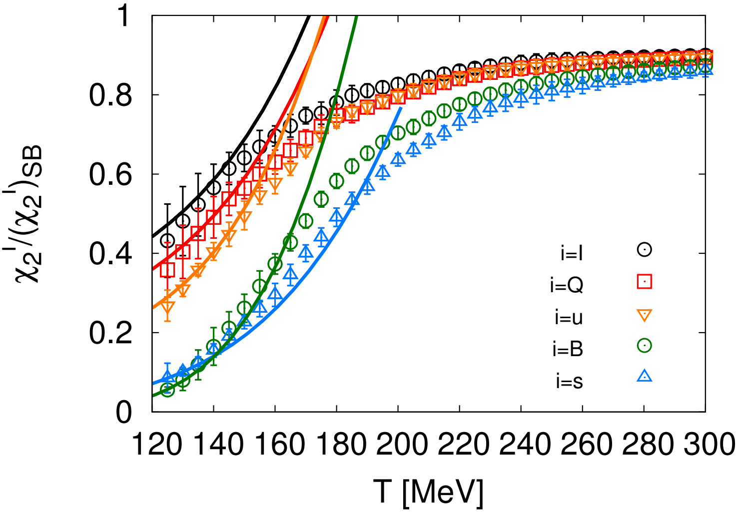
$$\frac{\partial \langle N_i \rangle}{\partial \mu_j} = T \frac{\partial^2 \log Z(T, V, \{\mu_q\})}{\partial \mu_j \partial \mu_i} = \frac{1}{T} (\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle)$$

The higher derivatives are the generalized quark number susceptibilities:

$$\chi_{i,j,k,l}^{u,d,s,c} = \frac{\partial^{i+j+k+l}(p/T^4)}{(\partial \hat{\mu}_u)^i (\partial \hat{\mu}_d)^j (\partial \hat{\mu}_s)^k (\partial \hat{\mu}_c)^l}$$

with $\hat{\mu}_q = \mu_q / T$.

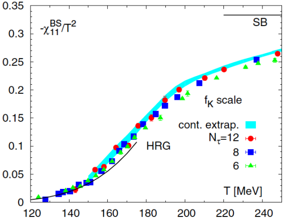
All diagonal fluctuations



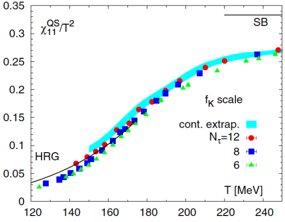
This result: [Wuppertal-Budapest 1112.4416] see also [HotQCD 1203.0784]

Off-diagonal susceptibilities

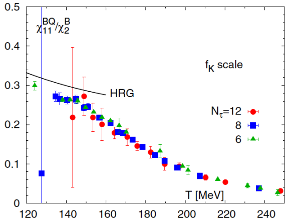
$-\chi_{11}^{BS}$ [HotQCD 1203.0784]



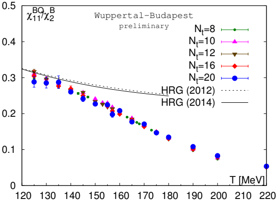
χ_{11}^{QS} [HotQCD 1203.0784]



χ_{11}^{BQ}/χ_2^B [HotQCD 1203.0784]



χ_{11}^{BQ}/χ_2^B [WB preliminary]



Fluctuations on the lattice

The partition function of the lattice gauge theory with staggered fermions is

$$Z = \int \mathcal{D}U \; e^{-S_g} (\det M_u(\mu_u))^{1/4} (\det M_d(\mu_d))^{1/4} (\det M_s(\mu_s))^{1/4} = \int \mathcal{D}U \; e^{-S_{\rm eff}}$$

where S_g is the gauge action. For $\mu > 0$ the determinant becomes complex.
Derivatives, however, are still accessible using $\mu = 0$ simulations.
First derivative of the free energy density:

$$\partial_i \log Z = \frac{1}{Z} \int \mathcal{D}U \; \partial_i e^{-S_{\rm eff}} = \langle A_i \rangle \; , \quad A_i = \frac{1}{4} \frac{d \log \det M_i(\mu_i)}{d\mu_i} = \frac{1}{4} {\rm Tr} M' M^{-1}$$

