

# Lattice QCD: From TMDs to OAM

Michael Engelhardt

New Mexico State University

In collaboration with:

B. Musch, P. Hägler, J. Negele, A. Schäfer

T. Bhattacharya, R. Gupta, B. Yoon

J. R. Green, S. Krieg, S. Meinel, A. Pochinsky, S. Syritsyn

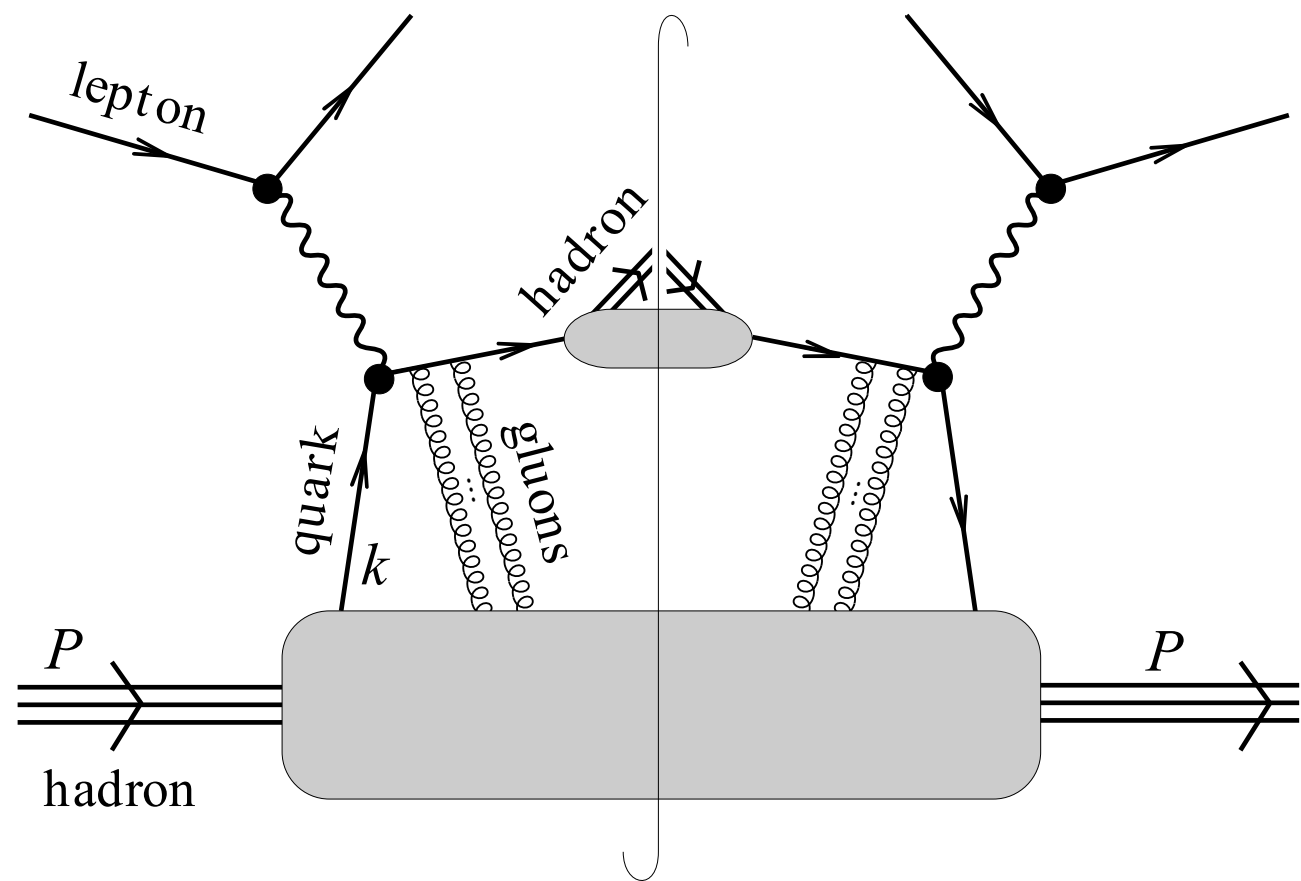
## Fundamental TMD correlator

$$\tilde{\Phi}_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \dots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma \mathcal{U}[0, \dots, b] q(b) | P, S \rangle$$

$$\Phi^{[\Gamma]}(x, k_T, P, S, \dots) \equiv \int \frac{d^2 b_T}{(2\pi)^2} \int \frac{d(b \cdot P)}{(2\pi) P^+} \exp(i x (b \cdot P) - i b_T \cdot k_T) \frac{\tilde{\Phi}_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \dots)}{\tilde{\mathcal{S}}(b^2, \dots)} \Big|_{b^+=0}$$

- “Soft factor”  $\tilde{\mathcal{S}}$  required to subtract divergences of Wilson line  $\mathcal{U}$
- $\tilde{\mathcal{S}}$  is typically a combination of vacuum expectation values of Wilson line structures
- Here, will consider only ratios in which soft factors cancel

## Gauge link structure motivated by SIDIS

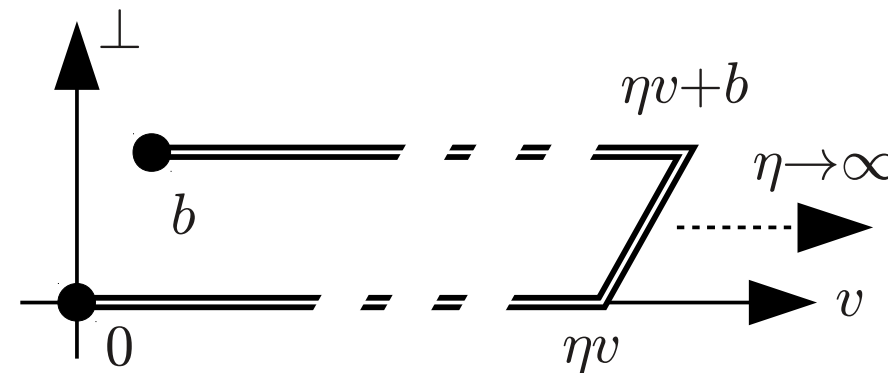


$$l + H(P) \longrightarrow l' + h(P_h) + X$$

Gauge link structure:

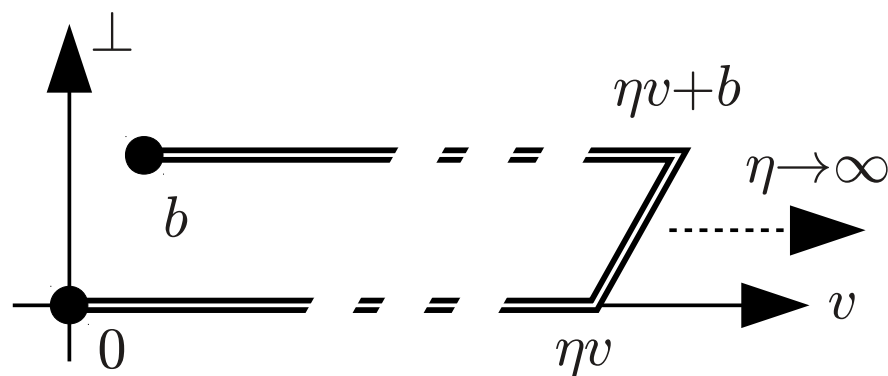
In matrix element  $\tilde{\Phi}_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \dots) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma \mathcal{U}[0, \dots, b] q(b) | P, S \rangle$

Staple-shaped gauge link  $\mathcal{U}[0, \eta v, \eta v + b, b]$



incorporates SIDIS final state effects

## Gauge link structure motivated by SIDIS



Beyond tree level: Rapidity divergences suggest taking staple direction slightly off the light cone. Approach of Aybat, Collins, Qiu, Rogers makes  $v$  space-like. Parametrize in terms of Collins-Soper parameter

$$\hat{\zeta} \equiv \frac{P \cdot v}{|P||v|}$$

Light-like staple for  $\hat{\zeta} \rightarrow \infty$ . Perturbative evolution equations for large  $\hat{\zeta}$ .

## Fundamental TMD correlator

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## Decomposition of $\Phi$ into TMDs

All leading twist structures:

$$\Phi[\gamma^+] = f_1 - \left[ \frac{\epsilon_{ij} k_i S_j}{m_H} f_{1T}^\perp \right] \text{odd}$$

$$\Phi[\gamma^+ \gamma^5] = \Lambda g_1 + \frac{k_T \cdot S_T}{m_H} g_{1T}$$

$$\Phi[i\sigma^i \gamma^5] = S_i h_1 + \frac{(2k_i k_j - k_T^2 \delta_{ij}) S_j}{2m_H^2} h_{1T}^\perp + \frac{\Lambda k_i}{m_H} h_{1L}^\perp + \left[ \frac{\epsilon_{ij} k_j}{m_H} h_1^\perp \right] \text{odd}$$

## TMD Classification

All leading twist structures:

$H$ $\downarrow$	$q \rightarrow$	U	L	T
U	$f_1$		$h_1^\perp$	← Boer-Mulders (T-odd)
L		$g_1$	$h_{1L}^\perp$	
T	$f_{1T}^\perp$	$g_{1T}$	$h_1 \quad h_{1T}^\perp$	

↑  
Sivers (T-odd)

## Decomposition of $\tilde{\Phi}$ into amplitudes

$$\tilde{\Phi}_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \hat{\zeta}, \mu) \equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma \mathcal{U}[0, \eta v, \eta v + b, b] q(b) | P, S \rangle$$

Decompose in terms of invariant amplitudes; at leading twist,

$$\begin{aligned} \frac{1}{2P^+} \tilde{\Phi}_{\text{unsubtr.}}^{[\gamma^+]} &= \tilde{A}_{2B} + im_H \epsilon_{ij} b_i S_j \tilde{A}_{12B} \\ \frac{1}{2P^+} \tilde{\Phi}_{\text{unsubtr.}}^{[\gamma^+ \gamma^5]} &= -\Lambda \tilde{A}_{6B} + i[(b \cdot P)\Lambda - m_H(b_T \cdot S_T)] \tilde{A}_{7B} \\ \frac{1}{2P^+} \tilde{\Phi}_{\text{unsubtr.}}^{[i\sigma^{i+} \gamma^5]} &= im_H \epsilon_{ij} b_j \tilde{A}_{4B} - S_i \tilde{A}_{9B} \\ &\quad - im_H \Lambda b_i \tilde{A}_{10B} + m_H[(b \cdot P)\Lambda - m_H(b_T \cdot S_T)] b_i \tilde{A}_{11B} \end{aligned}$$

(Decompositions analogous to work by Metz et al. in momentum space)



## Fourier-transformed TMDs

$$\tilde{f}(x, b_T^2, \dots) \equiv \int d^2 k_T \exp(ib_T \cdot k_T) f(x, k_T^2, \dots)$$

$$\tilde{f}^{(n)}(x, b_T^2, \dots) \equiv n! \left( -\frac{2}{m_H^2} \partial_{b_T^2} \right)^n \tilde{f}(x, b_T^2, \dots)$$

In limit  $|b_T| \rightarrow 0$ , recover  $k_T$ -moments:

$$\tilde{f}^{(n)}(x, 0, \dots) \equiv \int d^2 k_T \left( \frac{k_T^2}{2m_H^2} \right)^n f(x, k_T^2, \dots) \equiv f^{(n)}(x)$$

ill-defined for large  $k_T$ , so will not attempt to extrapolate to  $b_T = 0$ , but give results at finite  $|b_T|$ .

In this study, only consider first  $x$ -moments (accessible at  $b \cdot P = 0$ ), rather than scanning range of  $b \cdot P$ :

$$f^{[1]}(k_T^2, \dots) \equiv \int_{-1}^1 dx f(x, k_T^2, \dots)$$

→ Bessel-weighted asymmetries (Boer, Gamberg, Musch, Prokudin, JHEP 1110 (2011) 021)

## Relation between Fourier-transformed TMDs and invariant amplitudes $\tilde{A}_i$

Invariant amplitudes directly give selected  $x$ -integrated TMDs in Fourier ( $b_T$ ) space (showing just the ones relevant for Sivers, Boer-Mulders shifts), up to soft factors:

$$\tilde{f}_1^{[1](0)}(b_T^2, \hat{\zeta}, \dots, \eta v \cdot P) = 2\tilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P) / \tilde{S}(b^2, \dots)$$

$$\tilde{f}_{1T}^{\perp[1](1)}(b_T^2, \hat{\zeta}, \dots, \eta v \cdot P) = -2\tilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P) / \tilde{S}(b^2, \dots)$$

$$\tilde{h}_1^{\perp[1](1)}(b_T^2, \hat{\zeta}, \dots, \eta v \cdot P) = 2\tilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P) / \tilde{S}(b^2, \dots)$$

## Generalized shifts

Form ratios in which soft factors, ( $\Gamma$ -independent) multiplicative renormalization factors cancel

Boer-Mulders shift:

$$\langle k_y \rangle_{UT} \equiv m_H \frac{\tilde{h}_1^{\perp[1](1)}}{\tilde{f}_1^{[1](0)}} = \frac{\int dx \int d^2 k_T k_y \Phi[\gamma^+ + s^j i \sigma^{j+} \gamma^5](x, k_T, P, \dots)}{\int dx \int d^2 k_T \Phi[\gamma^+ + s^j i \sigma^{j+} \gamma^5](x, k_T, P, \dots)} \Big|_{s_T=(1,0)}$$

Average transverse momentum of quarks polarized in the orthogonal transverse (“ $T$ ”) direction in an unpolarized (“ $U$ ”) hadron; normalized to the number of valence quarks. “Dipole moment” in  $b_T^2 = 0$  limit, “shift”.

**Issue:**  $k_T$ -moments in this ratio singular; generalize to ratio of Fourier-transformed TMDs at *nonzero*  $b_T^2$ ,

$$\langle k_y \rangle_{UT}(b_T^2, \dots) \equiv m_H \frac{\tilde{h}_1^{\perp[1](1)}(b_T^2, \dots)}{\tilde{f}_1^{[1](0)}(b_T^2, \dots)}$$

(remember singular  $b_T \rightarrow 0$  limit corresponds to taking  $k_T$ -moment). “Generalized shift”.

## Generalized shifts from amplitudes

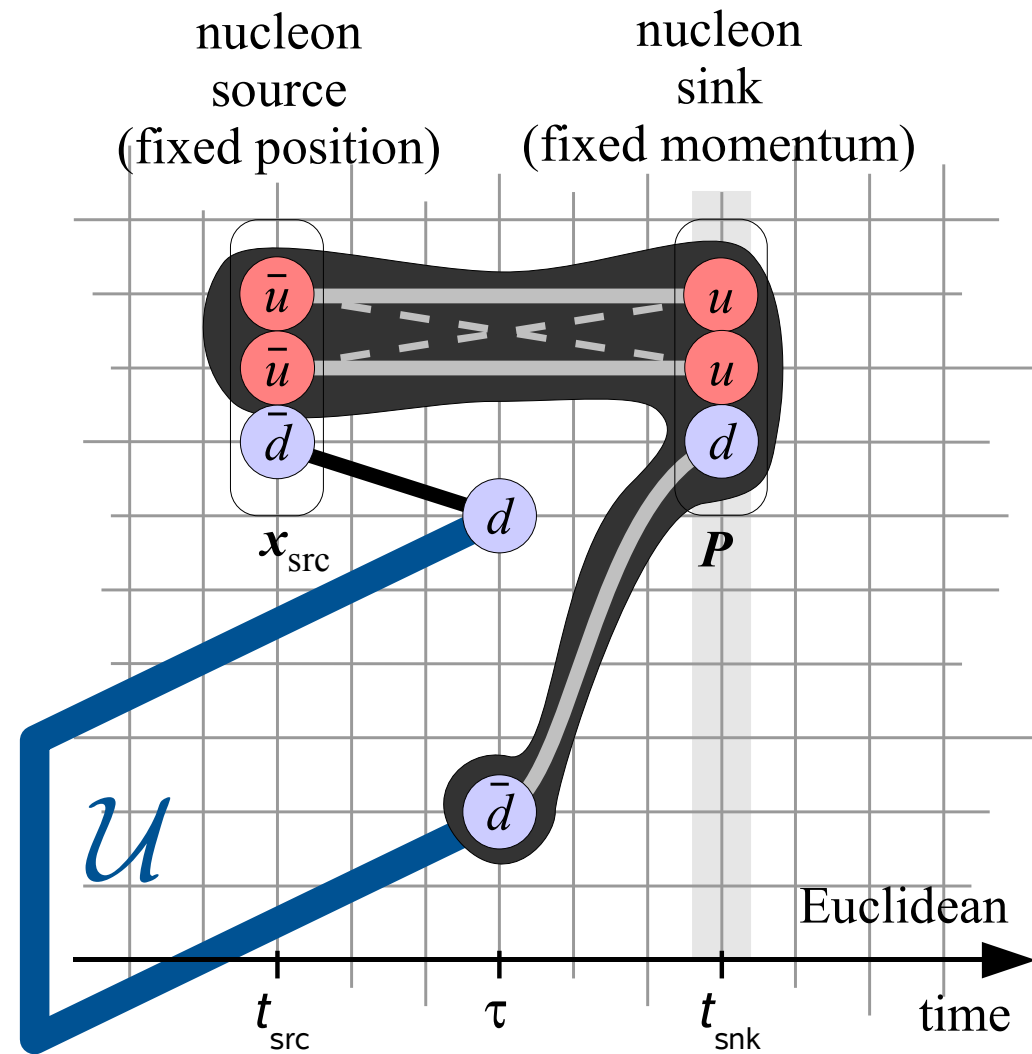
Now, can also express this in terms of invariant amplitudes:

$$\langle k_y \rangle_{UT}(b_T^2, \dots) \equiv m_H \frac{\tilde{h}_1^{\perp[1](1)}(b_T^2, \dots)}{\tilde{f}_1^{[1](0)}(b_T^2, \dots)} = m_H \frac{\tilde{A}_{4B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\tilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

Analogously, Sivers shift (in a polarized hadron):

$$\langle k_y \rangle_{TU}(b_T^2, \dots) = -m_H \frac{\tilde{A}_{12B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}{\tilde{A}_{2B}(-b_T^2, 0, \hat{\zeta}, \eta v \cdot P)}$$

## Lattice setup



- Evaluate directly  $\bar{\Phi}_{\text{unsubtr.}}^{[\Gamma]}(b, P, S, \hat{\zeta}, \mu)$   
 $\equiv \frac{1}{2} \langle P, S | \bar{q}(0) \Gamma \mathcal{U}[0, \eta v, \eta v + b, b] q(b) | P, S \rangle$
- Euclidean time: Place entire operator at one time slice, i.e.,  $b, \eta v$  purely spatial
- Since generic  $b, v$  space-like, no obstacle to boosting system to such a frame!
- **Parametrization of correlator in terms of  $\tilde{A}_i$  invariants** permits direct translation of results back to original frame; form desired  $\tilde{A}_i$  ratios.
- Use variety of  $P, b, \eta v$ ; here  $b \perp P, b \perp v$  (lowest  $x$ -moment, kinematical choices/constraints)
- Extrapolate  $\eta \rightarrow \infty, \hat{\zeta} \rightarrow \infty$  numerically.

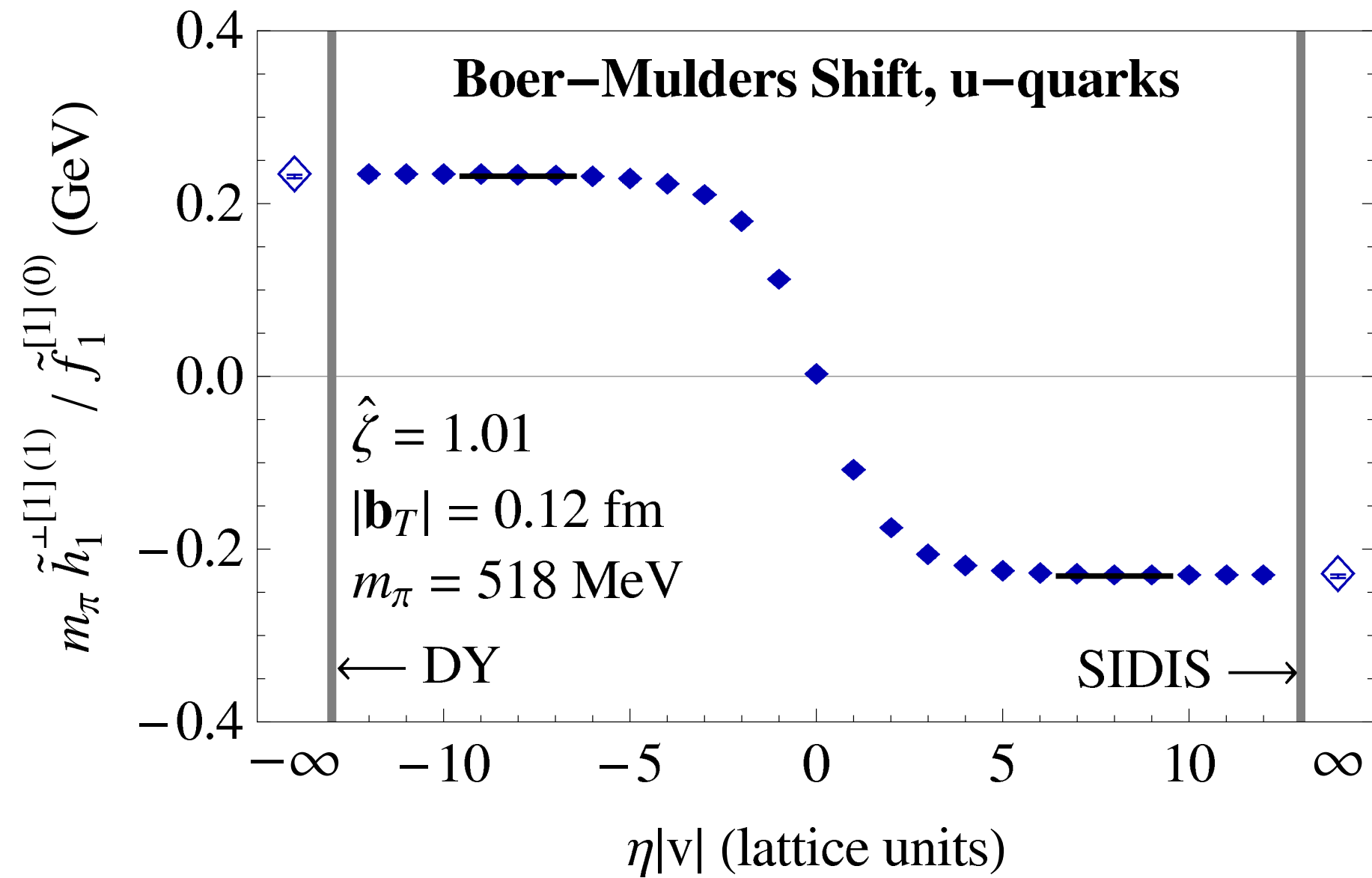
## Challenges

- The limit  $\hat{\zeta} \rightarrow \infty$ : Approaching the light cone
- Discretization effects, soft factor cancellation on the lattice in TMD ratios
- Progress toward the physical pion mass

Approaching the light cone (with a pion)

## Results: Boer-Mulders shift (pion)

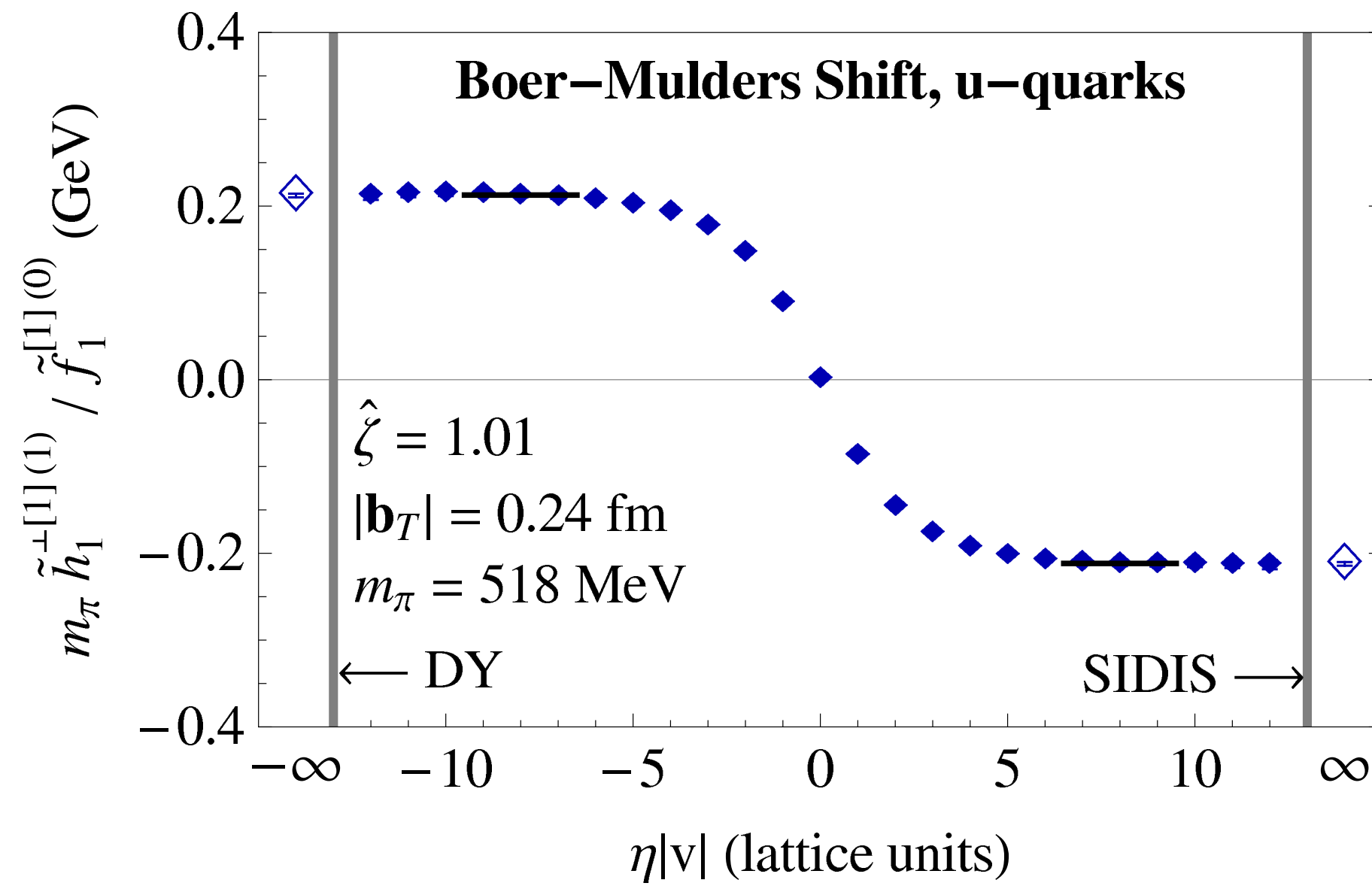
Dependence on staple extent; sequence of panels at different  $|b_T|$





