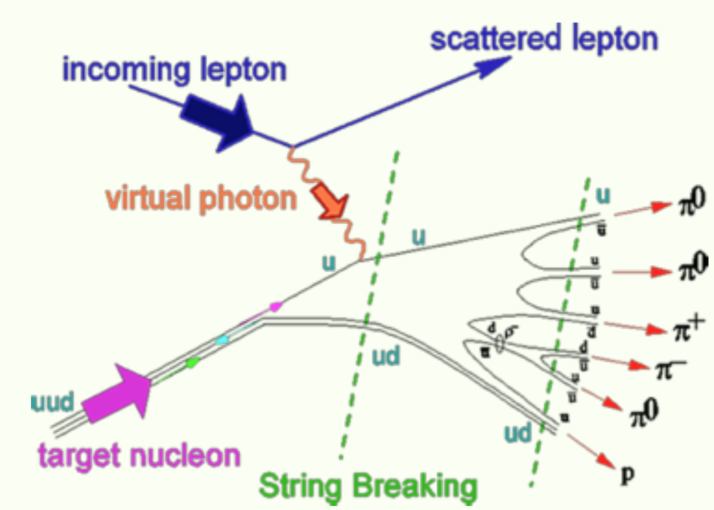
TMDs in *b*-Space Bessel Weighting

ECT*
EUROPEAN CENTRE FOR THEORETICAL STUDIES
IN NUCLEAR PHYSICS AND RELATED AREAS
TRENTO, ITALY

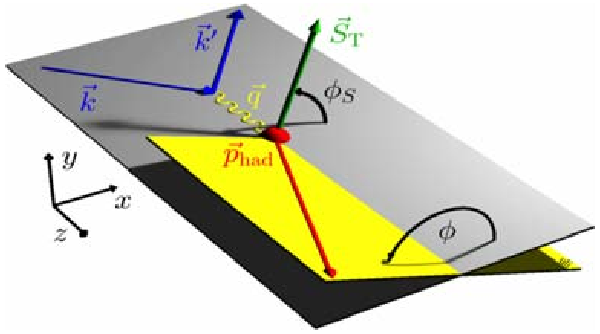
Parton TMDs at large x:
a window into parton dynamics in nucleon structure within QCD.

Leonard Gamberg Penn State Berks

Boer, LG, Musch,Prokudin JHEP 2011
M. Aghasyan, H. Avakian, E. De Sanctis, LG, M. Mirazita, B. Musch, A. Prokudin, P. Rossi JHEP 2015



Outline

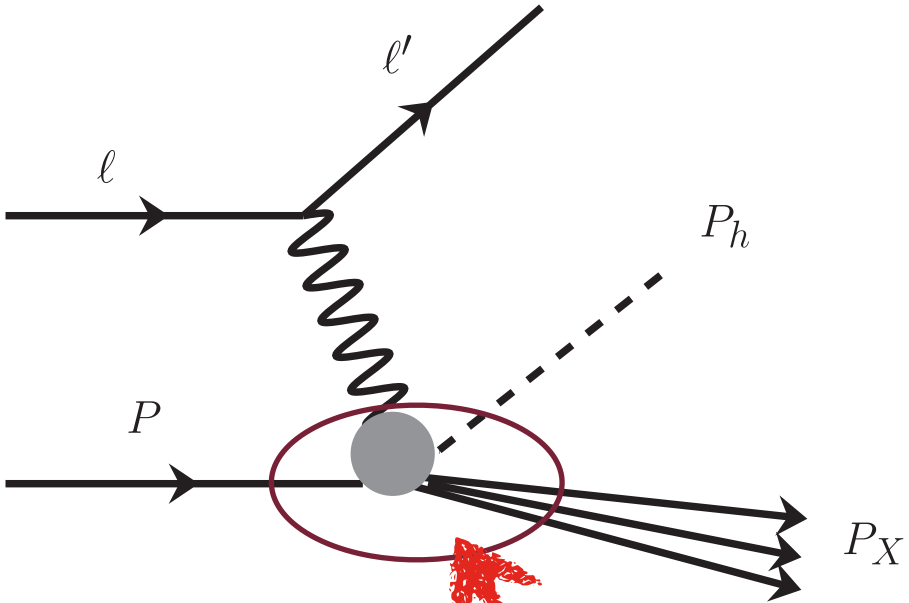


- TMDs in Fourier Space-*Exploit Bessel Moments/Weighting SIDIS cross section*
- *Use* Parton model to study topic & issues w/ conventional weighting
- Impact of studying *BW* and TMD evolution
- Sketch... Elements TMD Factorization-SIDIS see talk of Ted Rogers
- Cancellation of Universal & flavor indep. factors in BWAs
- A "case study" of BW of experimental observables $A_{LL}(b_T)$

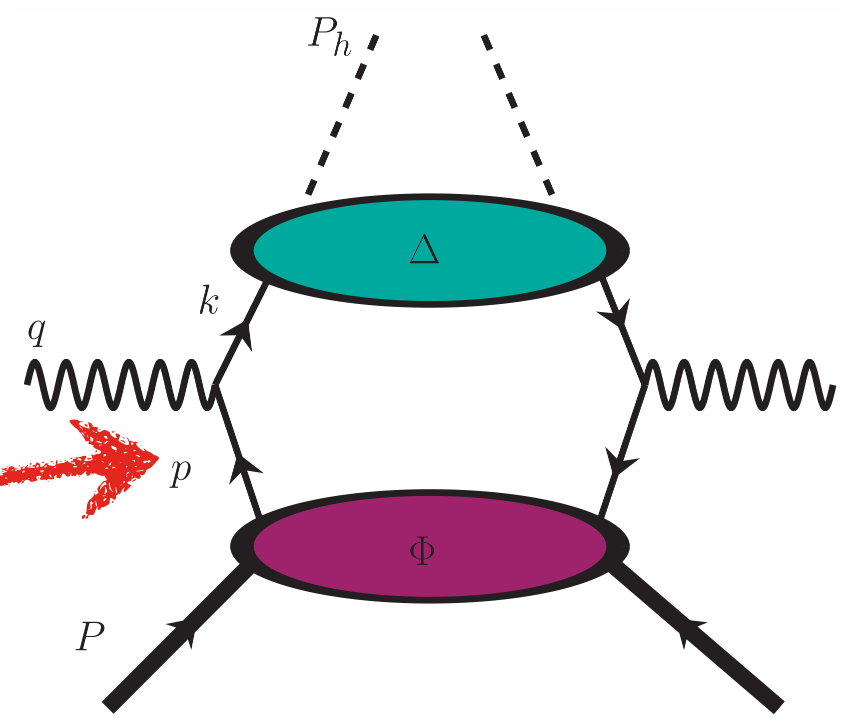
Comments on Weighting

- Weighting enables one disentangle in a model independent way the CS in terms of transverse momentum moments of TMDs
- Convert convolutions in the cross section into simple products
  not a new idea **Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98**
- Bessel Weighting solves problem of infinite contribution from large transverse momentum that arise using "conventional weighting"
  Boer, Gamberg,Musch,Prokudin JHEP 2011
- Explore impact these BWA have on studying the scale dependence of the SIDIS cross section at small to moderate transverse momentum where the TMD framework is expected to give a good description of the cross section **Boer, Gamberg, Musch,Prokudin JHEP 2011**

Parton Model SIDIS



Kotzinian NPB 95,
Mulders Tangermann NPB 96,
Boer & Mulders PRD 97
Bacchetta et al JHEP 08



1photon exchange
approx Factorize
Hadronic Tensor

$$\frac{d\sigma}{dx_B\, dy\, d\psi\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}|\, d|\boldsymbol{P}_{h\perp}|} = \frac{\alpha^2}{x_B\, y\, Q^2}\, \frac{y^2}{(1-\varepsilon)} \left(1+\frac{\gamma^2}{2x_B}\right) L_{\mu\nu}W^{\mu\nu},$$

Parton Model: P_T of hadron small sensitive to intrinsic transv. momentum of partons

$$W^{\mu\nu}(q,P,S,P_h) = \int \frac{d^2\mathbf{p}_T}{(2\pi)^2} \int \frac{d^2\mathbf{k}_T}{(2\pi)^2} \delta^2\left(\mathbf{p}_T - \frac{\mathbf{P}_{h\perp}}{z_h} - \mathbf{k}_T\right) \mathrm{Tr}\left[\Phi(x,\mathbf{p}_T)\gamma^\mu \Delta(z,\mathbf{k}_T)\gamma^\nu\right]$$

$$\Phi(x,\mathbf{p}_T) = \int dp^- \Phi(p,P,S)|_{p^+=x_B P^+}\ , \qquad \Delta(z,\mathbf{k}_T) = \int dk^- \Delta(k,P_h)|_{k^-=\frac{P^-}{z_h}}$$

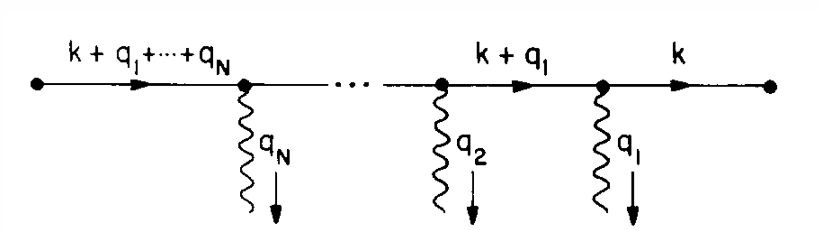
Small transverse momentum

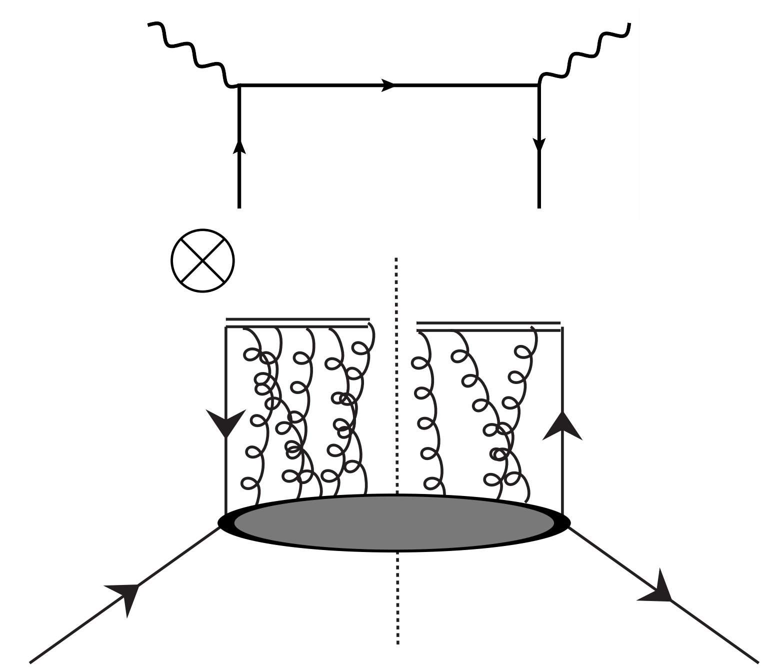
<u>Purely Kinematic-integrate over small momentum component</u>

"Gauge invariant extension" of parton model

Collins & Soper NPB193 (81) & Efremov, Radyushkin Theo. Math. Phys. 81... also Collins Found. PQCD 2011

respect gauge invariance **color** gauge invariance





Partonic picture Structure Functions momentum CONVOLUTION

$$\mathcal{C}\big[w\, f\, D\big] = x \sum_a e_a^2 \int d^2 \boldsymbol{p}_T\, d^2 \boldsymbol{k}_T\, \delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

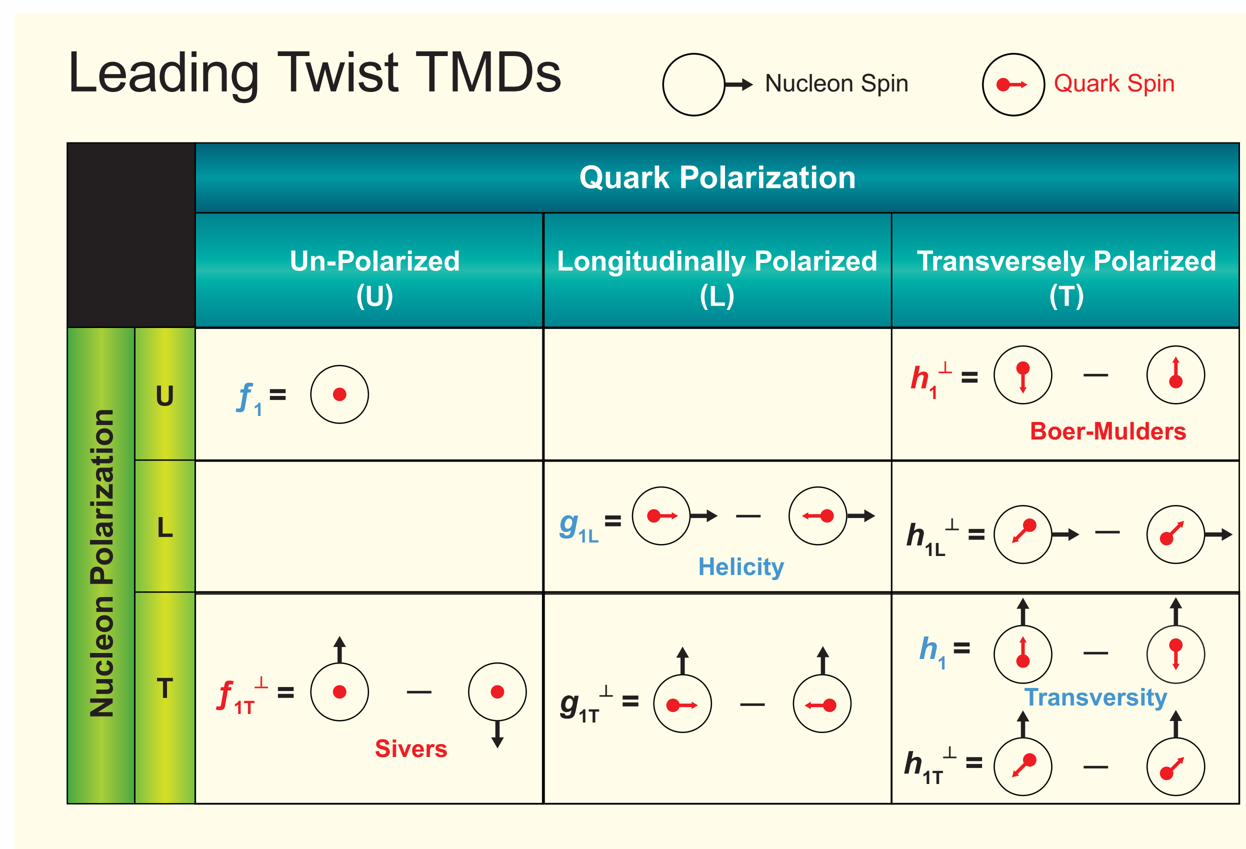
$$F_{UU,T} = \mathcal{C}\big[f_1 D_1\big],$$

$$F_{LL} = \mathcal{C}\big[g_{1L} D_1\big],$$

$$F_{UT,T}^{\sin(\phi_h - \phi_S)} = \mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\Big],$$

$$F_{UT}^{\sin(\phi_h + \phi_S)} = \mathcal{C}\Big[-\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{k}_T}{M_h} h_1 H_1^{\perp}\Big],$$