The 2nd derivative reads

$$\partial_i \partial_j \log Z \quad = \quad \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle + \delta_{ij} \langle dA_i / d\mu \rangle$$

In terms of physical derivatives

$$\begin{aligned}
\frac{d}{d\mu_B} &= \frac{1}{3}\partial_u + \frac{1}{3}\partial_d + \frac{1}{3}\partial_s \, , \\
\frac{d}{d\mu_Q} &= \frac{2}{3}\partial_u - \frac{1}{3}\partial_d - \frac{1}{3}\partial_s \, , \\
\frac{d}{d\mu_S} &= -\partial_s
\end{aligned}$$

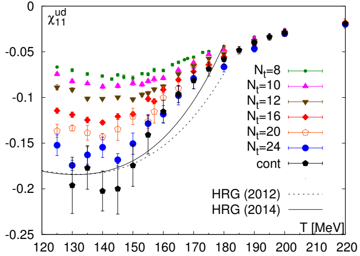
The fermion determinant $\det M = |\det M| e^{i\theta}$ has a fluctuating phase at $\mu > 0$: [Allton hep-lat/02040130]

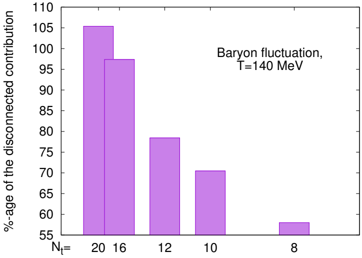
$$\theta = \frac{1}{4} N_f \mathrm{Im} \left[\mu \frac{\partial \ln \det M}{\partial \mu} + \frac{\mu^3}{3!} \frac{\partial \ln \det M}{\partial \mu} + \dots \right]$$

The fluctuation of $A = \partial \ln \det M / \partial \mu$ gives at LO for the phase:

$$\langle \theta^2 \rangle = -\frac{1}{9} \mu_B^2 L^3 T N_f^2 \chi_{11}^{ud}$$

(~ 1 at $\mu_B \approx 100$ MeV, with $T = T_c$ and $LT = 3$.)





[Wuppertal-Budapest 1507.04627]

Equation of state at finite density

Taylor approach: $\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \chi_2^B \frac{\mu_B^2}{2!T^2} + \chi_4^B \frac{\mu_B^4}{4!T^4} + \chi_6^B \frac{\mu_B^6}{6!T^6} + \dots$

$$\chi_n^B = \frac{\partial^n \log Z}{[\partial(\mu_B/T)]^n}$$

MILC and BNL-Bielefeld: $N_t = 6$ 3rd order [1003.5682, hep-lat/0512040]

In *heavy ion phenomenology* at RHIC $\mu_B > 0$ but the strangeness vanishes.

$$M_S = 0, \qquad M_Q = M_B \frac{Z}{A}, \qquad r = \frac{Z}{A} = \frac{79}{197} \approx 0.4 \quad \text{gold}$$

This turns χ_B into comlpicated mixed derivatives [BNL-Bielefeld 1208.1220].

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1. Taylor method

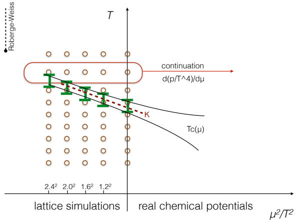
μ_B derivatives at $\mu_B = 0$ simulations;

more statistics are required

2. Analytical continuation

$\mu_B^2 \leq 0$ simulations;

careful analysis of systematics is required

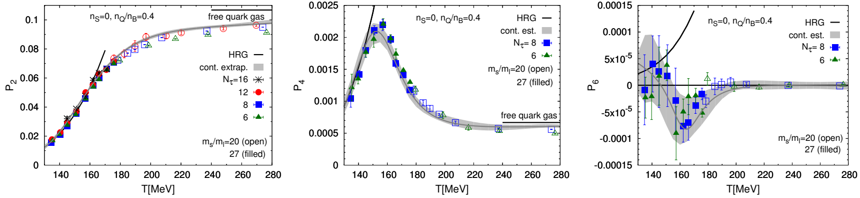


Roberge-Weiss temperature:

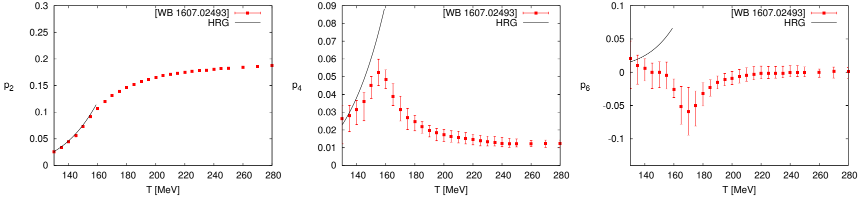
$T_{RW}^{2+1f} = 208(5)$ MeV [Bonati et al. 1602.01426]

Pressure coefficients

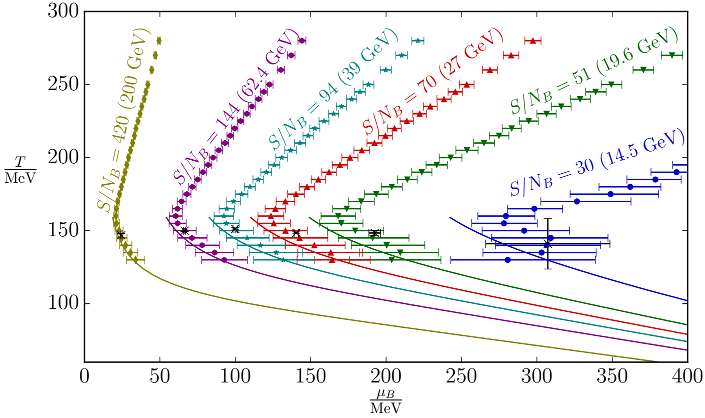
Taylor method: *mostly* $N_t = 8$, $\mathcal{O}(10^5)$ *configurations point,* [HotQCD: 18.00]



Analytical method: *continuum,* $\mathcal{O}(10^4)$ *configurations/point, errors include systematics* [Wuppertal-Budapest 1607.02493]

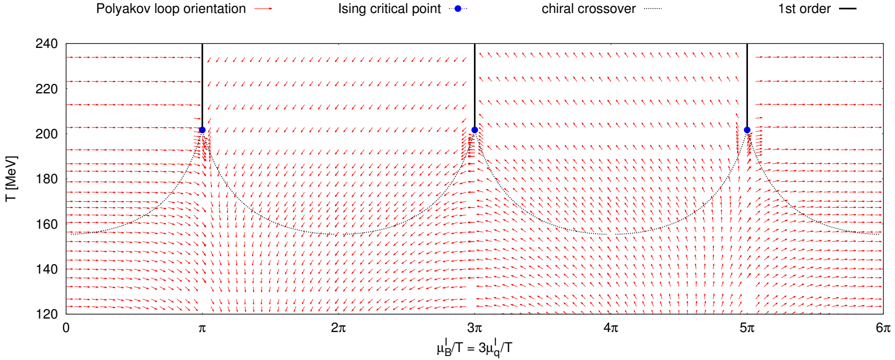


Adiabatic trajectories in the phase diagram



Well studied phase diagram: [Fodor &Katz hep-lat/0104001] [de Forcrand & Philipsen hep-lat/0205016] [Philipsen 0708.1293, Philipsen 1402.0838] [Cea et al hep-lat/0612018,0905.1292,1202.5700] [Bonati et al 1410.5758,1507.03571,1602.01426]

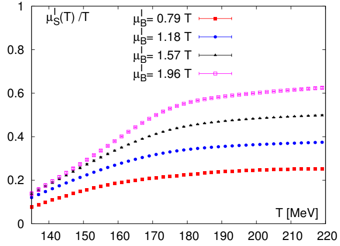
At imaginary μ_B there is no sign problem. The observables, including the crossover line, are analytical functions of μ_B^2.

