Hadron Structure in *Large Momentum Effective field Theory Approach*

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- Essential task in QCD: revealing hadron's properties in terms of quark and gluon (non-perturbative)

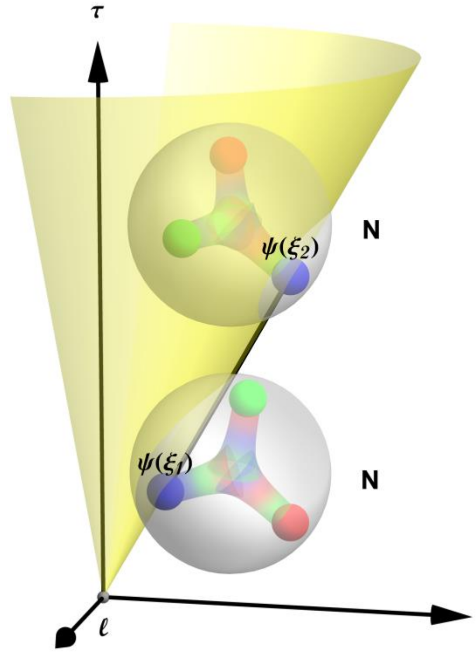
- Experiment: High-Energy scattering (DIS, D-Y, DVCS…) measure distribution functions

- Theory: QCD model, AdS/CFT, **lattice simulation (first principal calculation)**, ***La*rge *M*omentum *E*ffective field *T*heory (*LaMET*)**

X. Ji, PRL. **110** (2013) 262002,
Sci.China Phys.Mech.Astron. **57** (2014) 7, 1407-1412

High-Energy Scattering & Lattice Calculation Approach

- High-Energy scattering:



probe correlation at equal light-cone time (real-time dependence)

- Lattice can not directly simulate $\xi^{\pm} = \frac{(-i\tau \pm z)}{\sqrt{2}}$ with real τ, calculate Mellin moments instead. High moments needs fine lattice while computational cost $\sim a^{-7}$ [CP-PACS, JLQCD]

LaMET Approach

- Construct a quasi quantity $\tilde{O}$ that can be directly calculated on lattice (Euclidean)

- $\langle P | \tilde{O} | P \rangle$ depends on the momentum P of the external state (large but finite)

- Extract light-cone(IMF) quantity $\langle P_\infty | O | P_\infty \rangle$ by matching condition (fractorization formula)

$$\langle P | \tilde{O} | P \rangle (P) = Z(\mu, P) \otimes \langle P_\infty | O | P_\infty \rangle (\mu) + \mathcal{O}\left(P^{-n}\right)$$

UV controlled, perturbatively calculable

mass correction, higher-twist correction

Scattering Experiments vs. LaMET Approach

	High-Energy Scattering	**LaMET**
"observables"	Cross section	Quasi-quantities
Scale	Large momentum transfer (Q).	Hadron momentum (P).
Factorization	$\sigma = \sigma_H\left(x, Q^2\right) \otimes f\left(x, Q^2\right) + \mathcal{O}\left((Q)^{-n}\right)$	$\tilde{f}\left(P^z\right) = Z\left(\frac{P^z}{\mu}\right) \otimes f\left(\mu\right) + \mathcal{O}\left((P^z)^{-n}\right)$

E.g. PDF in LaMET Approach

- Definition

$$q(x) = \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\langle PS \left| \bar{\psi}(\tfrac{\xi^-}{2})\gamma^+ \mathcal{L}[\tfrac{\xi^-}{2}; -\tfrac{\xi^-}{2}]\psi(-\tfrac{\xi^-}{2}) \right| PS \right\rangle$$

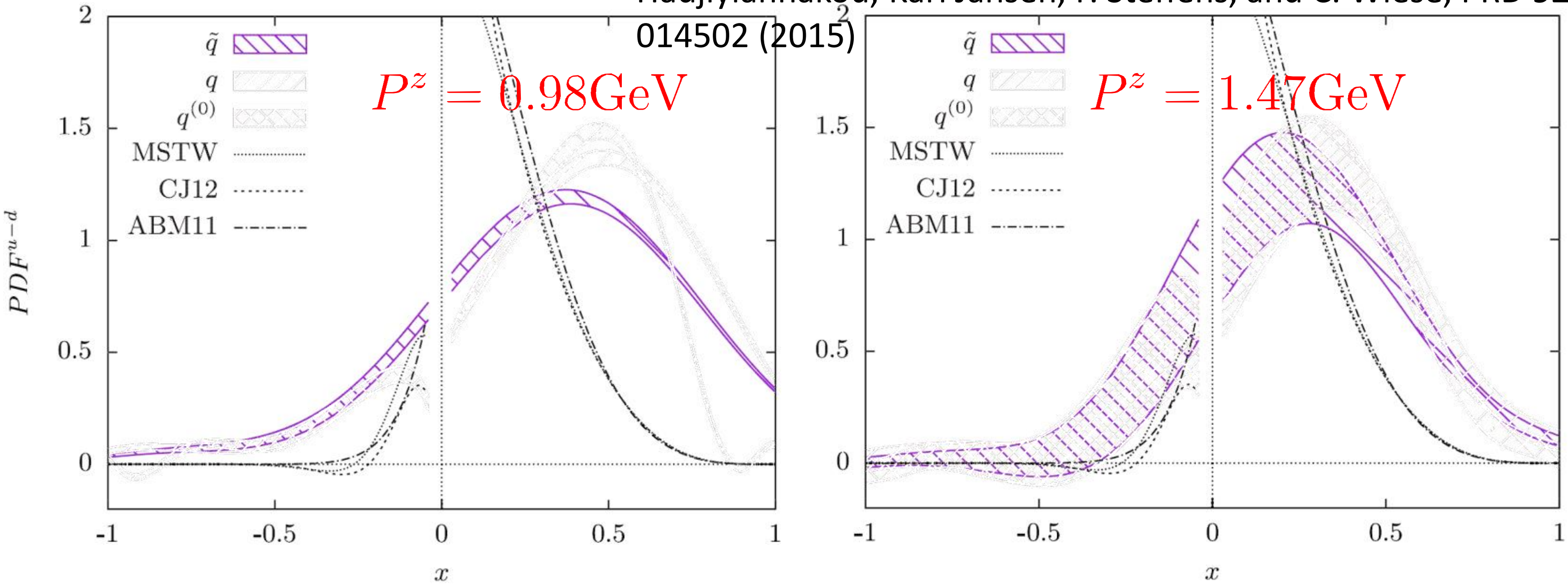
$$\tilde{q}(x) = \int \frac{dz}{2\pi} e^{ixp^z z} \langle PS \left| \bar{\psi}(\tfrac{z}{2})\gamma^z \mathcal{L}[\tfrac{z}{2}; -\tfrac{z}{2}]\psi(-\tfrac{z}{2}) \right| PS \rangle$$

pure spatial correlation

directly calculated on lattice, no prob. int..

- Lattice calculation

C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)



Matching Condition

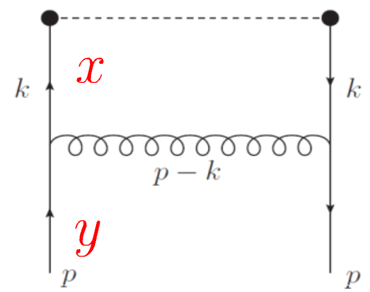
- Lattice “cross section” factorization

$$\tilde{q}(x) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}\right) q(y)$$

- Perturbative expansion

$$\tilde{q}(x) = \tilde{q}^{(1)}(x) + \delta\tilde{Z}_F^{(1)}\delta(1-x)\, q(x) = \tilde{q}^{(1)}(x) + \delta Z_F^{(1)}\delta(1-x)$$

- Matching factor



$$\mathcal{O}(\alpha_s^0): \quad Z^{(0)}\left(\xi, \frac{p^z}{\mu}\right) = \delta(1-\xi), \quad \xi = \frac{x}{y}$$

$$\mathcal{O}(\alpha_s): \quad Z^{(1)}\left(\xi, \frac{p^z}{\mu}\right) = \tilde{q}^{(1)}(\xi, p^z) - q^{(1)}(\xi, \mu) + \left[\delta\tilde{Z}_F(p^z) - \delta Z_F(\mu)\right]\delta(1-\xi)$$

X. Xiong, X. Ji, J.-H. Zhang, Y. Zhao, Phys. Rev. D **90**, 014051 (2014)
J.-W. Qiu, M.-Y. Qing, arXiv:1412.2688 [hep-ph]

PDF Matching @ One loop

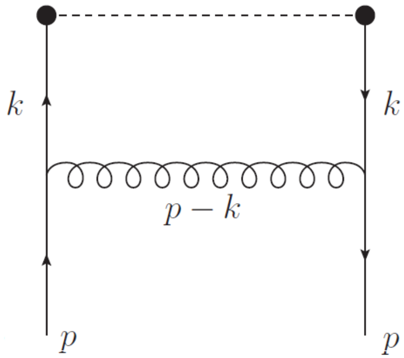
- gauge choice: $n\cdot A=0\rightarrow\mathcal{P}e^{i\int dn\cdot z\, n\cdot A}=1$

  IMF: $n\cdot A=A^{+},\ n^{2}=0$, Quasi: $n\cdot A=A^{z},\ n^{2}=-1$

$$D_{n}^{\mu\nu}(q)=\frac{-i}{q^{2}}\left(g^{\mu\nu}-\frac{q^{\mu}n^{\nu}+n^{\mu}q^{\nu}}{n\cdot q}+n^{2}\frac{q^{\mu}q^{\nu}}{n\cdot q^{2}}\right)$$

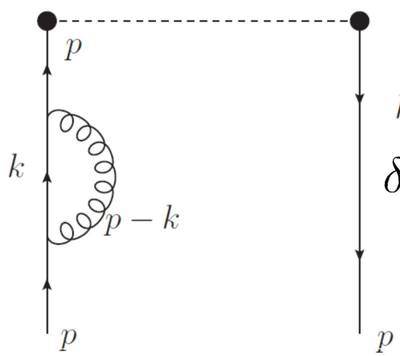
- momentum: $P^{\mu}=(P^{0},\mathbf{0}^{\perp},P^{z})$
- quark mass: m regularize collinear divergence
- massless gluon
- transverse cut-off: $\int_{0}^{\mu}dk_{\perp}$ regularize UV divergence (mimic lattice, breaks Lorentz symmetry, possibly breaks gauge symmetry)

Feynman Diagram ($\boldsymbol{n \cdot A = 0}$)