Sivers PRD 1990, Brodsky Hwang Schmidt 2002 PLB, Collins PLB 2002



SIDIS CS & leading and subleading twist structure functions

$$
\begin{aligned}
\frac{d\sigma}{dx_B\,dy\,d\psi\,dz_h\,d\phi_h\,dP_{h\perp}^2} &= \frac{\alpha^2}{x_B y Q^2}\,\frac{y^2}{2\,(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\Bigg\{F_{UU,T}+\varepsilon F_{UU,L}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h} \\
&+ \quad \varepsilon\cos(2\phi_h)\,F_{UU}^{\cos 2\phi_h}+\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,F_{LU}^{\sin\phi_h} \\
&+ \quad S_\parallel\left[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,F_{UL}^{\sin\phi_h}+\varepsilon\sin(2\phi_h)\,F_{UL}^{\sin 2\phi_h}\right] \\
&+ \quad S_\parallel\lambda_e\left[\sqrt{1-\varepsilon^2}\,F_{LL}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,F_{LL}^{\cos\phi_h}\right] \\
&+ \quad |\boldsymbol{S}_\perp|\Bigg[\sin(\phi_h-\phi_S)\left(F_{UT,T}^{\sin(\phi_h-\phi_S)}+\varepsilon\,F_{UT,L}^{\sin(\phi_h-\phi_S)}\right) \\
&+ \quad \varepsilon\,\sin(\phi_h+\phi_S)\,F_{UT}^{\sin(\phi_h+\phi_S)}+\varepsilon\,\sin(3\phi_h-\phi_S)\,F_{UT}^{\sin(3\phi_h-\phi_S)} \\
&+ \quad \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,F_{UT}^{\sin\phi_S}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\,F_{UT}^{\sin(2\phi_h-\phi_S)}\Bigg] \\
&+ \quad |\boldsymbol{S}_\perp|\lambda_e\Bigg[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\,F_{LT}^{\cos(\phi_h-\phi_S)}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,F_{LT}^{\cos\phi_S} \\
&+ \quad \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\,F_{LT}^{\cos(2\phi_h-\phi_S)}\Bigg]\Bigg\},
\end{aligned}
$$

Bacchetta et al JHEP 08

Observables SIDIS-CS expressed structure functions

$$\frac{d^6\sigma}{dx\,dy\,dz\,d\phi_h\,dP^2_{h\perp}} \sim \left\{F_{UU,T}\cdots+\ldots|S_\perp|\left(\sin(\phi_h-\phi_S)F_{UT,T}^{\sin(\phi_h-\phi_S)}+\sin(\phi_h+\phi_S)\varepsilon F_{UT}^{\sin(\phi_h+\phi_S)}\ldots\right)\ldots\right\}$$

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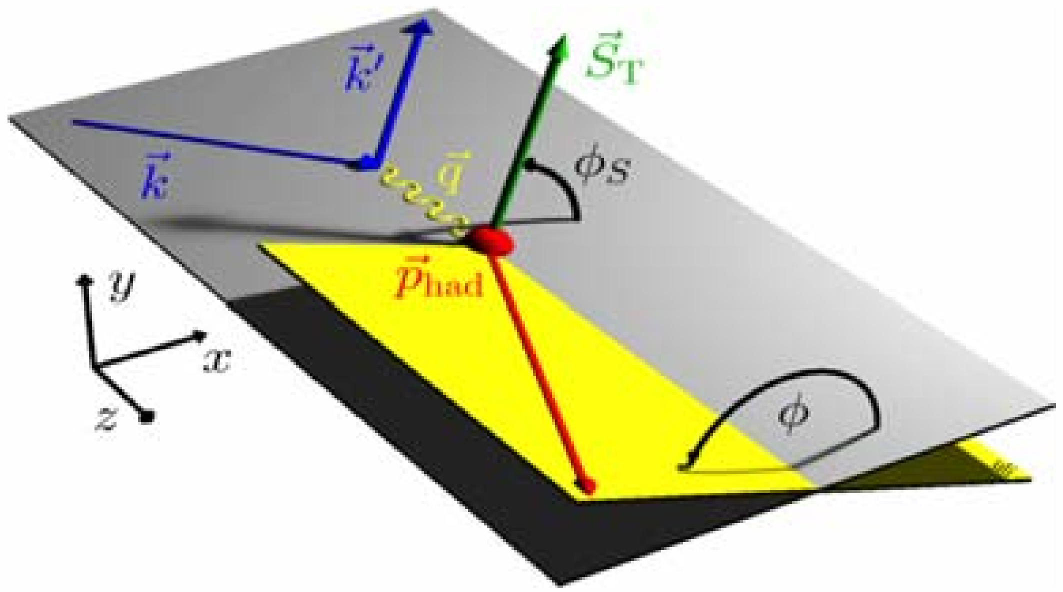


Spin asymmetry projected $\mathcal{P}$ from cross section

$$\mathcal{A}^{\mathcal{P}}_{XY} \equiv \frac{\int d\phi_h d\phi_S \mathcal{P}(\phi_h,\phi_S)\left(d\sigma^\uparrow - d\sigma^\downarrow\right)}{\int d\phi_h d\phi_S \left(d\sigma^\uparrow + d\sigma^\downarrow\right)}$$

XY-polarization e.g.

$$\mathcal{P}(\phi_h,\phi_S)=\sin(\phi_h-\phi_s)$$

Weighted asymmetries proposed: *model independent deconvolution* of CS in terms of moments of TMDs

Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98

Weighted asymmetries proposed *model independent deconvolution* of CS in terms of moments of TMDs

Kotzinian, Mulders PLB 97, Boer, Mulders PRD 98

e.g. $w_1(\boldsymbol{P}_{h\perp}) = \frac{|\boldsymbol{P}_{h\perp}|}{zM}$

$$A_{UT,T}^{w_1 \sin(\phi_h - \phi_S)} = 2\,\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}| d\phi_h\, d\phi_S\, w_1(|\boldsymbol{P}_{h\perp}|)\ \sin(\phi_h - \phi_S)\ \{d\sigma(\phi_h,\phi_S) - d\sigma(\phi_h,\phi_S+\pi)\}}{\int d|\boldsymbol{P}_{h\perp}|\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d\phi_S\, w_0(|\boldsymbol{P}_{h\perp}|)\ \{d\sigma(\phi_h,\phi_S) + d\sigma(\phi_h,\phi_S+\pi)\}},$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M}\sin(\phi_h - \phi_s)} = -2\,\frac{\sum_a e_a^2\ f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\ f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

Undefined w/o subtractions prescription-need regularization to subtract infinite contribution at large transverse momentum

Models studies ...
Gamberg, Golstein, Oganesyan PRD 2003
Conti Bacchetta Radici Eur.Phys.J. 2010

Problem with k_T moments

$$f_{1T}^{\perp(1)}(x) = \int d^2k_T \frac{k_T^2}{2M} f_{1T}^{\perp}(x, k_T)$$

Problem with k_T moments

$$f_{1T}^{\perp(1)}(x) = \int d^2k_T \frac{k_T^2}{2M} f_{1T}^{\perp}(x, k_T)$$

$$f_{1T}^{\perp}(x, k_T) \sim \frac{M^2}{\left(k_T^2 + M^2\right)^2}$$

- power counting ... Sivers tail

  Bacchetta et al. JHEP 08, Aybat, Collins, Rogers, Qiu PRD 2012

- Moment diverges

“Now for something completely different”

- Change the $w_1(\boldsymbol{P}_{h\perp}) = \frac{|\boldsymbol{P}_{h\perp}|}{zM}$ weight to a *Bessel function*

$$\frac{2\,J_1(|\boldsymbol{P}_{hT}|b_T)}{zMb_T}$$

- why on earth would you do that?!

Well... Traditional weighted asymmetry recovered ... UV divergent

$$\lim_{b_T \to 0} w_1 = 2J_1(|\boldsymbol{P}_{h\perp}|b_T)/zMb_T \longrightarrow |\boldsymbol{P}_{h\perp}|/zM$$

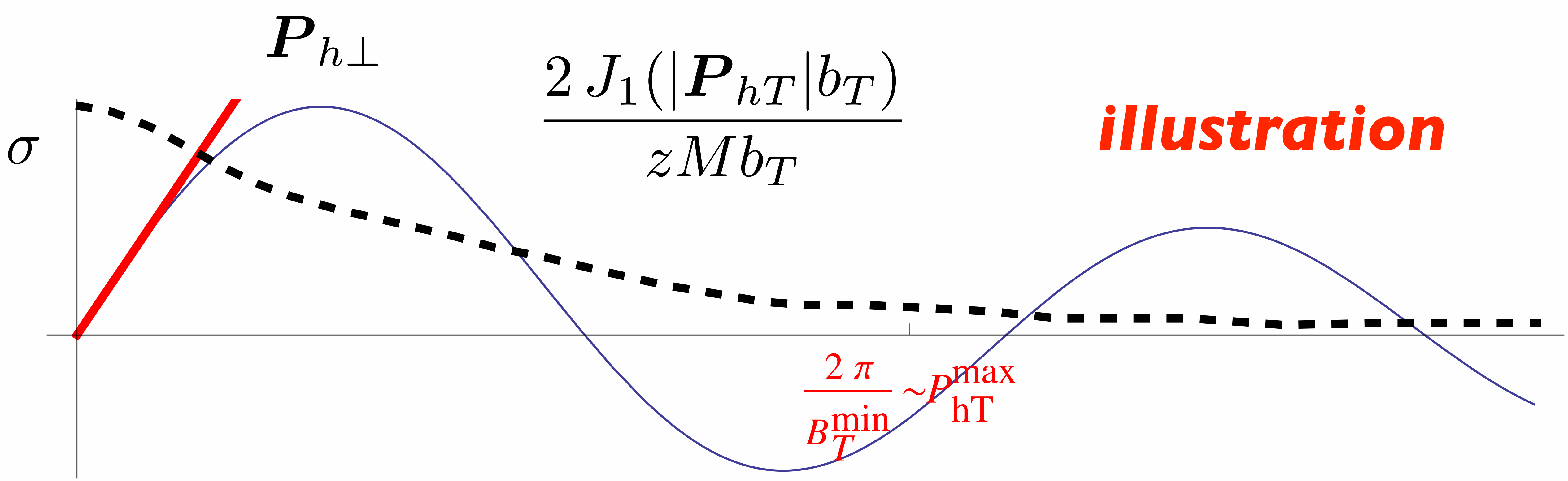
$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M}\sin(\phi_h-\phi_s)} = -2\frac{\sum_a e_a^2\ f_{1T}^{\perp(1)}(x)\ D_1^{a(0)}(z)}{\sum_a e_a^2\ f_1^{a(0)}(x)\ D_1^{a(0)}(z)}$$

undefined w/o regularization

Bacchetta et al. JHEP 08

More sensitive to low $\boldsymbol{P}_{h\perp}$ region

Bessel fnct serves as a lever arm to enhance the lower $\boldsymbol{P}_{h\perp}$ description and define finite "moments of TMDs" and cross section. For this need investigate the full TMD factorization formalsim in b_T-space



More formally this picture emerges from formalism on scale dependence of TMDs & TMD evolution

Comments on Weighting

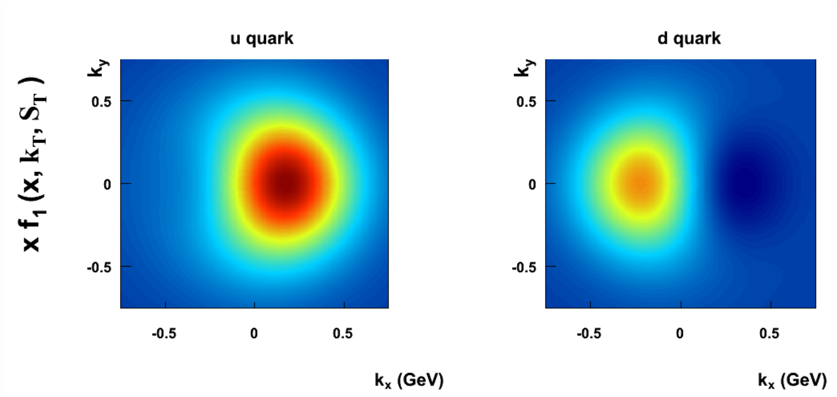
- Bessel weighting is a natural outgrowth of re-writing SIDIS cross section (or DY or e^+e^-) in "coordinate" b_T space
- *nb* the full treatment of this subject is to consider TMD evolution in "*b*-coordinate space". Seed of idea is in CSS work of 1981/1982 see Ted's Talk today on TMD Factorization

e.g. *BW* Example Sivers Function "parton model"