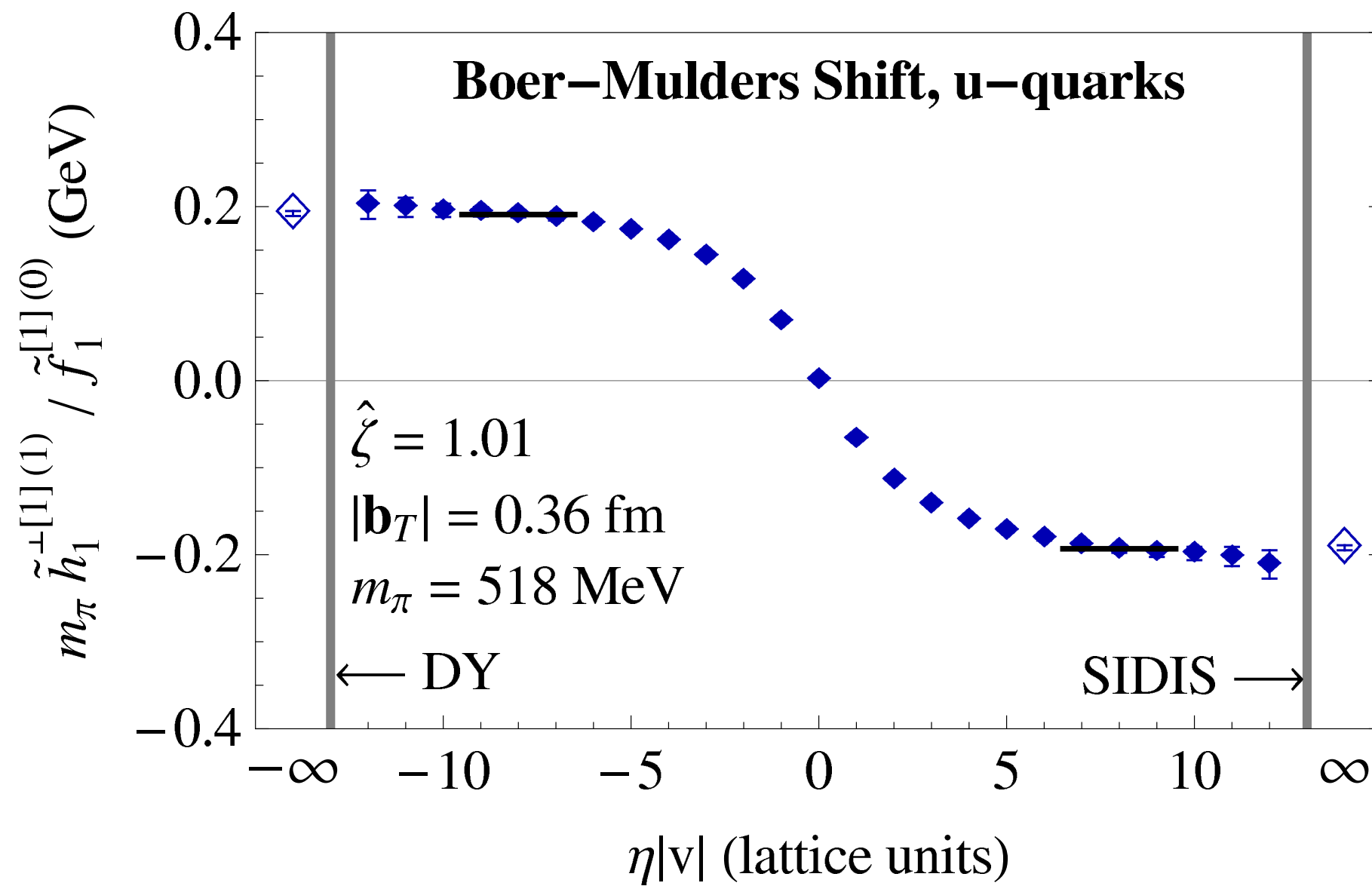
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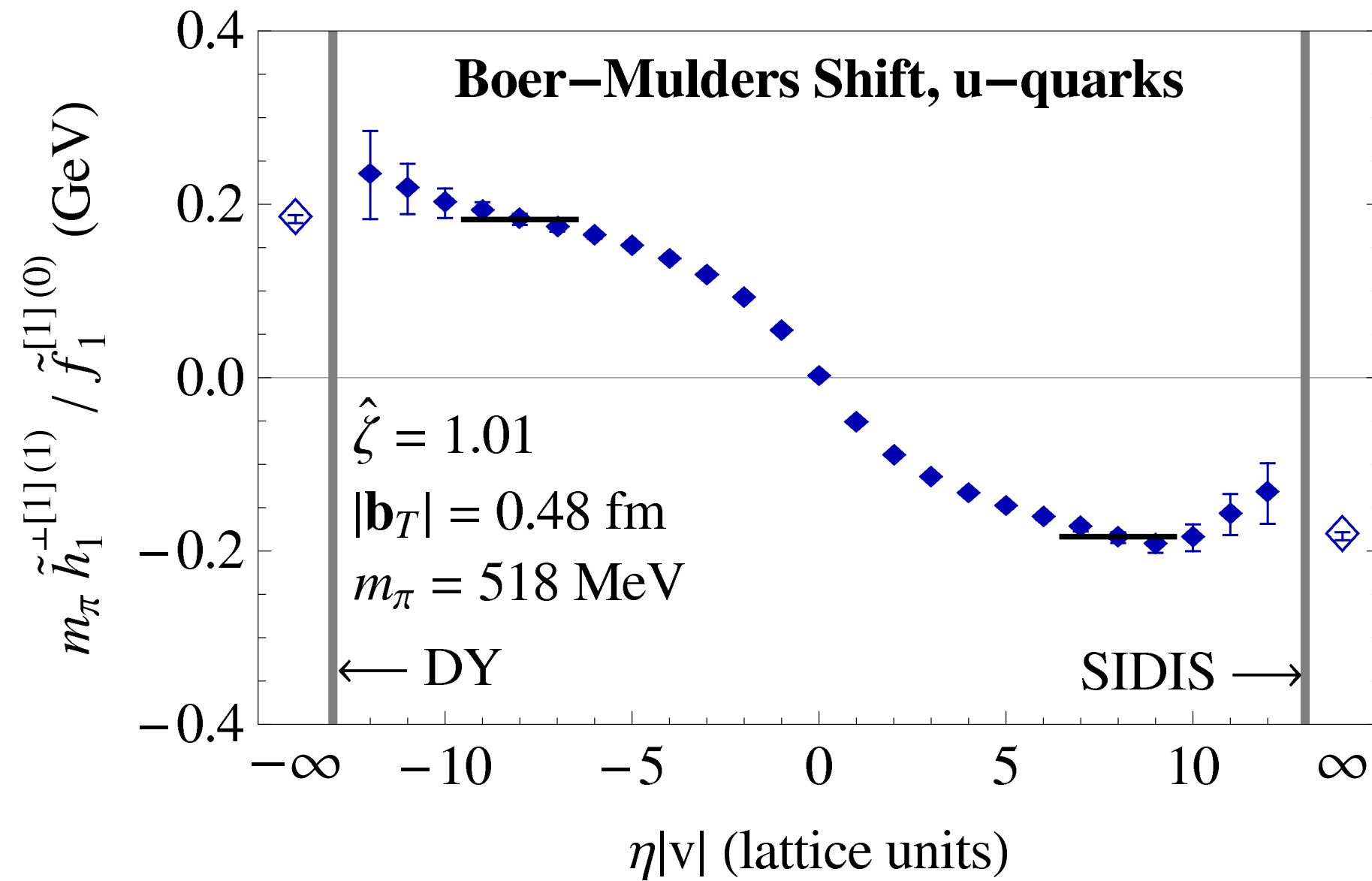
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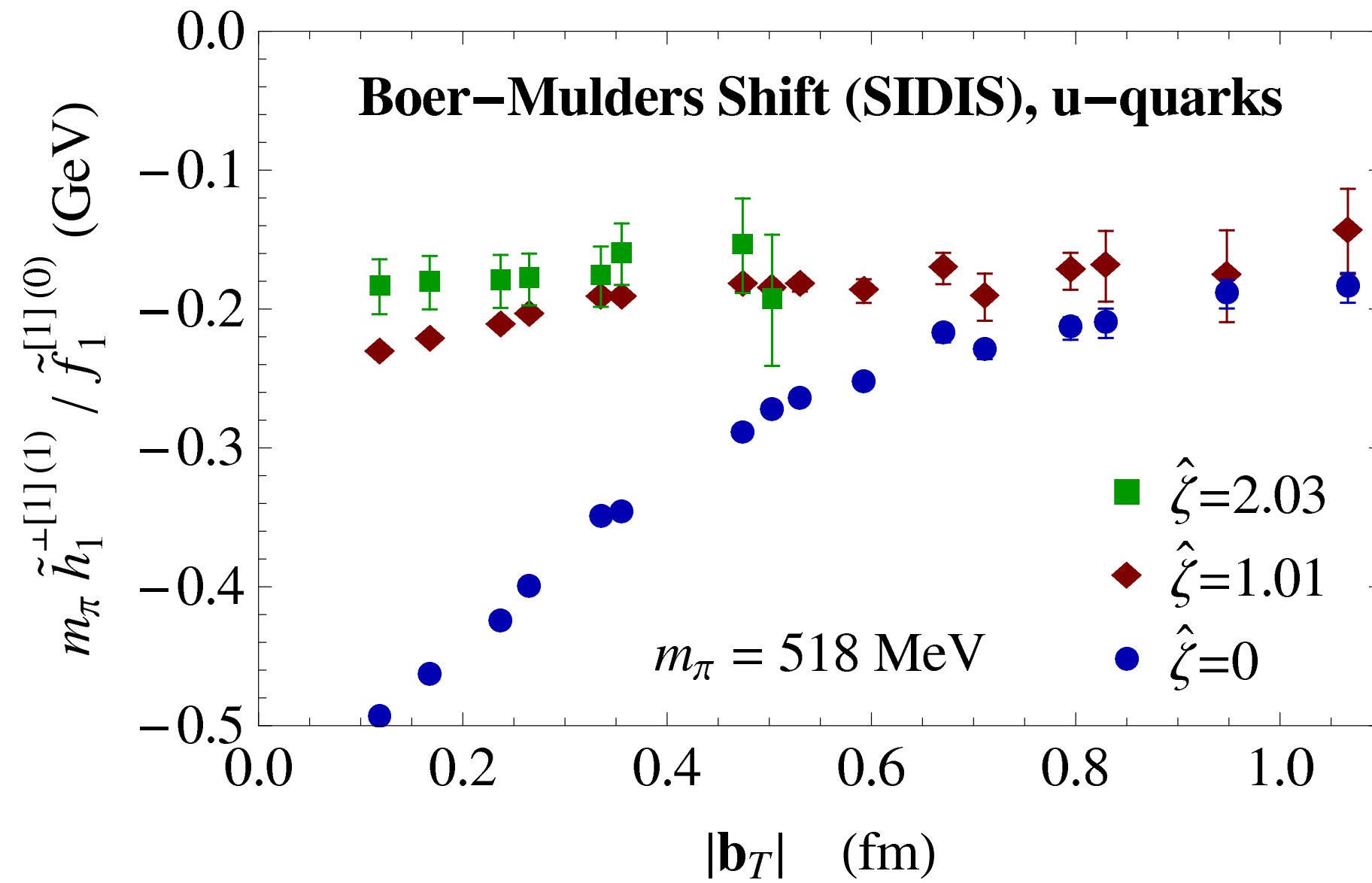
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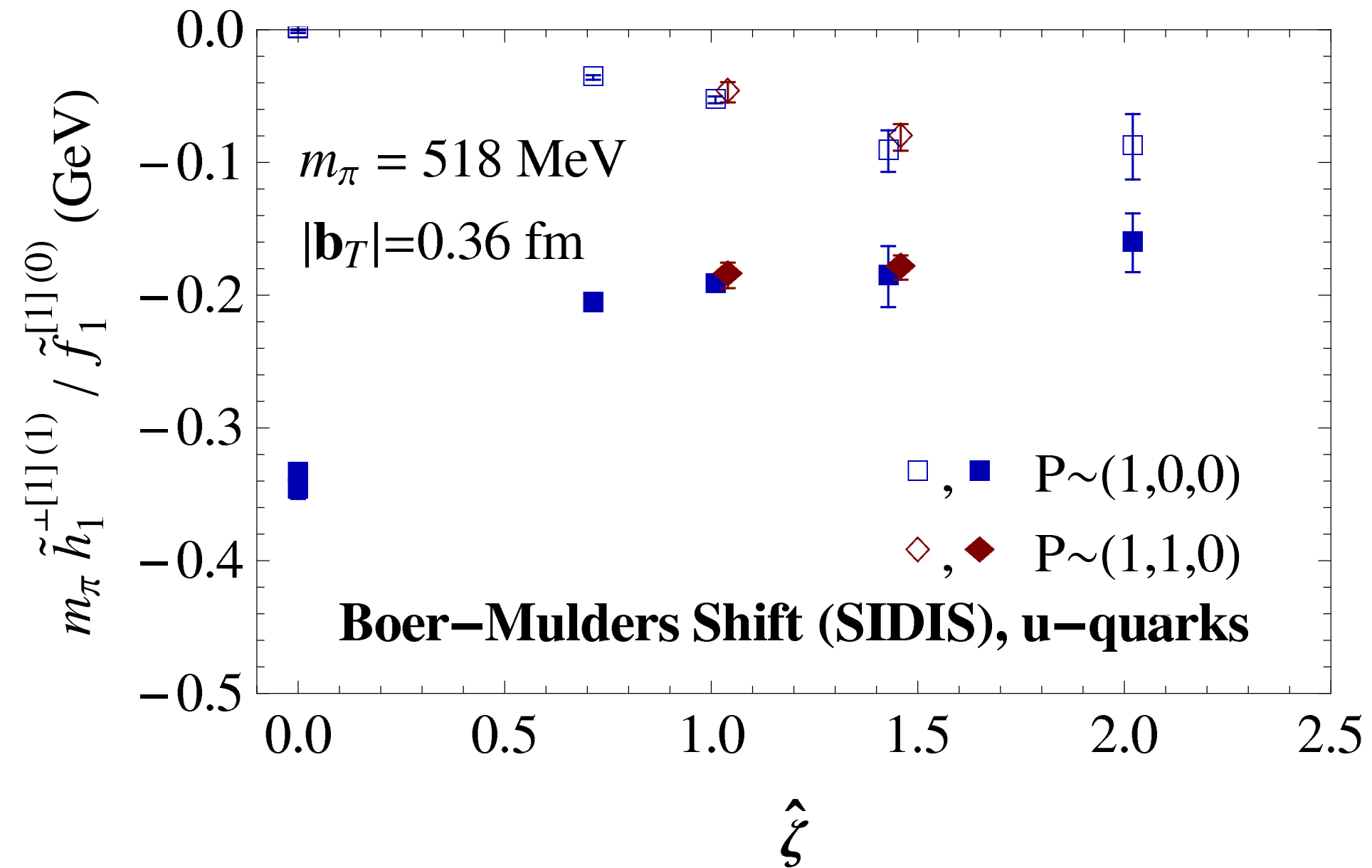
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Dependence of SIDIS limit on  $|b_T|$



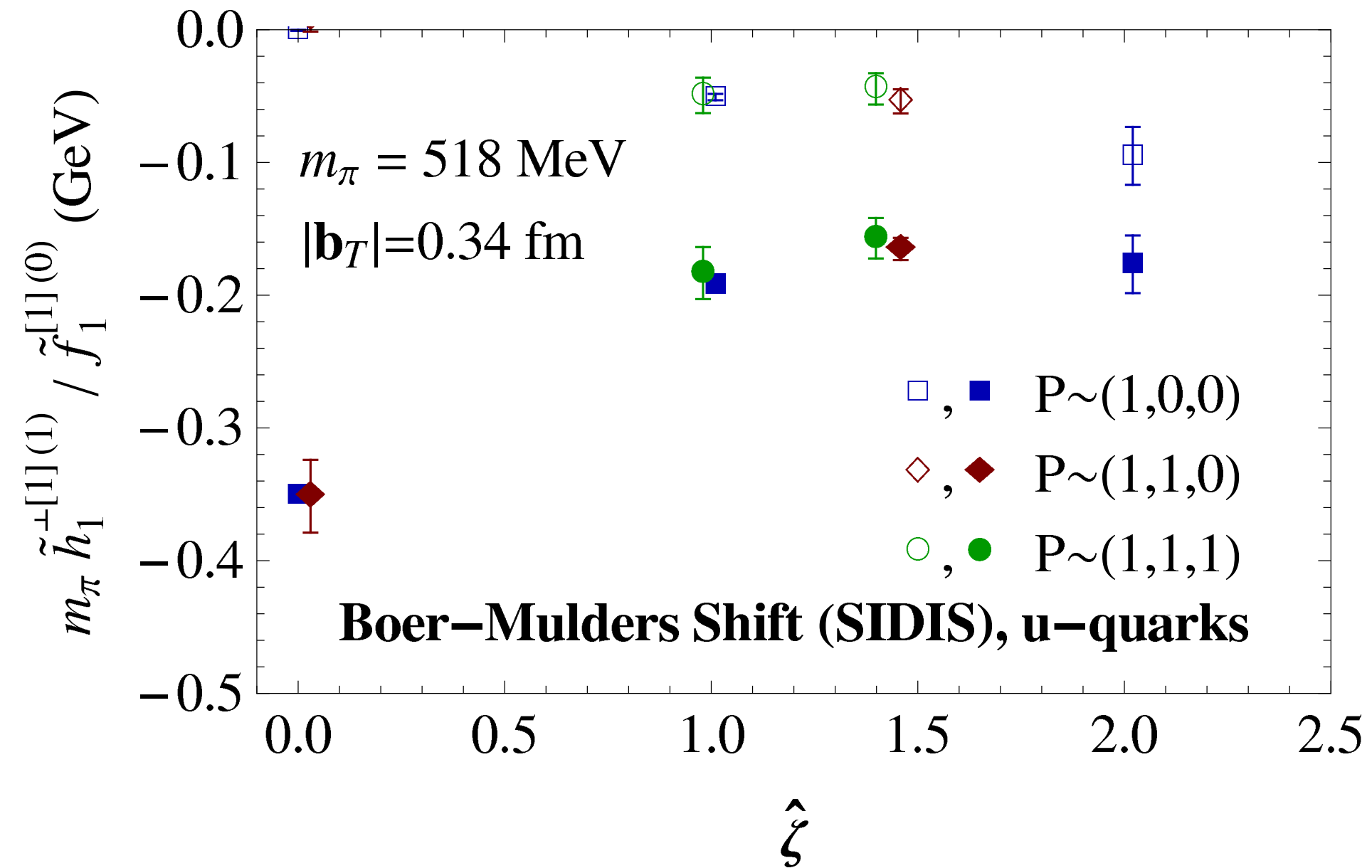
## Results: Boer-Mulders shift (pion)

Dependence of SIDIS limit on  $\hat{\zeta}$ ; open symbols: contribution  $\tilde{A}_4$  only



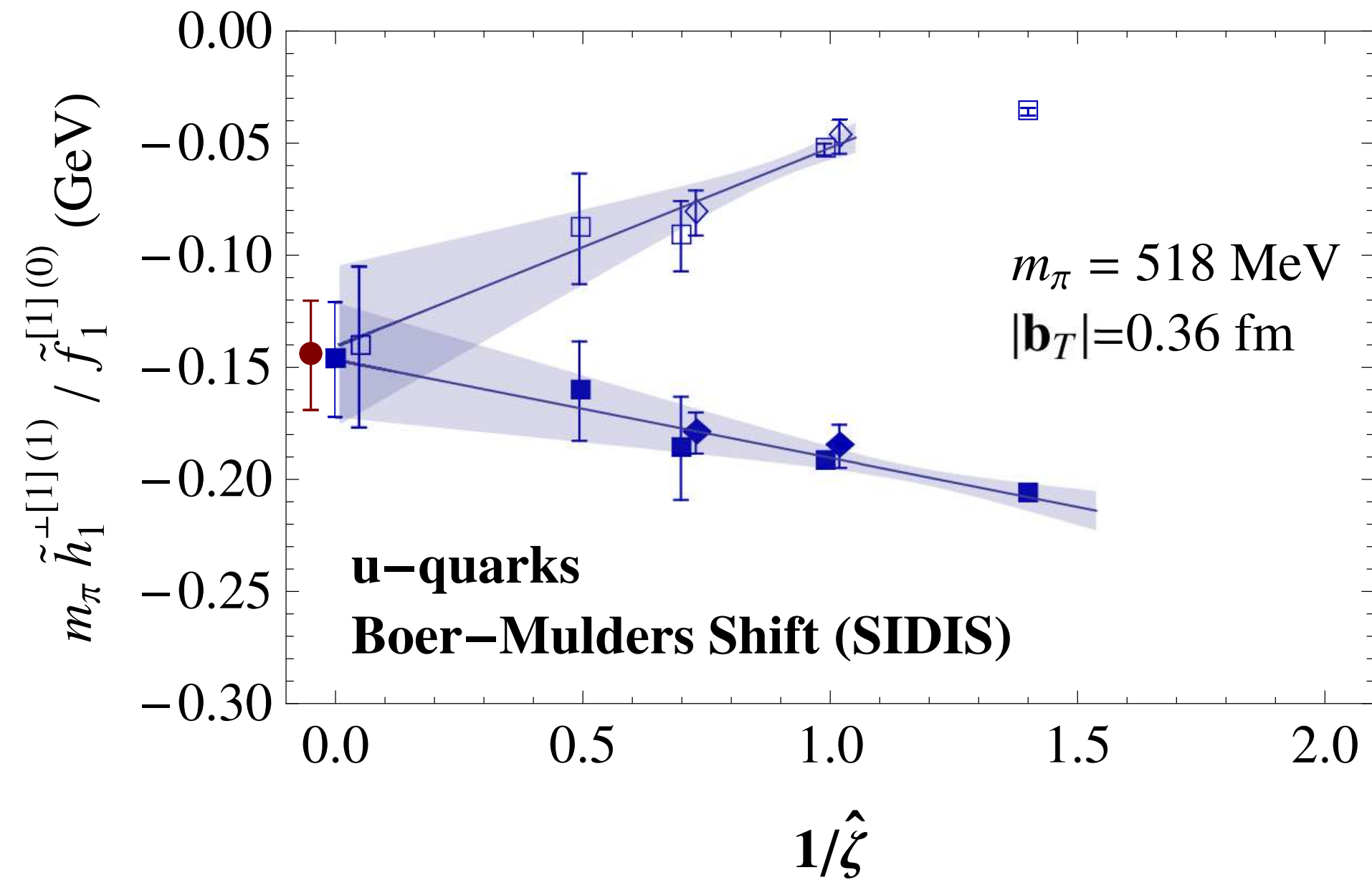
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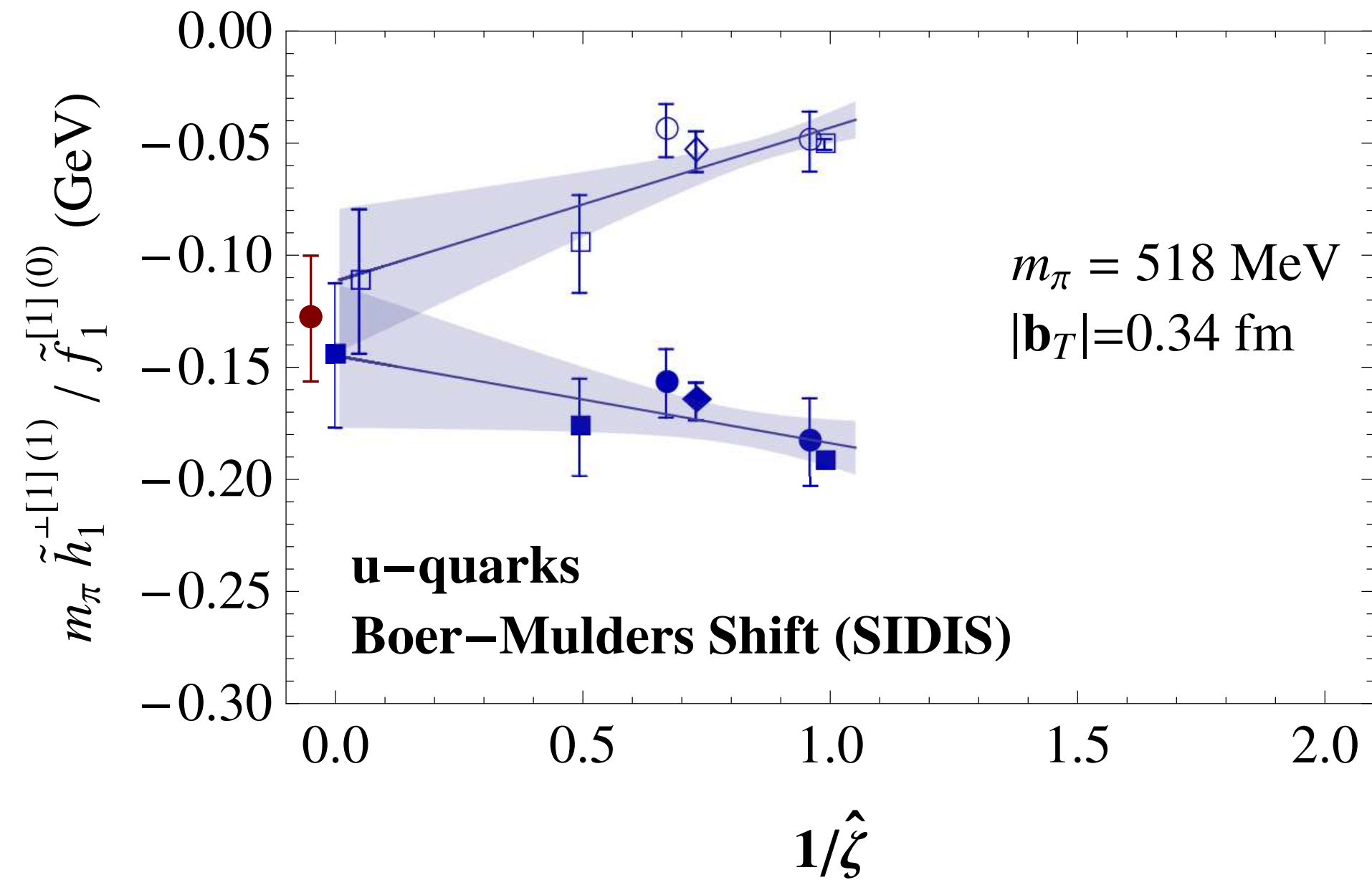
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Dependence of SIDIS limit on  $\hat{\zeta}$ ; fit function  $a + b/\hat{\zeta}$