- 

$$\mathcal{Q}(x, n \cdot P, \mu) \sim \int d^4k\, q\,(k, n \cdot P)\, \delta\left(x - \frac{n \cdot k}{n \cdot P}\right)$$

+



$$\delta \mathcal{Z}_F\,(n \cdot P, \mu)\, \delta(x-1) \sim \int d^4k\, \delta z_F\,(k, n \cdot P, \mu)\, \delta\,(x-1)$$

=

$$\mathcal{Q}^{(1)}(x, n \cdot P, \mu)$$

Quasi, IMF PDF @ One Loop

- Unpol. (helicity, transversity also completed)

$$\lim_{\mu\gg P^z} \mathcal{Q}^{(1)}(x,P^z,\mu)=\tilde{q}^{(1)}(x,\mu)$$

$$=\frac{\alpha_S C_F}{2\pi}\begin{cases} -\frac{1+x^2}{1-x}\ln\frac{x}{x-1}-1+\frac{\mu}{(1-x)^2P^z}\,, & x<0\,,\\ \frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2}+\frac{1+x^2}{1-x}\ln\frac{4x}{1-x}-\frac{4x}{1-x}+1+\frac{\mu}{(1-x)^2P^z}\,, & 0<x<1\,,\\ -\frac{1+x^2}{1-x}\ln\frac{x-1}{x}+1+\frac{\mu}{(1-x)^2P^z}\,, & x>1\,,\end{cases}$$

$$+\delta(x-1)\frac{\alpha_S C_F}{2\pi}\int dy\begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1+\frac{\mu}{(1-y)^2P^z}\,, & y<0\,,\\ \frac{1+y^2}{1-x}\ln\frac{(P^z)^2}{m^2}+\frac{1+y^2}{1-y}\ln\frac{4y}{1-y}-\frac{4y^2}{1-y}+1+\frac{\mu}{(1-y)^2P^z}\,, & 0<y<1\,,\\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1+\frac{\mu}{(1-y)^2P^z}\,, & y>1\,,\end{cases}$$

$$\lim_{P^z\gg\mu} \mathcal{Q}^{(1)}(x,P^z,\mu)=q^{(1)}(x)$$

$$=\frac{\alpha_S C_F}{2\pi}\begin{cases} 0\,, & x>1 \text{ or } x<0\,,\\ \frac{1+x^2}{1-x}\ln\frac{\mu^2}{m^2}-\frac{1+x^2}{1-x}\ln(1-x)^2-\frac{2x}{1-x}\,, & 0<x<1\,,\end{cases}$$

$$+\delta(x-1)\frac{\alpha_S C_F}{2\pi}\int dy\begin{cases} 0\,, & y>1 \text{ or } y<0\,,\\ -\frac{1+y^2}{1-y}\ln\frac{\mu^2}{m^2}+\frac{1+y^2}{1-y}\ln(1-y)^2+\frac{2y}{1-y}\,, & 0<y<1\,,\end{cases}$$

- Matching factor (unpolarized PDF)

$$Z^{(1)}\left(\xi,\frac{P^z}{\mu}\right)=\frac{\alpha_S C_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1}+1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & \xi>1\ ,\\ \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2}+\left(\frac{1+\xi^2}{1-\xi}\right)\ln\left[4\xi(1-\xi)\right]-\frac{2\xi}{1-\xi}+1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & 0<\xi<1\ ,\\ \left(\frac{1+xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi}-1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}, & \xi<0\ .\end{cases}$$

$$+\delta(1-\xi)\frac{C_F\alpha_S}{2\pi}\int dy\begin{cases}-\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1-\frac{\mu}{(1-y)^2P^z} & y>1\\ -\frac{1+y^2}{1-y}\frac{\ln\left(P_z^2\right)}{\mu^2}-\frac{1+y^2}{1-y}\ln\left[4y(1-y)\right]+\frac{4y^2-2y}{1-y}+1-\frac{\mu}{(1-y)^2P^z} & 0<y<1\\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1-\frac{\mu}{(1-y)^2P^z} & y<0\end{cases}$$

no $\ln(m)$, quasi/LC have same IR, match UV.

- Vector current conservation

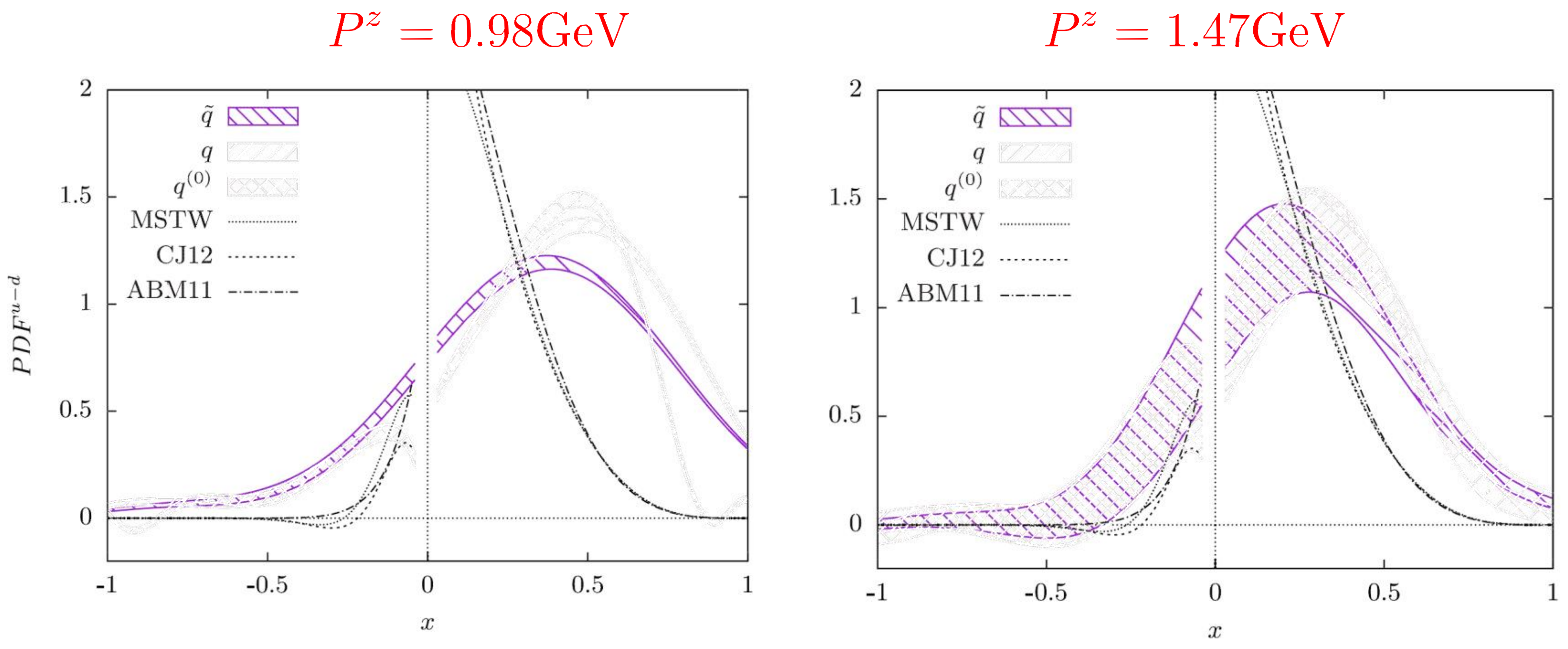
$$\int dx\,\tilde{q}^{(1)}(x)+\int dy\,\delta\tilde{Z}_F(y)=0$$ ⇒ gauge symmetry preserved

$$\int d\xi\,Z^{(1)}\left(\xi,\frac{P^z}{\mu}\right)=0$$ ⇒ Forms a plus-distribution

X. Xiong, X. Ji, J.-H. Zhang, Y. Zhao, Phys. Rev. D **90**, 014051 (2014)
J.-W. Qiu, M.-Y. Qing arXiv:1404.6860 [hep-ph]

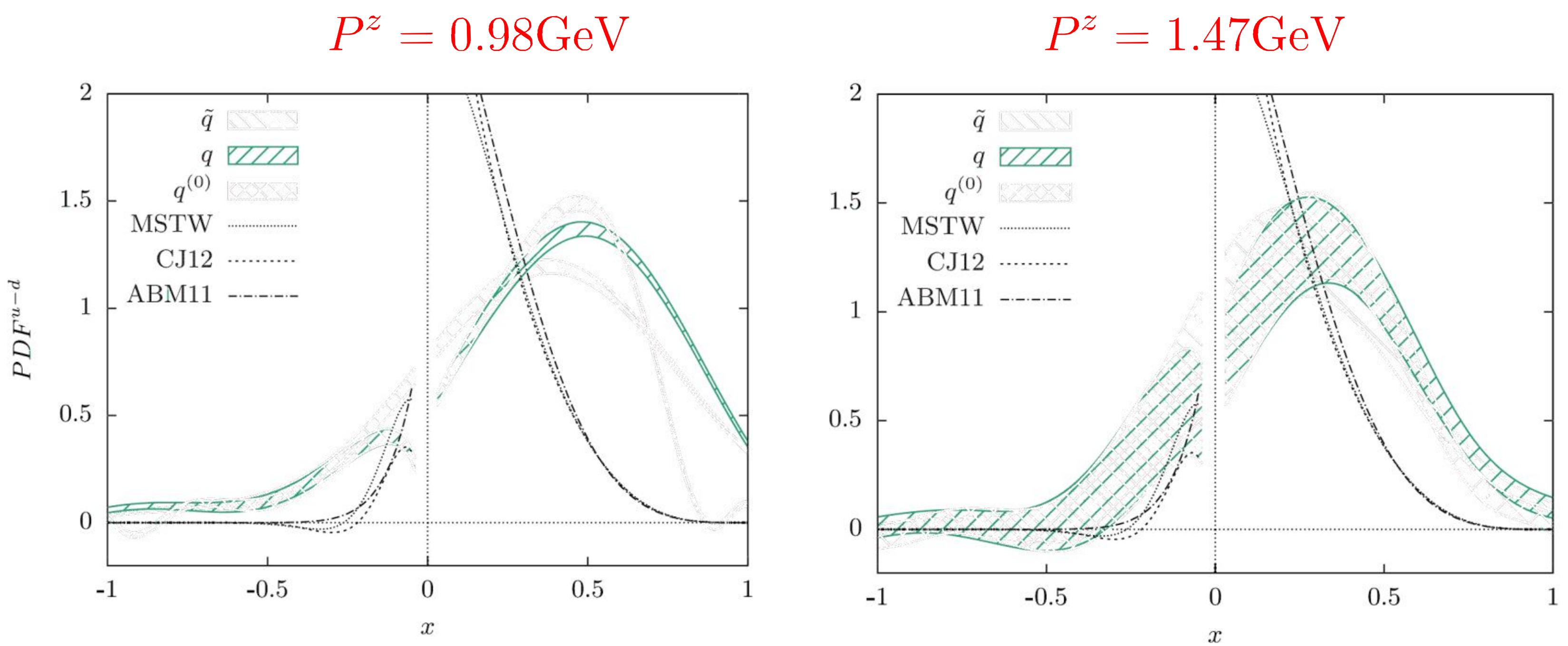
Lattice Quasi PDF Result

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 0145 (2015)