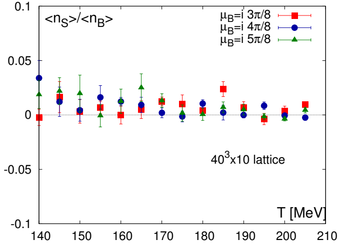


The phase diagram is perodic $\mu_B \to \mu_B + i2\pi T$, with simultaneous rotation between the Z(3) sectors.

Tuning to strangeness neutrality: $\langle n_S \rangle = 0$

We simulate each ensemble with an imaginary μ_B, μ_S pair such that $\langle n_S \rangle = 0$. This requires a non-trivial fine-tuning.





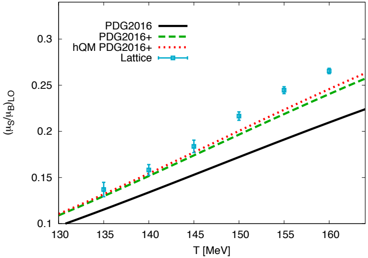
To compansated the remaining slight inaccuracies of the tuning and to achieve $\langle n_Q \rangle = 0.4 \langle n_B \rangle$ we correct all generalized quark number susceptibilities using higher order μ_Q and μ_S deviratives for each simulation point.

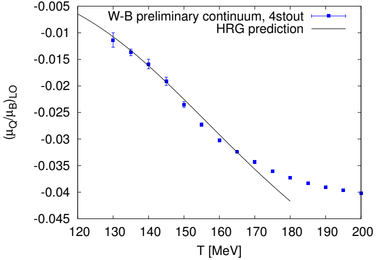
Strangeness and electric charge chemical potential

We have to calculate $\mu_Q(\mu_B)$ and $\mu_S(\mu_B)$ that are defined by the constraints of the experimental setup:

$$\langle S \rangle = 0 \qquad \langle Q \rangle / \langle B \rangle = Z^{Au}/A^{Au} \approx 0.4$$

Our latest continuum extrapolations with up to $N_\tau = 24$:

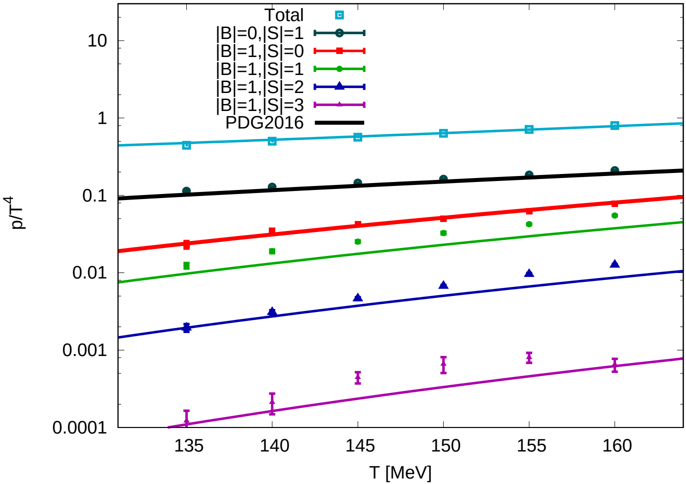




See also:
HotQCD: [1208.1220,1404.6511]
Wuppertal-Budapest: [1305.5161]

Strangeness sectors from the lattice

Lattice can calculate the partial pressure sector by sector.