Fourier Transform Convolution of Structure function

$$\mathcal{C}\left[w\,f\,D\right] = x\sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\left(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\right) w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$ <u>"dipole structure"</u>



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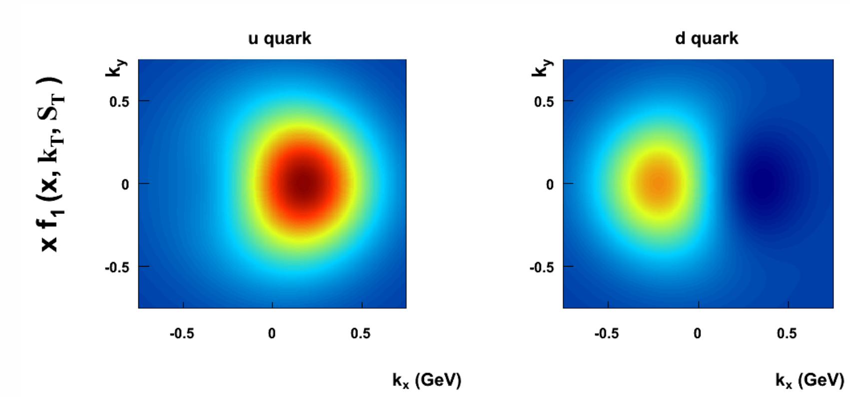


$$\delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big) = \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2}\, e^{i\boldsymbol{b}_T(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z)}$$

$$f_1(x, \boldsymbol{p}_T) = \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T \cdot \boldsymbol{p}_T}\ \tilde{f}_1(x, b_T)$$

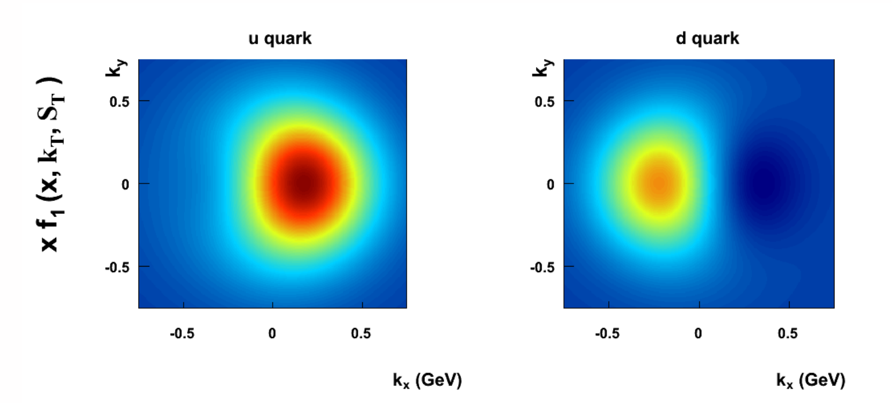
e.g. *BW* Example Sivers Function “parton model”

Fourier Transform Convolution of Structure function

$$\mathcal{C}\left[w\, f\, D\right] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\left(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\right) w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$

<u>“dipole structure”</u>



$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = -x_B \sum_a e_a^2 \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T|^2\, J_1(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|) M z\ \tilde{f}_{1T}^{\perp a(1)}(x, z^2\boldsymbol{b}_T^2)\, \tilde{D}_1^a(z, \boldsymbol{b}_T^2).$$

Boer, LG, Musch, Prokudin JHEP 2011

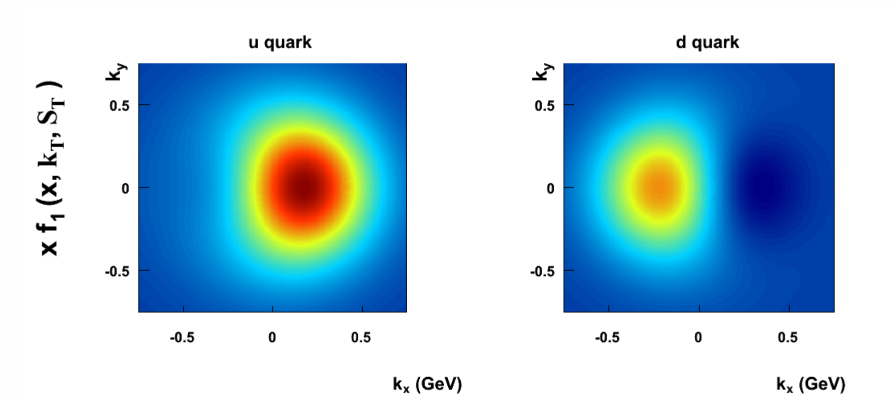
e.g. *BW* Example Sivers Function "parton model"

Fourier Transform Convolution of Structure function

$$\mathcal{C}\big[w\, f\, D\big] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}\big(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z\big)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, k_T^2)$$

$$F_{UT,T}^{\sin(\phi_h - \phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$

<u>"dipole structure"</u>



★ $$F_{UT,T}^{\sin(\phi_h - \phi_S)} = -x_B \sum_a e_a^2 \int \frac{d|\boldsymbol{b}_T|}{(2\pi)} |\boldsymbol{b}_T|^2 J_1(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|) M z\; \tilde{f}_{1T}^{\perp a(1)}(x, z^2 \boldsymbol{b}_T^2)\, \tilde{D}_1^a(z, \boldsymbol{b}_T^2).$$

$\tilde{f}_1$, $\tilde{f}_{1T}^{\perp(1)}$, and $\tilde{D}_1$ are Fourier Transf. of TMDs/FFs

convolution in momentum becomes product in *b*-space

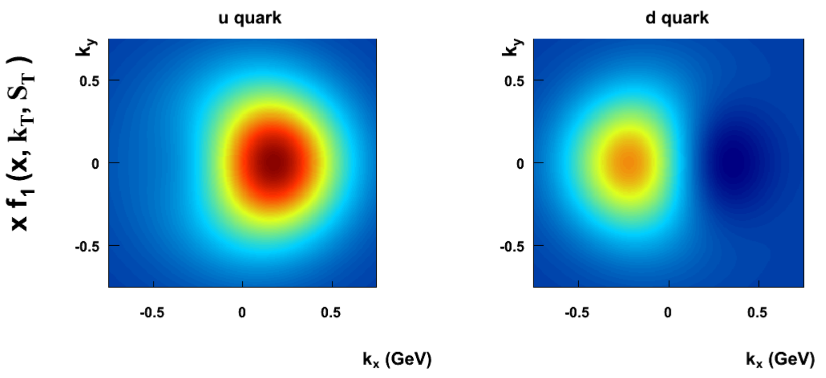
Boer, LG, Musch, Prokudin JHEP 2011

Comments on Weighting

From dipole structure,
the FT transform of the e.g. Sivers is first moment of Sivers TMD

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) \equiv \frac{2}{M^2}\frac{\partial}{\partial b_T^2}\tilde{f}_{1T}^{\perp}(x, b_T)$$

$$F_{UT,T}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{M} f_{1T}^{\perp} D_1\right],$$ <u>“dipole structure”</u>



Comments on Weighting

The FT transform of the e.g. Sivers asympt. reduces to first moment of Sivers TMD

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) \equiv \frac{2}{M^2} \frac{\partial}{\partial b_T^2} \tilde{f}_{1T}^{\perp}(x, b_T)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2\pi}{M^2} \int_0^\infty dk_T \, \frac{k_T^2}{b_T} \, J_1(k_T \, b_T) \; f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, b_T) = \frac{2}{M^2} 2\pi \int_0^\infty dk_T \, \frac{k_T^2}{2b_T} \, \frac{k_T \, b_T}{2} f_{1T}^{\perp}(x, k_T)$$

$$\lim_{b_T \to 0} \tilde{f}_{1T}^{\perp(1)}(x, 0) = f_{1T}^{\perp(1)}(x)$$

Pretzelocity and Collins

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\left[\frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\boldsymbol{p}_T\cdot\boldsymbol{k}_T)+\boldsymbol{p}_T^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)-4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)^2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)}{2M^2M_h}\,h_{1T}^{\perp}H_1^{\perp}\right]$$

Write in "cylindrical polar"- is traceless irreducible tensor no mixture of Bessel "J_3"

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = x_B\sum_a e_a^2\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|^4 J_3(|\boldsymbol{b}_T|\,|\boldsymbol{P}_{h\perp}|)\frac{M^2M_hz^3}{4}\,\tilde{h}_{1T}^{\perp a(2)}(x,z^2\boldsymbol{b}_T^2)\,\tilde{H}_1^{\perp a(1)}(z,\boldsymbol{b}_T^2)\,.$$

Simple product " $\mathcal{P}$ "

★ CS has simple interpretation--multipole expansion in terms of $b_T\,[\mathrm{GeV}^{-1}]$ conjugate to $\boldsymbol{P}_{h\perp}$

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\,dy\,d\phi_S\,dz_h\,d\phi_h\,|\boldsymbol{P}_{h\perp}|d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B\,y\,Q^2}\,\frac{y^2}{(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\int\frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T|\Big\{J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,T}+\varepsilon\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,L} \\
&+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos\phi_h}+\varepsilon\cos(2\phi_h)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+\lambda_e\,\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LU}^{\sin\phi_h} \\
&+S_\parallel\Big[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin\phi_h}+\varepsilon\sin(2\phi_h)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin 2\phi_h}\Big] \\
&+S_\parallel\lambda_e\Big[\sqrt{1-\varepsilon^2}\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}^{\cos\phi_h}\Big] \\
&+|\boldsymbol{S}_\perp|\Big[\sin(\phi_h-\phi_S)\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\left(\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)}+\varepsilon\,\mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)}\right) \\
&\quad+\varepsilon\,\sin(\phi_h+\phi_S)\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\quad+\varepsilon\,\sin(3\phi_h-\phi_S)\,J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} \\
&\quad+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin\phi_S} \\
&\quad+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(2\phi_h-\phi_S)}\Big] \\
&+|\boldsymbol{S}_\perp|\lambda_e\Big[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\,J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(\phi_h-\phi_S)} \\
&\quad+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\,J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos\phi_S} \\
&\quad+\sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\,J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(2\phi_h-\phi_S)}\Big]\Big\}
\end{aligned}
$$

Sivers

.)

Correlator w/explicit *spin orbit* correlations

$$\tilde{\Phi}^{[\gamma^+]}(x,\boldsymbol{b}_T) = \tilde{f}_1(x,\boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma}\, b_{T\rho} S_{T\sigma}\, M\, \tilde{f}_{1T}^{\perp(1)}(x,\boldsymbol{b}_T^2),$$

$$\tilde{\Phi}^{[\gamma^+\gamma^5]}(x,\boldsymbol{b}_T) = S_L\, \tilde{g}_{1L}(x,\boldsymbol{b}_T^2) + i\,\boldsymbol{b}_T\cdot\boldsymbol{S}_T M\, \tilde{g}_{1T}^{(1)}(x,\boldsymbol{b}_T^2)\,,$$

$$\begin{aligned}\tilde{\Phi}^{[i\sigma^{\alpha+}\gamma^5]}(x,\boldsymbol{b}_T) = {} & S_T^\alpha\, \tilde{h}_1(x,\boldsymbol{b}_T^2) + i\, S_L\, b_T^\alpha M\, \tilde{h}_{1L}^{\perp(1)}(x,\boldsymbol{b}_T^2) \\ & + \frac{1}{2}\left(b_T^\alpha b_T^\rho + \frac{1}{2}\,\boldsymbol{b}_T^2\, g_T^{\alpha\rho}\right) M^2\, S_{T\rho} \tilde{h}_{1T}^{\perp(2)}(x,\boldsymbol{b}_T^2) \\ & - i\,\epsilon_T^{\alpha\rho} b_{T\rho} M \tilde{h}_1^{\perp(1)}(x,\boldsymbol{b}_T^2)\,,\end{aligned}$$

See Talk of M. Engelhardt Studies of TMDs on the Lattice

N.B. b_T Transverse sep. of quarks in correlator

FT Structure Functions

$$\begin{aligned}
\mathcal{F}_{UU,T} &= \mathcal{P}[\tilde{f}_1^{(0)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} &= -\mathcal{P}[\tilde{f}_{1T}^{\perp(1)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{LL} &= \mathcal{P}[\tilde{g}_{1L}^{(0)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{LT}^{\cos(\phi_h-\phi_s)} &= \mathcal{P}[\tilde{g}_{1T}^{(1)}\ \tilde{D}_1^{(0)}]\,, \\
\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} &= \mathcal{P}[\tilde{h}_1^{(0)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UU}^{\cos(2\phi_h)} &= \mathcal{P}[\tilde{h}_1^{\perp(1)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UL}^{\sin(2\phi_h)} &= \mathcal{P}[\tilde{h}_{1L}^{\perp(1)}\ \tilde{H}_1^{\perp(1)}]\,, \\
\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} &= \frac{1}{4}\mathcal{P}[\tilde{h}_{1T}^{\perp(2)}\ \tilde{H}_1^{\perp(1)}].
\end{aligned}$$

$$\mathcal{P}[\tilde{f}^{(n)}\tilde{D}^{(m)}] \equiv x_B \sum e_a^2\,(zM|\boldsymbol{b}_T|)^n\,(zM_h|\boldsymbol{b}_T|)^m\,\tilde{f}^{a(n)}(x, z^2\boldsymbol{b}_T^2)\,\tilde{D}^{a(m)}(z, \boldsymbol{b}_T^2)\,.$$

"BW in Generalized Parton Model"

Bessel weighting-$\mathcal{P}$rojecting Sivers orthogonality Bessel Fncts.