Lattice Quasi PDF Result
+ $\mathcal{O}(\alpha_s)$ matching

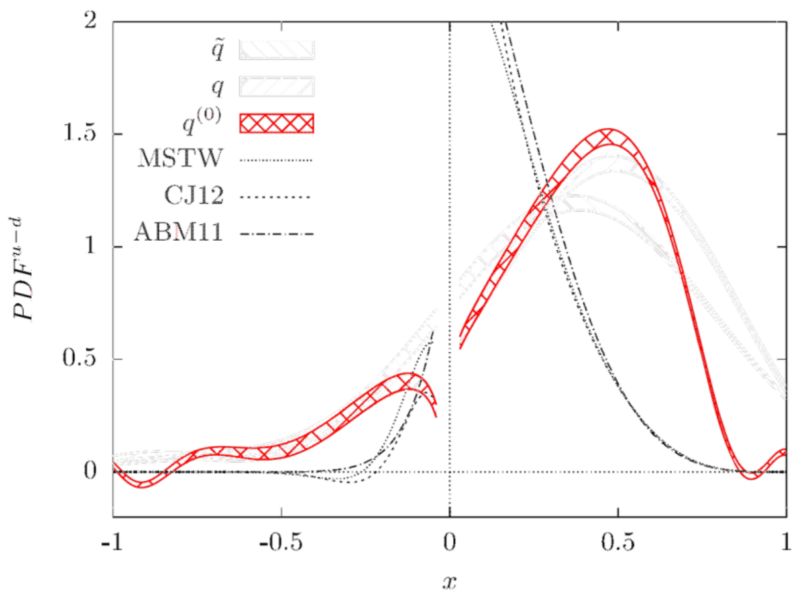
- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)



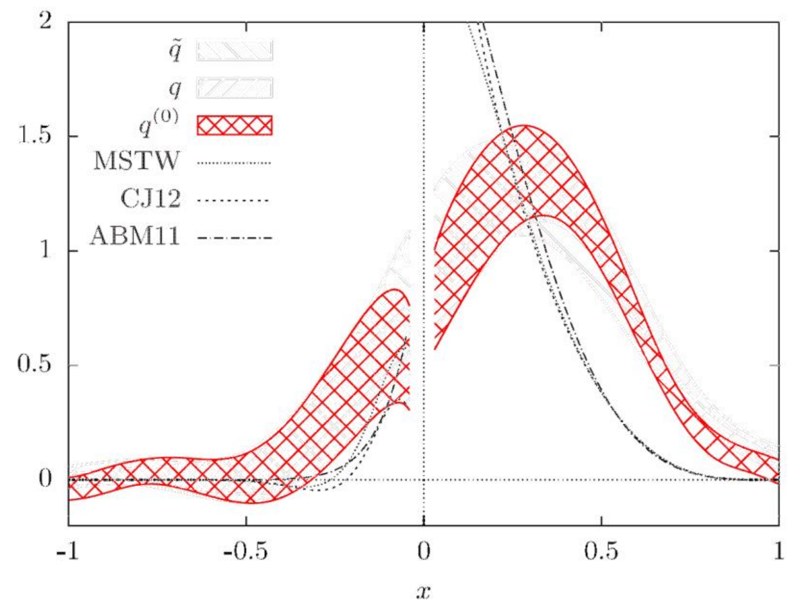
Lattice quasi PDF Results
$+\mathcal{O}(\alpha_s)$ matching+mass corrections

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)

$P^z = 0.98\text{GeV}$

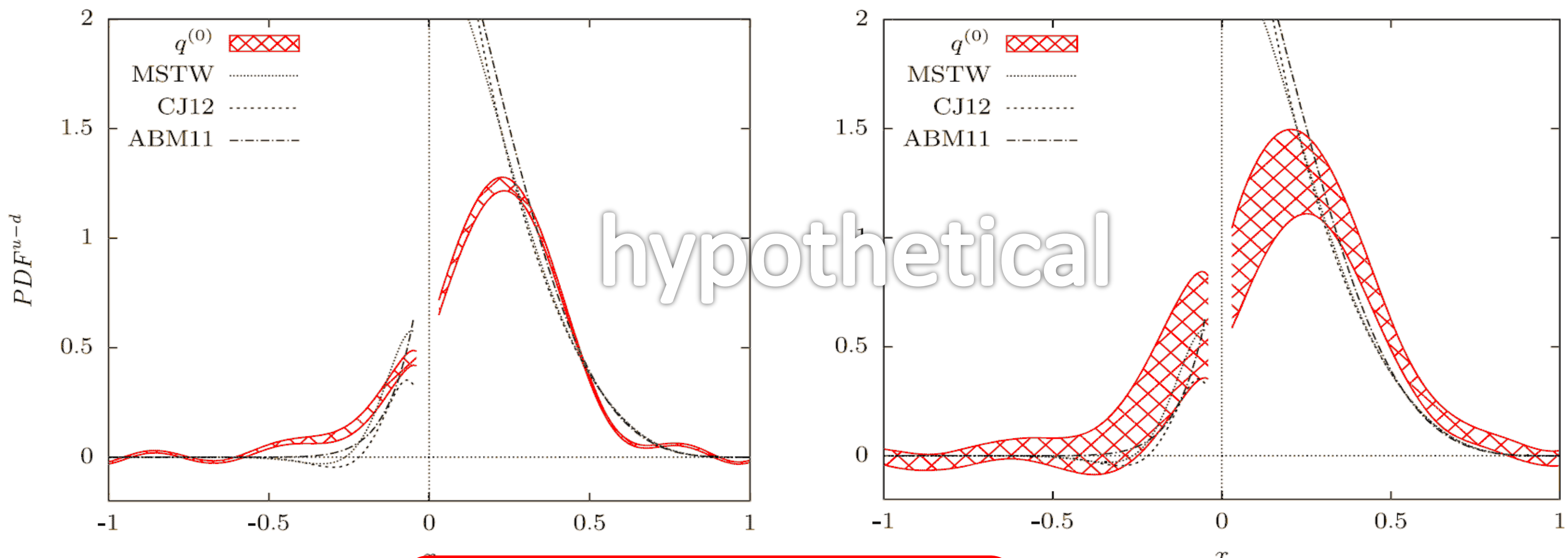


$P^z = 1.47\text{GeV}$



Lattice quasi PDF Results + mixed momentum setup

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)

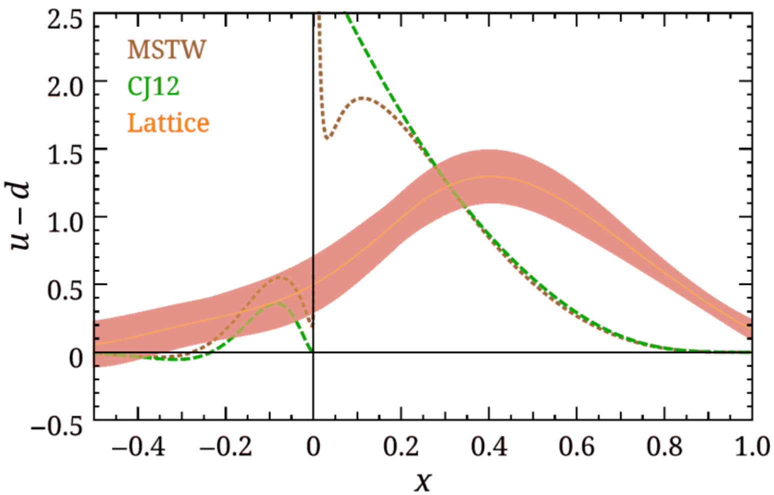


$$q^{(0)} = \mathcal{O}\left(\frac{m}{P^z}\right) + Z^{-1} \otimes \int \frac{dz}{2\pi} e^{iP^z z} \langle P | \cdots z, 0 \cdots | P \rangle$$

$P^z = 1.96\mathrm{GeV}$

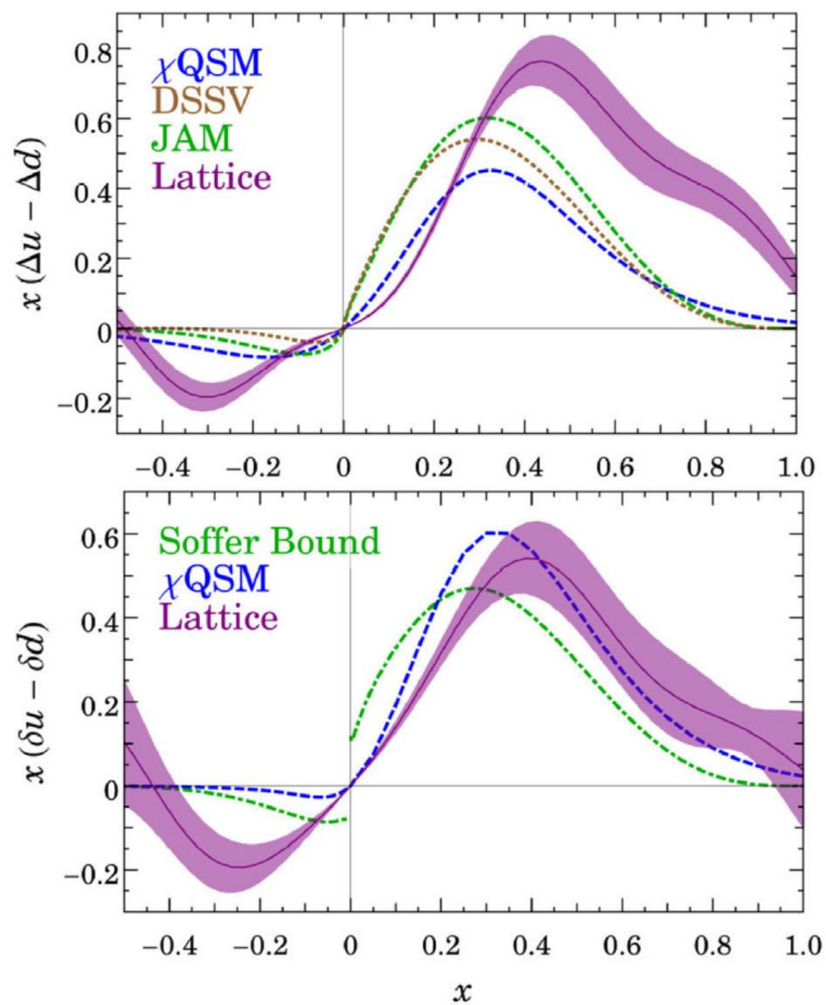
$P^z = 0.98, 1.47\mathrm{GeV}$

- H.-W. Lin, J.-W. Chen, S. D. Cohen and X. Ji, PRD **90**, 014051 (2014) ,
- H.-W. Lin, Few-Body Systems, Sept. 2015, Vol. **56**, Issue 6, 455-460