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Discretization effects:

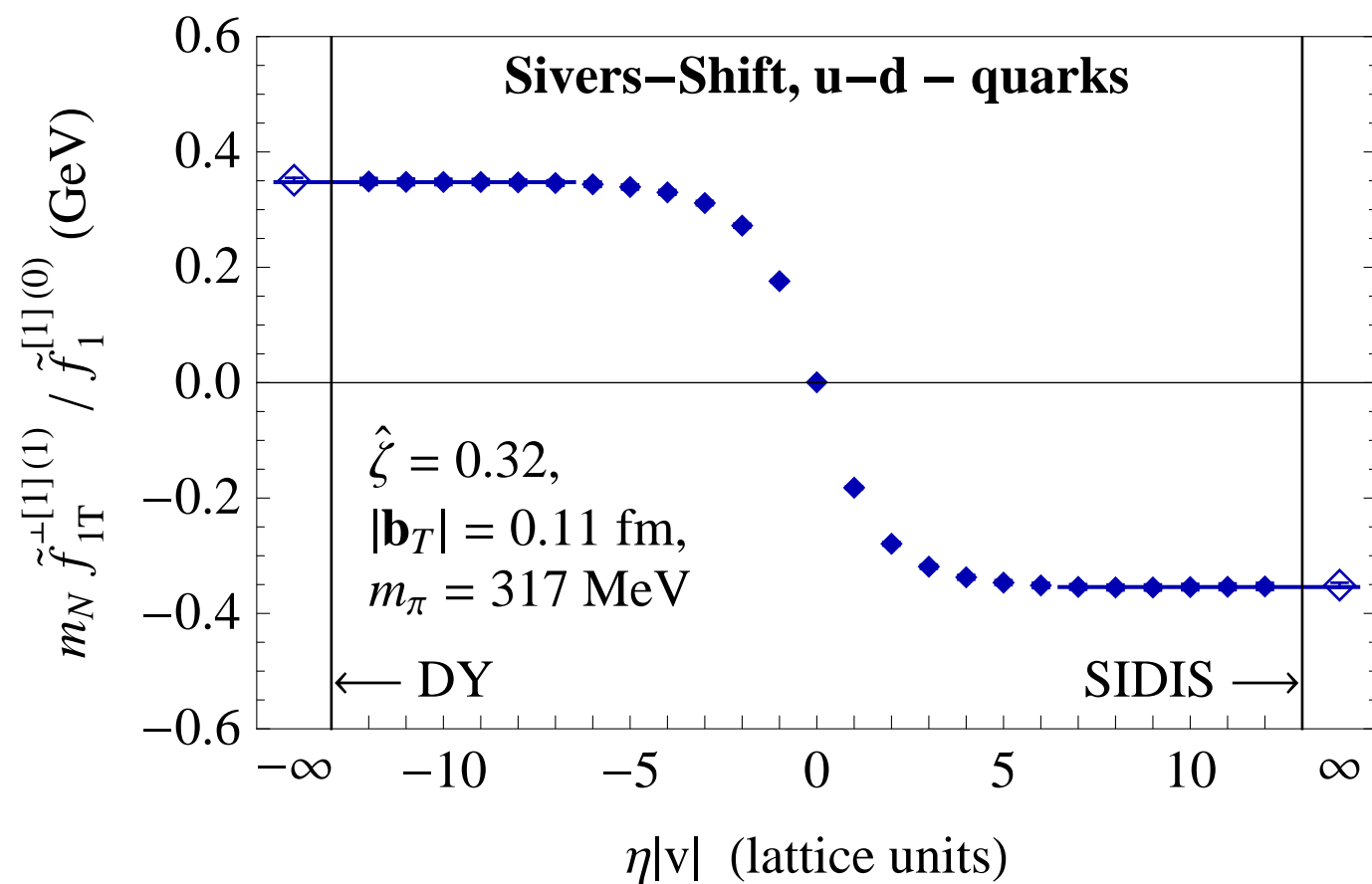
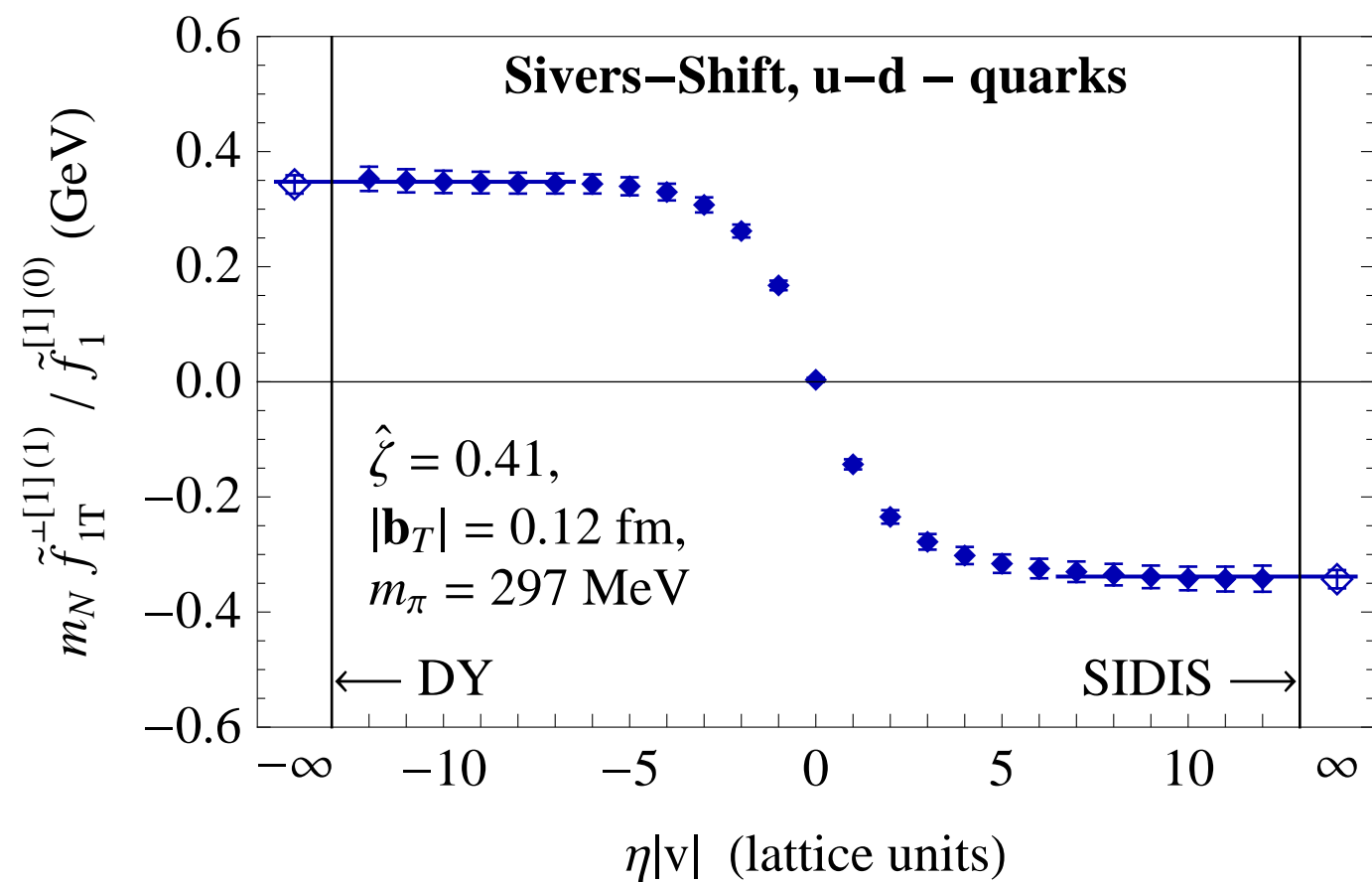
Comparison of

RBC/UKQCD DWF ensemble ( $m_\pi = 297 \text{ MeV}$ ,  $a = 0.084 \text{ fm}$ )

with clover ensemble ( $m_\pi = 317 \text{ MeV}$ ,  $a = 0.114 \text{ fm}$ )  
produced by K. Orginos and JLab collaborators

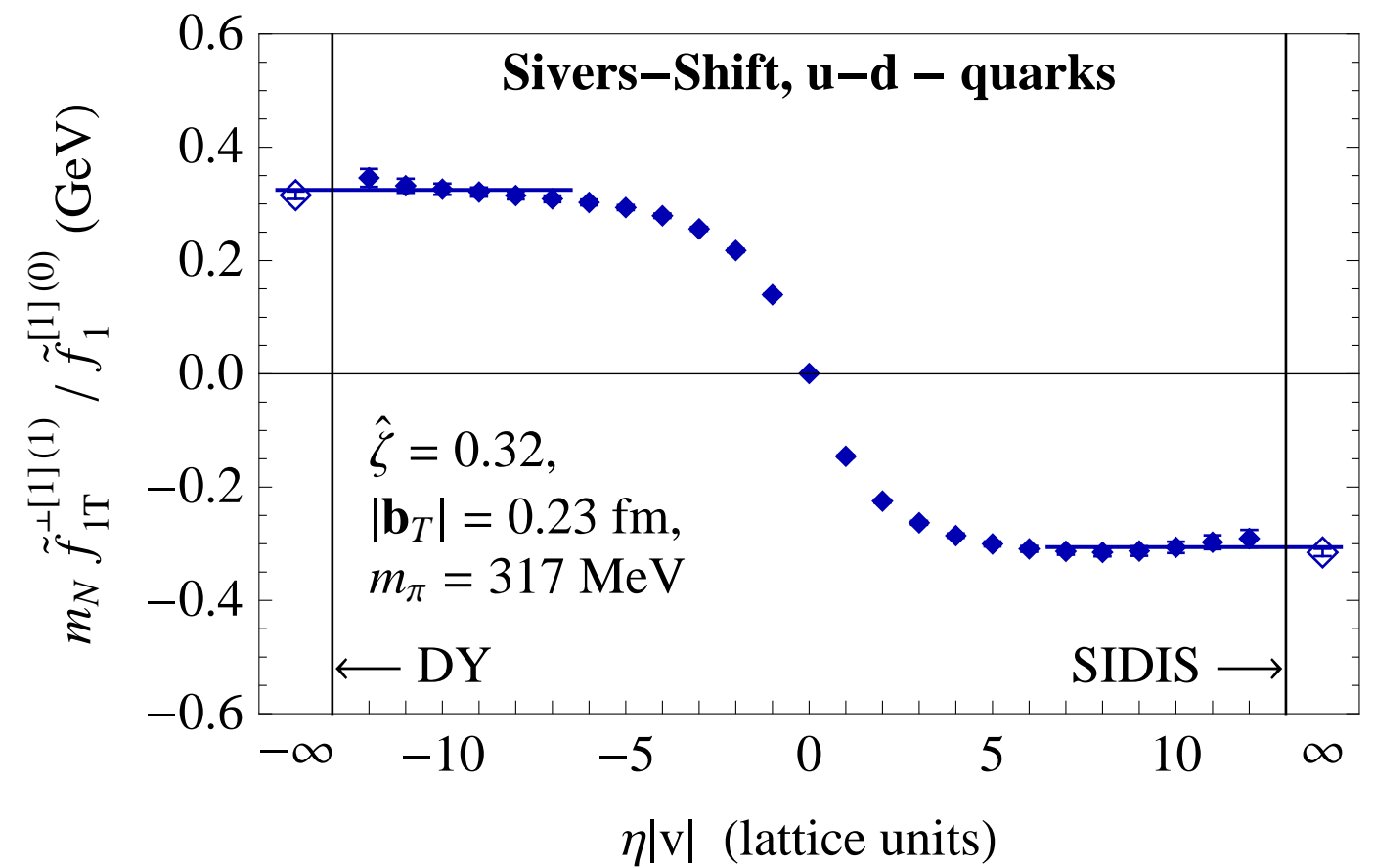
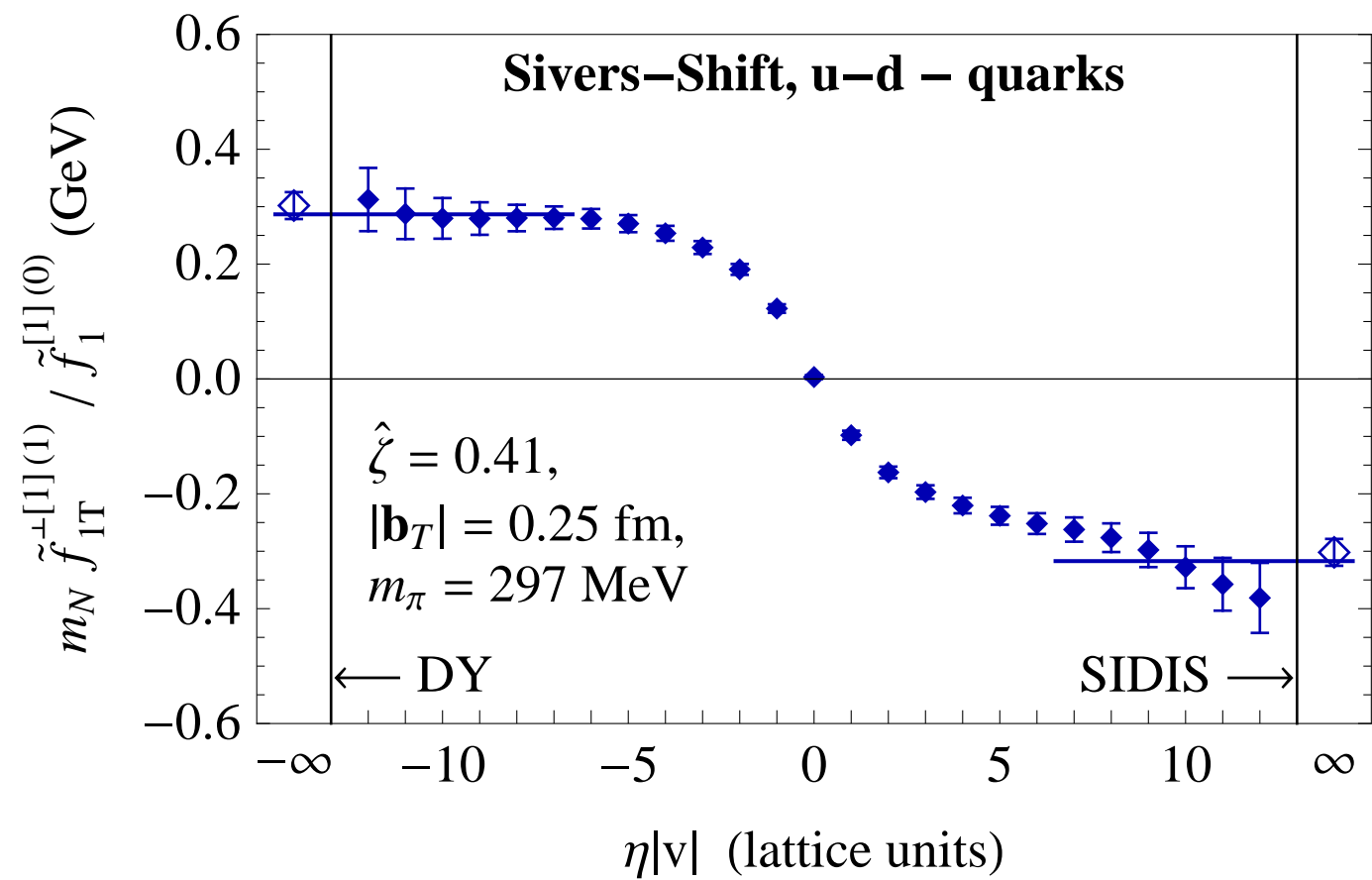
## Results: Sivers shift