$$\mathcal{W} = \sin(\phi_h - \phi_S)\frac{2\,J_1(|\boldsymbol{P}_{hT}|\mathcal{B}_T)}{zM\mathcal{B}_T}$$

$$A_{UT}^{\frac{2J_1(|\boldsymbol{P}_{hT}|b_T)}{zMb_T}\sin(\phi_h-\phi_S)}(b_T) =$$

$$2\frac{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\,\frac{2J_1(|\boldsymbol{P}_{hT}|b_T)}{zMb_T}\sin(\phi_h-\phi_S)\big(d\sigma^{\uparrow}-d\sigma^{\downarrow}\big)}{\int d|\boldsymbol{P}_{h\perp}|\,|\boldsymbol{P}_{h\perp}|\,d\phi_h\,d\phi_S\;\mathcal{J}_0^{\mathcal{B}_T}(|\boldsymbol{P}_{hT}|)\;(d\sigma^{\uparrow}+d\sigma^{\downarrow})}$$

$$A_{UT}^{\frac{2J_1(|\boldsymbol{P}_{hT}|b_T)}{zb_TM}\sin(\phi_h-\phi_s)}(b_T) = -2\frac{\sum_a e_a^2\,\tilde{f}_{1T}^{\perp(1)a}(x,z^2b_T^2)\,\tilde{D}_1^a(z,b_T^2)}{\sum_a e_a^2\;\tilde{f}_1^a(x,z^2b_T^2)\,\tilde{D}_1^a(z,b_T^2)}$$

Traditional weighted asymmetry recovered ... UV divergent

$$\lim_{b_T \to 0} w_1 = 2J_1(|\boldsymbol{P}_{h\perp}|b_T)/zMb_T \longrightarrow |\boldsymbol{P}_{h\perp}|/zM$$

$$A_{UT}^{\frac{|\boldsymbol{P}_{h\perp}|}{z_h M}\sin(\phi_h - \phi_s)} = -2\frac{\sum_a e_a^2 \; f_{1T}^{\perp(1)}(x) \; D_1^{a(0)}(z)}{\sum_a e_a^2 \; f_1^{a(0)}(x) \; D_1^{a(0)}(z)}$$

Bacchetta et al. JHEP 08

undefined w/o regularization

Part 2

- Studying BW and TMD evolution

- Explore impact these BWA have on studying the scale dependence of the SIDIS cross section at small to moderate transverse momentum where the TMD framework is expected to give a good description of the cross section **Boer, Gamberg, Musch,Prokudin JHEP 2011 & in progress**

- **SKETCH TMD EVOLUTION ....**

★ The usefulness of Fourier-Bessel transforms in studying the factorization as well as the scale dependence of transverse momentum dependent cross section has been known for over 30 years.

★ Is the natural language for TMD Evolution

★ Collins Soper (81), Collins, Soper, Sterman (85), Boer (01) (09) (13), Ji,Ma,Yuan (04), Collins-Cambridge University Press (11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers (11), Aybat, Prokudin, Rogers (11), Bacchetta, Prokudin (13), Sun, Yuan (13), Aidala, Field, Gamberg, Rogers (14),Collins Rogers 2015 ….

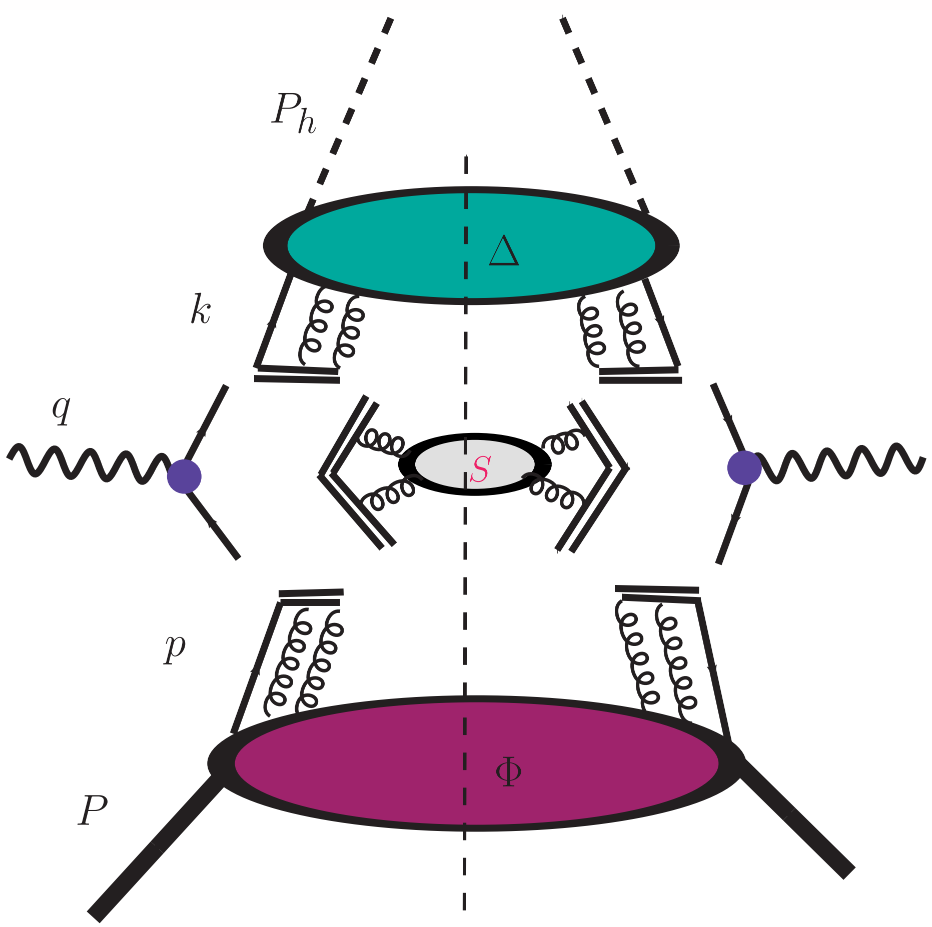
Comments

- Scale dependence in Collins-Soper evol. kernel has perturbative-short distance & non-perturbative (**NP**) large-distance content
- **Non-pertb. large-distance is *strongly universal* -many interesting predictions**
- Universal character can exploited in observables "Bessel Weighting"

(Boer Gamberg, Musch Prokudin JHEP 2011, Aghasyan, Avakian, Gamberg, Prokudin, Rossi et al 2015)

Review of TMD factorization

★ Collins Soper (81), Collins, Soper, Sterman (85), Boer (01) (09) (13), Ji,Ma,Yuan (04), Collins-Cambridge University Press (11), Aybat Rogers PRD (11), Abyat, Collins, Qiu, Rogers (11), Aybat, Prokudin, Rogers (11), Bacchetta, Prokudin (13), Sun, Yuan (13),Collins Rogers 2015 ....



- TMDs w/Gauge links: color invariant
- *Soft factor w/Gauge links*
- *Hard cross section*

- Extra divergences at one loop and higher
- Extra parameters needed to regulate light-cone, soft & collinear divergences
- Modifies convolution integral introduction of soft factor
- Some effects of evolution cancel in Bessel weighted asymmetries

b_T-space factorized cross section

$$
\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\text{SIDIS}}(\alpha_s(\mu), \mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T} \tilde{F}_{H_1}(x, b_*; \mu_b, \mu_b^2) \tilde{D}_{H_2}(z, b_*; \mu_b, \mu_b^2)
$$

$$
\exp\Bigg\{ - g_{\text{PDF}}(x, b_T; b_{\max}) - g_{\text{FF}}(z, b_T; b_{\max}) - 2g_K(b_T; b_{\max}) \ln\left(\frac{Q}{Q_0}\right)
$$

$$
+ 2\ln\left(\frac{Q}{\mu_b}\right)\tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{Q} \frac{d\mu'}{\mu'}\left[\gamma_{\text{PDF}}(\alpha_s(\mu'); 1) + \gamma_{\text{FF}}(\alpha_s(\mu'); 1) - 2\ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\Bigg\}
$$

$$
+ Y_{\text{SIDIS}}\,. \quad + P.S.C
$$

Collins 2011 (Cambridge Univ. Press)

Elements of TMD Fact. Cross section

- Y term serves to correct expression for structure function when $P_T \sim Q$
- Exponent contains both perturbative and non-perturbative content arising from TMD factorization $\longleftrightarrow$ evolution
- This structure is based upon earlier CS 81 & CSS 85 formalism & new treatment of soft factor and CSS equations.
  See also Collins 2011 Cambridge Press & Collins & Rogers PRD 2015

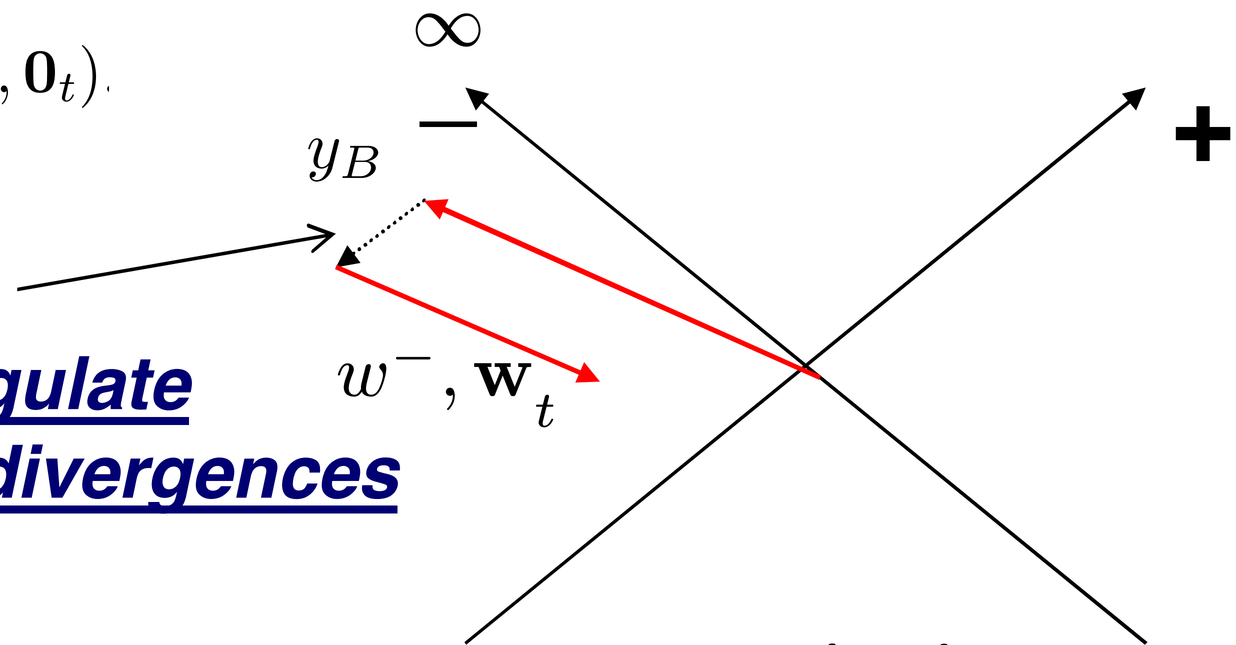
$$\frac{d\sigma}{dP_T^2} \propto \sum_{jj'} \mathcal{H}_{jj',\,\mathrm{SIDIS}}(\alpha_s(\mu),\mu/Q) \int d^2\boldsymbol{b}_T e^{i\boldsymbol{b}_T\cdot\boldsymbol{P}_T}\ \tilde{F}_{j/H_1}(x,b_T;\mu,\zeta_1)\,\tilde{D}_{H_2/j'}(z,b_T;\mu,\zeta_2)\ +\ Y_{\mathrm{SIDIS}}$$

In full QCD, the auxiliary parameters are exactly arbitrary and this is reflected in the the Collins-Soper (CS) equations for the TMD PDF, and the renormalization group (RG) equations

Collins arXiv: 1212.5974

Introduce rapidity scale parameter to regulate LC Divergences arising in Gauge link in bare TMD & soft factor

$$n_{\rm B} = (-e^{2y_{\rm B}}, 1, \mathbf{0}_t)$$

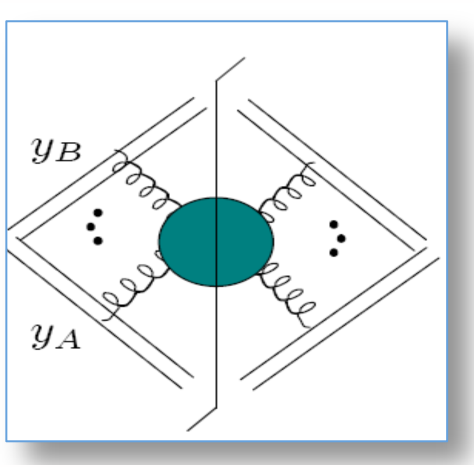


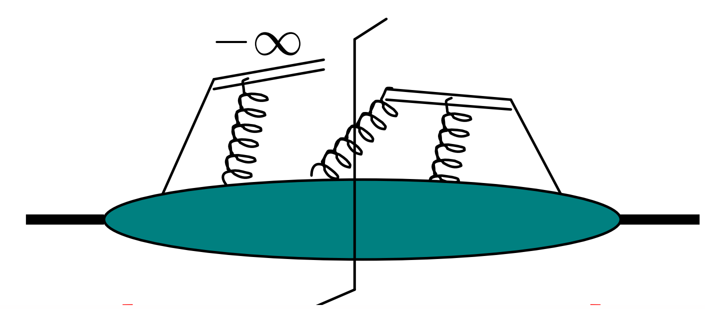
Tilt to regulate rapidity divergences

TMD tilted

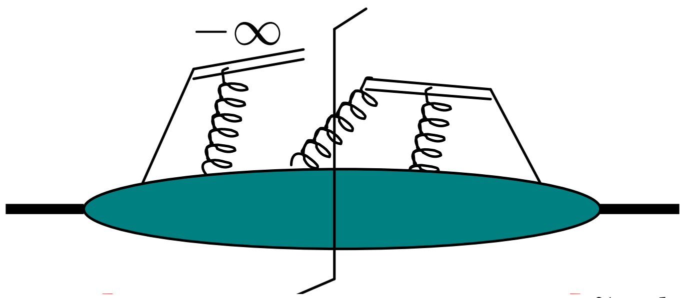
Collins Act Pol. 2003
Ji Ma Yuan 2004, 2005

y dependence





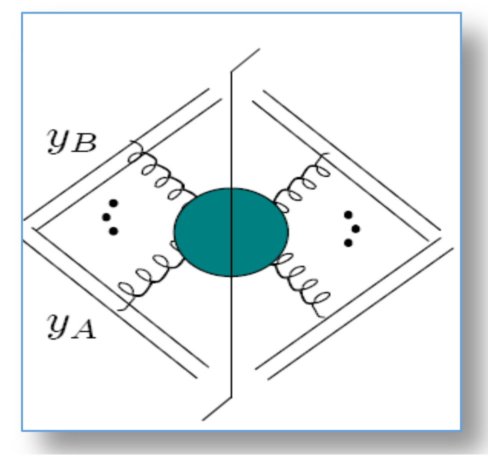
Evolution follows from their independence of rapidity scale



$$\tilde{F}_H^{\rm sub}(x, b_T; \mu, y_n) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}} \tilde{F}_H^{\rm unsub}(x, b_T; \mu, y_P - y_B) \sqrt{\frac{\tilde{S}(b_T; y_A, y_n)}{\tilde{S}(b_T; y_A, y_B)\tilde{S}(b_T; y_n, y_B)}}$$