$P^z = 0.43 n_z \mathrm{GeV}$

extrapolate mass correction to $P^z \to \infty$



Quasi PDF vs LC PDF

	LC PDF	**Quasi PDF**
Soft div.	$\left(\frac{\cdots}{1-x}\right)_+$	$\left(\frac{\cdots}{1-x}\right)_+$
Collinear div. & evolution	$\left[\frac{1+x^2}{1-x}\right]_+ \ln\frac{\mu^2}{m^2}$	$\left[\frac{1+x^2}{1-x}\right]_+ \ln\frac{(P^z)^2}{m^2}$
Support	$0 < x < 1$	$x \in R$
Interpretation	IMF, daughter parton's momentum larger than mother parton is suppressed By large P^+. Probability density	FMF, no $1/P^z$ suppression. No probability interpretation

Test of LaMET

- **Heavy Meson Distribution Amplitudes** *(perturbative test)*

Definition

$$-if_{\eta_c}P^+\Phi_{\eta_c}(x)=\int\frac{d\xi^-}{2\pi}\,e^{i\left(x-\frac{1}{2}\right)p^+\xi^-}\left\langle\eta_c\middle|\bar{\psi}\left(\frac{\xi^-}{2}\right)\gamma^+\gamma_5\mathcal{L}\left[\frac{\xi^-}{2};\frac{-\xi^-}{2}\right]\psi\left(\frac{-\xi^-}{2}\right)\middle|0\right\rangle$$

$$-if_{\eta_c}P^z\tilde{\Phi}_{\eta_c}(x)=\int\frac{dz}{2\pi}\,e^{-i\left(x-\frac{1}{2}\right)p^z z}\left\langle\eta_c\middle|\bar{\psi}\left(\frac{z}{2}\right)\gamma^z\gamma_5\mathcal{L}\left[\frac{z}{2};\frac{-z}{2}\right]\psi\left(\frac{-z}{2}\right)\middle|0\right\rangle$$

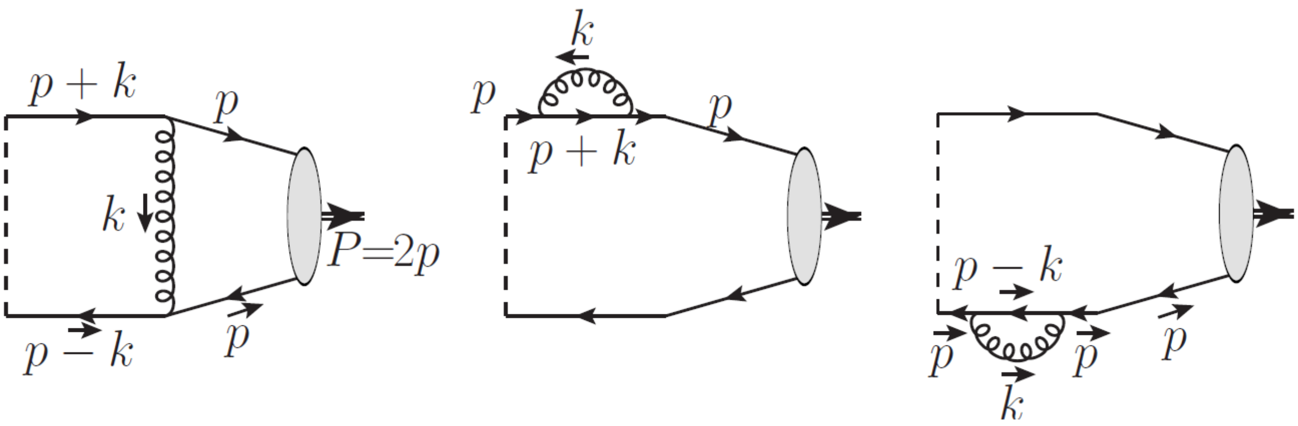
NRQCD refactorization of heavy meson DAs

$$\begin{pmatrix}\Phi(x,\mu)\\ \tilde{\Phi}(x,\mu,p^z)\end{pmatrix}=\sum_n\begin{pmatrix}\langle H\,|\mathcal{O}_n^{NRQCD}|\,0\rangle\\ \langle H\,|\mathcal{O}_n^{NRQCD}|\,0\rangle\end{pmatrix}\begin{pmatrix}\phi^n(x,\mu)\\ \tilde{\phi}^n(x,\mu,p^z)\end{pmatrix}$$

Perturbativly calculable coefficient function (UV), compare quasi v.s IMF ➡ p^z needed to recover IMF DA

NR behavior, same IR between quasi and IMF ($v_{\mathrm{rel.}}^n$) expansion, n=0,1,.. : s,p wave DA

- s-wave DA @ 1-loop



- DR(IR) + Cut-off(UV) Hybrid regularization

$$\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}\int d^{2-2\epsilon}k_\perp = \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}\int_0^{\Lambda} dk_\perp k_\perp^{1-2\epsilon}$$

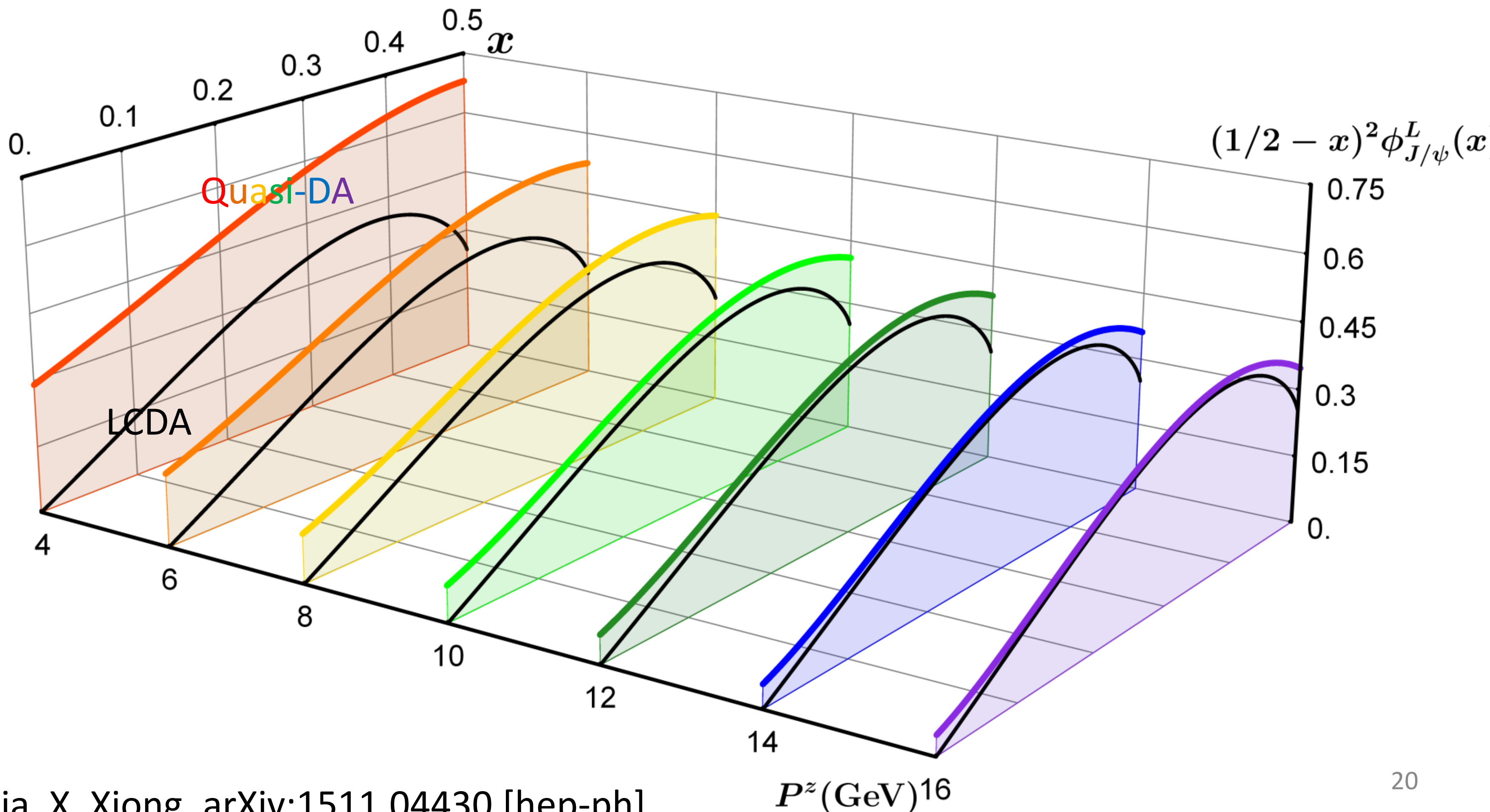
- all $\epsilon^{-1} + \ln\mu^2$ are cancelled and $(1-2x)^{-1}, (1-2x)^{-2}$ are regularized to +,++ distributions, no IR pole

  ➡ NRQCD factorize IR into long range matrix element

- Numerical Results of DA @ 1-loop

charmonium: $J/\psi^L, J/\psi^T, \eta_c$'s s-wave $\tilde{\phi}(x,\mu,p^z)$, $\phi(x,\mu)$