Dependence on staple extent; sequence of panels at different  $|b_T|$



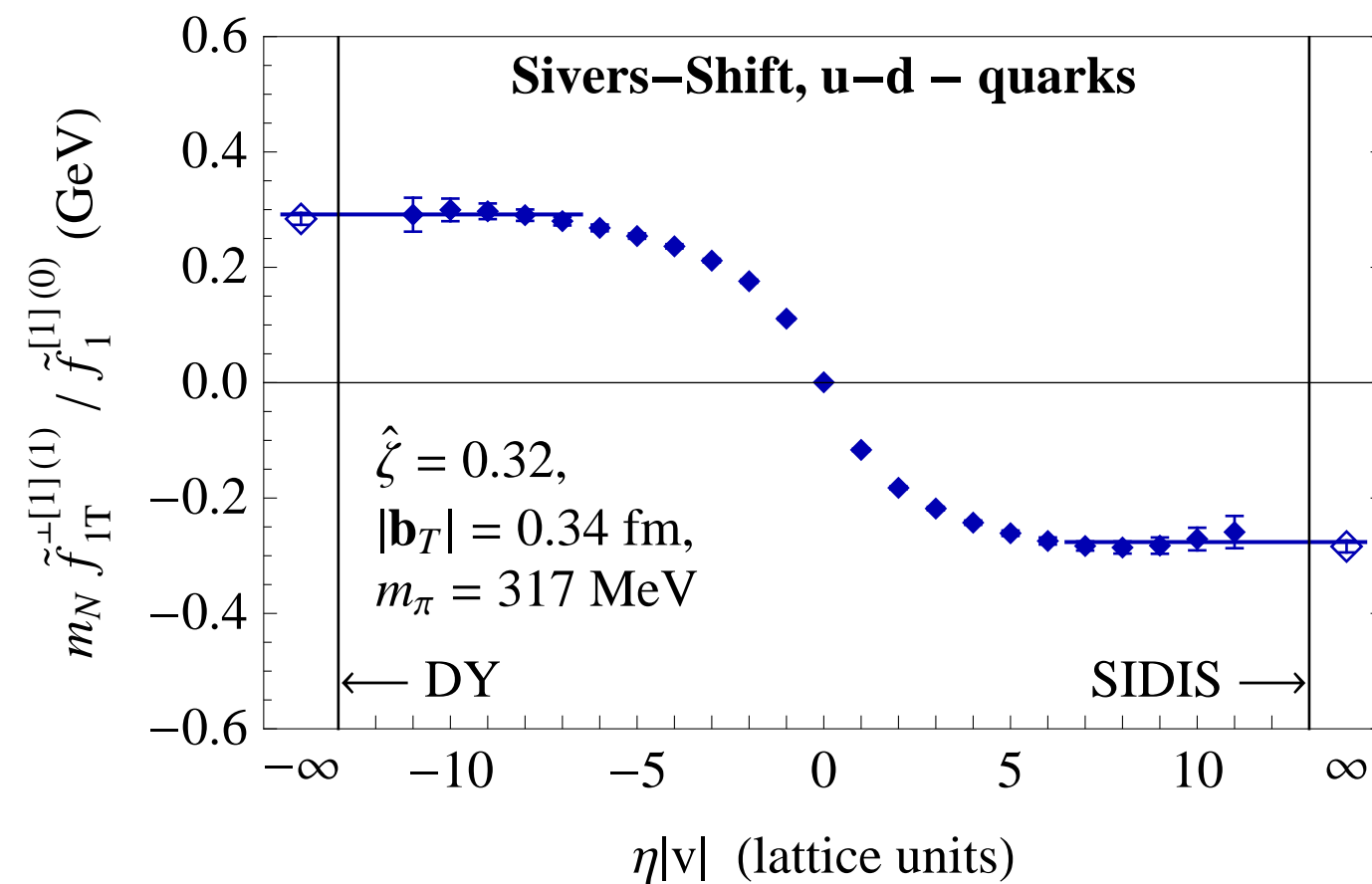
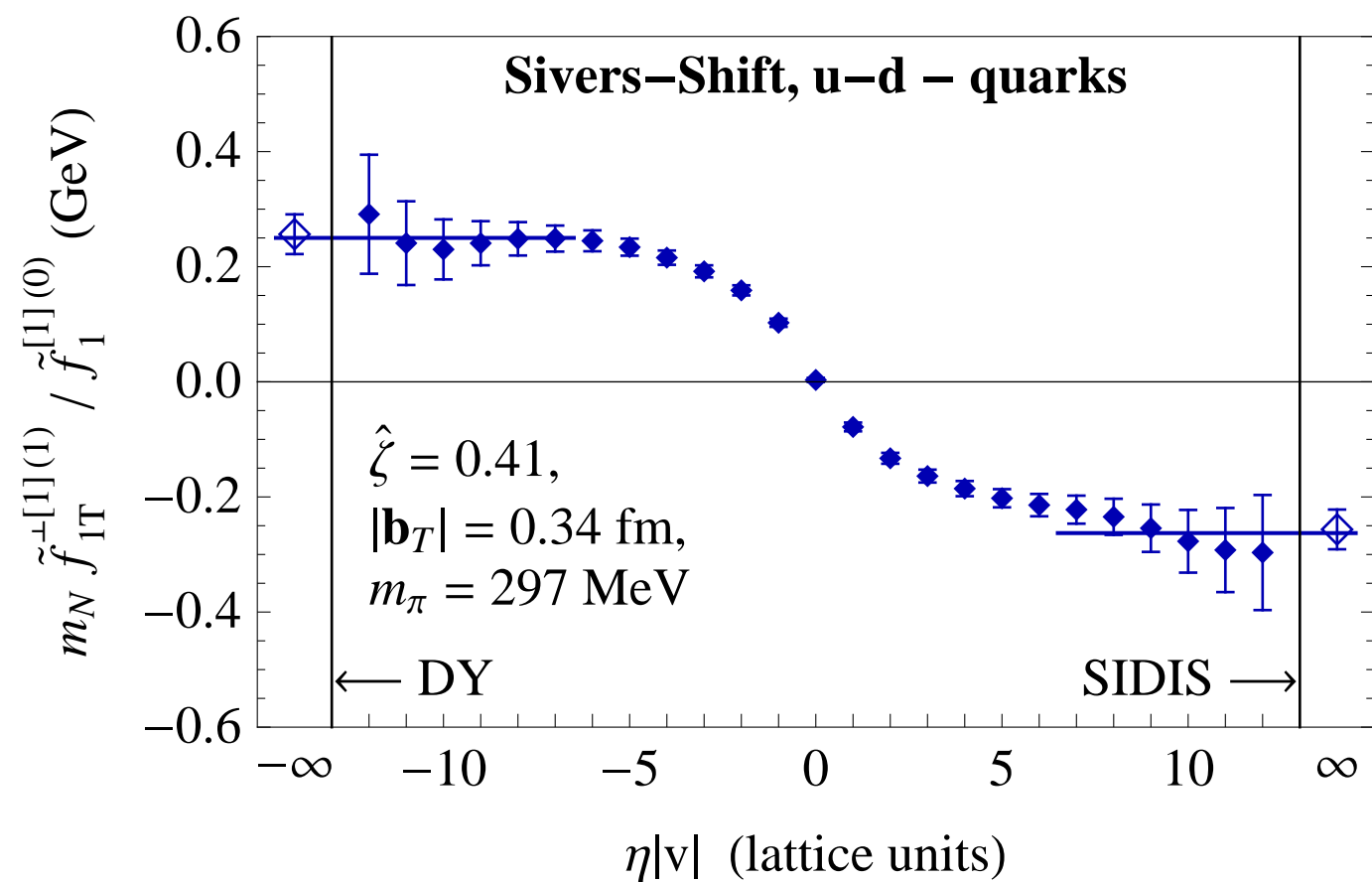
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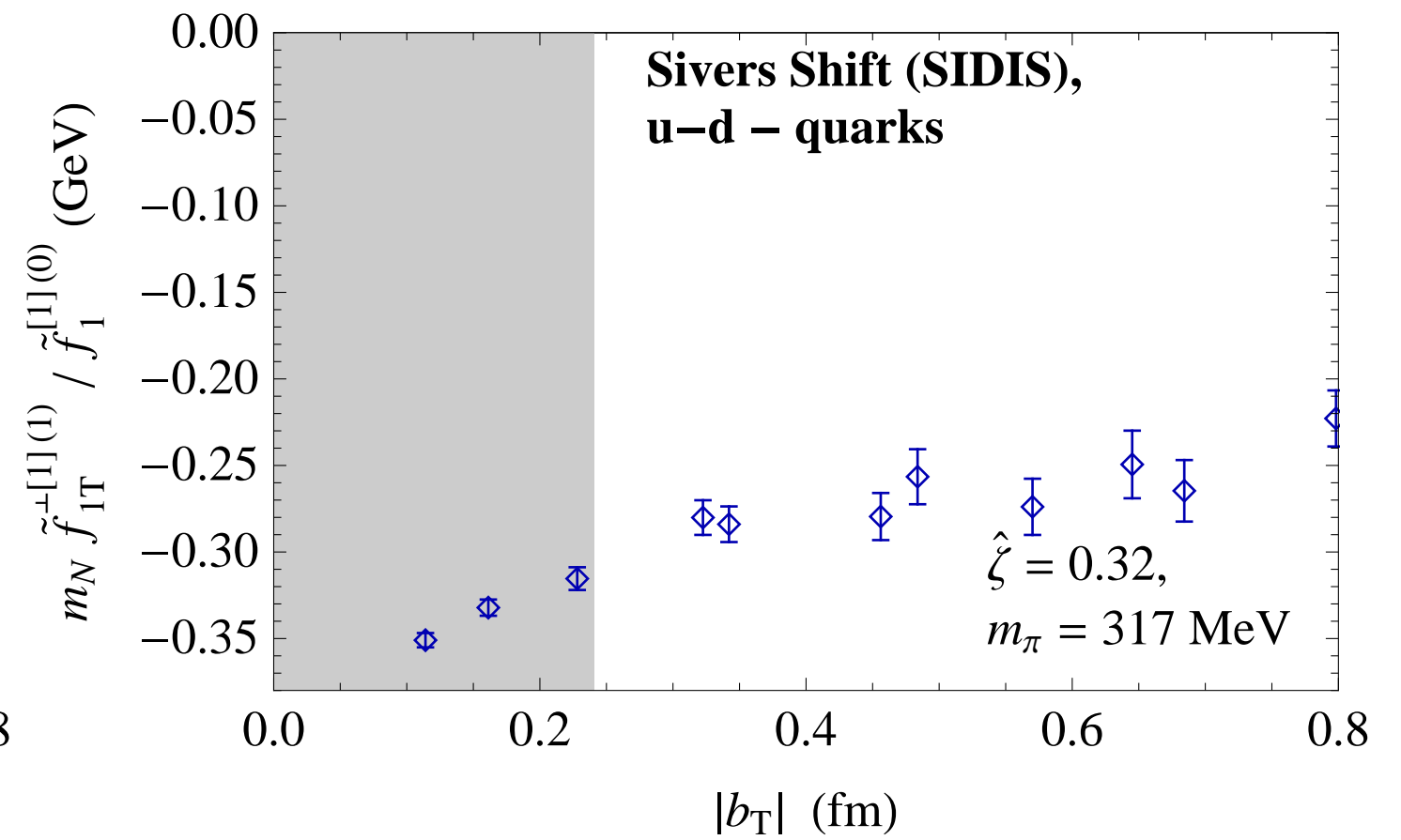
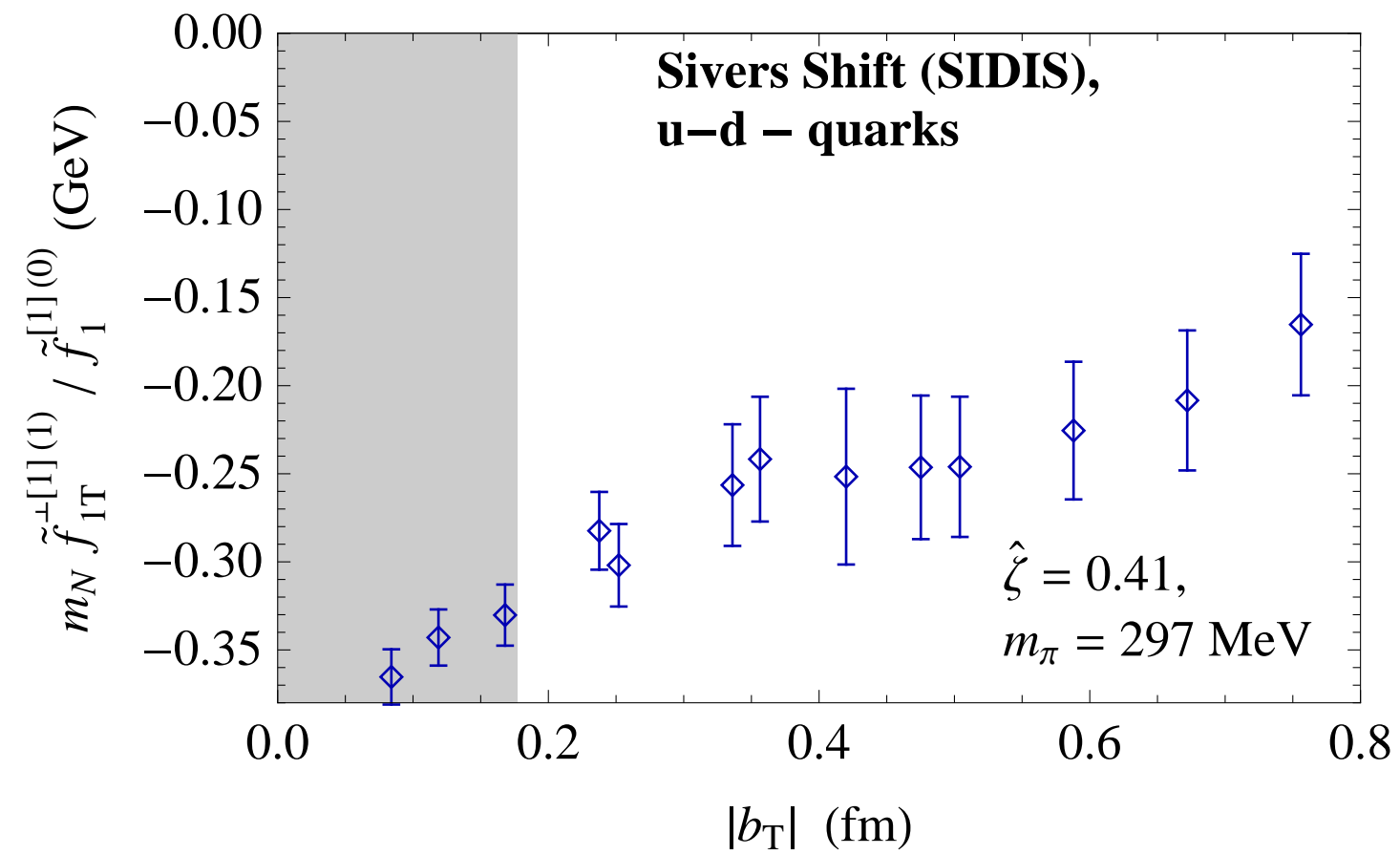
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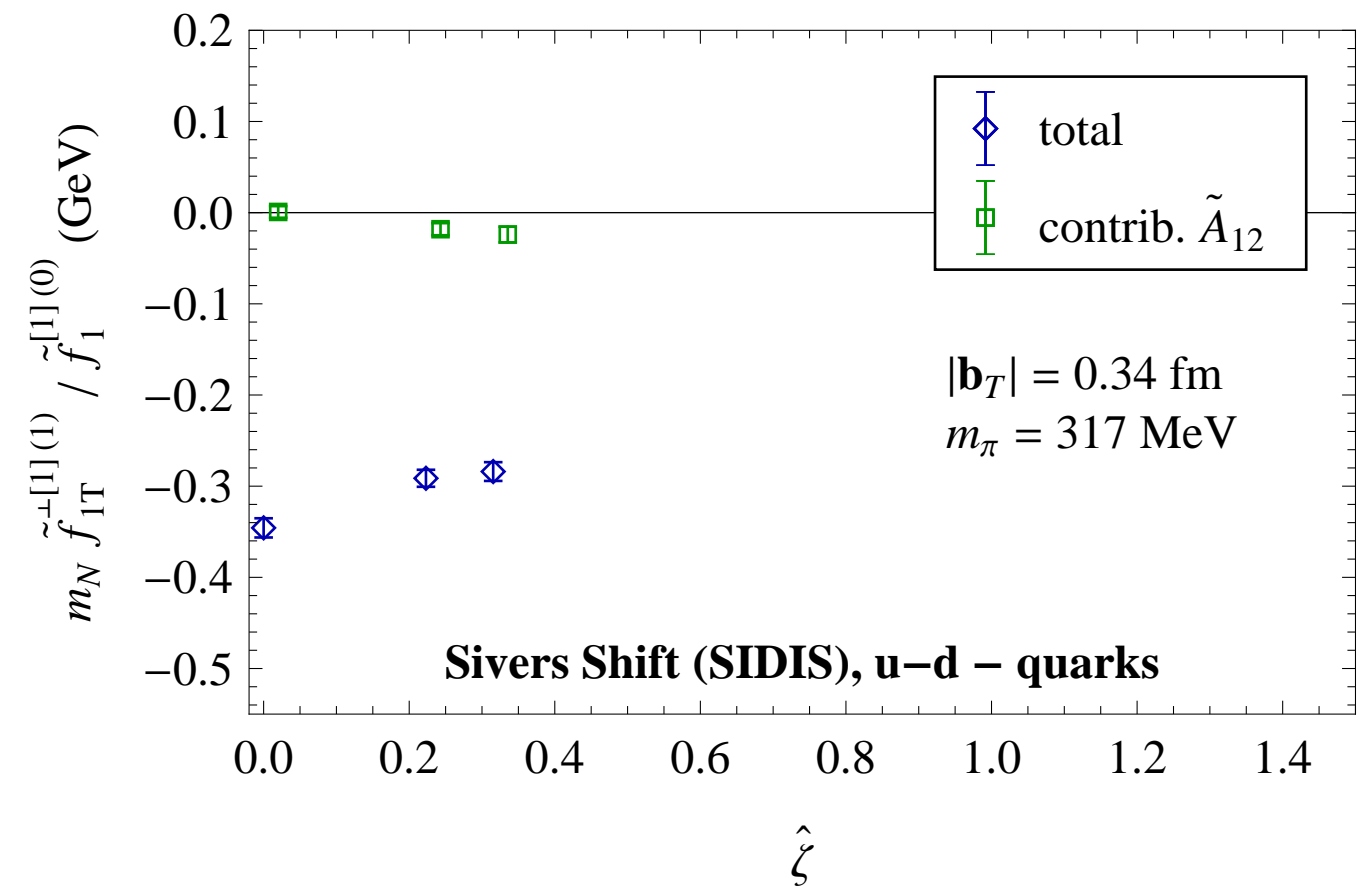
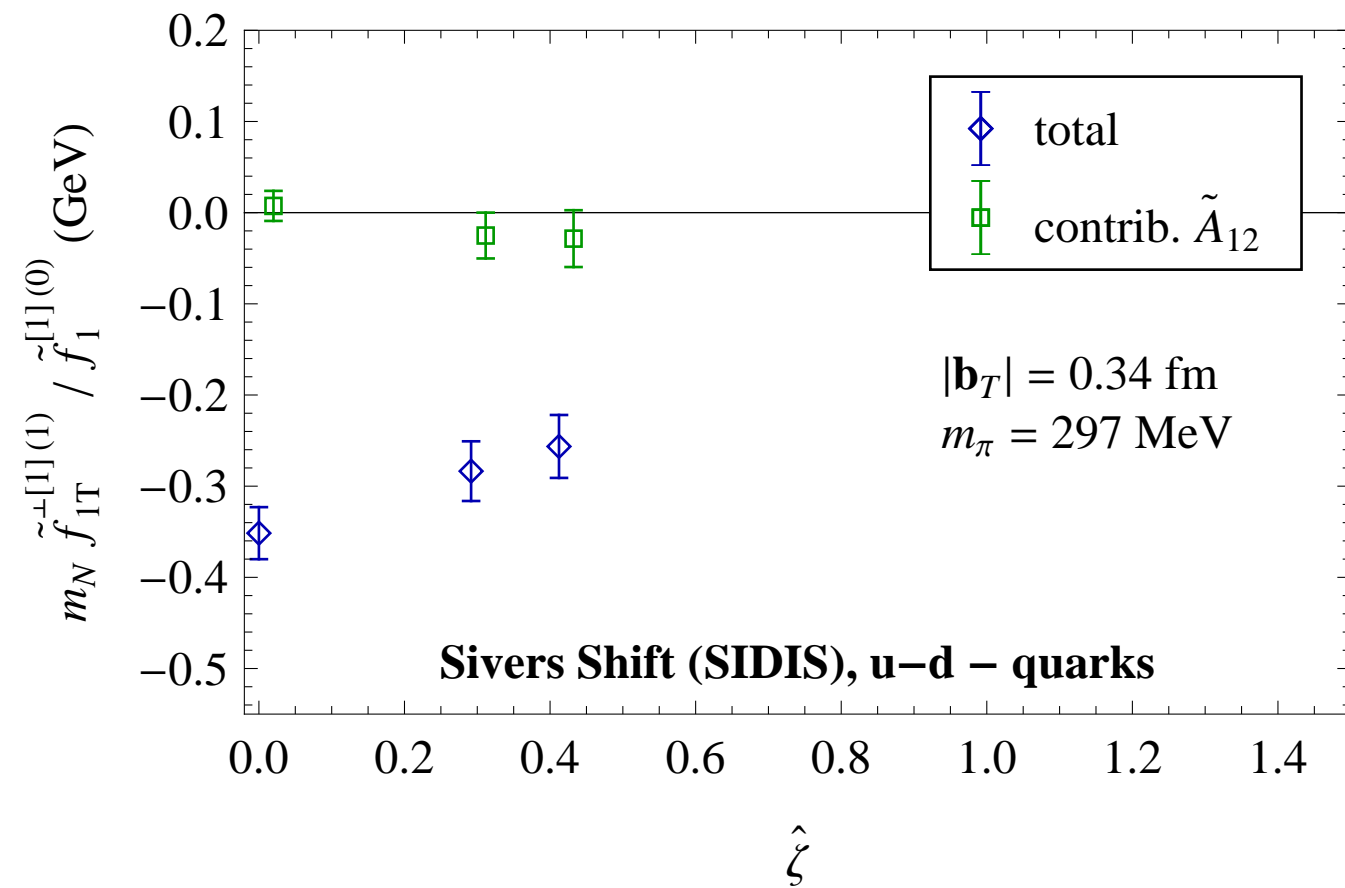
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Dependence of SIDIS limit on  $|b_T|$



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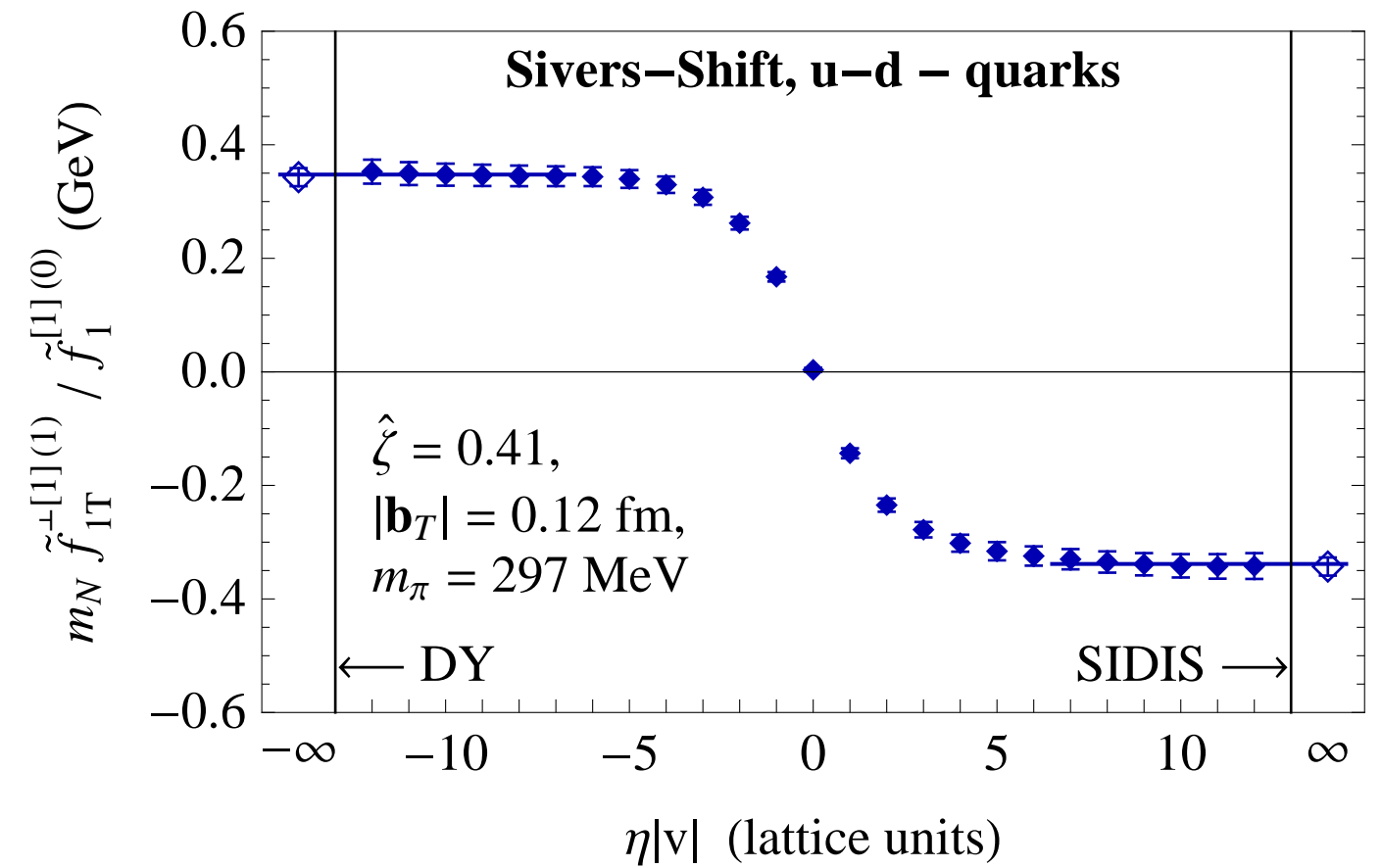
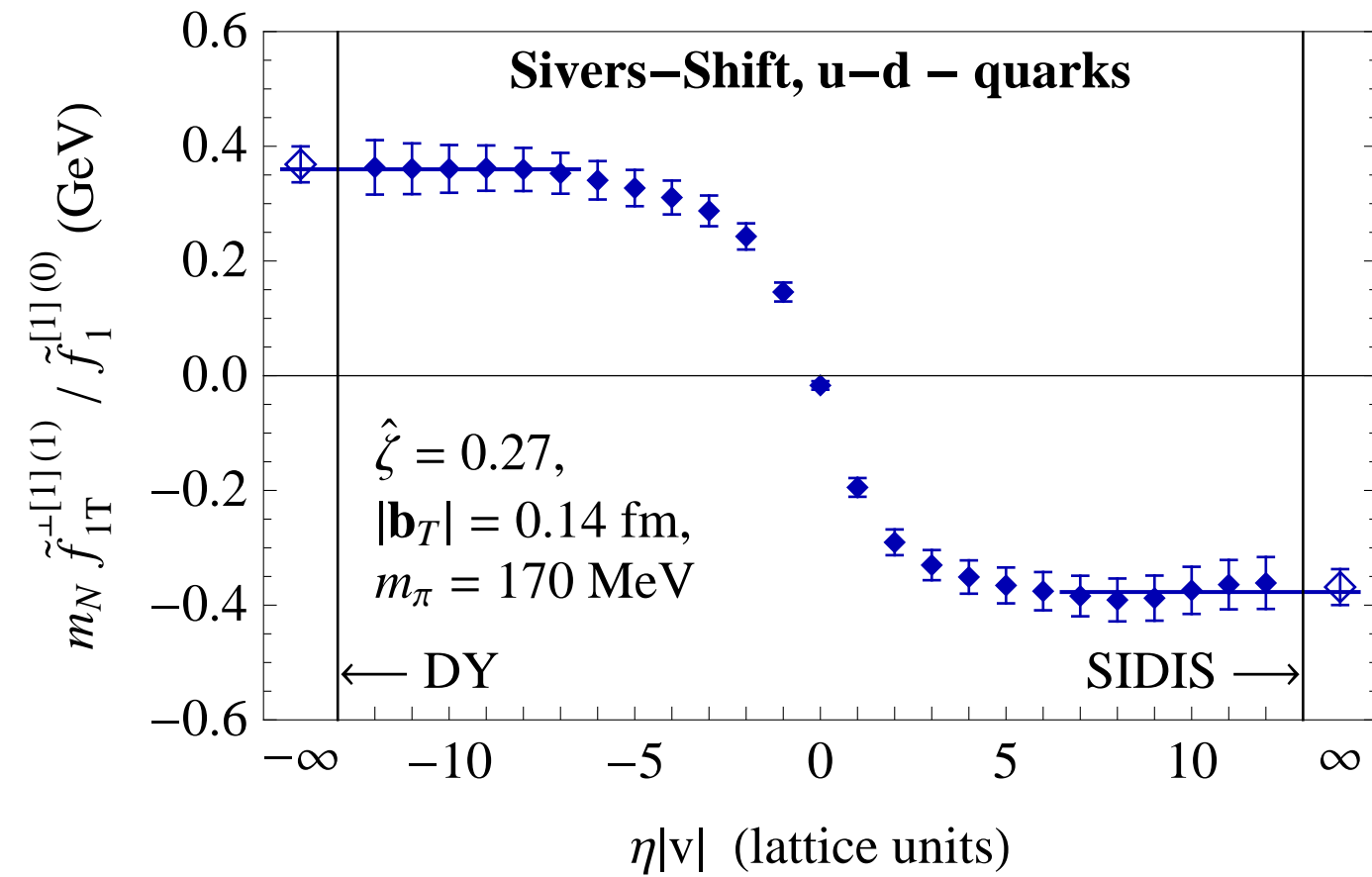
Dependence of SIDIS limit on  $\hat{\zeta}$