From operator definition get

Collins-Soper Equation:



$$-\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$

$$\tilde{K}(b_T; \mu) = \frac{1}{2}\frac{\partial}{\partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}$$

Soft factor further "repartitioned"
This is done to

1) cancel LC divergences in "unsubtracted" TMDs
2) separate "right & left" movers i.e. full factorization
3) remove double counting of momentum regions

Along with .... Renormalization group Equations

$$\frac{d\tilde{K}}{d\ln\mu} = -\gamma_K(g(\mu))$$

$$\frac{d\ln\tilde{F}(x, b_T; \mu, \zeta)}{d\ln\mu} = -\gamma_F(g(\mu); \zeta/\mu^2)$$

RGE:
get anomalous
for *F* & *K*

Solve Collins Soper & RGE eqs. to obtain "evolved TMDs"

Evolved TMDs

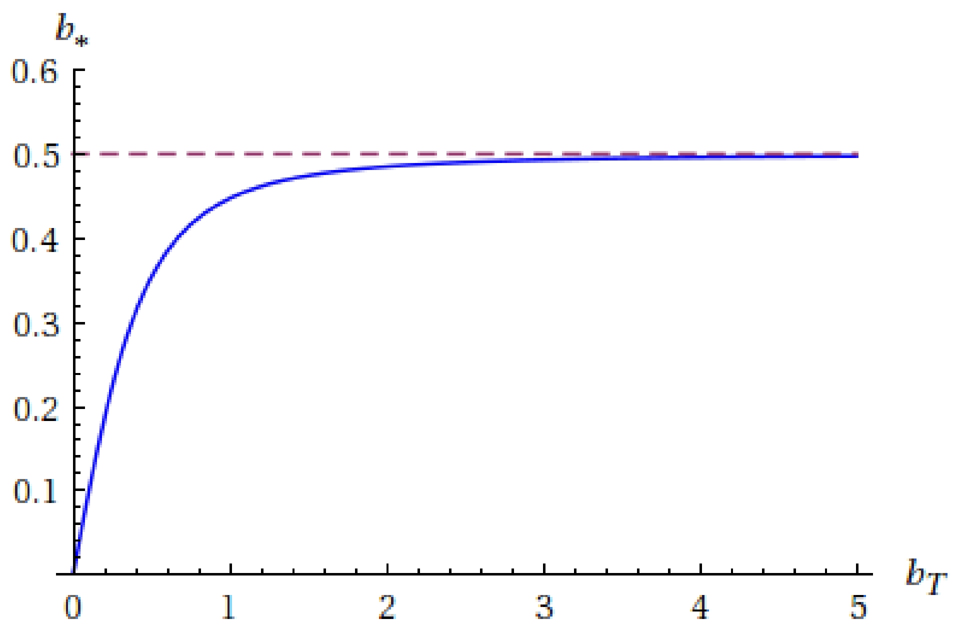
- One then “segregates” small b_T (Pert) & Large b_T - (non-perturbative)

One TMD factorization entire range of P_T or b_T

Collins Soper Sterman NPB 85

- Maximizes the perturbative content while providing a TMD factorized cross section that is applicable over the entire range of P_T

$$\mathbf{b}_* = \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}, \qquad \mu_b = \frac{C_1}{b_*}.$$



Separate the pert/non-perturbative part of $\tilde{K}(b_T, \mu)$

Solve RGE:

Collins Soper Sterman NPB 85

$$\tilde{K}(b_T;\mu) = \tilde{K}(b_*;\mu_b) - \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(g(\mu')) - g_K(b_T)$$

$$\mathbf{b}_* = \frac{\mathbf{b}_T}{\sqrt{1 + b_T^2/b_{\max}^2}}, \qquad \mu_b = \frac{C_1}{b_*}.$$

$b_{\max}$ chosen so that b_* doesn't go too far beyond the pertb. region maximize perturbative content in evolving TMDs and cross section

Structure Function *in terms of TMDs in QCD*

$$\mathcal{F}_{UU}(x,z,b,Q^2) = \sum_a \tilde{F}^a_{H1}(x,b_T,\mu,\zeta_F)\tilde{D}^a_{H2}(z_h,b_T,\mu,\zeta_D)H_{UU}(Q^2,\mu^2)$$

Evolved Structure Function & *TMDs in b-space*

$$\mathcal{F}_{UU}(x,z,b,Q^2) = \sum_a \tilde{F}^a_{H1}(x,b_T,\mu,\zeta_F)\tilde{D}^a_{H2}(z_h,b_T,\mu,\zeta_D)H_{UU}(Q^2,\mu^2)$$

Non-perturbative large b_T behavior

Totally universal related to derivative of soft factor independent of x & hadron

$$\tilde{F}_{H_1}(x,b_T;Q,Q^2) = \tilde{F}_{H_1}(x,b_*;\mu_b,\mu_b^2)\exp\Bigg\{-g_1(x,b_T;b_{\max}) - g_K(b_T;b_{\max})\ln\left(\frac{Q}{Q_0}\right) + \ln\left(\frac{Q}{\mu_b}\right)\tilde{K}(b_*;\mu_b) + \int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_{\mathrm{PDF}}(\alpha_s(\mu');1) - \ln\left(\frac{Q}{\mu'}\right)\gamma_K(\alpha_s(\mu'))\right]\Bigg\}$$

perturbative small b_T behavior

The functions have good perturbative behavior at entire range of b_T

Unpolarized and Sivers evolve in same way !!!

Recall correlator in *b*-space From Bessel Transform

$$\tilde{\Phi}^{[\gamma^+]}(x, \boldsymbol{b}_T) \;=\; \tilde{f}_1(x, \boldsymbol{b}_T^2) - i\,\epsilon_T^{\rho\sigma} b_{T\rho} S_{T\sigma}\, M \tilde{f}_{1T}^{\perp(1)}(x, \boldsymbol{b}_T^2)$$

$$\frac{\partial \tilde{\phi}^i_{f/P}(x, \mathbf{b}_\mathrm{T}; \mu, \zeta_F)\epsilon_{ij} S_T^j}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T; \mu)\tilde{\phi}^i_{f/P}(x, \mathbf{b}_\mathrm{T}; \mu, \zeta_F)\epsilon_{ij} S_T^j.$$

Collins Soper Equation

Sivers Structure Function

$$\mathcal{F}_{UT}(x,z,b,Q) = \tilde{f}^{(1)}_{1T\,i/P}(x,b_\star;\mu_b)\tilde{D}_{H/j}(z,b_\star;\mu_b)e^{-S^{pert}(b_*,Q)}e^{-S^{NP}_{UT}(b,Q,x,z)}H_{UT}$$

★ Abyat, Collins, Qiu, Rogers PRD (11), $b_* = \frac{b}{\sqrt{1+(b/b_{max})^2}}$

$$e^{-S^{NP}_{UT}}(b,Q,x,z) = \exp\left\{-\left[g_1(x,b_T;b_{\max}) + g_2(z,b_T;b_{\max}) + 2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)\right]\right\}_{UT}$$

Non perturbative factor contribution must be fit

CSS NPB 85

Sivers *BWA*: Cancellation of Universal *NP* and flavor blind hard contributions

When $\Lambda^2_{QCD} \ll P_h^2 \ll Q^2$

$$\mathcal{A}_{UT}(x,z,b,Q^2) = \frac{\tilde{f}_{1T}^{\perp(1)}(x, z^2\boldsymbol{b}^2, \mu_0^2, Q_0)\tilde{D}_1(z_h, \boldsymbol{b}^2, \mu_0^2, Q_0)\cancel{\tilde{H}_{UT}(\mu_0^2, Q_0)}e^{-S^{\mathrm{pert}}(b_*,Q)}\cancel{e^{-2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)}}}{\tilde{f}_1(x, z^2\boldsymbol{b}^2, \mu_0^2, Q_0)\tilde{D}_1(z_h, \boldsymbol{b}^2, \mu_0^2, Q_0)\cancel{\tilde{H}_{UU}(\mu_0^2, Q_0)}e^{-S^{\mathrm{pert}}(b_*,Q)}\cancel{e^{-2g_k(b_T)\ln\left(\frac{Q}{Q_0}\right)}}}$$

BWA less sensitivity to TMD Evolution
Prediction of TMD factorization & Evolution

Boer, Gamberg, B. Musch, A. Prokudin....

Bessel Weighting of experimental observables

- What good is all of this?
- Test the idea
- How?
- We used a MC
- So first re-write BWA for an “experiment”

Part 3

A First pheno study of BW of Experimental Observables

Studies of transverse momentum dependent parton distributions and Bessel weighting JHEP *2015*

M. Aghasyan,[a,b] **H. Avakian,**[c] **E. De Sanctis,**[a] **L. Gamberg,**[d] **M. Mirazita,**[a] **B. Musch,**[e] **A. Prokudin**[c] **and P. Rossi**[a,c]

New experimental tool to study the 3-D nucleon content to the SIDIS cross section that minimizes the transverse momentum model dependencies inherent in conventional extractions of TMDs.

So lets consider the Bessel Weighted double spin Asymm in *b*-space

$$S_{||}\lambda_e = \pm 1$$

★ First we project out the Structure functions going into asymmetry from Multipole expansion

$$
\begin{aligned}
&\frac{d\sigma}{dx_B\, dy\, d\phi_S\, dz_h\, d\phi_h\, |\boldsymbol{P}_{h\perp}| d|\boldsymbol{P}_{h\perp}|} = \\
&\frac{\alpha^2}{x_B y Q^2}\frac{y^2}{(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x_B}\right)\int \frac{d|\boldsymbol{b}_T|}{(2\pi)}|\boldsymbol{b}_T| \Big\{ J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,T} + \varepsilon\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU,L} \\
&+ \sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos\phi_h} + \varepsilon\cos(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UU}^{\cos(2\phi_h)} \\
&+ \lambda_e\sqrt{2\,\varepsilon(1-\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LU}^{\sin\phi_h} \\
&+ S_\parallel\Big[\sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin\phi_h} + \varepsilon\sin(2\phi_h)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UL}^{\sin 2\phi_h}\Big] \\
&+ S_\parallel\lambda_e\Big[\sqrt{1-\varepsilon^2}\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL} + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_h\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LL}^{\cos\phi_h}\Big] \\
&+ |\boldsymbol{S}_\perp|\Big[\sin(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\Big(\mathcal{F}_{UT,T}^{\sin(\phi_h-\phi_S)} + \varepsilon\,\mathcal{F}_{UT,L}^{\sin(\phi_h-\phi_S)}\Big) \\
&\qquad + \varepsilon\,\sin(\phi_h+\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(\phi_h+\phi_S)} \\
&\qquad + \varepsilon\,\sin(3\phi_h-\phi_S)\, J_3(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(3\phi_h-\phi_S)} \\
&\qquad + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin\phi_S} \\
&\qquad + \sqrt{2\,\varepsilon(1+\varepsilon)}\,\sin(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{UT}^{\sin(2\phi_h-\phi_S)}\Big] \\
&+ |\boldsymbol{S}_\perp|\lambda_e\Big[\sqrt{1-\varepsilon^2}\,\cos(\phi_h-\phi_S)\, J_1(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(\phi_h-\phi_S)} \\
&\qquad + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos\phi_S\, J_0(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos\phi_S} \\
&\qquad + \sqrt{2\,\varepsilon(1-\varepsilon)}\,\cos(2\phi_h-\phi_S)\, J_2(|\boldsymbol{b}_T||\boldsymbol{P}_{h\perp}|)\,\mathcal{F}_{LT}^{\cos(2\phi_h-\phi_S)}\Big]\Big\} \qquad .)
\end{aligned}
$$

Where the Parton Model Structure Functions in *b*-space are ...

$$\mathcal{F}_{UU,T} = x \sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)\,, \;\; \mathcal{F}_{LL} = x \sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)$$

Where the Parton Model Structure Functions in *b*-space are ...

$$\mathcal{F}_{UU,T} = x \sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)\,, \;\; \mathcal{F}_{LL} = x \sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)$$

Remind ourselves of Asymmetry in "*b*" space for double longitudinal polarized process

$$S_{||} \lambda_e = \pm 1$$

So Bessel Weighted double spin Asymm in *b*-space

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)} \equiv \frac{\tilde{\sigma}_{LL}(b_T)}{\tilde{\sigma}_{UU}(b_T)} = \sqrt{1-\varepsilon^2}\, \frac{\sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)}{\sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2) \tilde{D}_1^a(z, b_T^2)}$$

Project the Structure functions from differential cross section

$$d\Phi \equiv dx\, dy\, d\psi\, dz\, dP_{h\perp} P_{h\perp}$$

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left(\frac{d\sigma^+}{d\Phi} + \frac{d\sigma^-}{d\Phi} \right) = K(x,y) \mathcal{F}_{UU,T}$$

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left(\frac{d\sigma^+}{d\Phi} - \frac{d\sigma^-}{d\Phi} \right) = K(x,y) \sqrt{1-\varepsilon^2} \mathcal{F}_{LL}$$

Let us re-write cross section in terms of events

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left(\frac{1}{\mathcal{N}_0^+} \frac{dn^+}{d\Phi} + \frac{1}{\mathcal{N}_0^-} \frac{dn^-}{d\Phi} \right) = K(x,y) \mathcal{F}_{UU,T}$$

$$\int dP_{h\perp} P_{h\perp} J_0(b_T P_{h\perp}) \left(\frac{1}{\mathcal{N}_0^+} \frac{dn^+}{d\Phi} - \frac{1}{\mathcal{N}_0^-} \frac{dn^-}{d\Phi} \right) = K(x,y) \sqrt{1-\varepsilon^2} \mathcal{F}_{LL}$$

$dn^{\pm}$ are the number of events in a differential phase space volume, $\boldsymbol{d\Phi}$, and $\mathcal{N}_o^{\pm}$ is the standard normalization factor, that is the product of the number of beam and target particles with ± polarization per unit target area. We assume that the experiment has been set up such that $\mathcal{N}_o^+ = \mathcal{N}_o^-$.