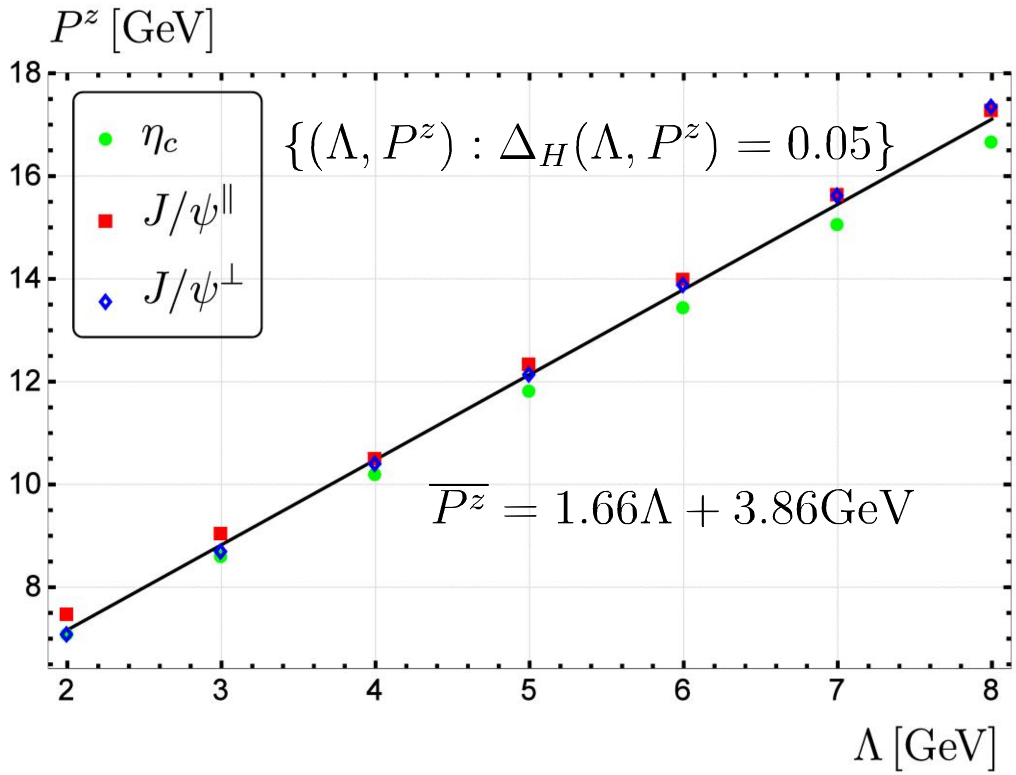
e.g. $\phi^L_{J/\psi} : m_c = 1.4\text{GeV}, \Lambda = 3\text{GeV}$



Yu. Jia, X. Xiong, arXiv:1511.04430 [hep-ph]

- *Degree of Resemblance*

$$\Delta_H(P^z) = \frac{\int_0^{\frac{1}{2}} dx\ (1-2x)^4 \left(\tilde{\phi}-\phi\right)^2}{\int_0^{\frac{1}{2}} dx\ (1-2x)^4\,\phi^2}$$



- Provide some information on setting lattice spacing parameter and estimating the correction needed

Yu. Jia, X. Xiong, arXiv:1511.04430 [hep-ph]

- **2-D Large N_c QCD** *(nonperturbative test)*

  a. Theoretical laboratory

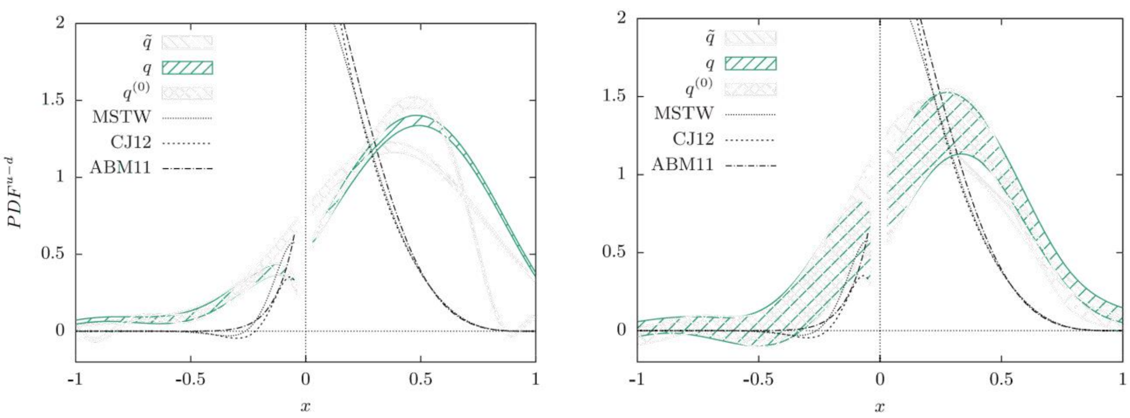
  b. Exactly solvable (nonperturbatively, numerically)

  c. No physical (transverse) gluon in 2-D, simple Fock state wave function for mesons

Motivation:

a. To test LaMET nonperturbatively.

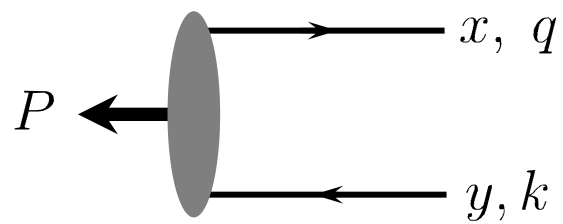
b. To understand the role of perturbative matching



Lattice quasi-PDF result + $\mathcal{O}(\alpha_s)$ matching

Meson in 2-D Large N_c QCD

- Fock State



- IMF: 't Hooft Equation

mass spectrum; light-cone wave function

$$\left(\frac{m^2-2f}{x}+\frac{m^2-2f}{1-x}-M^2\right)\phi_+(x)=2f\,⨍_0^1 dy\,\frac{\phi_+(y)}{(x-y)^2}$$

$f=\frac{g^2N_c}{4\pi}$

- Quasi: Bethe–Salpeter Equation

active quark; meson momentum; dispersion relation; Spector quark; wave function

$$\left(\omega(q)+\omega(P-q)\mp P^0(P)\right)\Phi_\pm(P,q)=f\,⨍\frac{dk}{(q-k)^2}\left(\xi_1(q,k)\,\Phi_\pm(P,k)-(\xi_2(q,k)\,\Phi_\mp(P,k)\right)$$

- Calculation Setup

Effective loop expansion in f/m_q^2 : coupling constant in 2-D has mass dimension $[f]=[m^2]$

a. Heavy quark: $m_q^2 = 8.9f \gg f$

first heavy quark limit, then large N_c limit:

test perturbative matching

b. Light-quark: $m_q^2 = 0.065f \ll f$

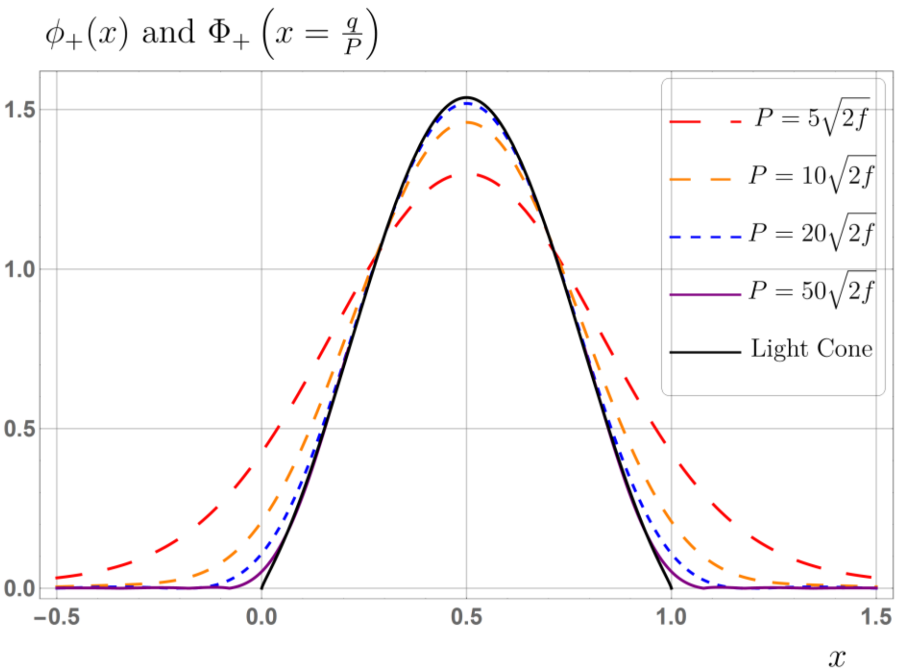
nonperturbatively matching needed!

Numerical Result

- Light-Cone Wave function & quasi wave function with boost

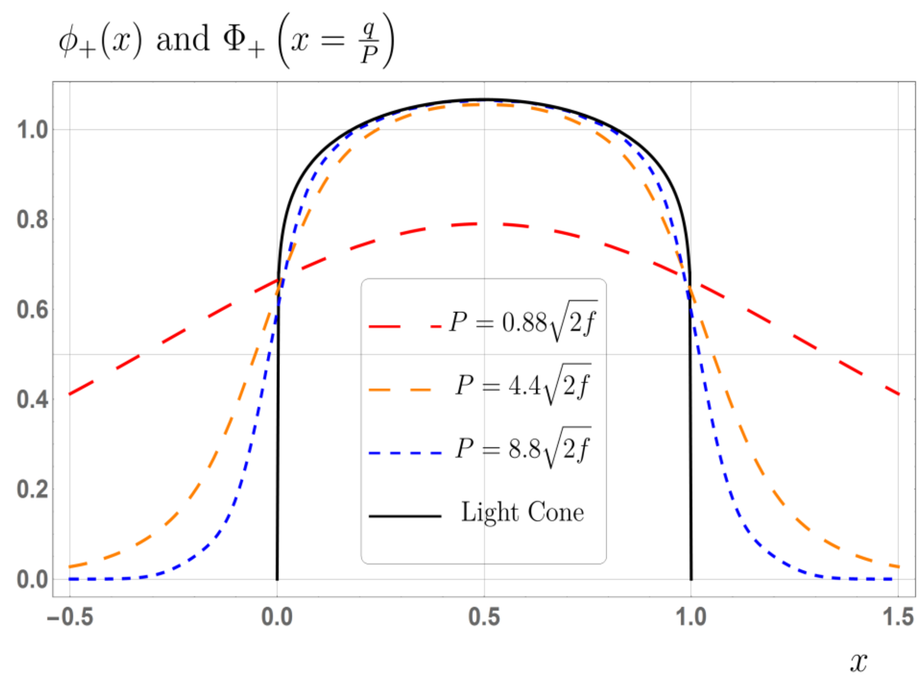
Heavy quark case

$$m_q = 2.11\sqrt{2f},\ m_q^2 \gg f$$



Light quark case

$$m_q = 0.18\sqrt{2f},\ m_q^2 \ll f$$



Conclusion (Roadmap)