Dependence on the pion mass

## Results: Sivers shift

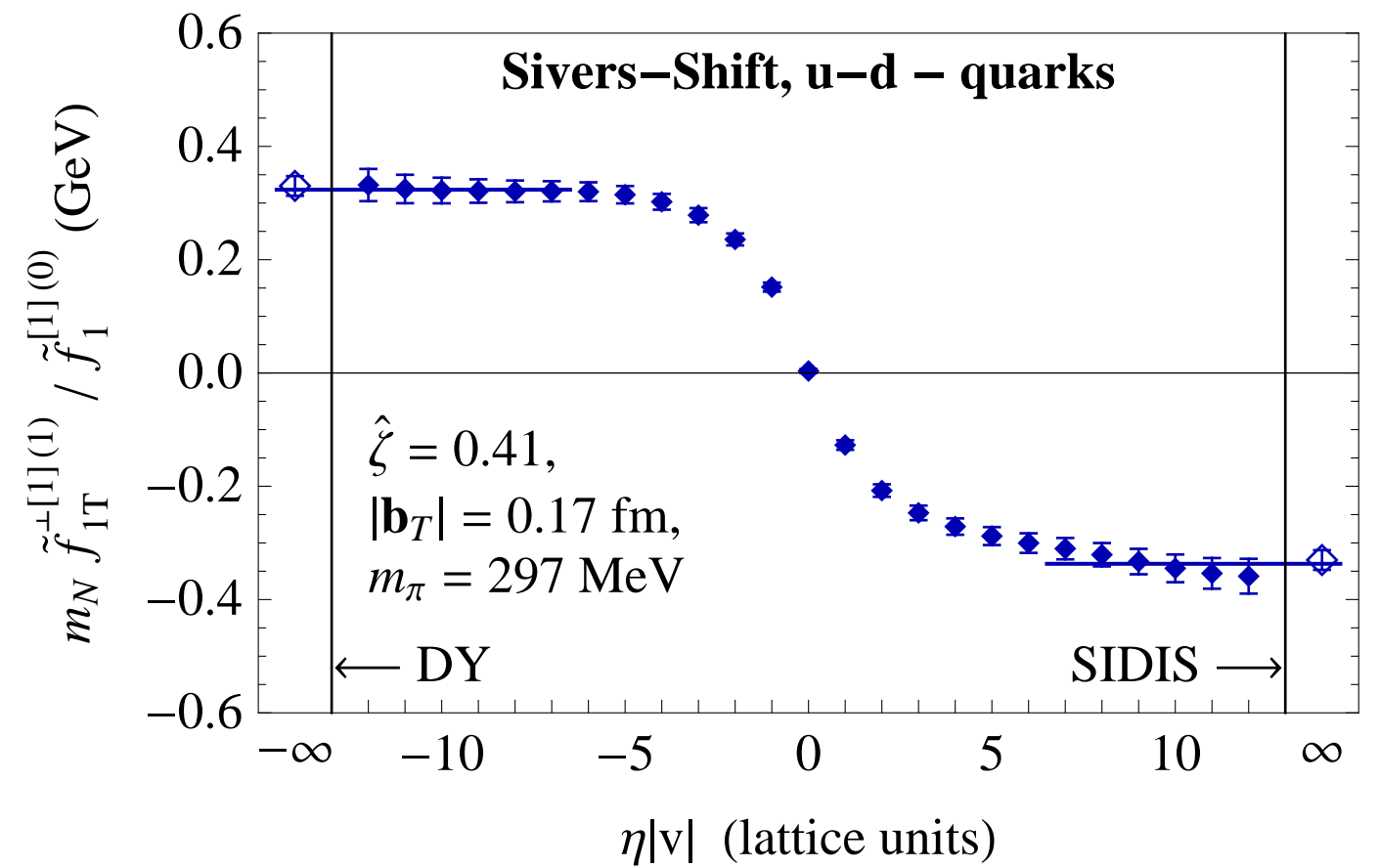
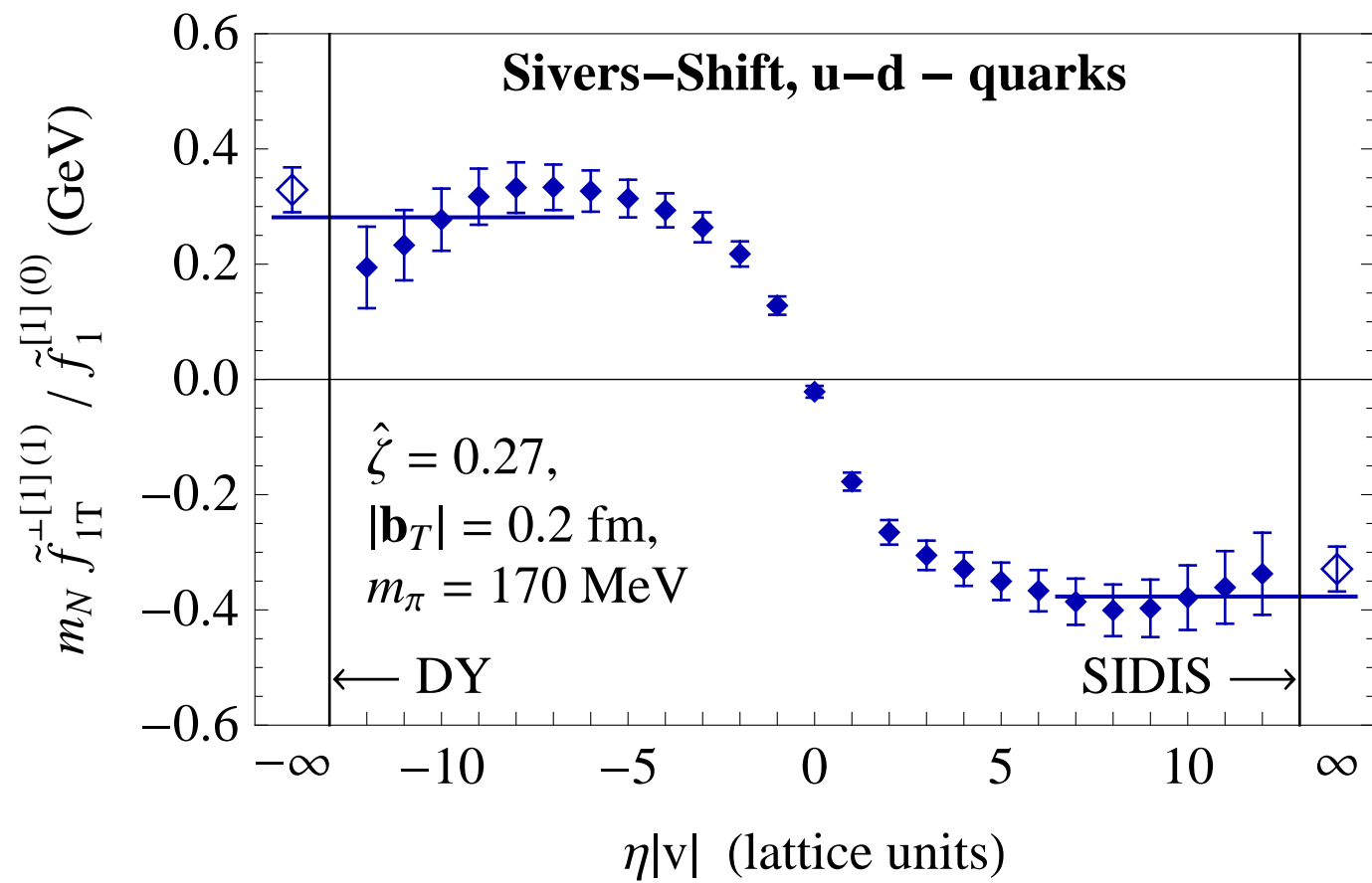
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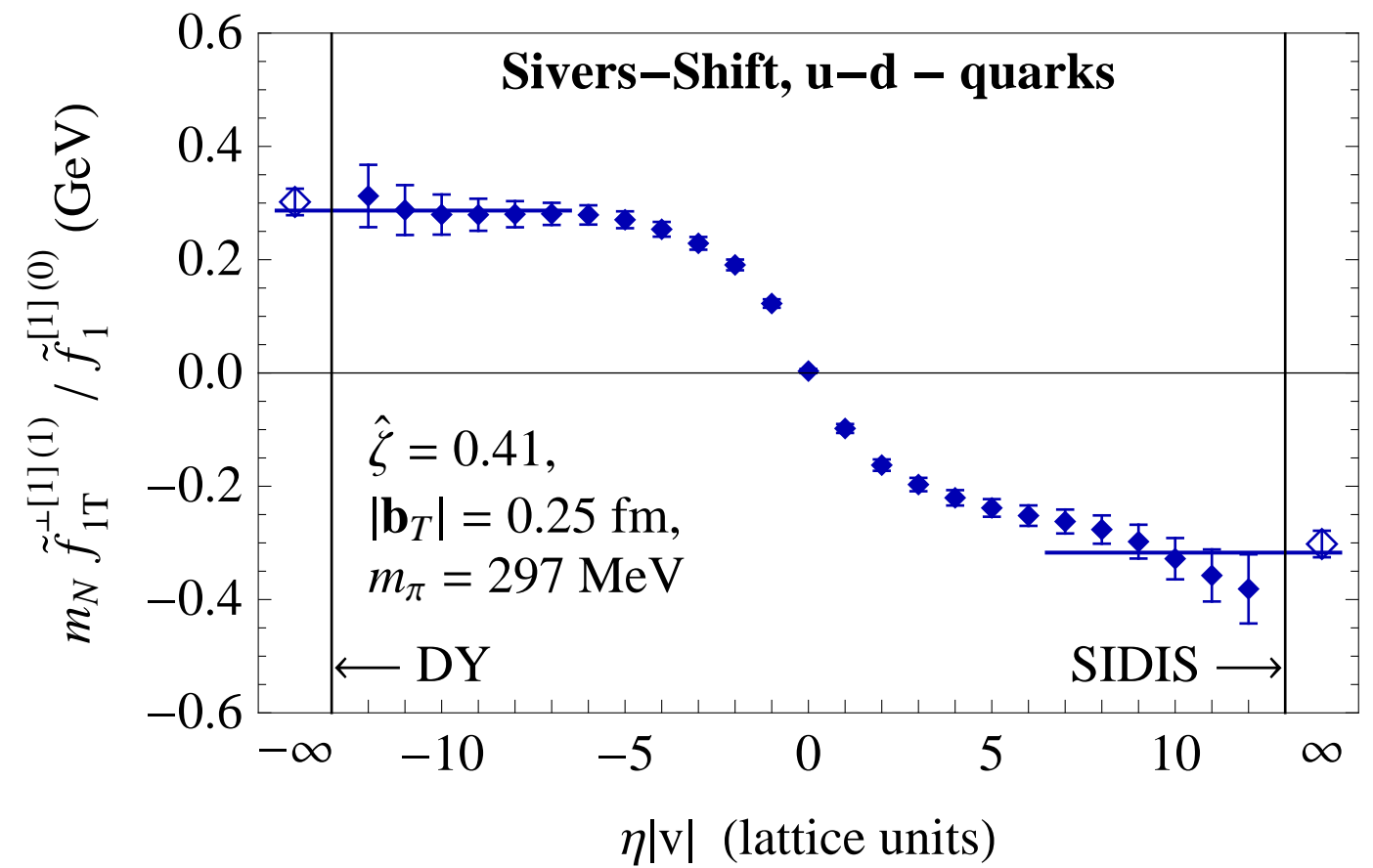
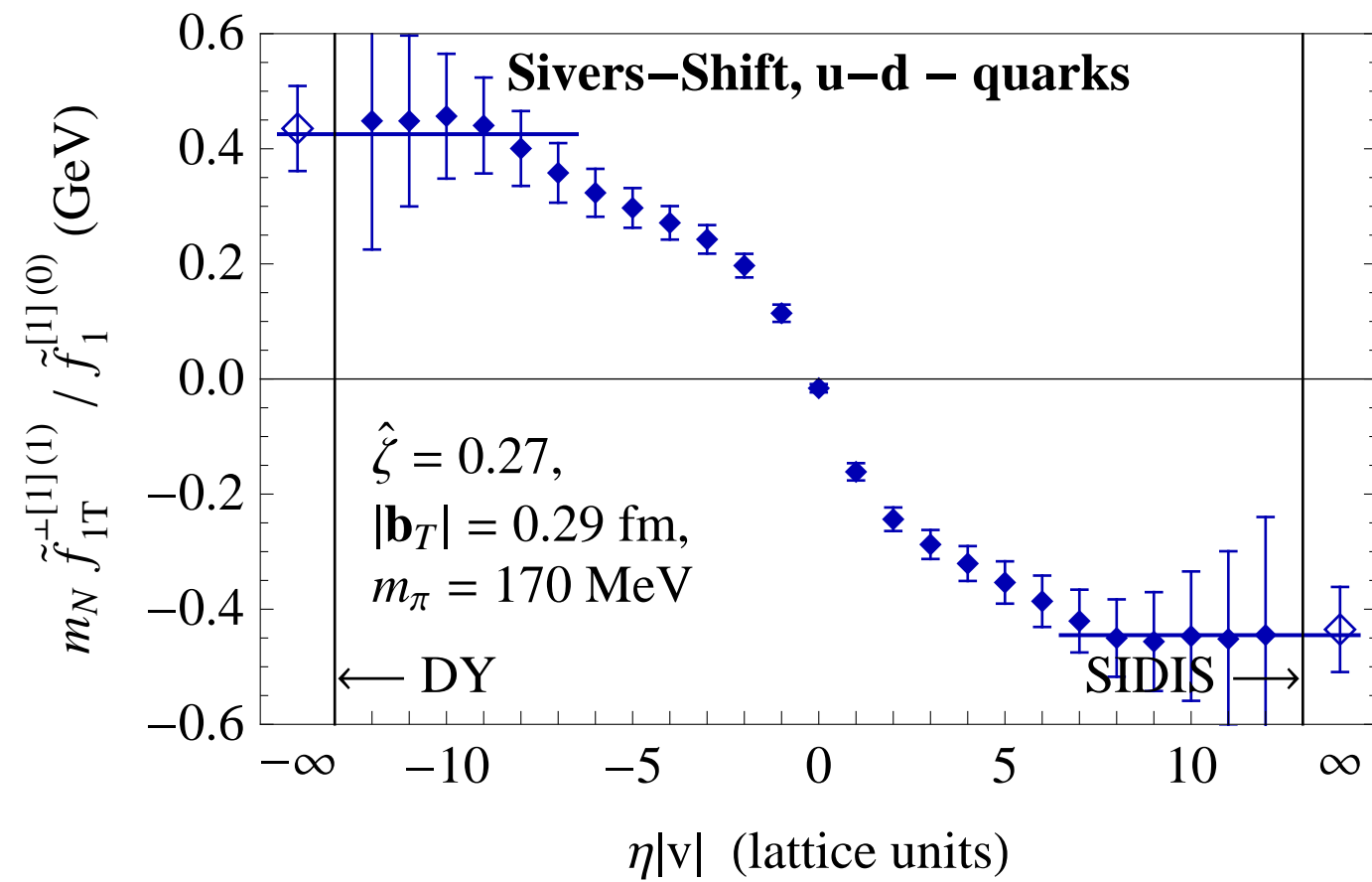
## Results: Siverson shift

Dependence on staple extent; sequence of panels at different  $|b_T|$



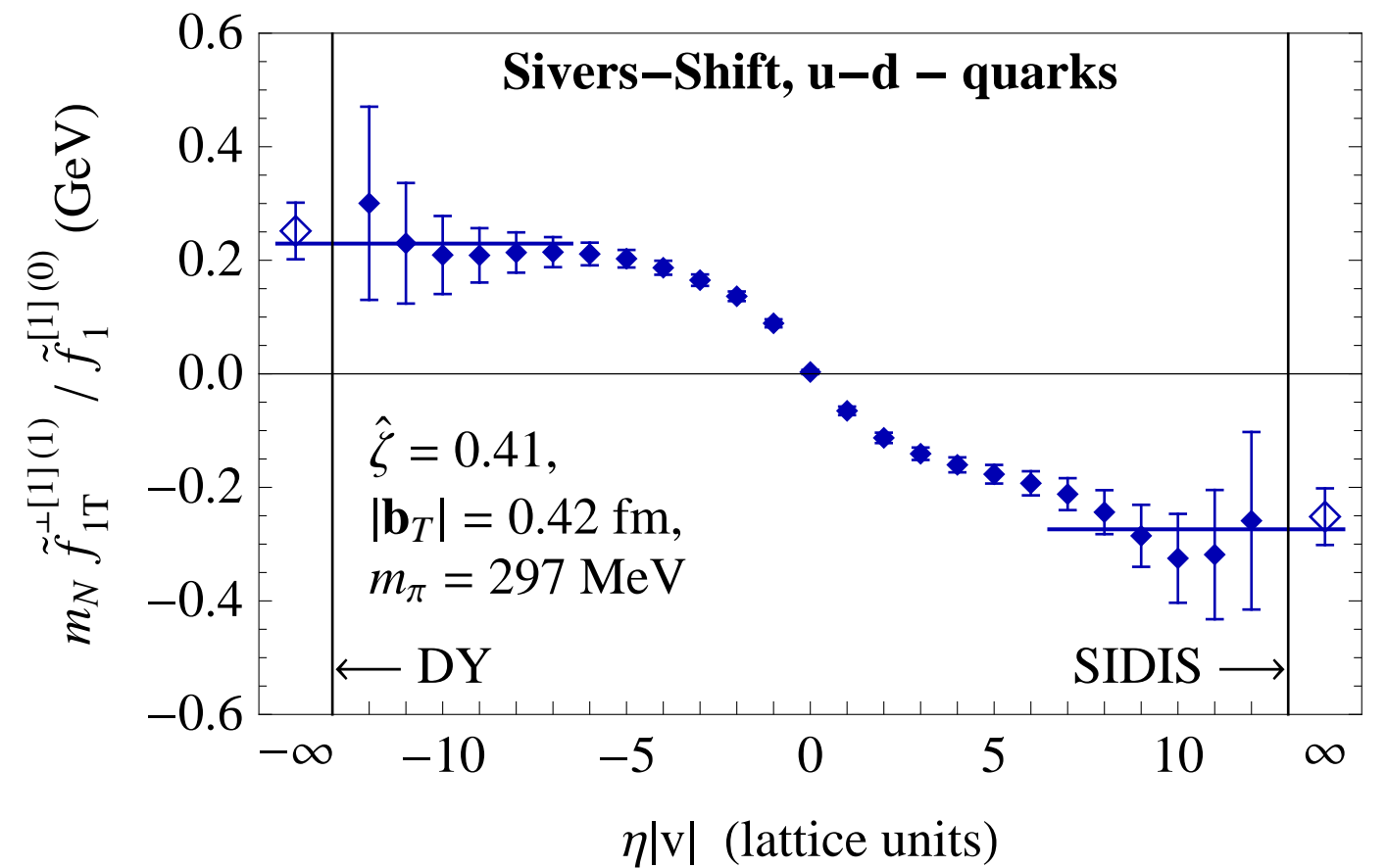
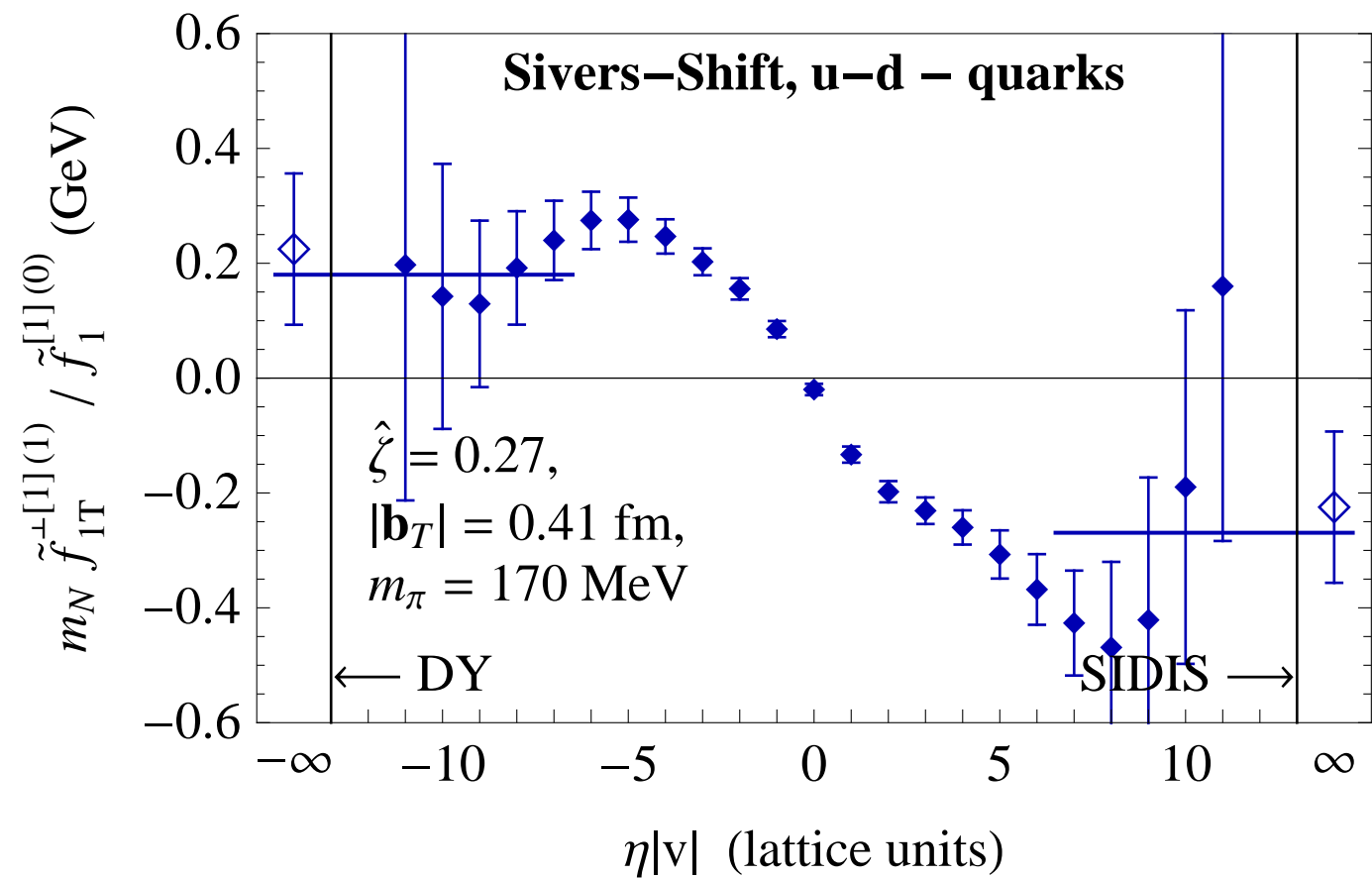
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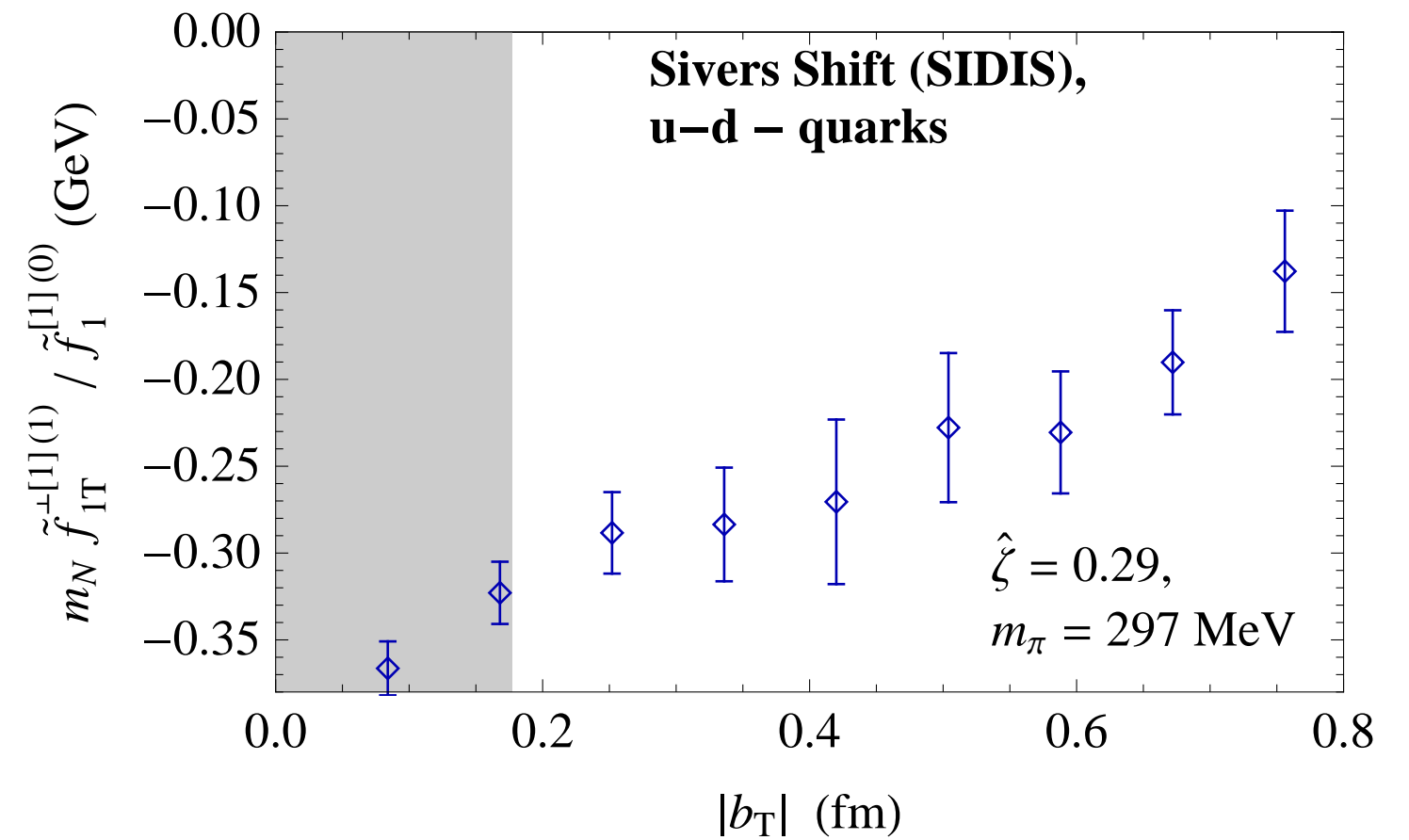
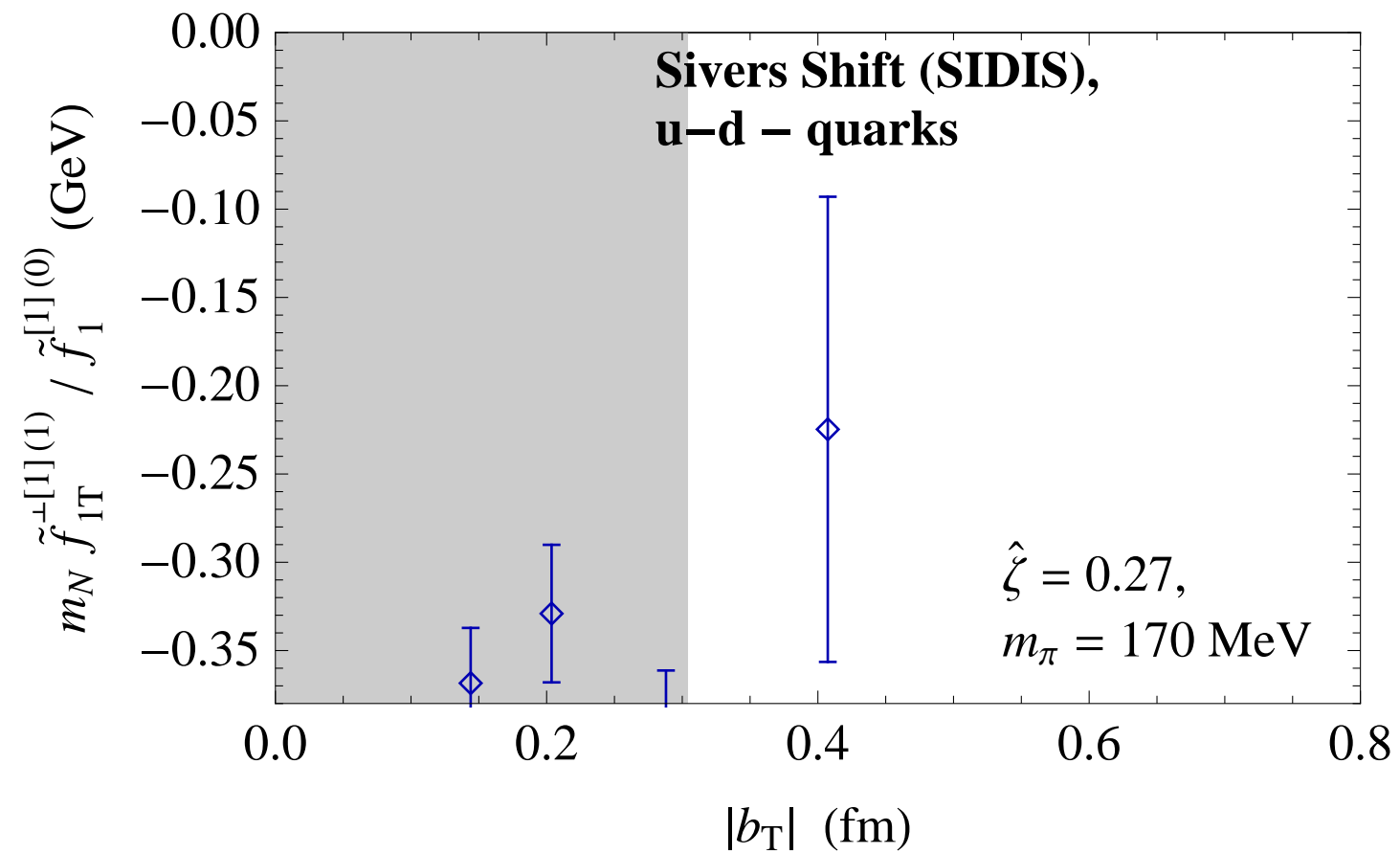
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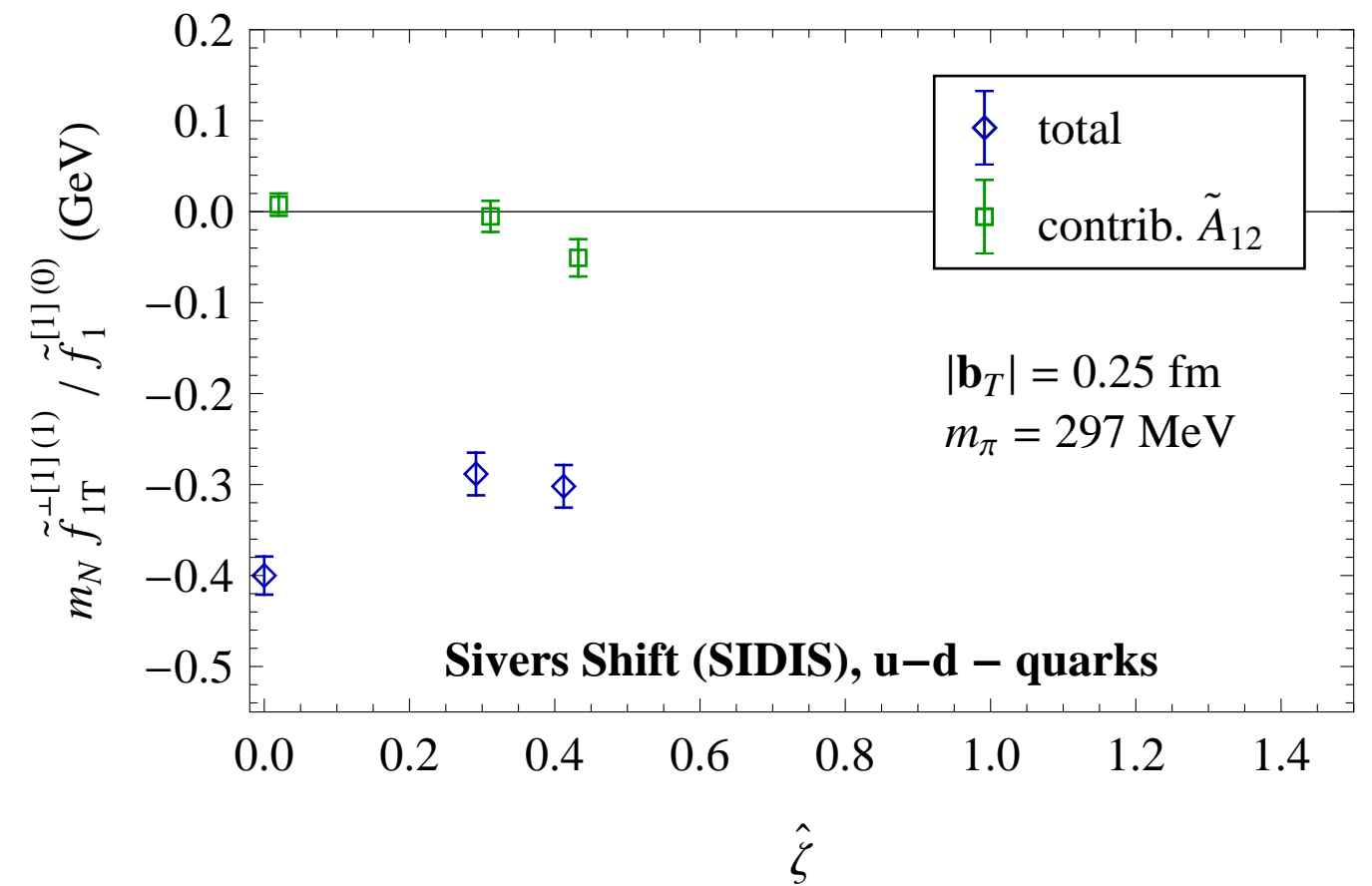
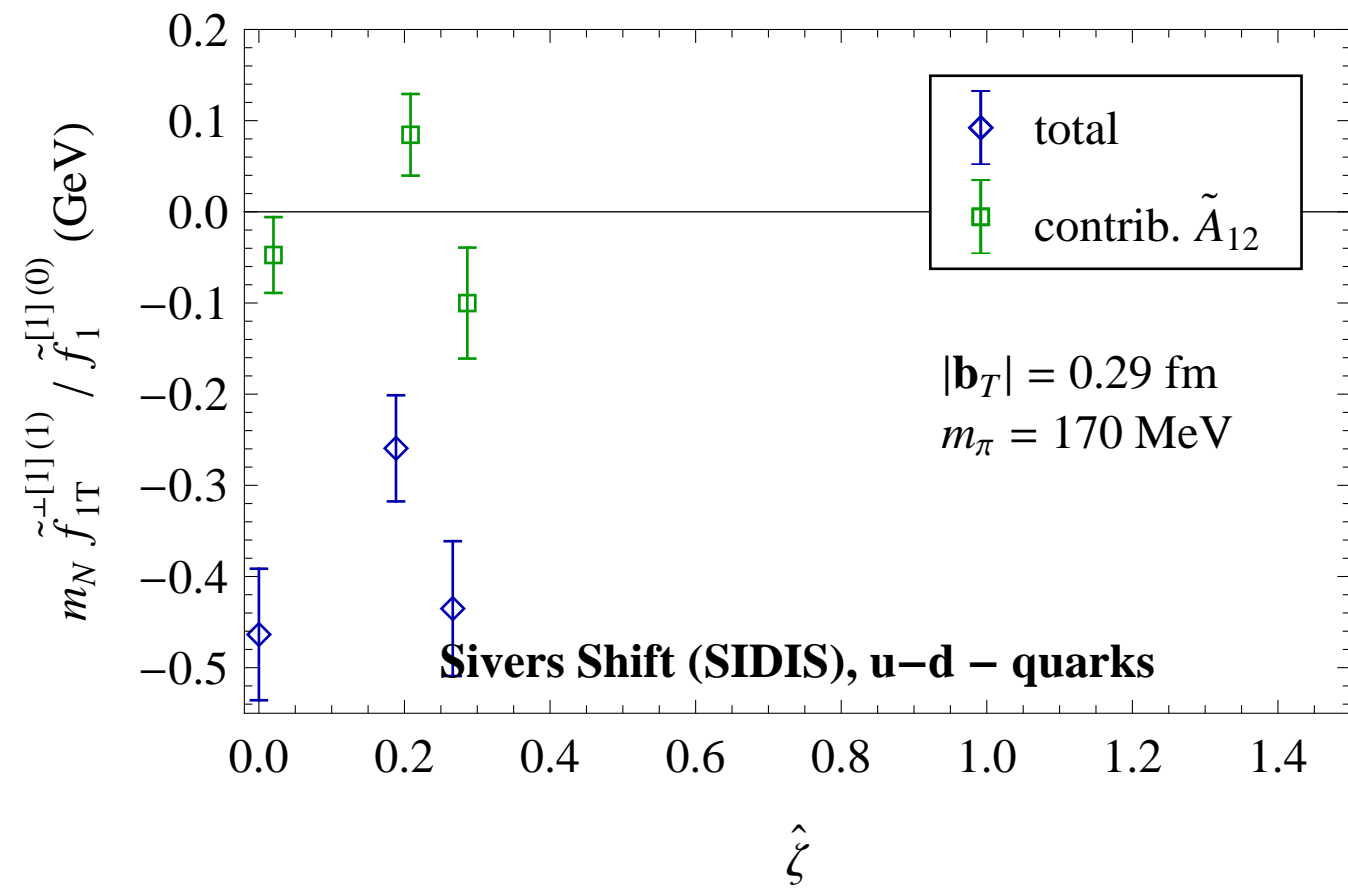
## Results: Sivers shift