Next discretize differential cross section

$$d\Phi \equiv dx\, dy\, d\psi\, dz\, dP_{h\perp} P_{h\perp} \longrightarrow \Delta\Phi \equiv \Delta x\, \Delta y\, \Delta z\, \Delta P_{h\perp} P_{h\perp}$$

And re-do/reconsider the projecting e.g.

$$\int dP_{h\perp} P_{h\perp} J_0(B_T P_{h\perp}) \frac{dn^{\pm}}{d\Phi} \Longrightarrow \sum_{i\,\epsilon\,\mathrm{bin}[x,y,z]} J_0(B_T P_{h\perp\, i}) \frac{\Delta n^{\pm}}{\Delta x\, \Delta y\, \Delta z}$$

Sum over events in bin to sum over events

$$K(x,y)\,\sqrt{1-\varepsilon^2}\,\mathcal{F}_{LL}(B_T) =$$

$$\Rightarrow \left\{ \sum_{j \text{ events}}^{N^+} J_0(B_T P_{h\perp\, j}) \quad - \quad \sum_{j \text{ events}}^{N^-} J_0(B_T P_{h\perp\, j}) \right\}$$

Experimental procedure to BWA for double longitudinal beam/target polarization

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\tilde{\sigma}^+(b_T) - \tilde{\sigma}^-(b_T)}{\tilde{\sigma}^+(b_T) + \tilde{\sigma}^-(b_T)}$$

$$= \frac{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})} \equiv \frac{\tilde{S}^+ - \tilde{S}^-}{\tilde{S}^+ + \tilde{S}^-}$$

j are indices for the sums on events and $N^{\pm}$ are the number of events, for positive/negative products of lepton and nucleon helicities and at given x, y and z, and where $S^{\pm}$ indicate the sum over events for ± helicities.

Method....

- Every time you have an event at a P_h plug in the value of P_h and get a value for, $J_n(b\ P_h)$ and then perform the sums
- Test this idea w/ a Monte Carlo

Developed a differential Monte Carlo based on parton model to test the Bessel Weighting

$$\ell(l) + N(P) \to \ell(l') + h(P_h) + X,$$

Kotzinian NPB 1995
Mulders Tangerman NPB 1996
Bacchetta et al. JHEP 2006
Anselmino et al. PRD 71 2005

$$\frac{d\sigma}{dx dy dz d^2\boldsymbol{p}_\perp d^2\boldsymbol{k}_\perp d\phi_{l'}} = 2\,K(x,y) J(x,Q^2,\boldsymbol{k}_\perp^2) \times x \sum_a e_a^2 \left[f_{1,a}(x,\boldsymbol{k}_\perp^2) D_{1,a}(z,\boldsymbol{p}_\perp^2) + \lambda\sqrt{1-\varepsilon^2} g_{1L,a}(x,\boldsymbol{k}_\perp^2) D_{1,a}(z,\boldsymbol{p}_\perp^2) \right]$$

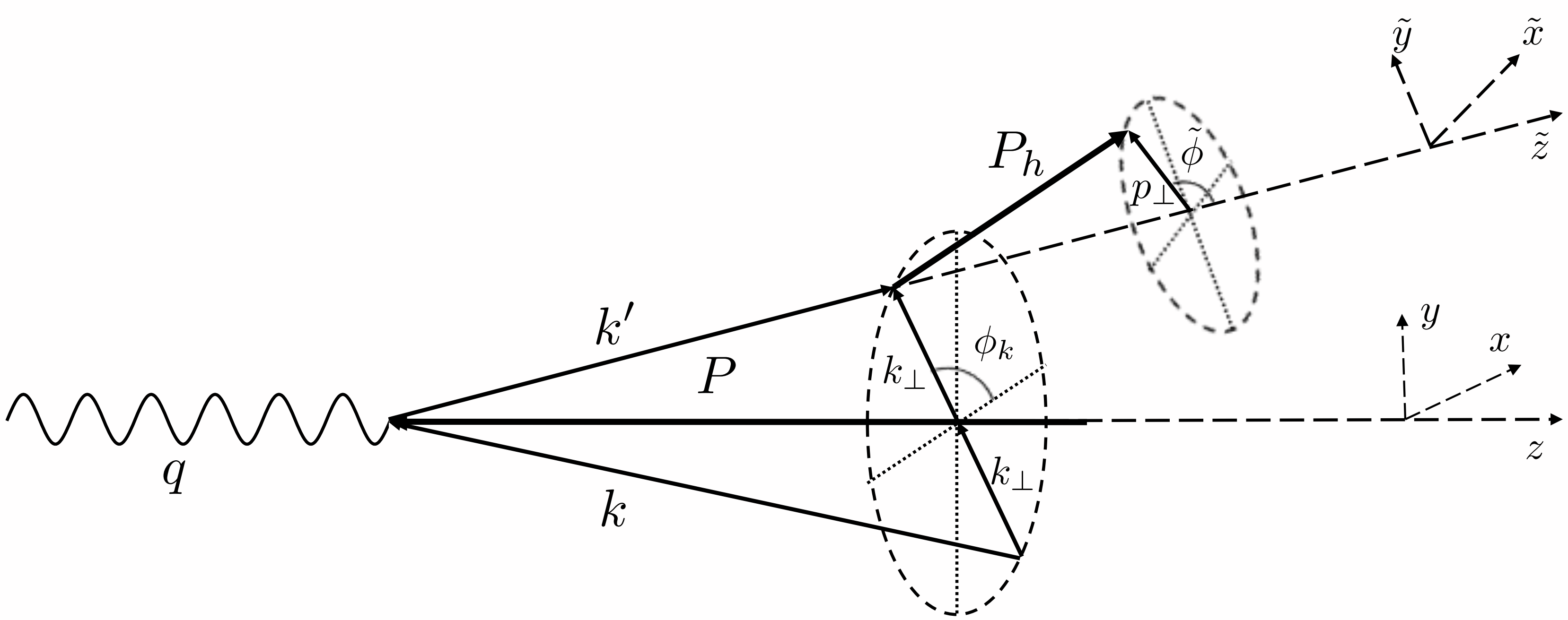


Figure 1. Kinematics of the process. q is the virtual photon, k and k' are the initial and struck quarks, $k_\perp$ is the quark transverse component. P_h is the final hadron with a $p_\perp$ component, transverse with respect to the fragmenting quark k' direction.

Input distributions to MC

$$f_1(x, \boldsymbol{k}_\perp^2) = f_1(x) \frac{1}{\langle k_\perp^2(x) \rangle_{f_1}} \exp\left(- \frac{\boldsymbol{k}_\perp^2}{\langle k_\perp^2(x) \rangle_{f_1}} \right), \tag{3.9}$$

$$g_{1L}(x, \boldsymbol{k}_\perp^2) = g_{1L}(x) \frac{1}{\langle k_\perp^2(x) \rangle_{g_1}} \exp\left(- \frac{\boldsymbol{k}_\perp^2}{\langle k_\perp^2(x) \rangle_{g_1}} \right), \tag{3.10}$$

$$D_1(z, \boldsymbol{p}_\perp^2) = D_1(z) \frac{1}{\langle p_\perp^2(z) \rangle} \exp\left(- \frac{\boldsymbol{p}_\perp^2}{\langle p_\perp^2(z) \rangle} \right), \tag{3.11}$$

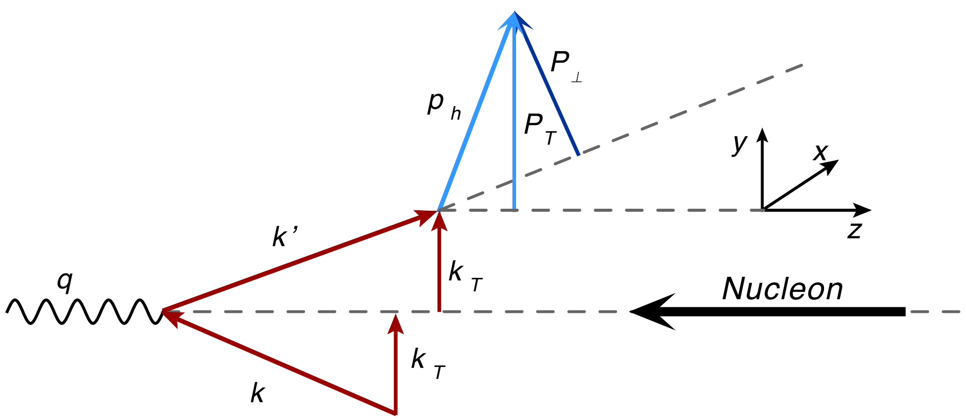
$$\langle k_\perp^2(x) \rangle = C\, x(1-x) \qquad \langle p_\perp^2(z) \rangle = D\, z(1-z)$$

$$C = 0.54\,\mathrm{GeV}^2 \ \text{ and } \ D = 0.5\,\mathrm{GeV}^2.$$

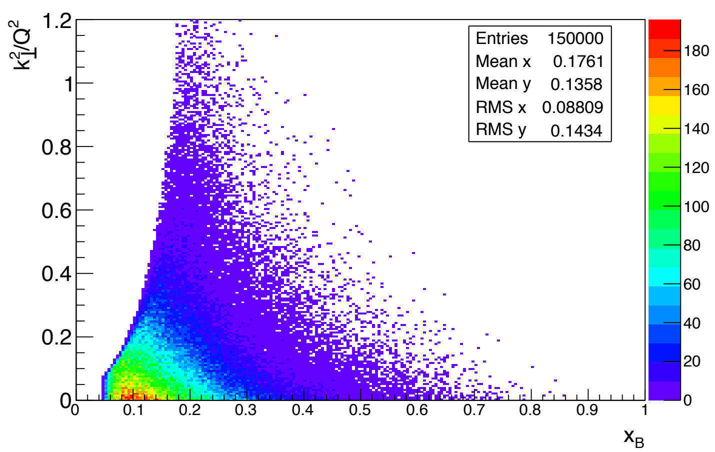
The generator we construct is implemented with on-shell initial partons with four momentum conservation imposed.

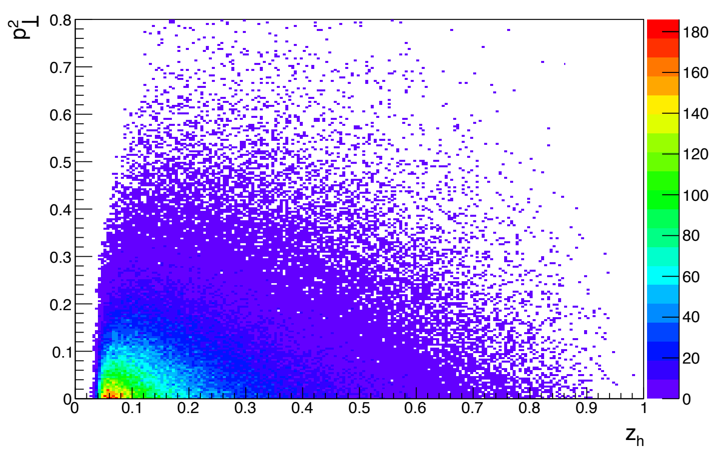
The limitations due to available phase space integration will modify the reconstructed distributions with respect to the input