Discrepancy to HRG: *indication for undiscovered resonances?*

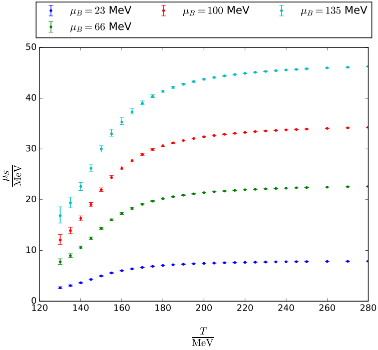
poster by Paolo Parotto

Strangeness and electric charge chemical potential

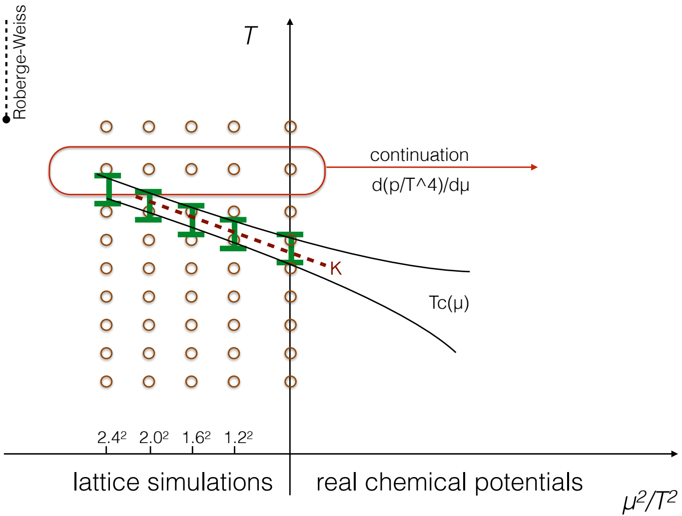
We have to calculate $\mu_Q(\mu_B)$ and $\mu_S(\mu_B)$ that are defined by the constraints of the experimental setup:

$$\langle S \rangle = 0 \qquad \langle Q \rangle / \langle B \rangle = Z^{Au}/A^{Au} \approx 0.4$$

In the continuum limit, extrapolated to finite μ_B beyond leading order:



Analytic continuation



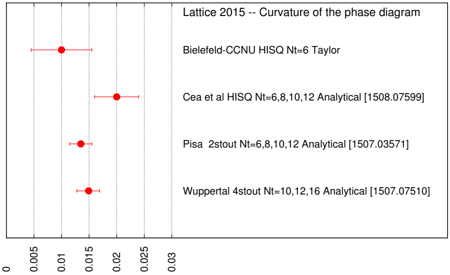
Many exploratory studies: [de Forcrand & Philipsen hep-lat/0205016]

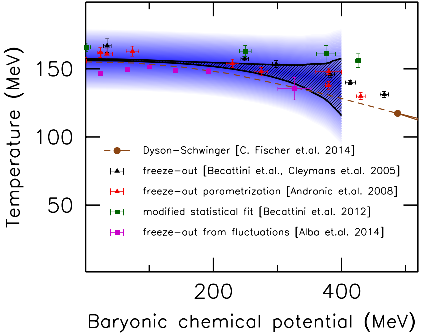
[Philipsen 0708.1293] [Philipsen 1402.0838] [Cea et al hep-lat/0612018,0905.1292,1202.5700]

The $T_c(\mu_B)$ can be expandedd around $\mu_B = 0$ (Taylor method) or found through analytical continuation with $\mathrm{Im}\ \mu_B > 0$.

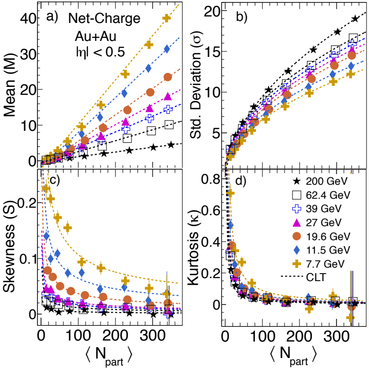
$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa \left(\frac{\mu_B}{T_c(\mu)} \right)^2 + \ldots$$

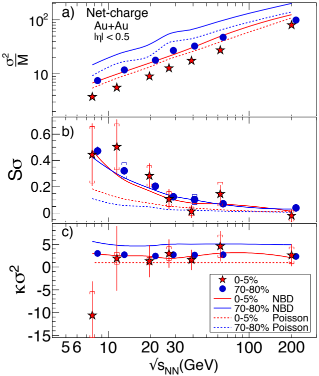
Lattice 2015, Kobe:





At RHIC **STAR** has measured the mean, variance, skewness and kurtosis of the event-by-event **net charge** distribution at various energies and centralities.