- Quasi-PDF

finite

$$\tilde{q}(x, P^z, \mu) = \int \frac{dz}{2\pi} e^{ixp^z z} \langle P | \bar{\psi}(\tfrac{z}{2})\gamma^z \mathcal{L}[\tfrac{z}{2}; -\tfrac{z}{2}]\psi(-\tfrac{z}{2}) | P \rangle$$

calculated on lattice

Lattice renormalization

- Matching

$$\tilde{q}\left(x, P^z, \mu\right) \otimes Z^{(-1)}\left(x, P^z, \mu\right)$$

1-loop continuum completed

Lattice perturbation

Non-perturbative

- Nucleon mass & higher-twist corrections

$$q\left(x, \mu\right) = \tilde{q}\left(y, P^z, \mu\right) \otimes Z^{(-1)}\left(\frac{x}{y}, P^z, \mu\right)$$

$$+ \mathcal{O}\left(\frac{M_N^n}{(P^z)^n}\right)$$

major correction
currently $M_N / P^z \approx 1$

$$+ \mathcal{O}\left(\frac{\Lambda_{QCD}^n}{(P^z)^n}\right)$$

Higher twist,
Lattice calculable

Ultimate goal: direct lattice determination of light-cone distributions

- Quasi/LC distribution share the same IR
- Analog: quasi-TMD (softer factor subtraction), quasi-GPD (matching)…
- Matching only controlled by UV (perturbative)
- Preliminary lattice results provide confidence
- Renormalization of quasi distributions

  X.D. Ji, J.H. Zhang PRD **92**, 034006 (2015)

  Future paper by J.W. Qiu and Y.Q. Ma

Thanks

Backup Slides

- Space like correlation function $\neq$ static, does not depend on time

$$\langle q_1 | e^{iHt} \bar{\psi}\left(\frac{z}{2}\right) \gamma^z \Gamma \mathcal{L}[\frac{z}{2}; \frac{-z}{2}] \psi\left(\frac{-z}{2}\right) e^{-iHt} | q_2 \rangle = e^{i(E_1 - E_2)t} \langle \cdots \rangle$$

Forward case: no time dependence

Off-forward case: fixed time

Light-cone case:

$$H = P^- \rightarrow e^{i\left(P_1^- - P_2^-\right)\xi^+} \sim 1 + \mathcal{O}\left(\frac{m^2 \xi^+}{P^+}\right)$$

Mass and Higher-Twist Correction

- Mass correction at $\mathcal{O}(M^2/(P^z)^2)$

  $$\tilde{q}(x, P^z, \mu) = \int \frac{dz}{2\pi} e^{ixp^z z} \langle P | \bar{\psi}(\tfrac{z}{2}) \gamma^z \mathcal{L}[\tfrac{z}{2}; -\tfrac{z}{2}] \psi(-\tfrac{z}{2}) | P \rangle$$

  series expansion

  $$\langle P | \bar{\psi}(0) \gamma^z \mathcal{L}[0, z] \psi(z) | P \rangle = \frac{1}{2P^z} \sum_n \frac{(-iz)^n}{(n)!} \langle P | \bar{\psi}(0) \gamma^z (iD^z)^n \psi(0) | P \rangle$$

  ignore trace of operator (higher-twist correction)

  e.g. $\langle P \left| g^{zz} \bar{\psi}(0) (iD^z)^i (\gamma^\mu iD_\mu) (iD^z)^{n-i} \psi(0) \right| P \rangle \sim \mathcal{O}\left(\frac{\Lambda^n_{QCD}}{(P^z)^n} \right)$

  gives

  $$\langle P | \bar{\psi}(0) \gamma^z (iD^z)^n \psi(0) | P \rangle = 2 a_n \left[P^{(\mu_0} \cdots P^{\mu_n)} - \mathrm{tr}\left(P^{(\mu_0} \cdots P^{\mu_n)} \right) \right] \Big|_{\mu_i = z}$$

  Mellin moments of quasi PDF $\int dx x^n \tilde{q}(x)$

the trace of matrix element is

$$\mathrm{tr}\left(P^{(\mu_0}\cdots P^{\mu_n)}\right)=\sum_{i=1}^{n}\frac{g^{\mu_0\mu_i}P^2}{4}P^{(\mu_1}\cdots P^{\mu_{i-1}}\cdots P^{\mu_{i+1}}\cdots P^{\mu_n)}+\mathcal{O}\left(\frac{M^4}{(P^z)^4}\right)$$

taking $\mu_i = z$ gives

$$\mathrm{tr}\left(\cdots\right)=-n\frac{M^2}{4\left(P^z\right)^2}\left(P^z\right)^{n+1}$$

Therefore

$$\frac{1}{2P^z}\sum_n\frac{(-iz)^n}{(n)!}\langle P\left|\bar{\psi}\left(0\right)\gamma^z\left(iD^z\right)^n\psi\left(0\right)\right|P\rangle$$

$$=\sum_n\frac{(-iz)^n}{n!}2a_n\left(P^z\right)^n\left[1+n\frac{M^2}{4\left(P^z\right)^2}\right]+\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{\left(P^z\right)^2},\frac{M^4}{\left(P^z\right)^4}\right)$$

$$=\sum_n\frac{(-i\lambda z)^n}{n!}2a_n\left(P^z\right)^n+\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{\left(P^z\right)^2},\frac{M^4}{\left(P^z\right)^4}\right)$$

with $\lambda=1+\frac{M^2}{4\left(P^z\right)^2}$

Fourier Transform to x space

$$\tilde{q}(x,P^z,\mu)\to\lambda^{-1}\tilde{q}\left(\lambda^{-1}x\right)$$

Matching Condition

- Lattice "cross section" factorization

$$\tilde{q}(x) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}\right) q(y)$$

- Perturbative expansion

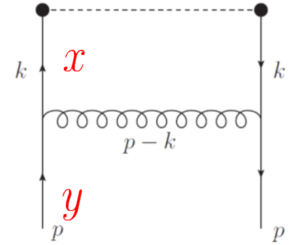
$$\tilde{q}(x) = \tilde{q}^{(1)}(x) + \delta\tilde{Z}_F^{(1)}\delta(1-x) \qquad q(x) = \tilde{q}^{(1)}(x) + \delta Z_F^{(1)}\delta(1-x)$$

$$\delta\tilde{Z}_F^{(1)}\delta(1-x) + \tilde{q}^{(1)}(x)$$

$$= \int_0^1 \frac{dy}{y}\delta\left(\frac{x}{y}-1\right)\left[\delta Z_F^{(1)}\delta(1-y) + q^{(1)}(y)\right] + \int_0^1 \frac{dy}{y} Z^{(1)}\left(\frac{x}{y}, \frac{P^z}{\mu}\right)\delta(1-y)$$

$$= \delta Z_F\delta(1-x) + q^{(1)}(x) + Z^{(1)}\left(x, \frac{P^z}{\mu}\right).$$

- Matching factor



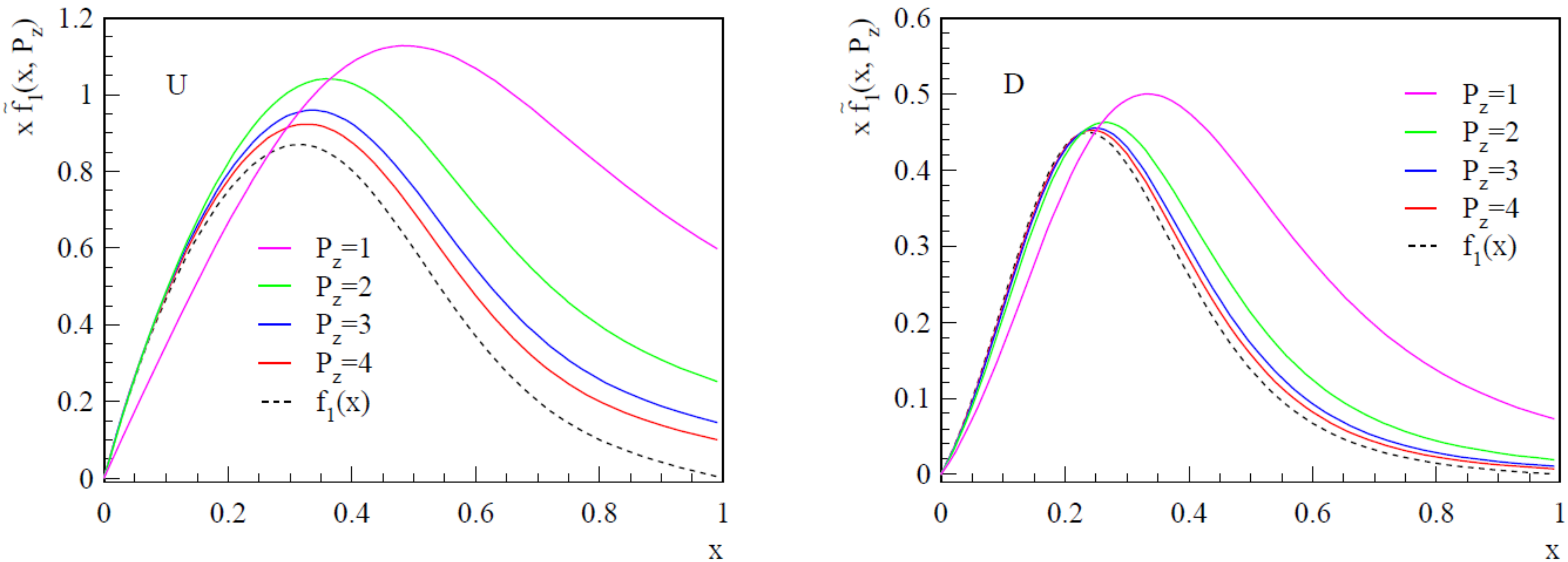
$$\mathcal{O}(\alpha_s^0): \ \boldsymbol{Z^{(0)}\left(\xi, \frac{p^z}{\mu}\right) = \delta(1-\xi)}, \quad \xi = \frac{x}{y}$$

$$\mathcal{O}(\alpha_s): \ \boldsymbol{Z^{(1)}\left(\xi, \frac{p^z}{\mu}\right) = \tilde{q}^{(1)}(\xi, p^z) - q^{(1)}(\xi, \mu) + \left[\delta\tilde{Z}_F(p^z) - \delta Z_F(\mu)\right]\delta(1-\xi)}$$

Diquark Model Results of quasi PDF

- L. Gamberg, Z. B. Kang, I. Vitev and H. Xing, PLB **743**, 112 (2015)