Dependence of SIDIS limit on  $|b_T|$



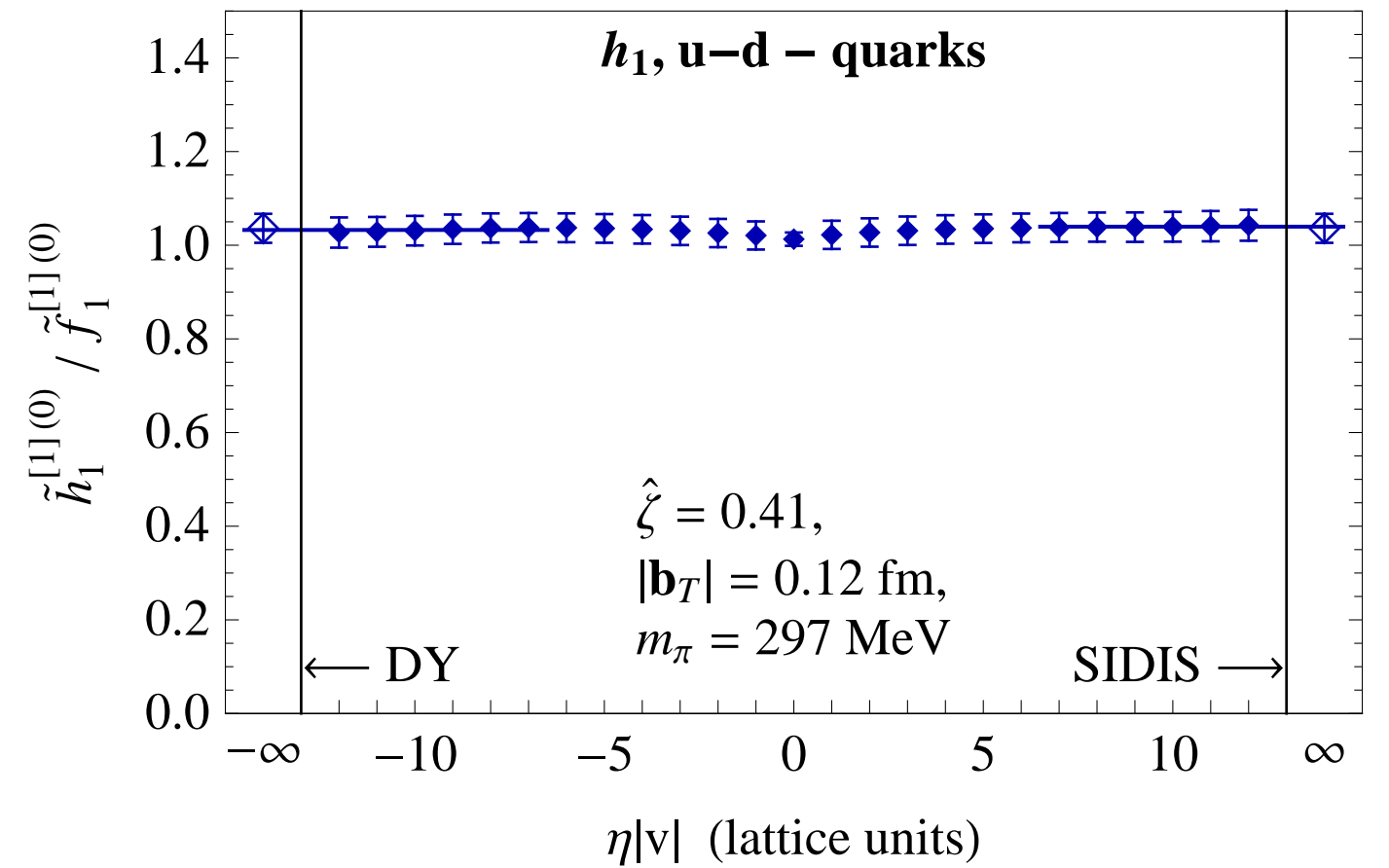
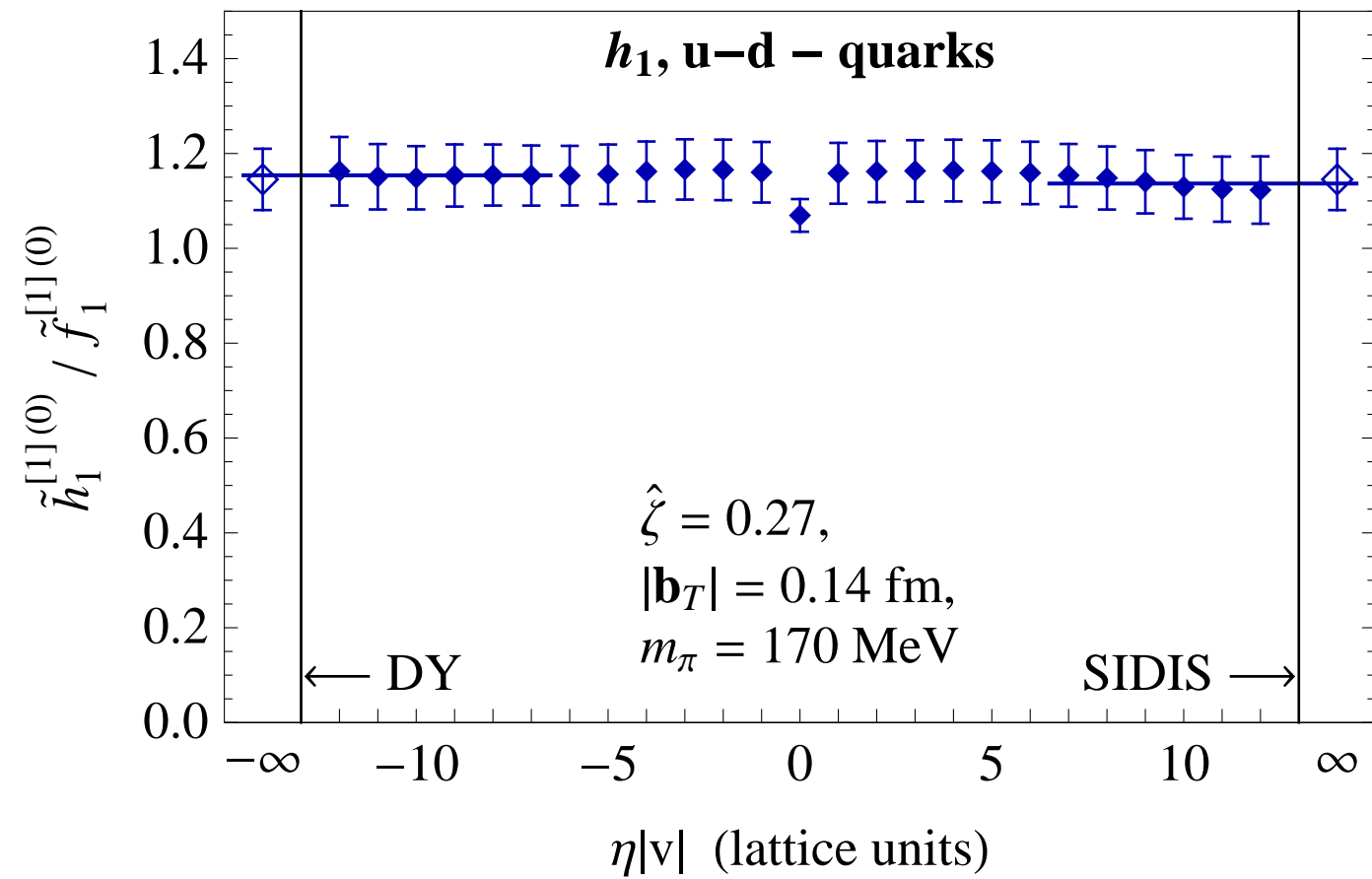
## Results: Sivers shift

Dependence of SIDIS limit on  $\hat{\zeta}$



## Results: Transversity

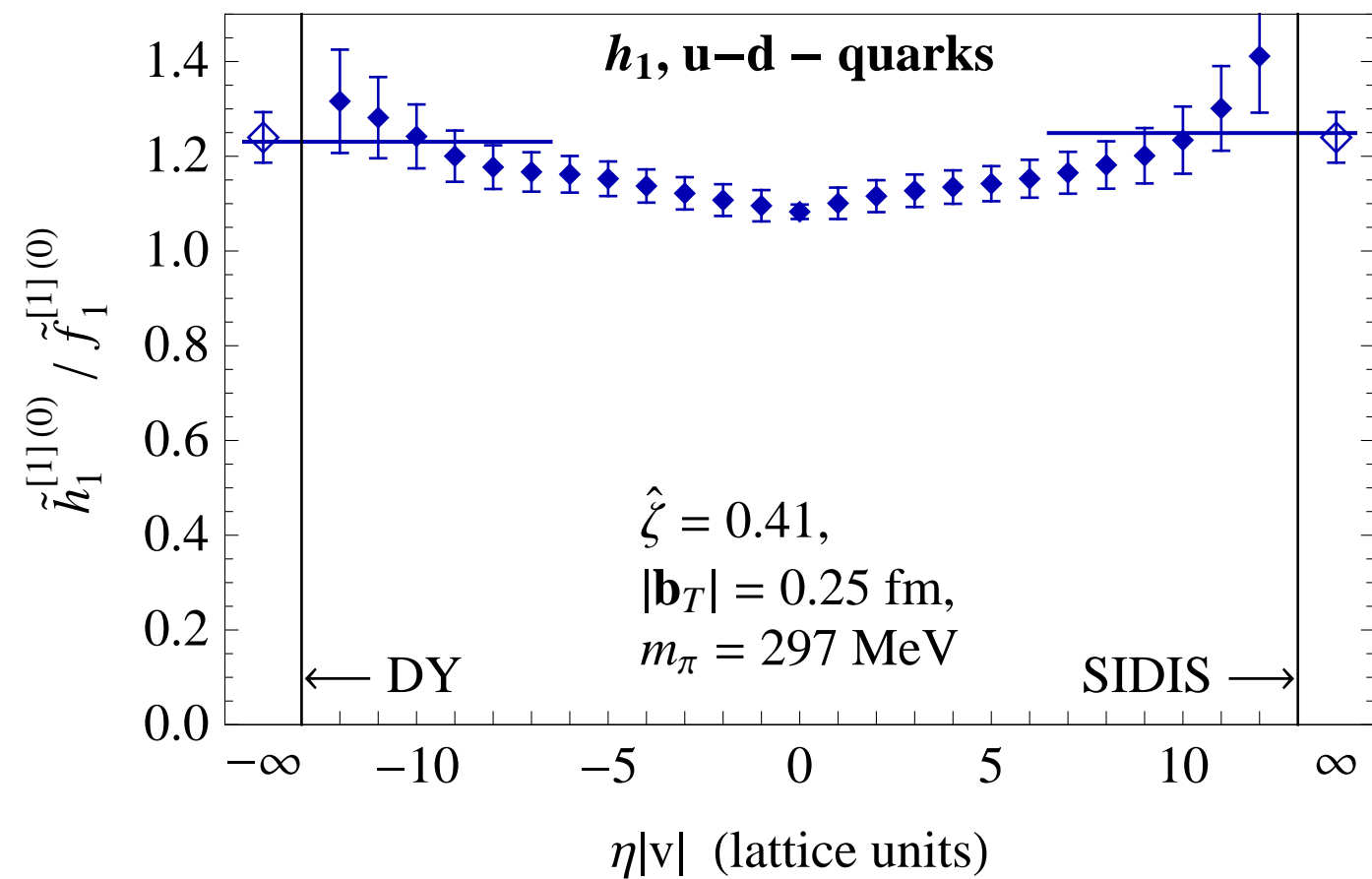
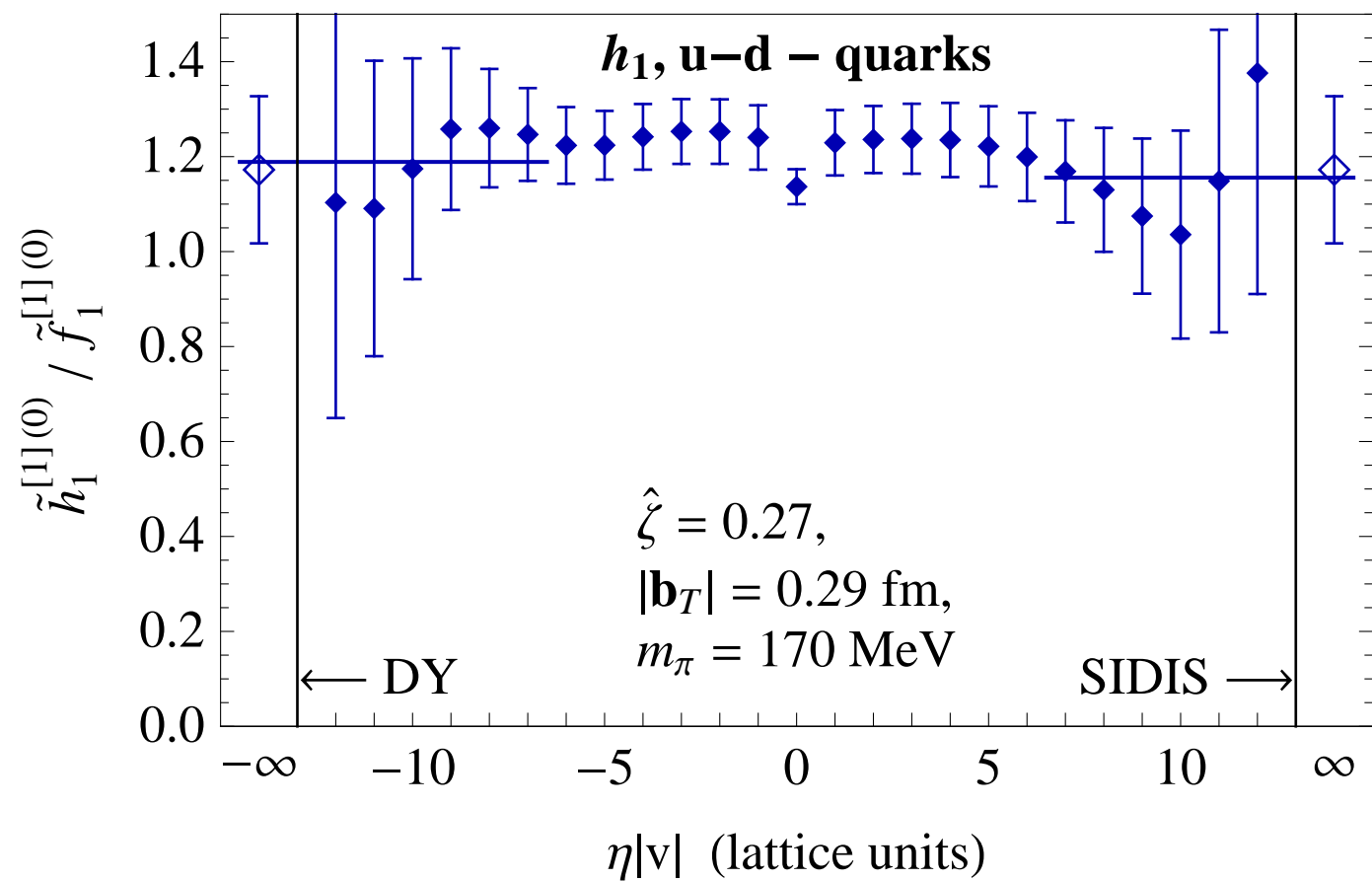
Dependence on staple extent; sequence of panels at different  $|b_T|$