[STAR: 1402.1558]

We have to calculate $\mu_Q(\mu_B)$ and $\mu_S(\mu_B)$ that are defined by the constraints of the experimental setup:

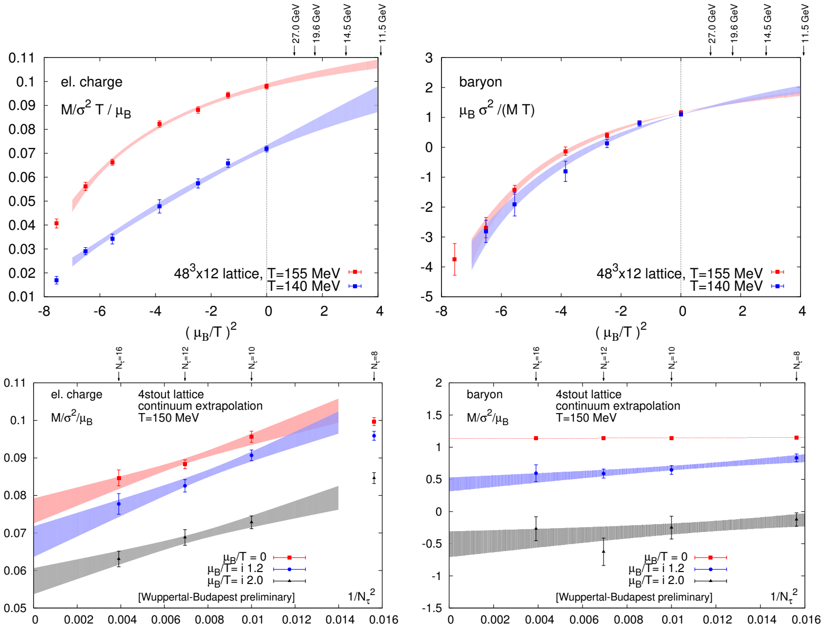
$$\langle S\rangle = 0 \qquad \langle Q\rangle/\langle B\rangle = Z^{Au}/A^{Au} \approx 0.4$$

$$\begin{aligned}
M_Q &= V\chi_1^Q(\mu_B) = V\mu_B\left[\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}\right] + \mathcal{O}(V\mu_B^3) \\
\sigma_Q^2 &= V\chi_2^Q(\mu_B) = V\chi_B^Q + \mathcal{O}(V\mu_B^2) \\
S_Q\sigma_Q^3 &= V\chi_3^Q(\mu_B) = V\mu_B\left[\chi_{13}^{BQ} + \chi_{31}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_4^Q\frac{d\mu_Q}{d\mu_B}\right] + \mathcal{O}(V\mu_B^3) \\
\kappa\sigma_Q^4 &= V\chi_4^Q(\mu_B) = V\chi_B^Q + \mathcal{O}(V\mu_B^2)
\end{aligned}$$

To leadig order, at infinitesimal μ_B:

$$\frac{M_Q}{\mu_B\sigma_Q^2} = \frac{\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}}{\chi_2^Q}; \quad \frac{S_Q\sigma_Q^3}{M_Q} = \frac{\chi_{13}^{BQ} + \chi_{31}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_4^Q\frac{d\mu_Q}{d\mu_B}}{\chi_{11}^{BQ} + \chi_{11}^{QS}\frac{d\mu_S}{d\mu_B} + \chi_2^Q\frac{d\mu_Q}{d\mu_B}}$$