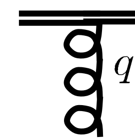
$$\mu^2 = 0.3\mathrm{GeV}^2$$



Gauge Invariance

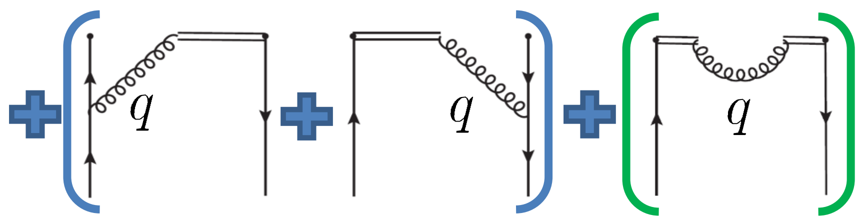
- Preserved by gauge link.
- $n \cdot A = 0$ And Feynman gauge and gauge

$$D_n^{\mu\nu}(q) = \frac{-i}{q^2}\left(g^{\mu\nu} - \left[\frac{q^\mu n^\nu + n^\mu q^\nu}{n \cdot q}\right] + \left[n^2 \frac{q^\mu q^\nu}{n \cdot q^2}\right]\right)$$

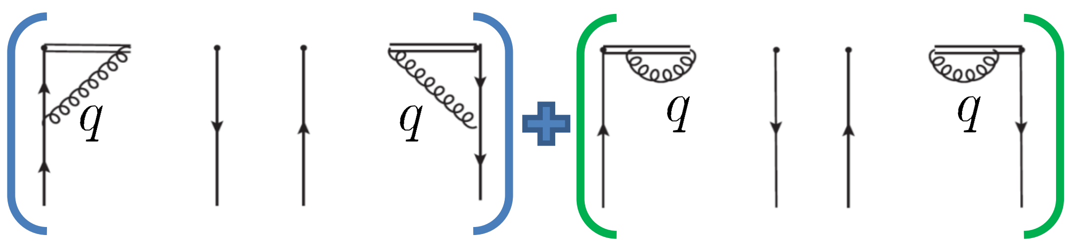


$$\sim \frac{n^\mu}{n \cdot q \pm i\epsilon}$$

$$D_F^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2}$$



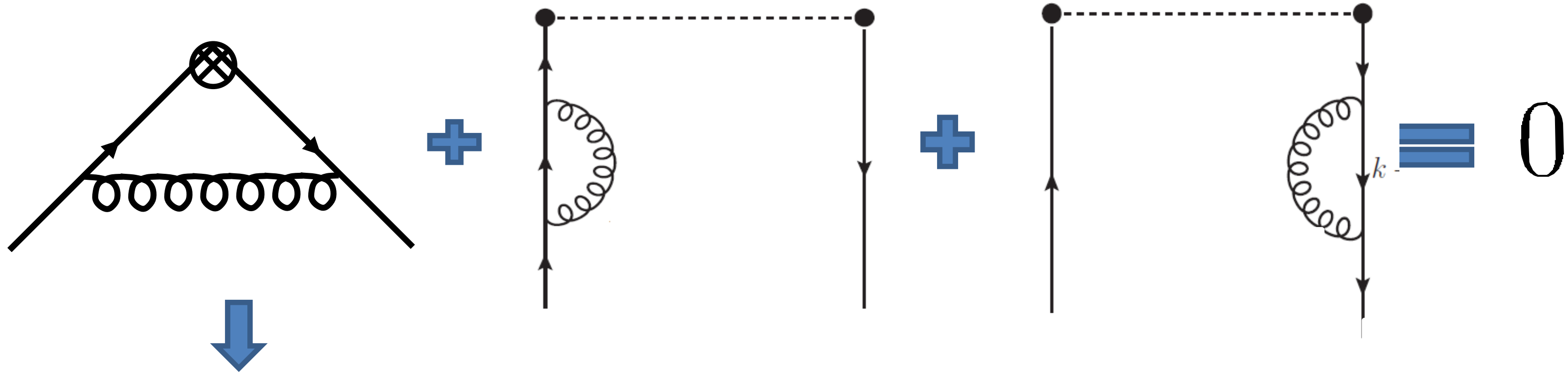
for $q^{(1)}(x)$

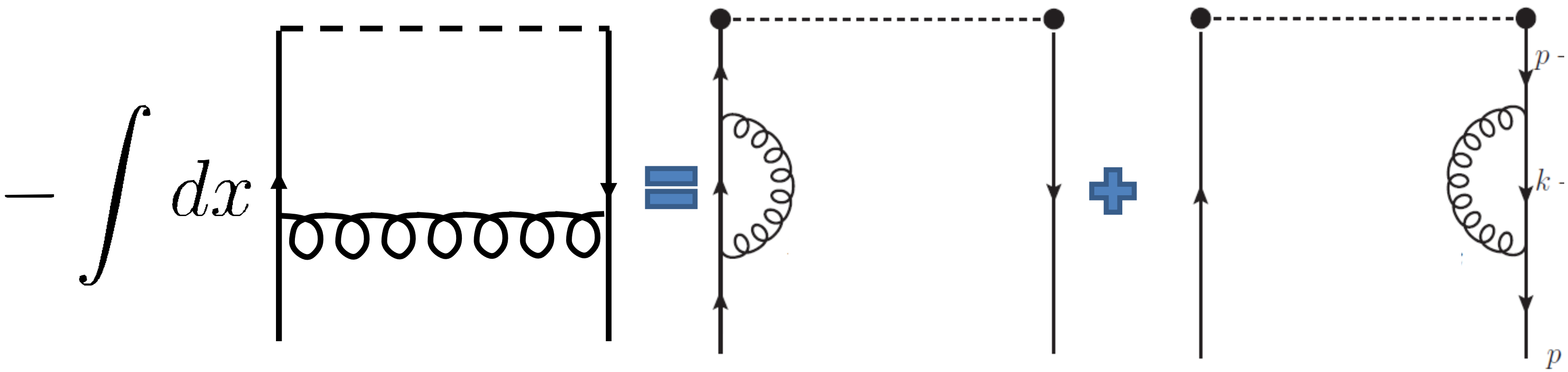


for $\delta Z_F^{(1)} \delta(1-x)$

Gauge Invariance

- Start from vector current conservation





$$q(x) - \delta Z_F \delta\,(x-1) = q(x) - \delta\,(x-1)\int dy q(y)$$

k^-(LC)/ k^0(Qujasi)-integral

Performed by Cauchy residue theorem

quark, gluon propagator (linear in k^-,quadratic in k^0)

$k^2-m^2+i\epsilon=2k^+k^- - \boldsymbol{k}_\perp^2-m^2+i\epsilon \quad (p-k)^2+i\epsilon=2\left(P^+-k^+\right)\left(P^- -k^-\right)-\boldsymbol{k}_\perp^2+i\epsilon$

	$P^- + \frac{-(\boldsymbol{P}_\perp-\boldsymbol{k}_\perp)^2+i\epsilon}{2(1-x)P^+}$	$\frac{\boldsymbol{k}_\perp^2+m^2-i\epsilon}{2xP^+}$	$\int dk^- [\cdots]$
$x<0$	$+$	$+$	0
$0<x<1$	$+$	$-$	$\neq 0$
$x>1$	$-$	$-$	0

$k^2-m^2+i\epsilon=\left(k^0\right)^2-\boldsymbol{k}_\perp^2-(k^z)^2-m^2+i\epsilon \quad (p-k)^2+i\epsilon=\left(P^0-k^0\right)^2-(P^z-k^z)^2-\boldsymbol{k}_\perp^2+i\epsilon$

always one k^0 pole on upper/lower plane

Calculation Example

- Feynman Part $D_n^{\mu\nu}(q) = \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu n^\nu + n^\mu q^\nu}{n\cdot q} + n^2 \frac{q^\mu q^\nu}{n\cdot q^2}\right)$

$$q^{(1)}(x) = \frac{1}{P^z}\int \frac{d^4k}{(2\pi)^4}\bar{u}(P)\left(-ig_s \mathrm{t}_a \gamma^\mu\right)\frac{i}{\not{k}-m+i\epsilon}\gamma^z \frac{i}{\not{k}-m+i\epsilon}\left(-ig_s \mathrm{t}_\mathrm{b}\gamma^\nu\right)$$

$$\times \frac{-ig_{\mu\nu}}{(P-k)^2+i\epsilon}\delta\left(k^z - xP^z\right)u(P)+\cdots$$

$$= \int \frac{d^2k_\perp}{(2\pi)^4}\frac{g_s^2 C_F \pi P^z}{\sqrt{k_\perp^2+(1-x)^2 P_z^2}\left[2P^0\sqrt{k_\perp^2+(1-x)^2 P_z^2}-(1-2x)P_z^2+m^2-P_0^2\right]}$$

$$- \frac{g_s^2 C_F \pi P^z}{\sqrt{k_\perp^2+x^2P_z^2+m^2}\left[2P^0\sqrt{k_\perp^2+x^2P_z^2+m^2}+2xP_z^2+m^2+P_0^2-P_z^2\right]}$$

$$\sim \frac{P^z}{\sqrt{P_z^2+m^2}}\ln\frac{\sqrt{P_z^2+m^2}\sqrt{\mu^2+(1-x)^2P_z^2}-(1-x)P_z^2}{\sqrt{P_z^2+m^2}\sqrt{(1-x)^2P_z^2}-(1-x)P_z^2}+\cdots$$

- $P^z \to \infty$

$$q(x) \to \int_0^{\mu} \frac{d^2k_\perp}{(2\pi)^4} \begin{cases} \frac{2g_s^2 C_F \pi}{k_\perp^2 + m^2(1-x)^2} & 0 < x < 1 \\ \mathcal{O}\left(\frac{1}{P^z}\right)^n & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \frac{g_s^2 C_F}{8\pi^2} \ln\left[\frac{\mu^2 + m^2(1-x)^2}{m^2(1-x)^2}\right] & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$= q_{LC}(x)$$

Can be calculated directly using light-cone coordinates

Same collinear, different UV → perturbative matching

- $\mu \to \infty$

$$q(x) \to \begin{cases} \frac{g_s^2 C_F}{8\pi^2} \ln\left[\frac{(P^z)^2}{m^2}\right] + \text{non-ln}\left(\frac{P^z}{m}\right) \text{ terms} & 0 < x < 1 \\ \text{non-ln}\left(\frac{P^z}{m}\right) \text{ terms} & \text{Otherwise} \end{cases}$$

$$= q_{quasi}(x)$$

E.g.2: GPD

- Definition

$$P^z \int \frac{dz}{2\pi} e^{-ixp^z z} \langle p + \tfrac{\Delta}{2}, S \left| \bar{\psi}(-\tfrac{z}{2}) \gamma^z \mathcal{L}[-\tfrac{z}{2}; \tfrac{z}{2}] \psi(\tfrac{z}{2}) \right| p - \tfrac{\Delta}{2}, S \rangle$$

$$= \mathcal{H}(x, \xi, \Delta^2) \bar{U}(p + \tfrac{\Delta}{2}) \gamma^z U(p - \tfrac{\Delta}{2}) + \mathcal{E}(x, \xi, \Delta^2) \bar{U}(p + \tfrac{\Delta}{2}) \frac{i\sigma^{z\rho}\Delta_\rho}{2m} U(p - \tfrac{\Delta}{2})$$