Kinematic inside MC

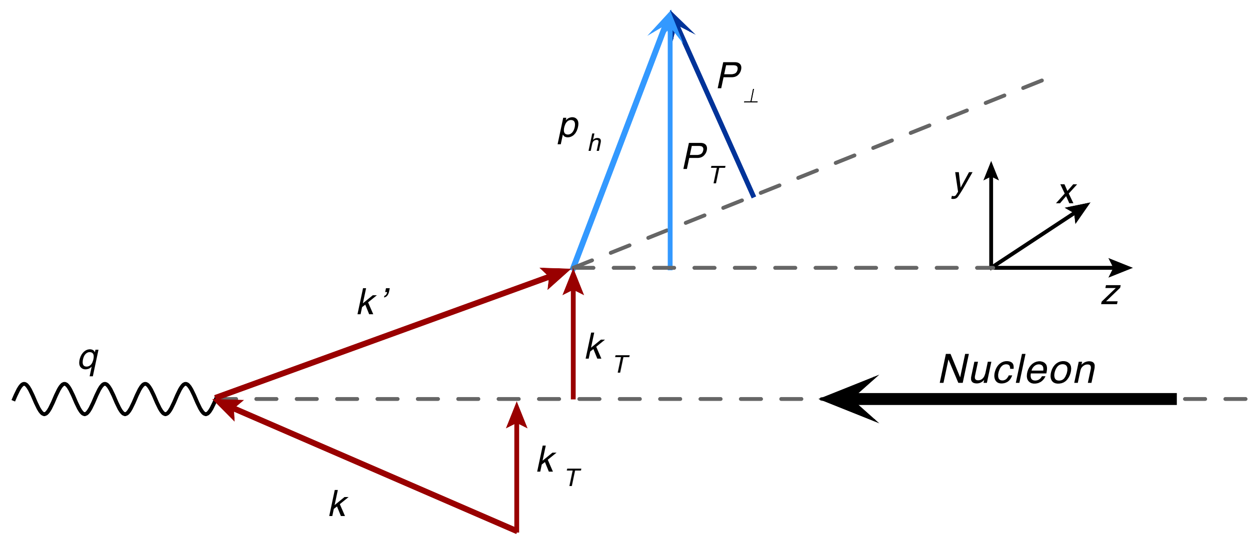


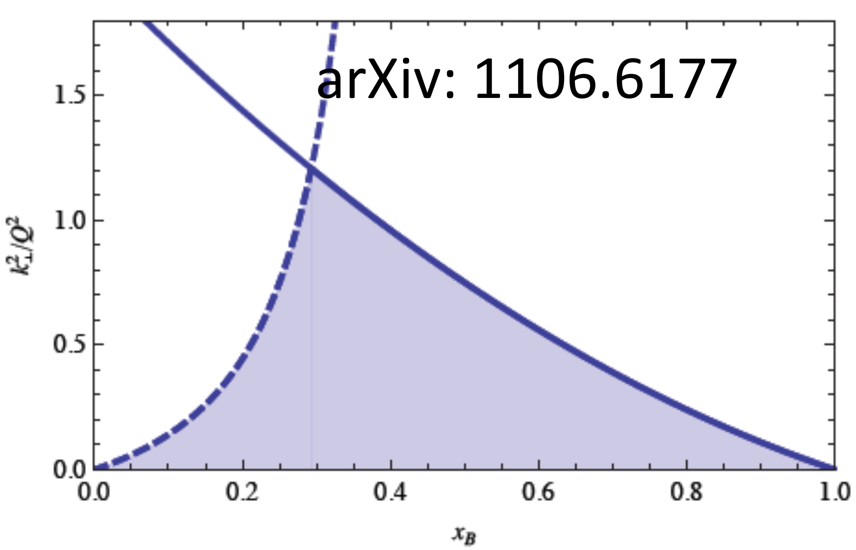
Kinematics inside MC



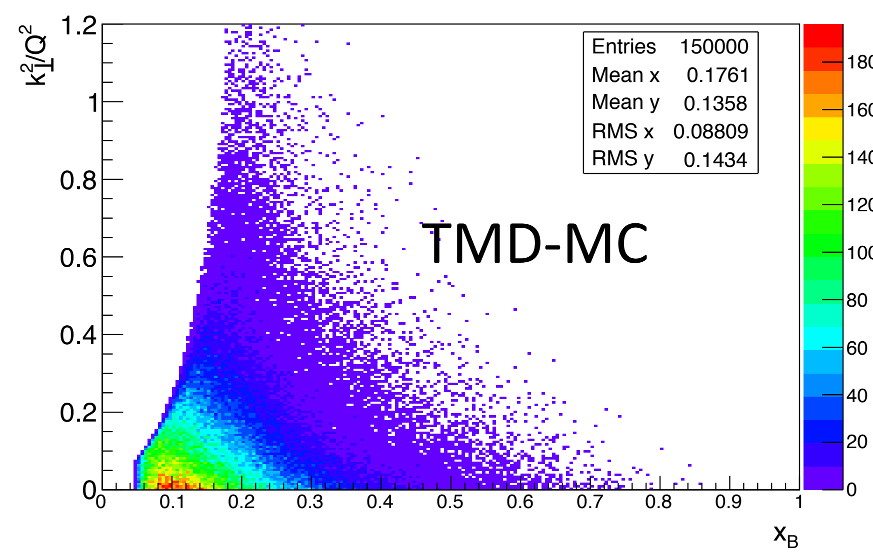


Kinematic inside MC





Boglione, Melis Prokudin
Phys.Rev. D84 (2011) 034033



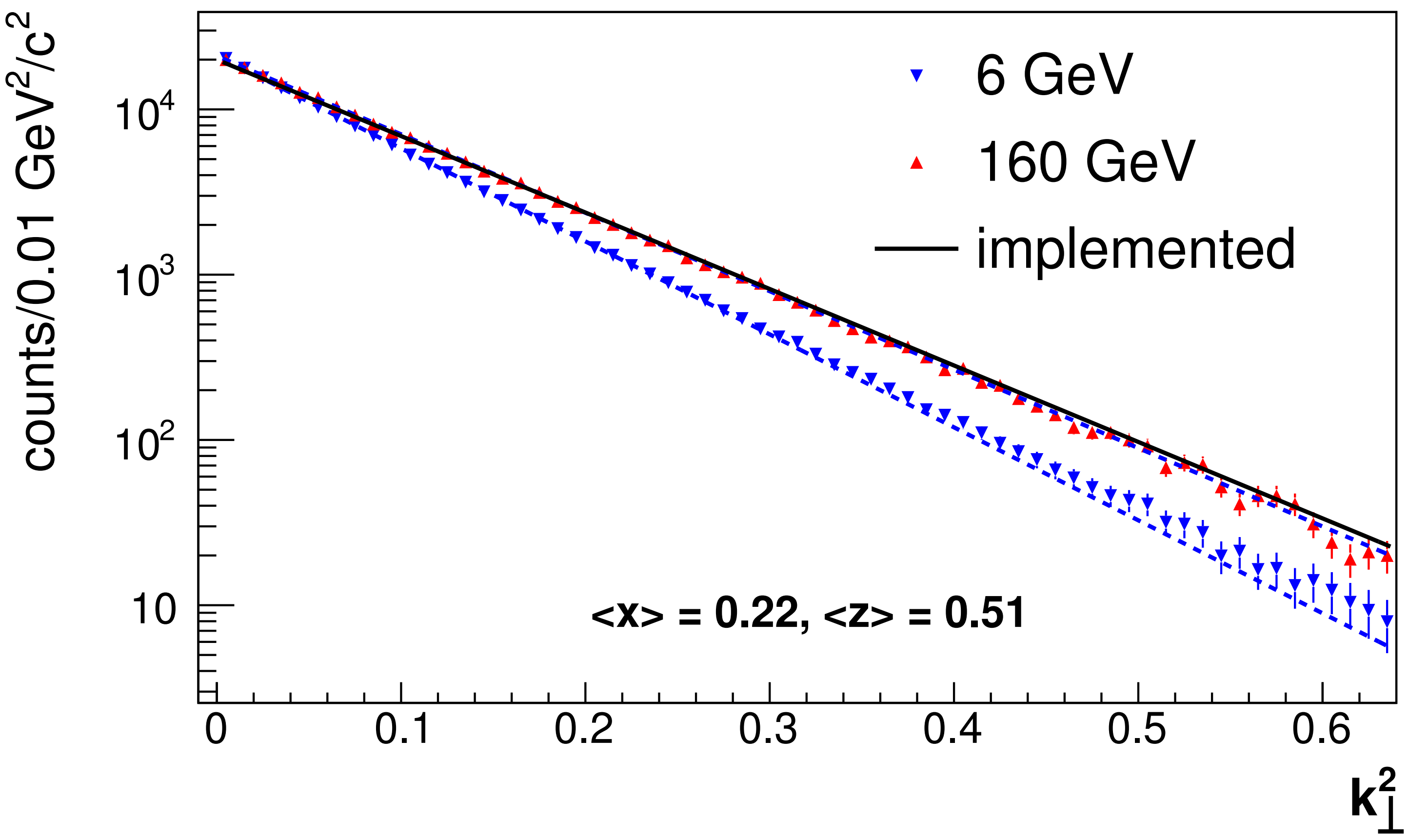
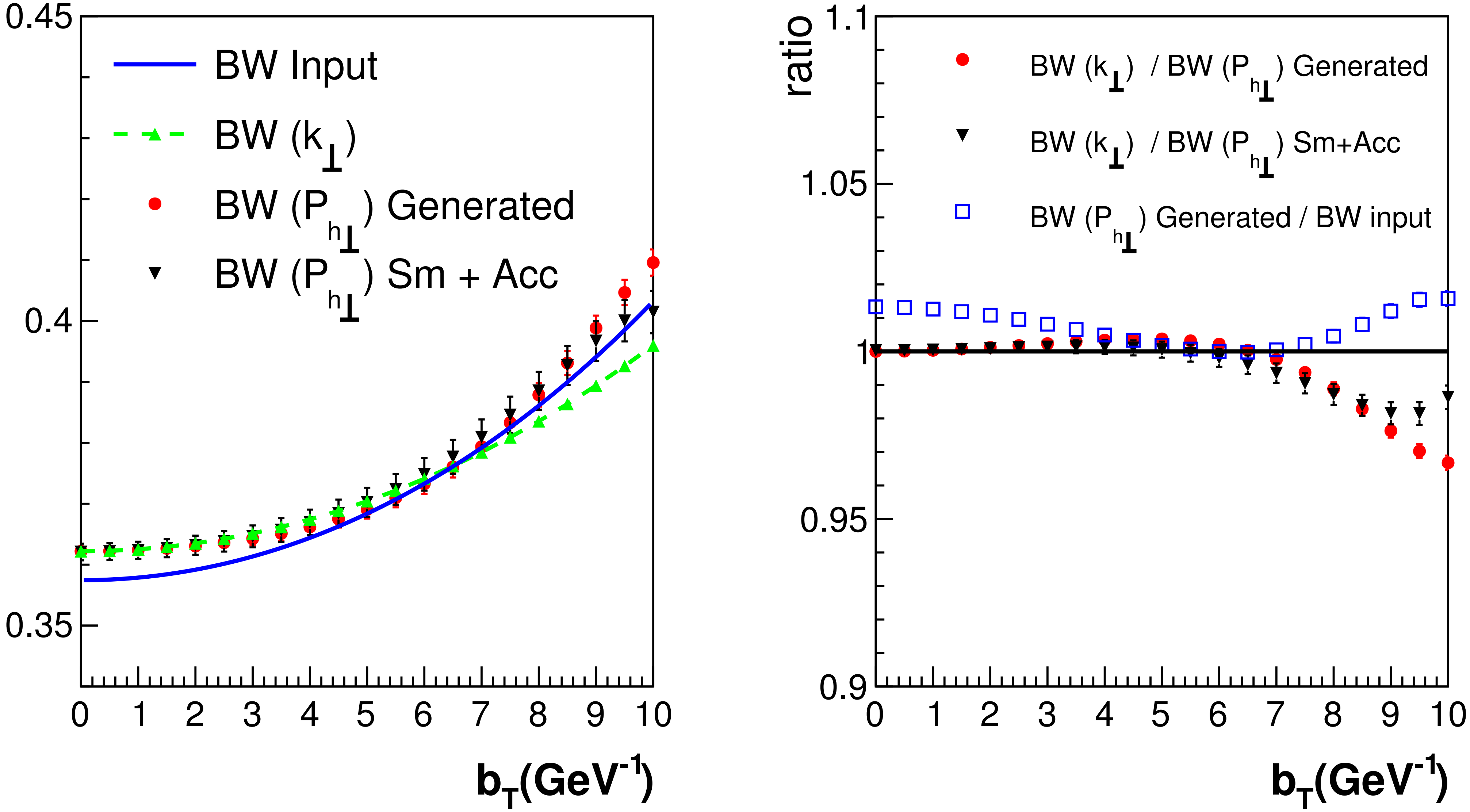


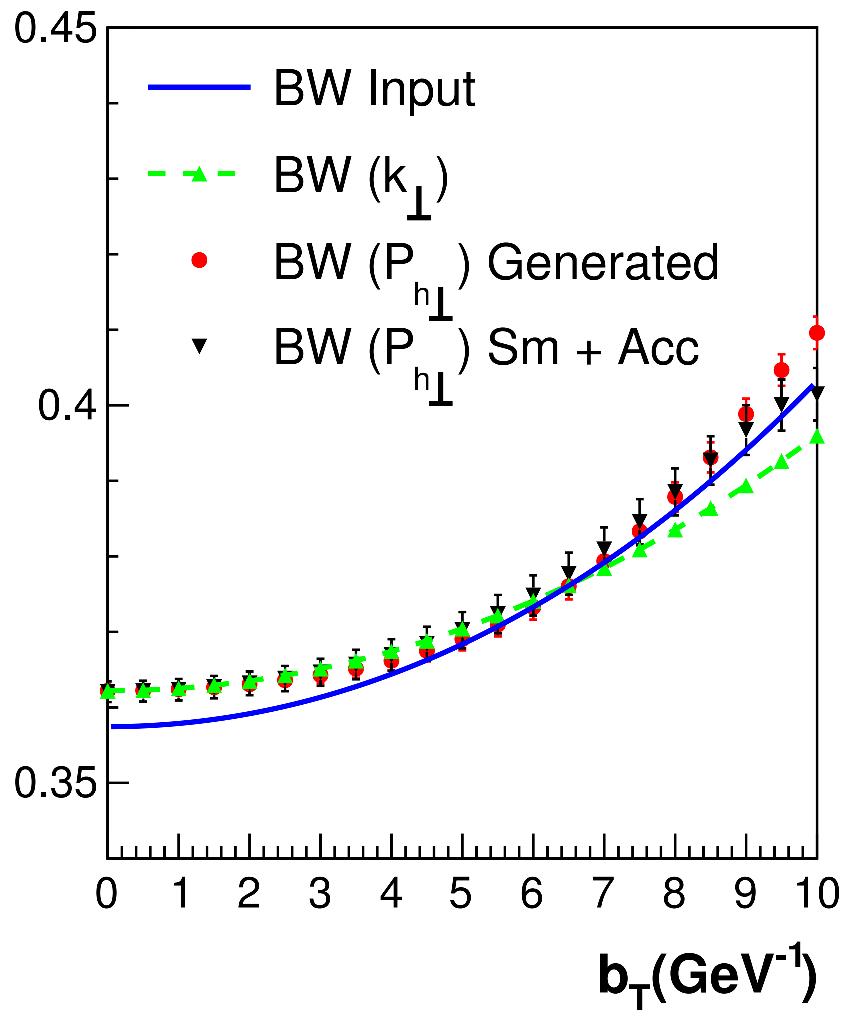
Figure 2. (Color online) The solid line is the Gaussian input distribution implemented using eq. (3.9), with red triangles coming from the Monte Carlo at 160 GeV initial lepton energy, blue triangles coming from the Monte Carlo at 6 GeV. The dashed line represents the fit to the Monte Carlo distributions which returned values of $C = 0.527\,\mathrm{GeV}^2$ and $C = 0.444\,\mathrm{GeV}^2$ at 160 GeV and 6 GeV respectively.