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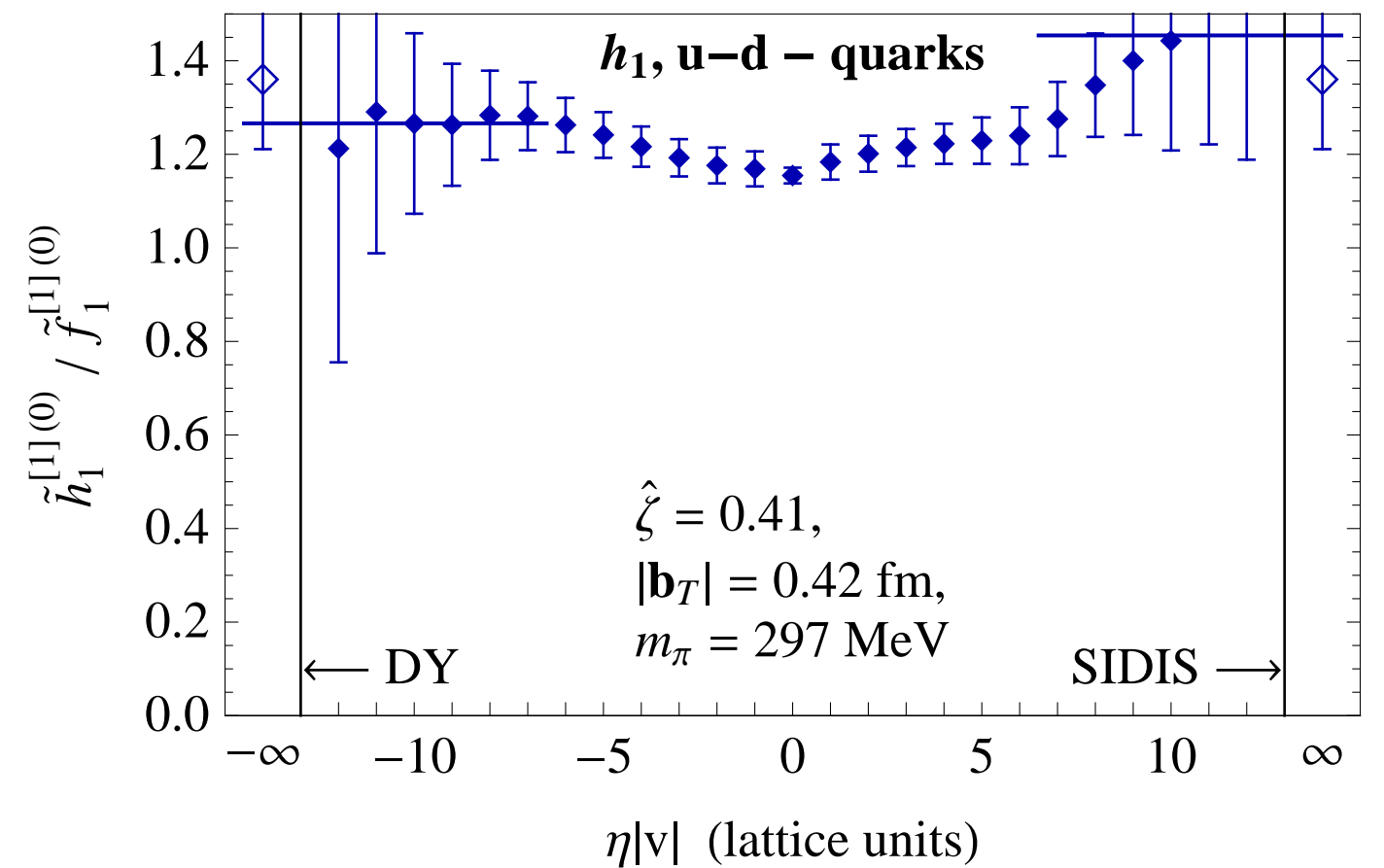
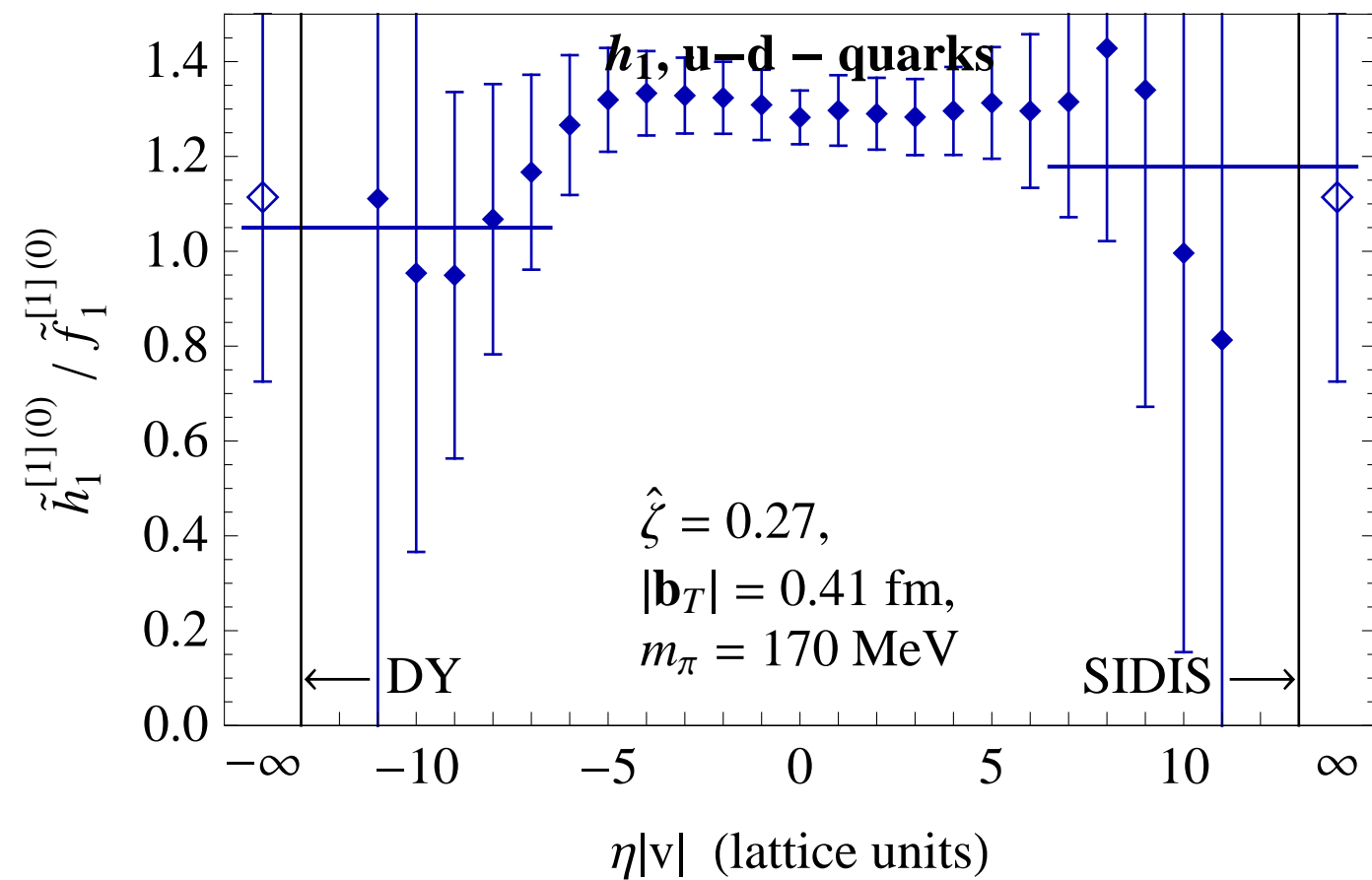
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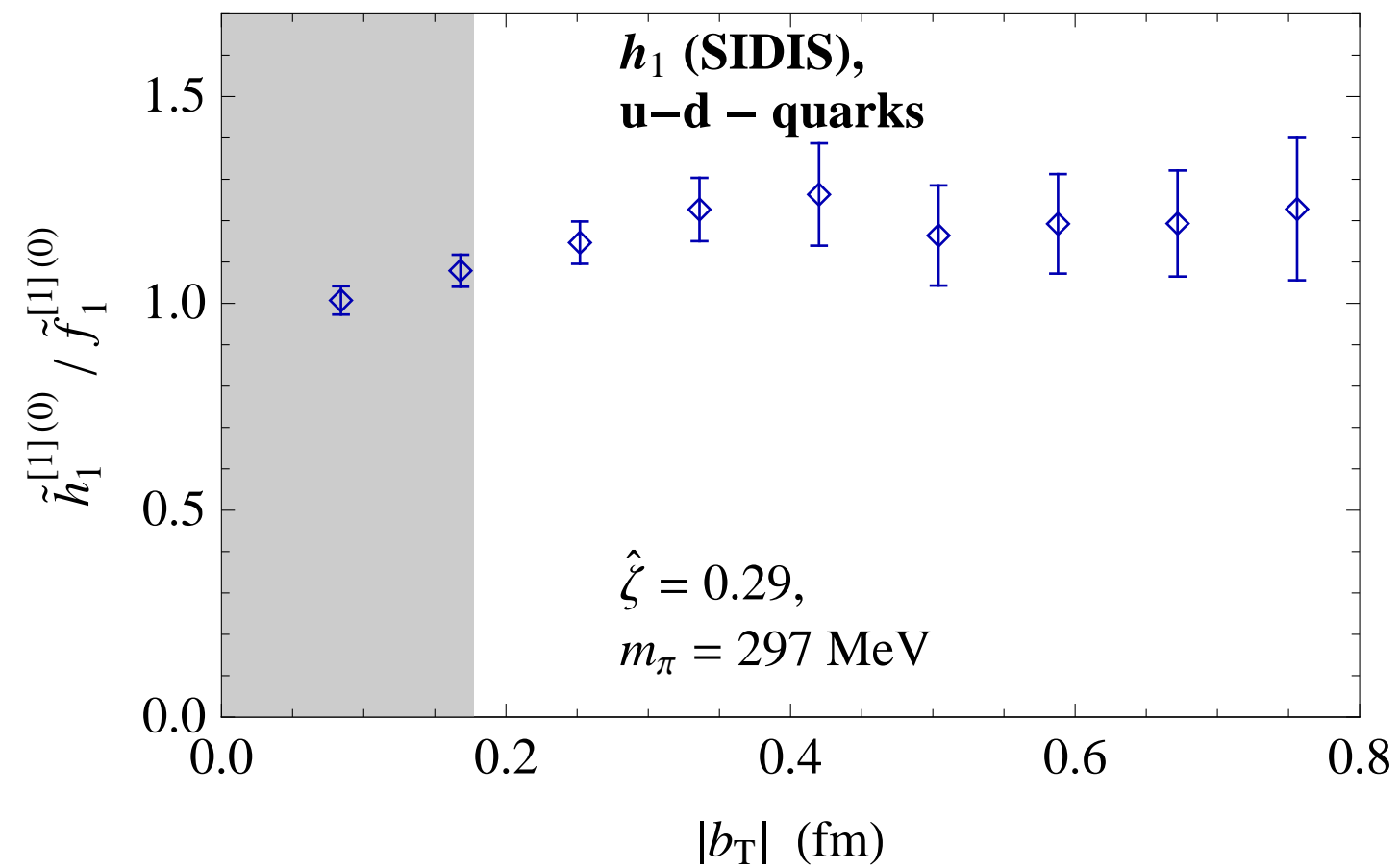
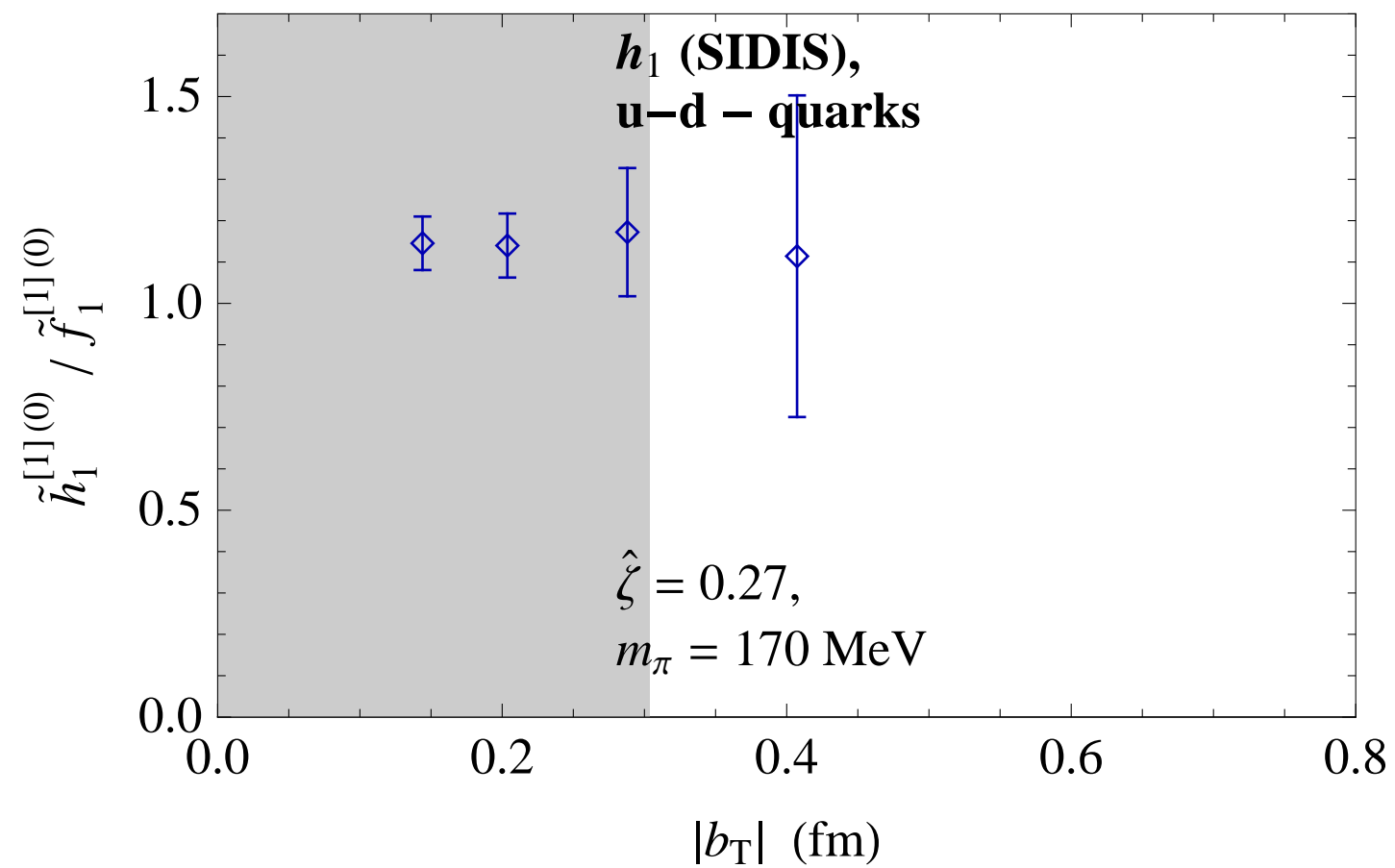
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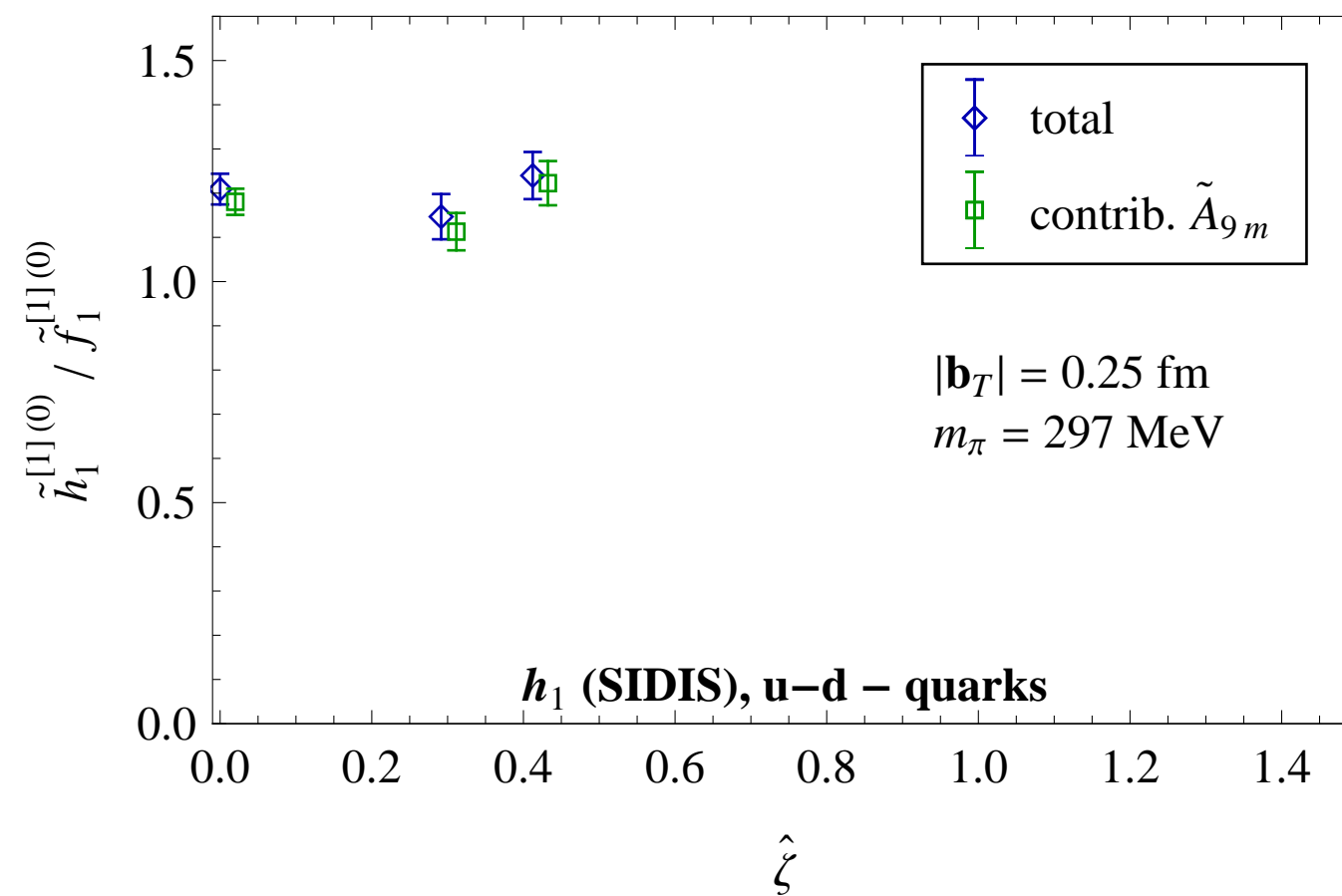
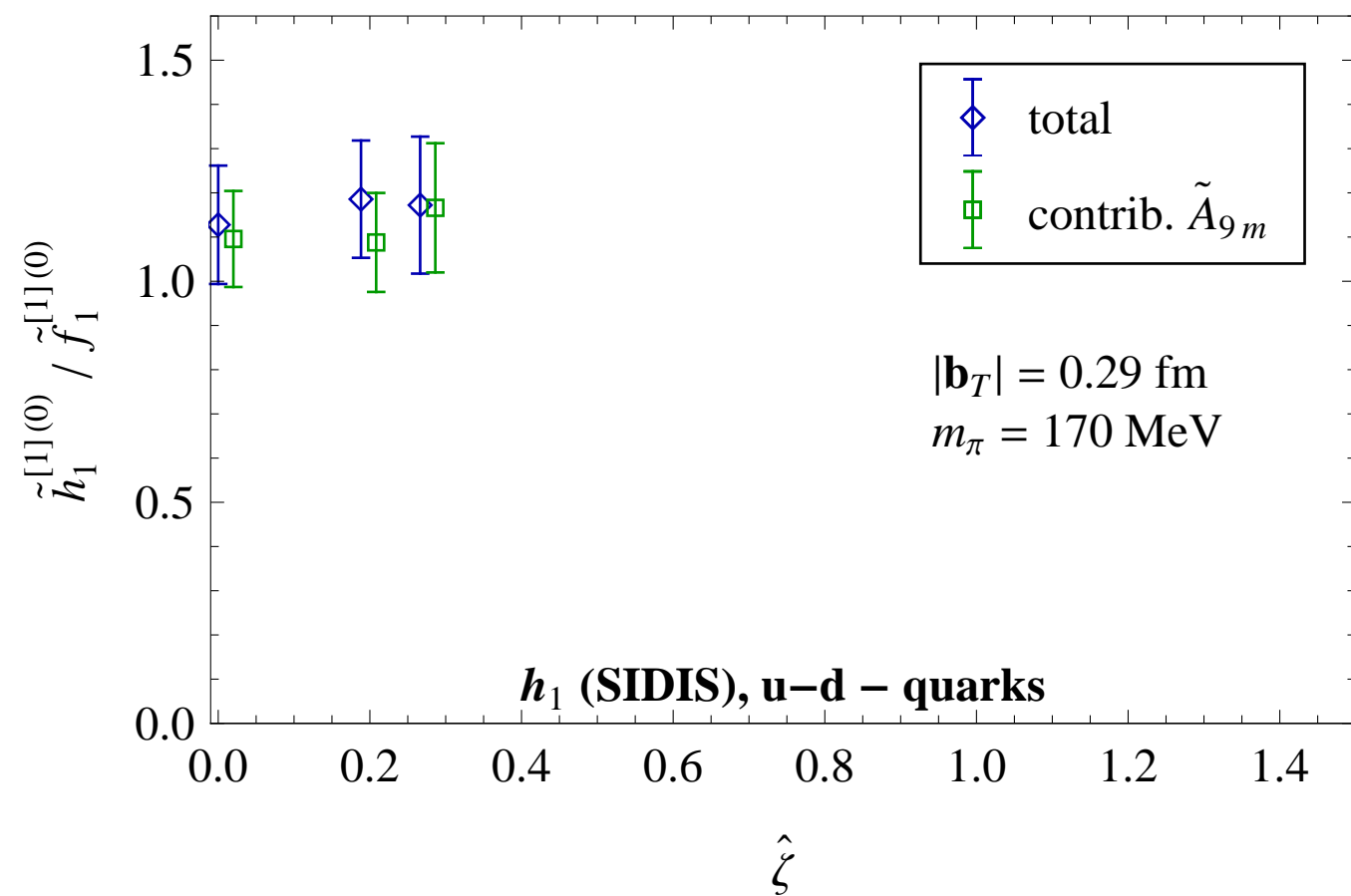
## Results: Transversity

Dependence of SIDIS/DY limit on  $|b_T|$



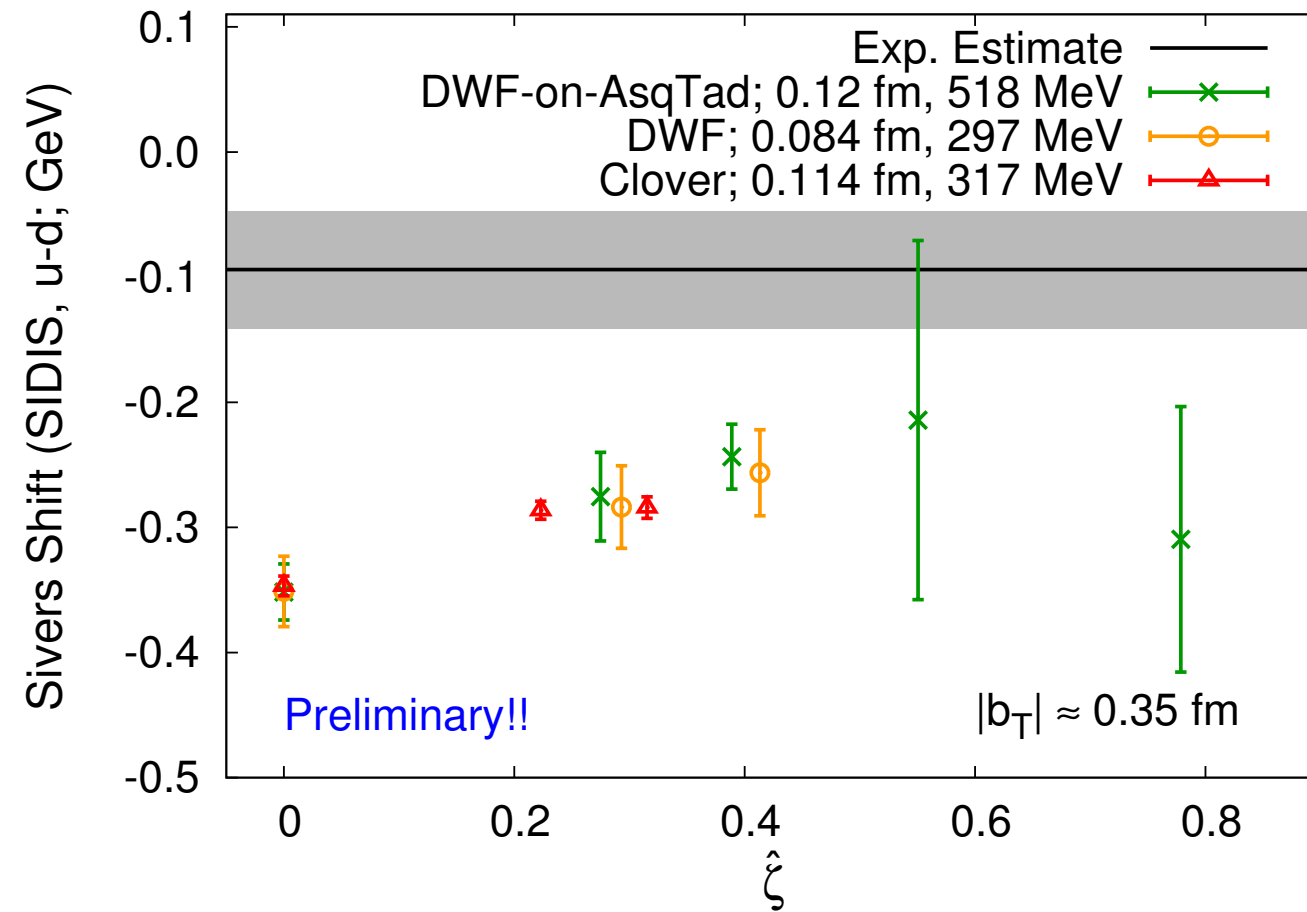
## Results: Transversity

Dependence of SIDIS/DY limit on  $\hat{\zeta}$



## Results: Sivers shift summary

Dependence of SIDIS limit on  $\hat{\zeta}$



Experimental value from global fit to HERMES, COMPASS and JLab data,  
M. Echevarria, A. Idilbi, Z.-B. Kang and I. Vitev, Phys. Rev. D 89 (2014) 074013

## Quark Orbital Angular Momentum

$$L_3^{\mathcal{U}} = \int dx \int d^2 k_T \int d^2 r_T (r_T \times k_T)_3 \mathcal{W}^{\mathcal{U}}(x, k_T, r_T) \quad \text{Wigner distribution}$$

$$= - \int dx \int d^2 k_T \frac{k_T^2}{m^2} F_{14}(x, k_T^2, k_T \cdot \Delta_T, \Delta_T^2) \Big|_{\Delta_T = 0} \quad \text{Generalized momentum-dependent parton distribution (GTMD)}$$

$$= 2 \epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle \Big|_{b^+ = b^- = 0, \Delta_T = 0, b_T \rightarrow 0}$$

Y. Hatta, M. Burkardt:

Staple-shaped  $\mathcal{U}[-b/2, b/2] \longrightarrow$  Jaffe-Manohar OAM

Straight  $\mathcal{U}[-b/2, b/2] \longrightarrow$  Ji OAM

Connection to GTMDs –

A. Metz, M. Schlegel, C. Lorcé,

B. Pasquini ...

## Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{2\epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle}{\langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle} \Big|_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}$$

$$P = p - \Delta_T/2, \quad P' = p + \Delta_T/2, \quad p, S \text{ in 3-direction, } p \rightarrow \infty$$