- Convention

$$p^\mu = \left(p^0, \mathbf{0}^\perp, p^z\right), \ \Delta^\mu = \left(\Delta^0, \Delta^1, 0, \Delta^z\right), \ x = \frac{k^z}{p^z}, \ \ \xi = \frac{\Delta^z}{p^z}, \ \ t = \Delta^2$$

- Tree level $H^{(0)}(x, \xi, t) = \delta(x - 1), \ E^{(0)}(x, \xi, t) = 0$

- Properties of GPD

Forward limit : $H(x, 0, 0) = f(x)$

Polynomiality: Lorentz symmetry

One-loop GPD results

- Finite P^z, quasi-GPD & Infinite P^z , light-cone GPD
  gluon exchange diagram, only leading log terms
- E.g. unpolarized (long. and trans. pol. completed)

$$\tilde{H}^{(1)}(x,\xi,t,\mu,p^z)\vee H^{(1)}(x,\xi,t,\mu,p^z) =$$

$$\frac{\alpha_S C_F}{2\pi}\begin{cases} \cdots + \frac{\mu}{(1-x)^2 p^z} \vee 0 & x < -\xi \\ \frac{x+\xi}{2\xi(1+\xi)}(1+\frac{2\xi}{1-x})\ln\frac{p_z^2}{-t}\vee\ln\frac{\mu^2}{-t} + \cdots + \frac{\mu}{(1-x)^2 p^z} & -\xi < x < \xi \\ \frac{1+x^2-2\xi^2}{(1-x)(1-\xi^2)}\ln\frac{p_z^2}{-t}\vee\ln\frac{\mu^2}{-t} + \cdots + \frac{\mu}{(1-x)^2 p^z} & \xi < x < 1 \\ \cdots + \frac{\mu}{(1-x)^2 p^z} \vee 0 & x > 1, \end{cases}$$

$$\tilde{E}^{(1)}(x,\xi,t,\mu) = E^{(1)}(x,\xi,t,\mu) =$$

$$\frac{\alpha_S C_F}{2\pi}\frac{m^2}{-t}\begin{cases} \frac{2(x-\xi)}{1+\xi}\ln\left(\frac{-t}{m^2}\right) + \cdots & -\xi < x < \xi \\ \frac{4(x+\xi^2)}{1-\xi^2}\ln\left(\frac{-t}{m^2}\right) + \cdots & \xi < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

X. Ji, A. Schäfer, X. Xiong, J-H. Zhang, PRD**92** (2015) 014039

- Forward limit

first take forward limit $\xi, t \to 0$

then $m \to 0$ recover PDF from an finite $t,\ m$ result

$\xi, t \to 0$ and $m \to 0$ DO NOT commute

e.g.

$$\ln\left(m^2 - \frac{t}{4}\right) \to \ln\left(-\frac{t}{4}\right)$$

$$\ln\left(m^2 - \frac{t}{4}\right) \to \ln\left(m^2\right)$$

- Polynomiality

taking moments of $\int dx\, x^n \int \frac{dz}{2\pi} e^{-ixp^z z} \langle p+\frac{\Delta}{2} | \bar{\psi}(-\frac{z}{2}) \gamma^z \mathcal{L}[-\frac{z}{2};\frac{z}{2}] \psi(\frac{z}{2}) | p-\frac{\Delta}{2} \rangle$

$$n_{\mu_0} n_{\mu_1} \cdots n_{\mu_n} \left\langle P+\frac{\Delta}{2} \left| \bar{\psi}(0)\, \gamma^{\mu_0} i \overleftrightarrow{D}^{\mu_1} \cdots i \overleftrightarrow{D}^{\mu_n} \psi(0) \right| P-\frac{\Delta}{2} \right\rangle$$

$$\sim C(t)\,(n\cdot P)\cdots(n\cdot P)\,(n\cdot\Delta)\cdots(n\cdot\Delta) \sim \sum_i C_i(t)\xi^i$$

In 1-loop GPD, only H, E's $\ln\left(\frac{\mu^2}{-t}\right), \ln\left(\frac{P_z^2}{-t}\right)$ terms satisfy polynomiality (transverse cut-off breaks Lorentz Symmetry, but $\ln(\mu^2)$ terms are the same as DR)

- Meson DA from GPD

$$\left\langle q_1 \left| \bar{\psi}\left(\frac{z}{2}\right) \gamma^z \gamma^5 \mathcal{L}\left[\frac{z}{2};\frac{-z}{2}\right] \psi\left(\frac{-z}{2}\right) \right| q_2 \right\rangle$$

crossing symmetry

$$\left\langle q_1\, \bar{q}_2 \left| \bar{\psi}\left(\frac{z}{2}\right) \gamma^z \gamma^5 \mathcal{L}\left[\frac{z}{2};\frac{-z}{2}\right] \psi\left(\frac{-z}{2}\right) \right| 0 \right\rangle$$

- Polynomiality Results

$$H^{n+1}(\xi,t)=\sum_{i=0}^{[n/2]}(2\xi)^i A^q_{n+1,2i}(t)+\text{mod }(n,2)(2\xi)^{n+1}C^q_{n+1}(t)$$

$$E^{n+1}(\xi,t)=\sum_{i=0}^{[n/2]}(2\xi)^i B^q_{n+1,2i}(t)-\text{mod }(n,2)(2\xi)^{n+1}C^q_{n+1}(t)$$

$$\begin{aligned}H^{n+1}(\xi,t)=&\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\begin{cases}\frac{1}{2k^2+3k+1}\sum_{i=0}^{k}\xi^{2i} & n=2k\\ \frac{1}{2k^2+5k+3}\sum_{i=0}^{k}\xi^{2i} & n=2k+1\end{cases}\\ &+\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\sum_{i=0}^{n}\left(\begin{array}{c}n\\ i\end{array}\right)(-\xi)^{n-i}(1+\xi)^{i+1}\int_0^1 d\chi\,\frac{\chi^{i+1}}{(1-\chi)_+}\\ &+\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\sum_{i=0}^{n}\left(\begin{array}{c}n\\ i\end{array}\right)\xi^{n-i}(1-\xi)^{i+1}\int_0^1 d\chi\,\frac{\chi^{i+1}}{(1-\chi)_+}\\ &-\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{m^2}\right)\int_0^1 d\chi\left[(1-\chi)+\frac{2\chi}{(1-\chi)_+}\right]\end{aligned}$$

$$\begin{aligned}E^{n+1}(\xi,t)=&\frac{C_F\alpha_s}{2\pi}\frac{m^2}{-t}\ln\left(\frac{-t}{m^2}\right)\begin{cases}\frac{2(4k+3)}{(2k+1)(k+1)}\sum_{i=1}^{k}\xi^{2i}+\frac{2}{(k+1)} & n=2k\\ \frac{2(4k+5)\xi^2}{(2k+3)(k+1)}\sum_{i=0}^{k}\xi^{2i}+\frac{4}{2k+3} & n=2k+1\end{cases}\\ &-\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{m^2}\right)\int_0^1 d\chi\left[(1-\chi)+\frac{2\chi}{(1-\chi)_+}\right]\end{aligned}$$

+,++ Distribution

- +-distribution

$$\int_0^{\frac{1}{2}} dx \left[\frac{f(x)}{\frac{1}{2}-x}\right]_+ g(x) = \int_0^{\frac{1}{2}} dx \, \frac{f(x)\left[g(x) - g\left(\frac{1}{2}\right)\right]}{\frac{1}{2}-x}$$

$$\left[\frac{f(x)}{\frac{1}{2}-x}\right]_+ = \frac{f(x)}{\frac{1}{2}-x} - \delta\left(x - \frac{1}{2}\right) \int_0^{\frac{1}{2}} dxy \frac{f(y)}{\frac{1}{2}-y}$$

example: DGLAP evolution kernel

- ++ - distribution

$$\int_0^{\frac{1}{2}} dx \left[\frac{f(x)}{\left(\frac{1}{2}-x\right)^2}\right]_{++} g(x) = \int_0^{\frac{1}{2}} dx \, \frac{f(x)\left[g(x) - g'(\frac{1}{2})(x-\frac{1}{2}) - g\left(\frac{1}{2}\right)\right]}{\left(\frac{1}{2}-x\right)^2}$$

Plus-Distribution

- Plus-distribution (only make sense when convoluted)

$$\int_0^1 dx \left[\frac{f(x)}{1-x}\right]_+ g(x) = \int_0^1 dx \, \frac{f(x)\left[g(x) - g\left(1\right)\right]}{1-x}$$

$$\left[\frac{f(x)}{1-x}\right]_+ = \frac{f(x)}{1-x} - \delta\left(x-1\right)\int_0^1 dy \frac{f(y)}{1-y}$$

Plus-distribution regularized pole@ $x=1$

gluon momentum $P-k \sim 1-x=0$, soft gluon emission(IR)

- 't Hooft Equation

$$\left(\frac{m^2-2f}{x}+\frac{m^2-2f}{1-x}-M^2\right)\phi_+(x)=2f\,⨍_0^1 dy\frac{\phi_+(y)}{(x-y)^2}$$

basis expansion $\phi_+(x)=C_i\varphi_i(x)$ gives

$$\boldsymbol{A}(m,M)\cdot\boldsymbol{C}=\boldsymbol{0}$$

nontrivial solution requires $\det\left[\boldsymbol{A}(m,M)\right]=0$

- B-S Equation parameters

$$p\cos(\theta(p))-m\sin(\theta(p))=\frac{f}{2}⨍\frac{dk}{(p-k)^2}\sin(\theta(p)-\theta(k))$$

$$\omega(p)=m\cos(\theta(p))+p\sin(\theta(p))+\frac{f}{2}⨍\frac{dk}{(p-k)^2}\cos(\theta(p)-\theta(k))$$

$$\xi_1(p,q,k)=\cos\frac{\theta(p)-\theta(k)}{2}\cos\frac{\theta(p-q)-\theta(p-k)}{2}$$

$$\xi_2(p,q,k)=\sin\frac{\theta(p)-\theta(k)}{2}\sin\frac{\theta(p-q)-\theta(p-k)}{2}$$