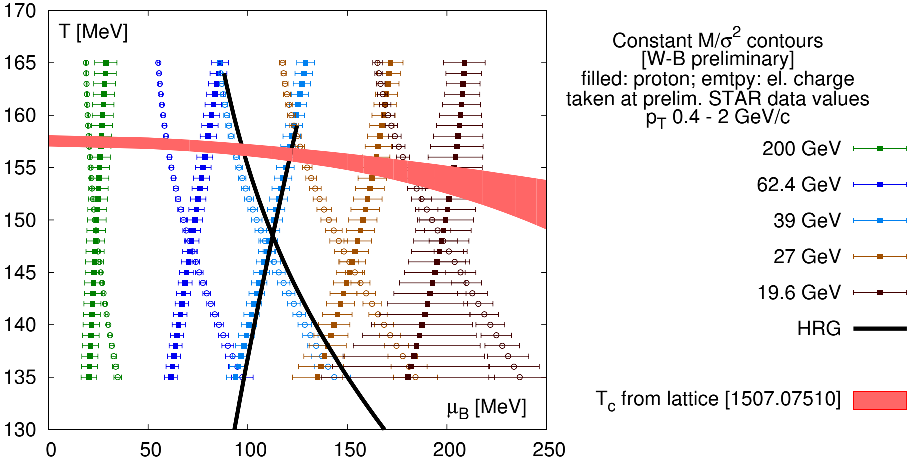
Extending to imaginary μ_B



Contours of constant fluctuations on the phase diagram

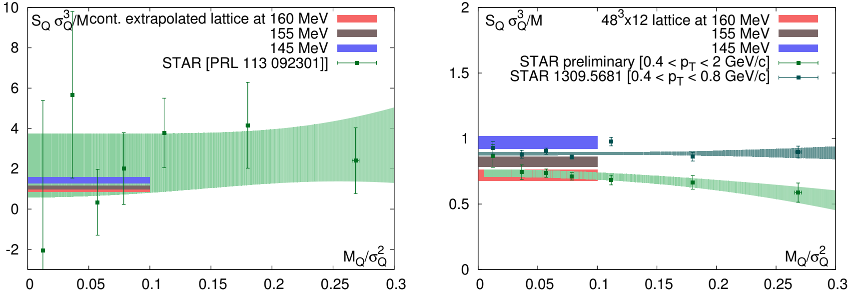
As a realistic example we take the preliminary STAR proton and electric charge fluctuation data [1309.5681,1402.1558].

$\sqrt{s_{NN}}$ [GeV]	200	62.4	39	27	19.6
σ_p^2/M_p	7.10(1)(138)	2.46(0)(11)	1.70(0)(5)	1.31(0)(2)	1.12(0)(1
σ_Q^2/M_Q	80.2(2)(1)	27.32(7)(3)	17.46(3)(1)	12.71(3)(1)	8.95(3)(1



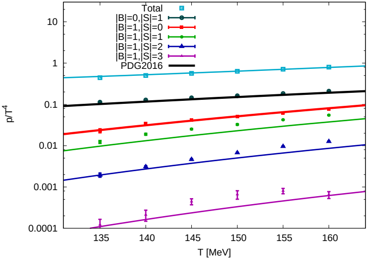
Higher order fluctuations vs expermient

Lattice can calculate the baryon and electric charge skewness at small chemical potentials. Here we show the preliminary Wupperal-Budapest results together with STAR data.



STAR's new preliminary proton skewness is not compatible with the low temperatures from the M/σ^2 analysis.

- Lattice QCD is making good progress in exploring thermodynamics (T_c, equation of state) at small and intermediate chemical potentials ($\mu_B/T < 2$). *Sufficient range for most RHIC energies*
- Low order fluctuations are well established, even at small μ_B, higher order fluctuations are subject of current research.
- Higher order fluctuations can be explored with large statistics $\mu_B = 0$ simulations (BNL-Bielefeld way) or with studying the response to imaginary chemical potential (Wuppertal-Budapest strategy).
- Imaginary chemical potentials are also useful for precision tests of the HRG model and various spectra.
  Here: The pressure contribution of $|S|$=0,1,2,3 mesons and baryons.



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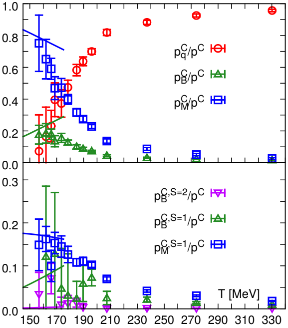
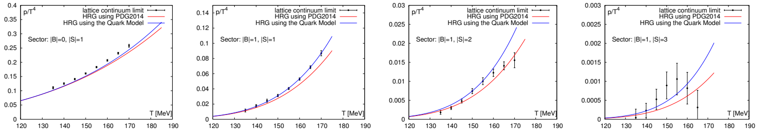