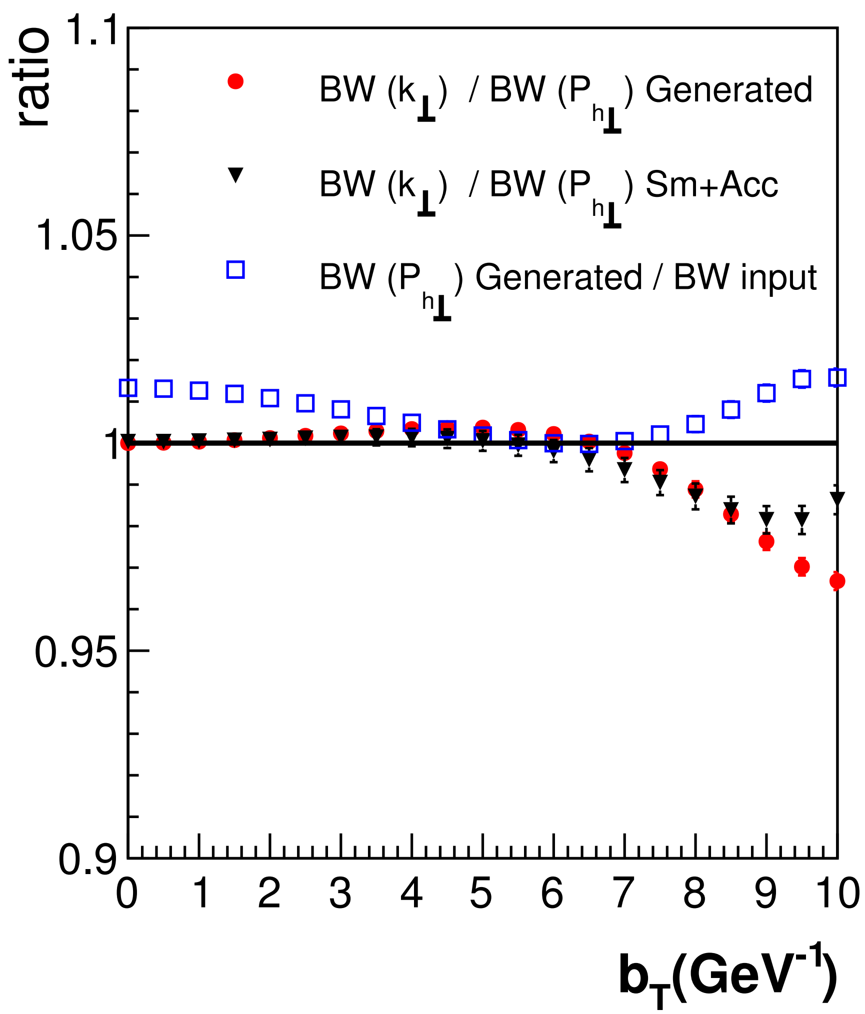
well ...????



$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) \; = \sqrt{1-\varepsilon^2}\,\frac{\sum_a e_a^2 \tilde{g}_{1L}^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}{\sum_a e_a^2 \tilde{f}_1^a(x, z^2 b_T^2)\tilde{D}_1^a(z, b_T^2)}$$

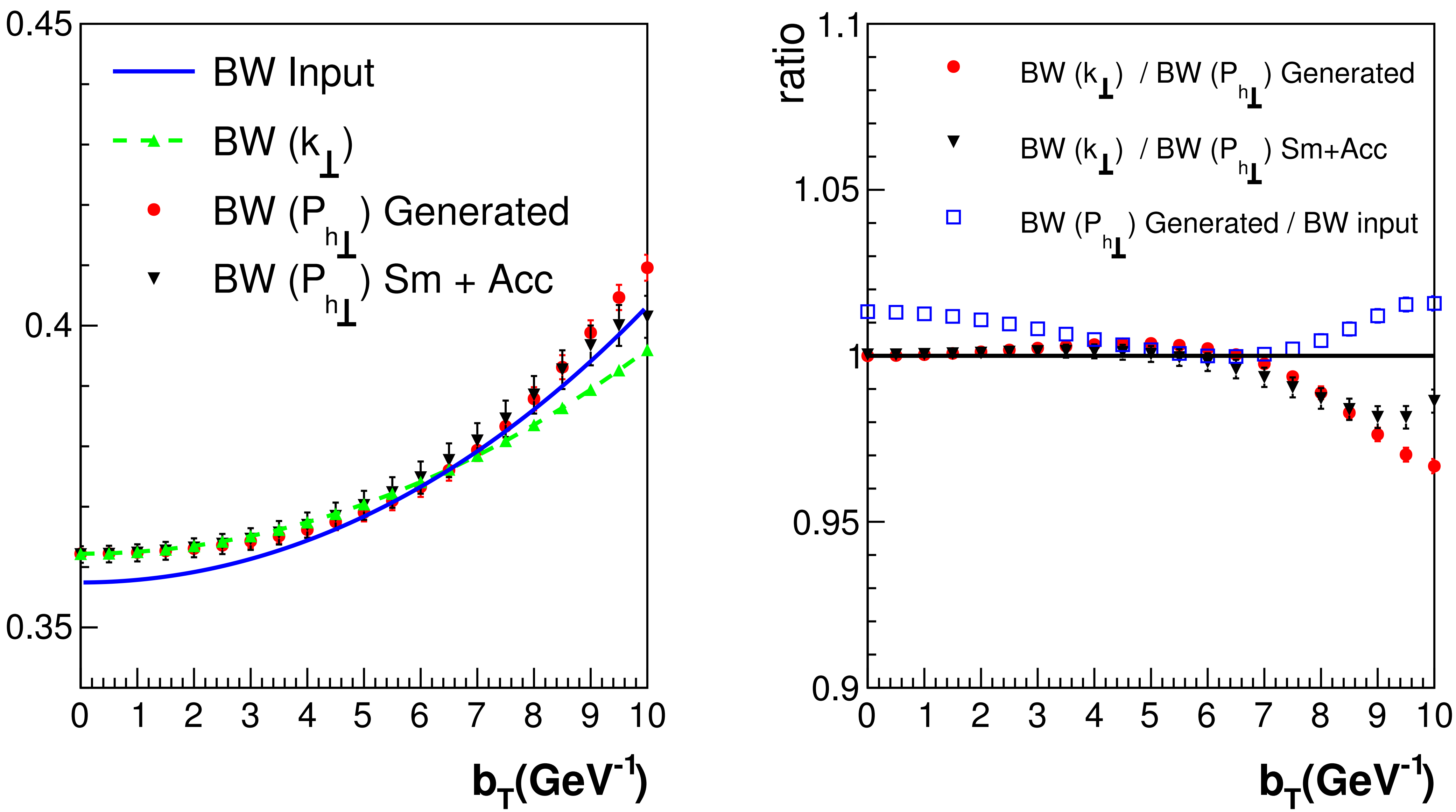
The blue curve labeled "BW Input", is the asymmetry calculated analytically using the right hand side of Eq and the Fourier transformed input distribution functions





Compare w/ the Monte Carlo generated distribution using Eq (full red points) labeled "BW(Ph⊥) Generated",

$$A_{LL}^{J_0(b_T P_{h\perp})}(b_T) = \frac{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T P_{h\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T P_{h\perp j}^{[-]})}$$



Compare w/ the Monte Carlo generated distribution using Eq (green) labeled "BW(k⊥) Generated",

$$a_{LL}^{J_0(b_T k_\perp)}(b_T) \equiv \quad \sqrt{1-\varepsilon^2}\frac{\tilde{g}_{1L}(b_T)}{\tilde{f}_1(b_T)} = \frac{\sum_j^{N^+} J_0(b_T k_{\perp j}^{[+]}) - \sum_j^{N^-} J_0(b_T k_{\perp j}^{[-]})}{\sum_j^{N^+} J_0(b_T k_{\perp j}^{[+]}) + \sum_j^{N^-} J_0(b_T k_{\perp j}^{[-]})}$$

Bo-Qiang Ma and Zhun Lu PRD 87 2013 model calculation

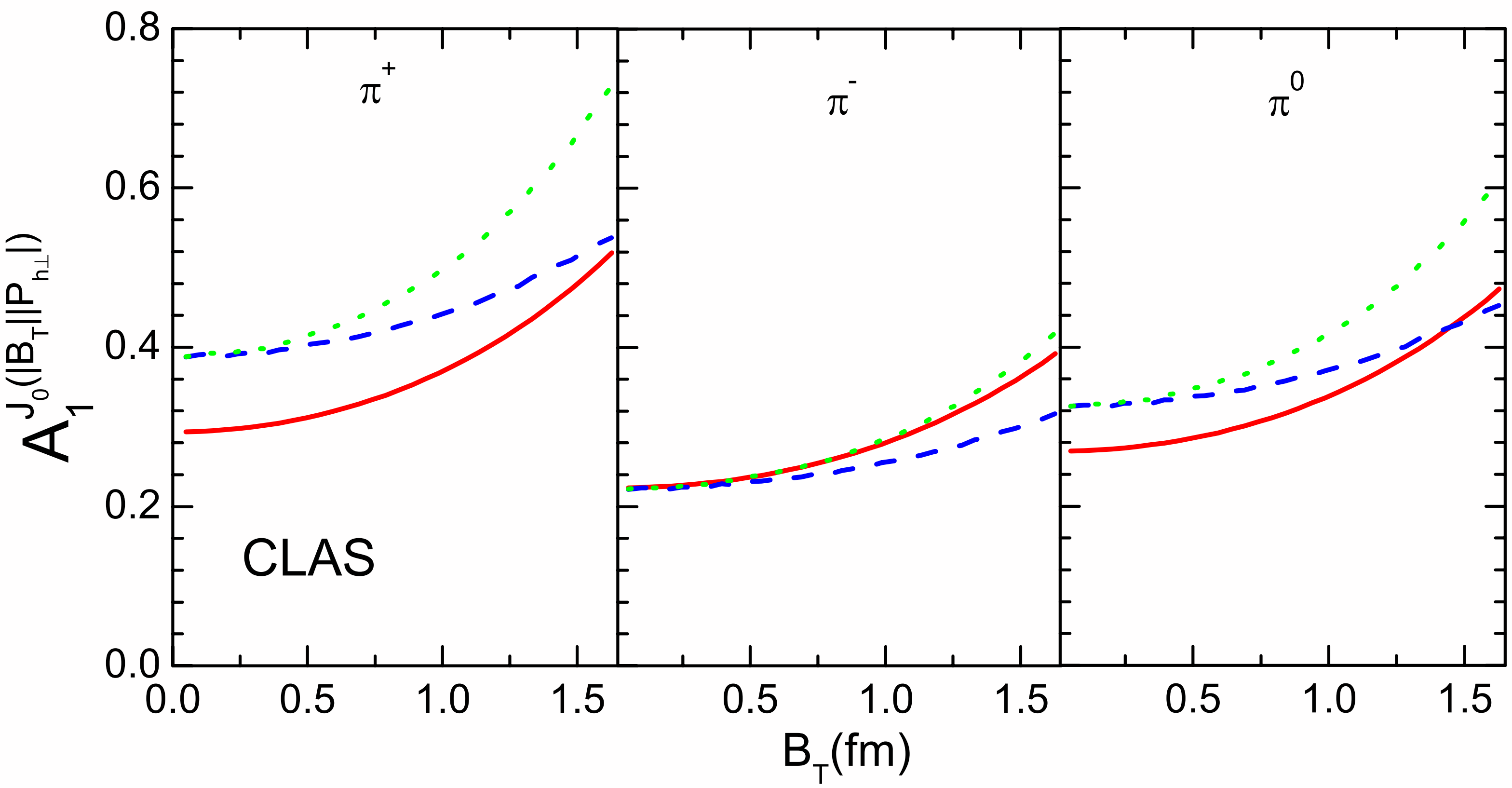


FIG. 5 (color online). The Bessel-weighted DSAs $A_1^{J_0(|\mathcal{B}_T||\boldsymbol{P}_{h\perp}|)}$ for π^+, π^-, and π^0 productions as functions of $\mathcal{B}_T$ at CLAS. The solid lines are from approach 2 of the light-cone diquark model, while the dashed line and the dotted lines are from the Gaussian ansatz for the TMD helicity distributions with $\langle p_T^2 \rangle_g^q = 0.17$ GeV and 0.10 GeV^2, respectively.

Conclusions cont.

- Propose generalized Bessel Weights to study 3-D structure of the nucleon

- Bessel Weighting solves problem of infinite contribution from large transverse momentum that arise from using "conventional weighting

- Provides a regularization of infinite contributions at lg. transverse momentum when $\mathcal{B}_T^2$ is non-zero

- Soft, Hard CS, eliminated from weighted asymmetries, Sudakov dpnds coupling of *b* & *Q*

- Possible to compare observables at different scales.... could be useful for an EIC

Conclusions cont.

- New experimental tool to study the TMD content at to the SIDIS that minimize the transverse momentum model dependencies inherent in conventional extractions of TMDs.
- Impact for Lattice calculation of moments of TMDS, B. Musch, Ph. Hagler, M. Engelhardt, J.W. Negele, A. Schafer 2011-2015