Y. Hatta, M. Burkardt:

Staple-shaped  $\mathcal{U}[-b/2, b/2] \longrightarrow$  Jaffe-Manohar OAM

Straight  $\mathcal{U}[-b/2, b/2] \longrightarrow$  Ji OAM

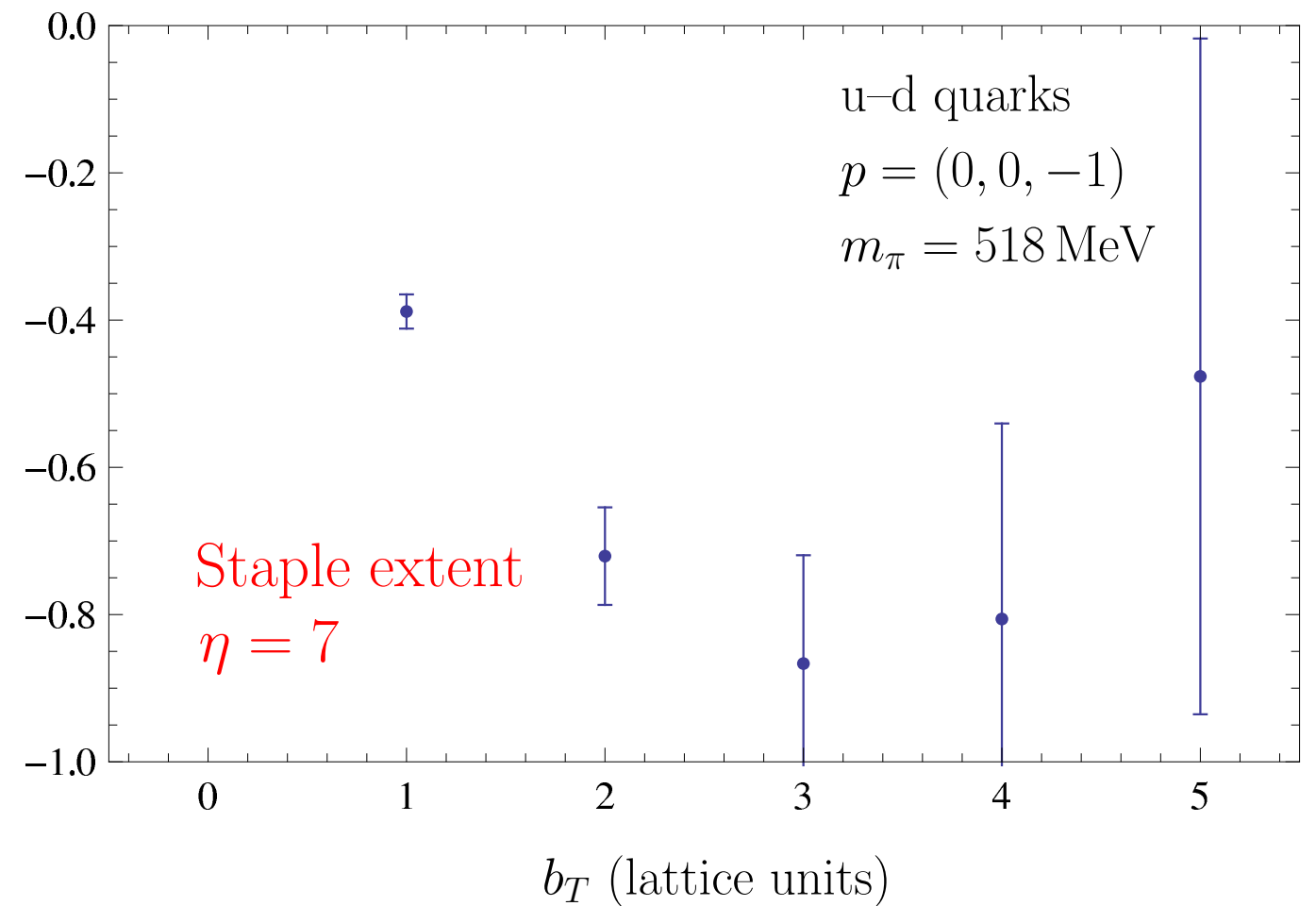
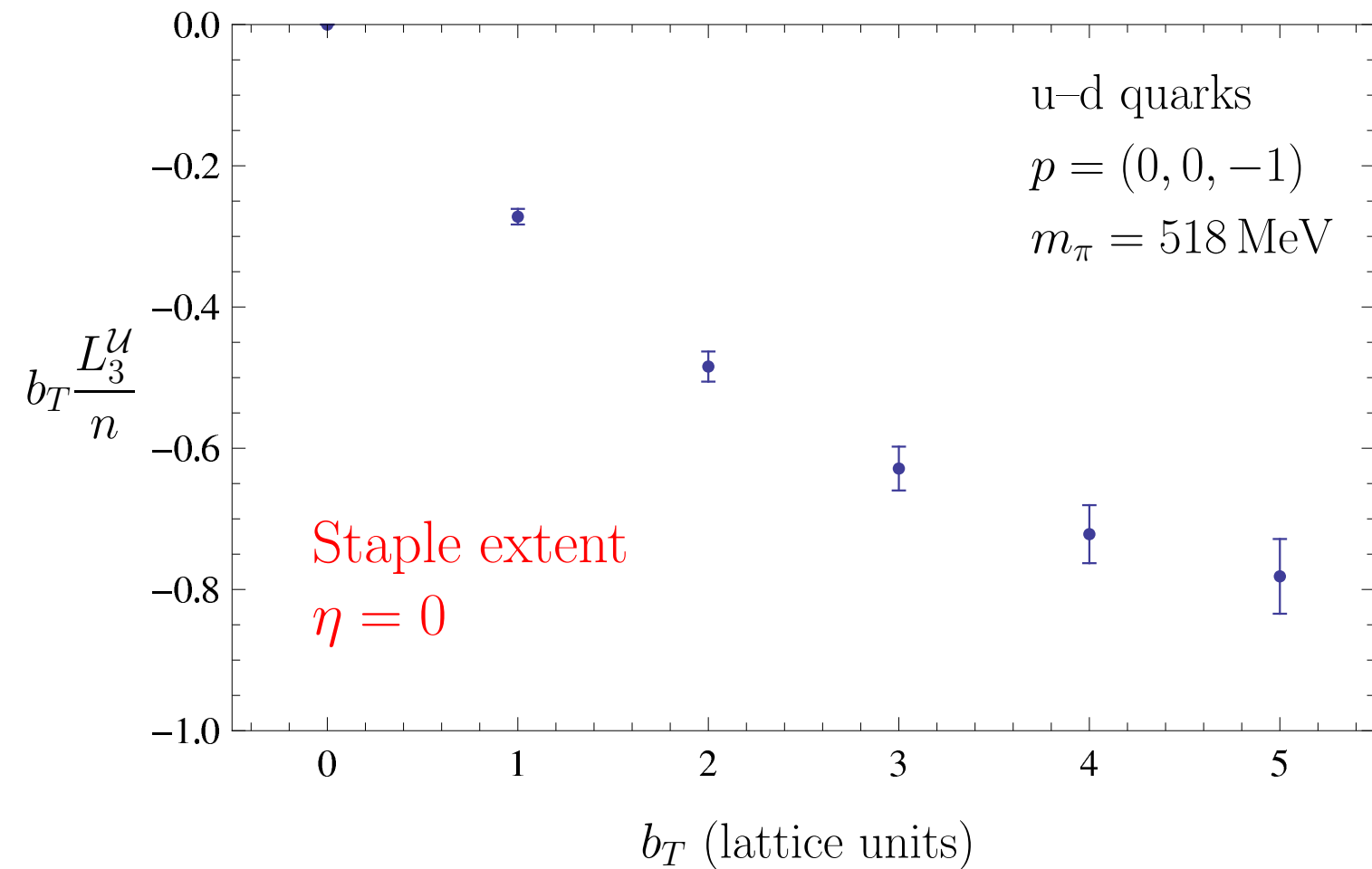
M. Polyakov et al., S. Liuti et al. (arXiv:1601.06117):

Connection to twist three

Planned: direct lattice calculation of Ji OAM at twist three

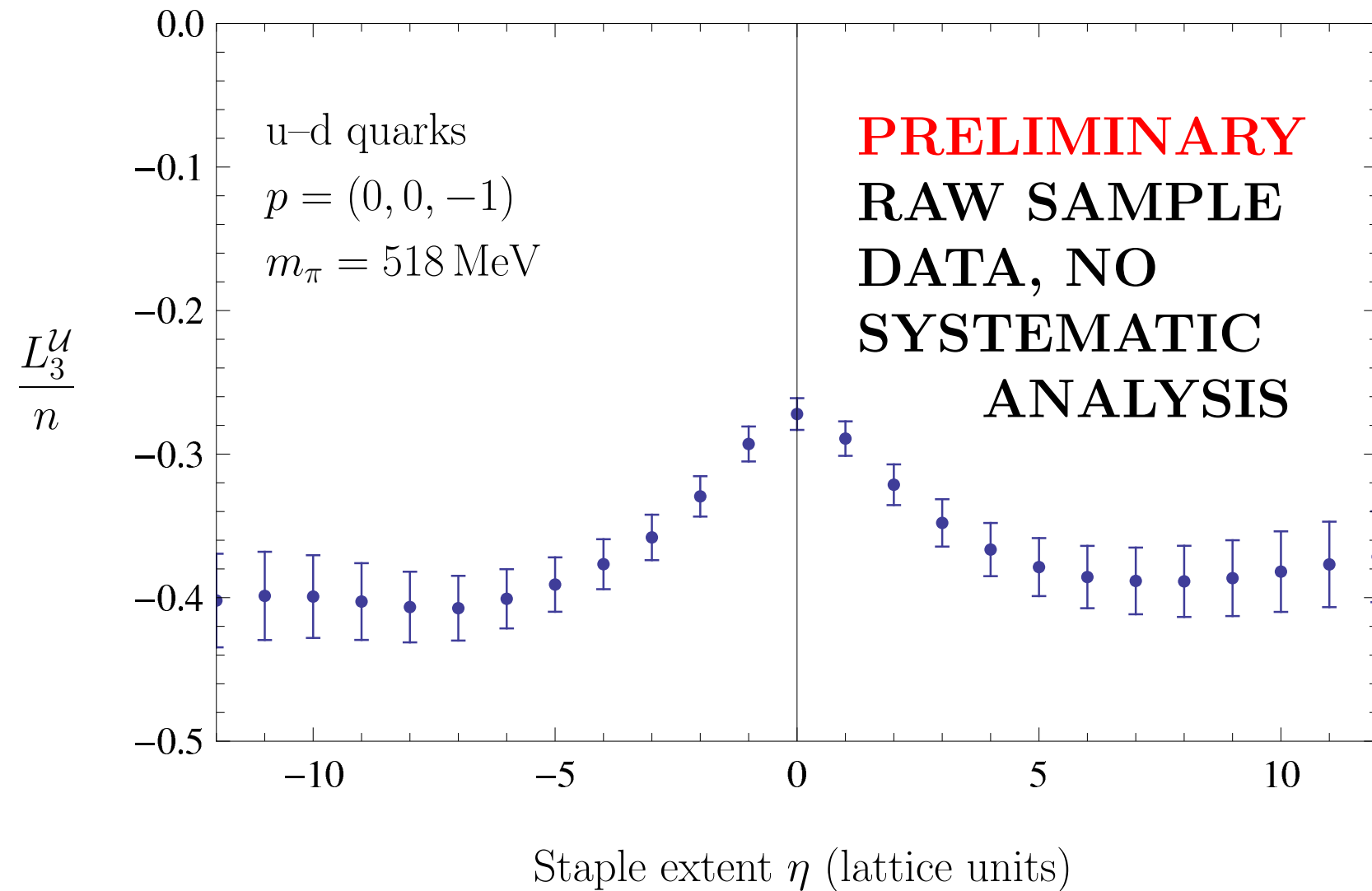
## Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{2\epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}{\langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}$$



## Quark orbital angular momentum in units of the number of valence quarks

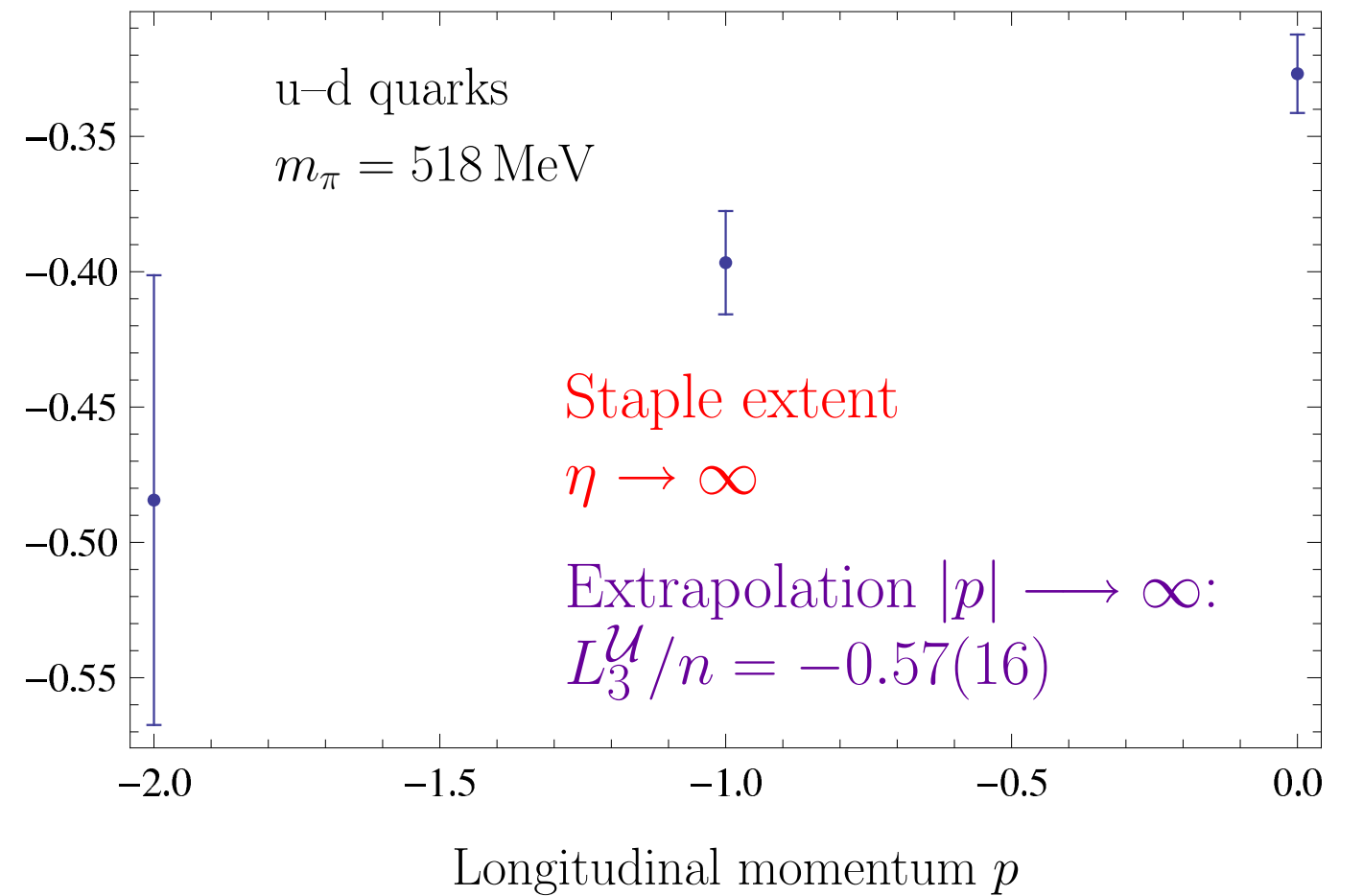
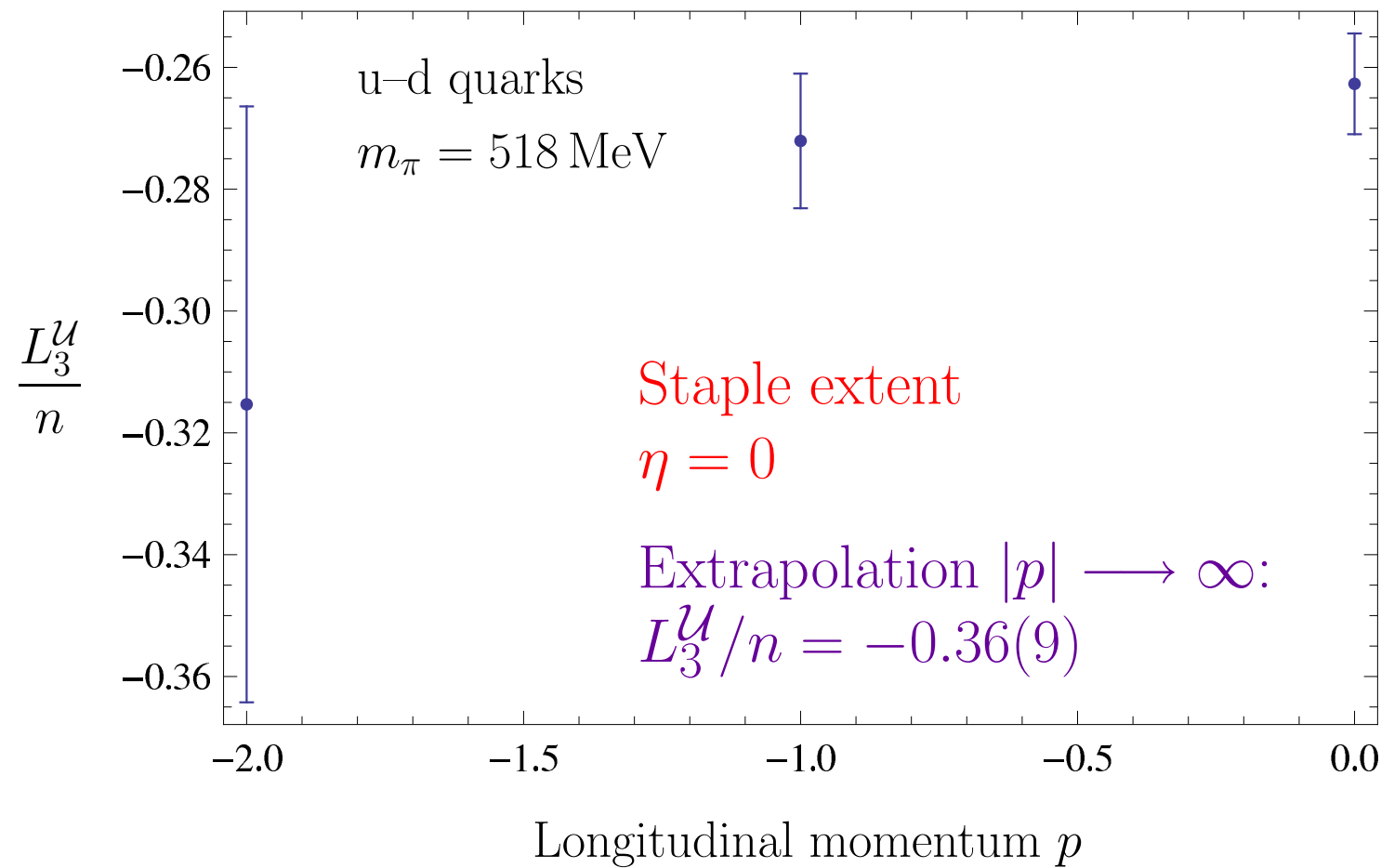
$$\frac{L_3^{\mathcal{U}}}{n} = \frac{2\epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}{\langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}$$





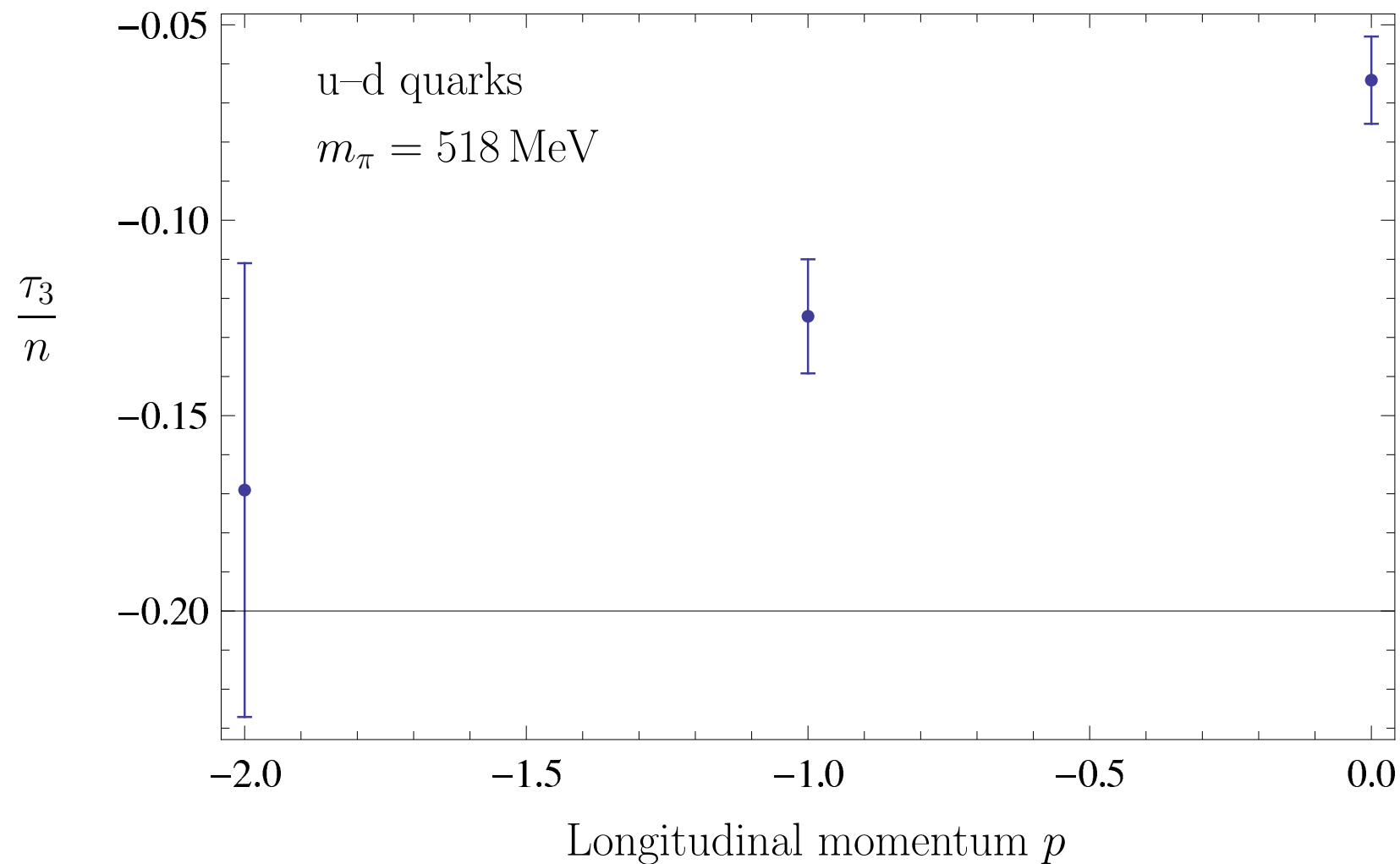
## Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{2\epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}{\langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}$$



## Quark orbital angular momentum in units of the number of valence quarks

$$\frac{L_3^{\mathcal{U}}}{n} = \frac{2\epsilon_{ij} \frac{\partial}{\partial b_{T,i}} \frac{\partial}{\partial \Delta_{T,j}} \langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}{\langle P, S | \bar{\psi}(-b/2) \gamma^+ \mathcal{U}[-b/2, b/2] \psi(b/2) | P', S \rangle |_{b^+=b^-=0, \Delta_T=0, b_T \rightarrow 0}}$$



Burkardt's torque,  
 $\tau_3/n = L_3^{\mathcal{U}}/n|_{\eta \rightarrow \infty} - L_3^{\mathcal{U}}/n|_{\eta=0}$

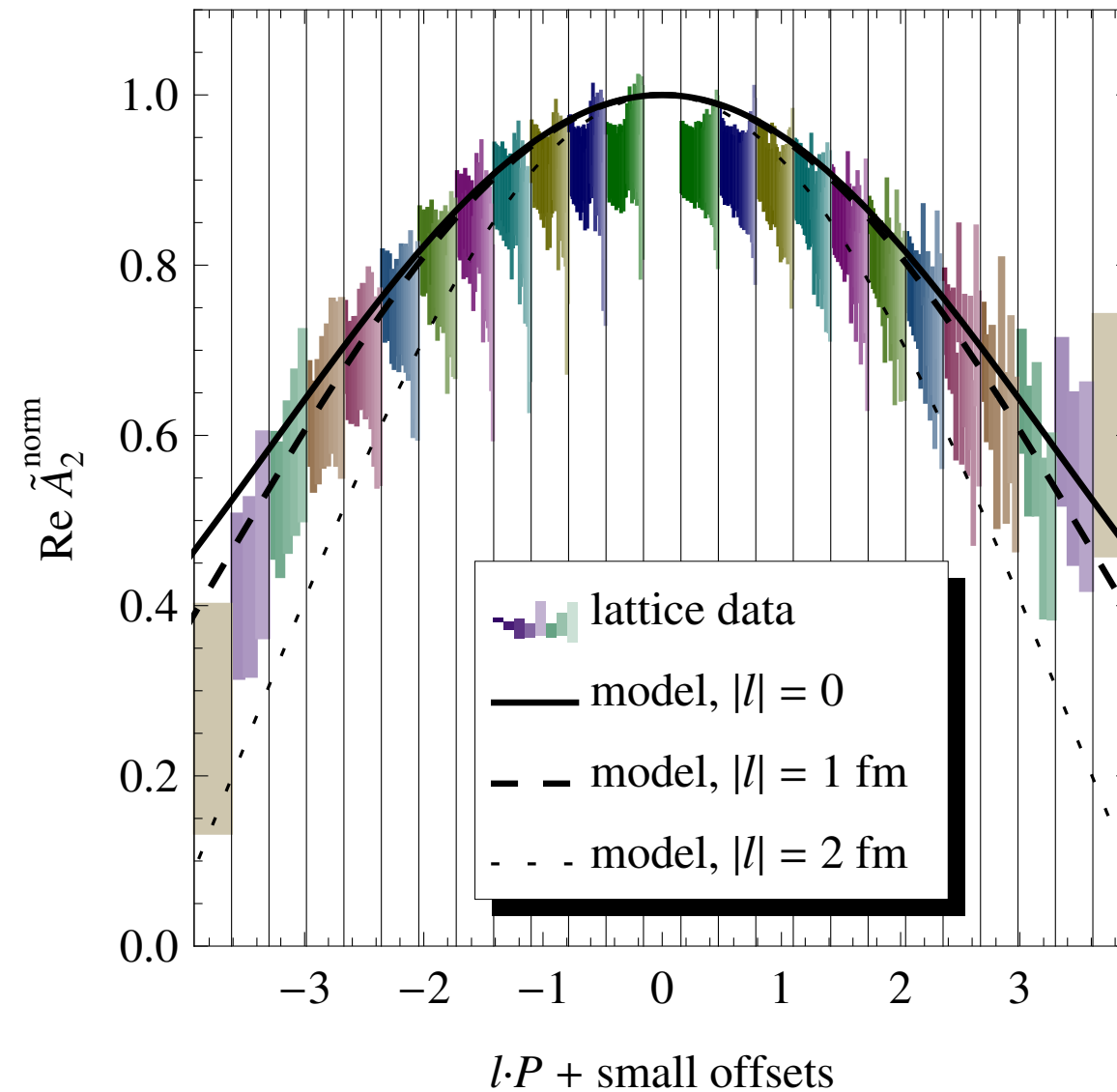
Extrapolation  $|p| \rightarrow \infty$ :  
 $\tau_3/n = -0.21(11)$

## Conclusions and Outlook

- Continued exploration of TMDs using bilocal quark operators with staple-shaped gauge link structures; exploration of challenges posed by  $\hat{\zeta} \rightarrow \infty$  limit, discretization effects, physical limit.
- Considered appropriate ratios of Fourier-transformed TMDs (“shifts”) in which soft factors, multiplicative renormalization constants cancel.
- These observables show no statistically significant variation under the considered changes of action, lattice spacing and pion mass, except at very short distances.
- Generalization to mixed transverse momentum / transverse position observables (Wigner functions) gives direct access to quark orbital angular momentum; analysis in progress. **Jaffe-Manohar orbital angular momentum evaluated for the first time.**
- Current production:  $x$ -dependence of the Sivers shift.

## Accessing Bjorken-x dependence

(Fourier transform of)  
unpolarized distribution,  
up quarks, normalized to  
unity at  $l \cdot P = 0$



From: B. Musch, P. Hägler,  
J. Negele and A. Schäfer,  
Phys. Rev. **D 83** (2011)  
094507.

Lattice:  $m_\pi = 625 \text{ MeV}$   
Model curves: Spectator  
diquark model

$l \cdot P$ : Variable Fourier conjugate to Bjorken  $x$

# Relation to Ji Large Momentum Effective Theory (LaMET)

Phenomenology

Lattice QCD