Search for open charm bound states above T_c

$$p^C(T, \mu_C, \mu_B) = p^C_q(T) \cosh(\hat{\mu}_C + \hat{\mu}_B/3) + p^C_B(T) \cosh(\hat{\mu}_C + \hat{\mu}_B) + p^C_M(T) \cosh(\hat{\mu}_C) \ ,$$

[Mukherjee et al 1509.08887]

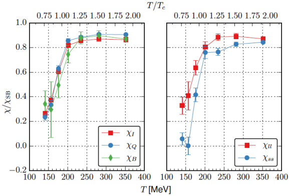
Search for resonances not covered by the Particle Data Book below T_c



[BNL-Bielefeld 1404.4043, 1404.6511; Wuppertal-Budapest prelim (S.B. Lattice16)]

Quark number susceptibilities

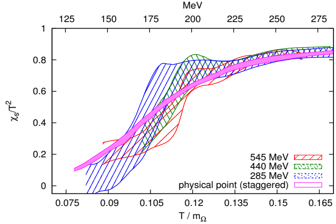
anisotropic wilson $m_\pi = 392$ MeV



[FASTSUM 1309.6253,1412.6411]

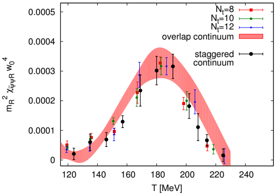
isotropic wilson in continuum

$m_\pi = 545, 440, 285$ MeV.



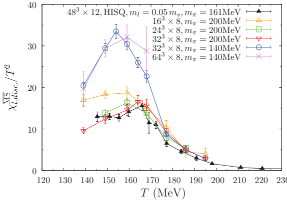
[WB 1205.0440,1504.03676]

Chiral susceptibility



Overlap fermions ($m_\pi = 350$ MeV)

[WB 1204.4089].



Domain wall ($m_\pi = 135$ MeV)

[HotQCD 1402.5175]

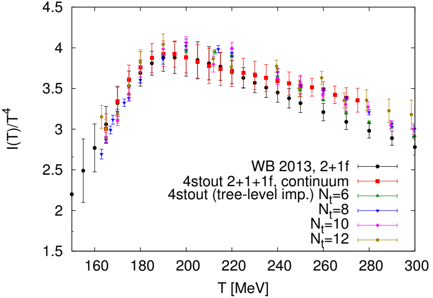
Caveats

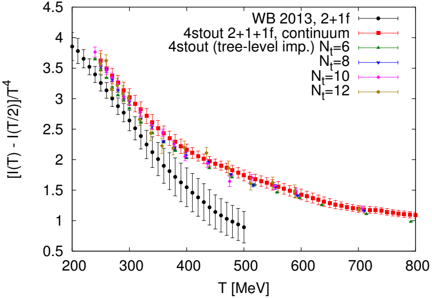
- *Effects due to volume variation because of finite centrality bin width*
  Experimentally corrected by centrality-bin-width correction method
- *Finite reconstruction efficiency*
  Experimentally corrected based on binomial distribution
  [A. Bzdak, V. Koch, PRC (2012)]
- *Spallation protons*
  Experimentally removed with proper cuts in p_T
- *Canonical vs Grand Canonical ensemble*
  Experimental cuts in the kinematics and acceptance
  [V. Koch, S. Jeon, PRL (2000)]
- *Proton multiplicity distributions vs baryon number fluctuations*
  Numerically very similar once protons are properly treated
  [M. Asakawa and M. Kitazawa], [PRC (2012), M. Nahrgang et al., 1402.1238]
- *Final-state interactions in the hadronic phase* [J.Steinheimer et al., PRL (2013)]
  Consistency between different charges = fundamental test

$2+1+1$ flavor equation of state – lattice data

For low temperatures $I(T)/T^4$ is calculated using vacuum subtraction.

For higher temperatures $[I(T) - I(T/2)]/T^4$ is calculated and continuum extrapolated.





The final trace anomaly is calculated from the continuum extrapolated terms using the formula

$$\frac{I(T)}{T^4} = \sum_{k=0}^{n-1} 2^{-4k} \frac{I(T/2^k) - I(T/2^{k+1})}{(T/2^k)^4} + 2^{-4n} \frac{I(T/2^n)}{(T/2^n)^4}$$