Two-Pion Matrix Elements in SU(2) Chiral Perturbation Theory: Analytic Structures and Implications for Lattice QCD

Yonggang Ren,

Andrew Jackura

WILLIAM & MARY

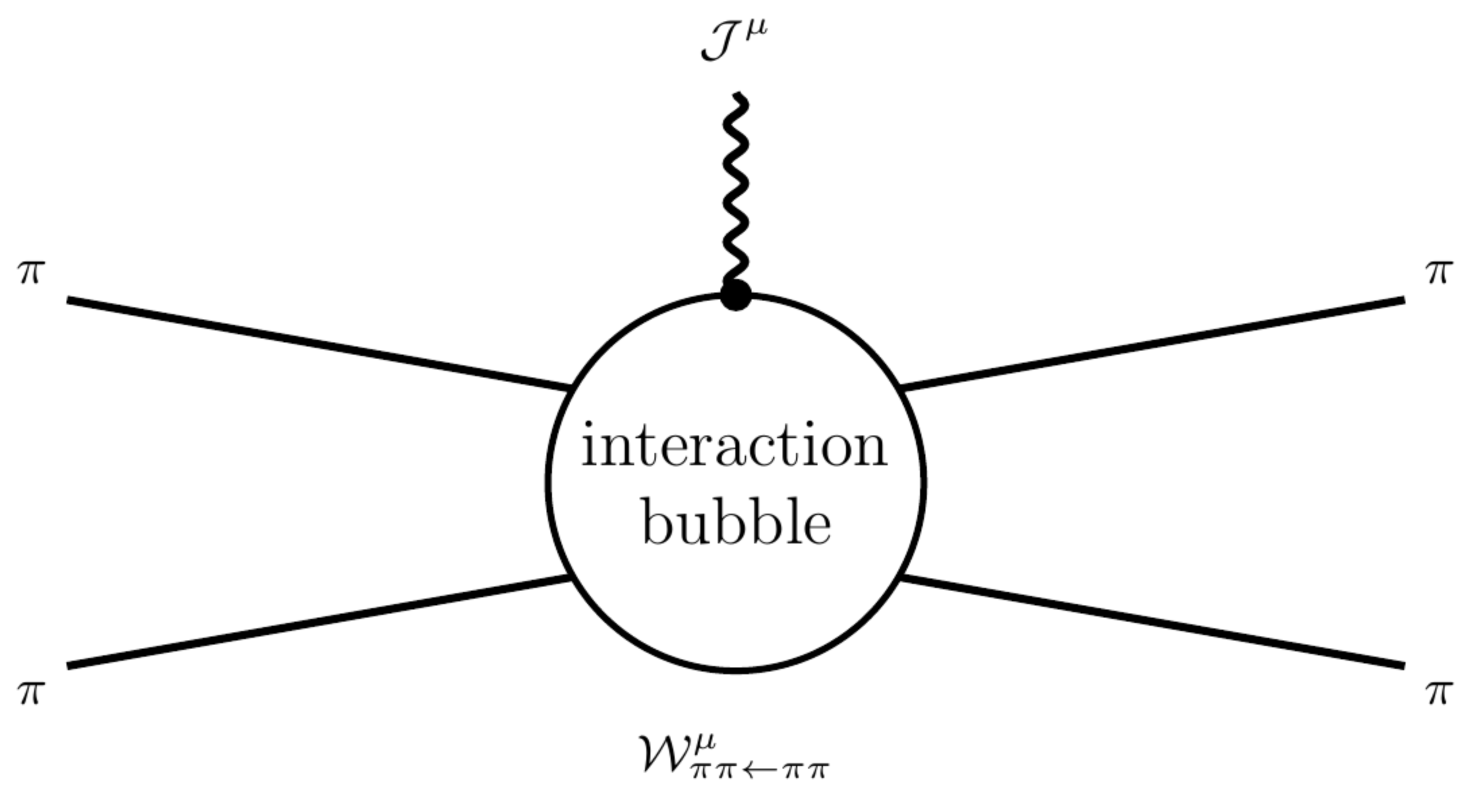
CHARTERED 1693

Contents

Two-particle matrix element

$$\pi\pi + \mathcal{J}^\mu \to \pi\pi, \qquad \langle \pi\pi, \mathrm{out} | \mathcal{J}^\mu | \pi\pi, \mathrm{in} \rangle$$

- interacting pi-pi
- in/out rescattering
- finite-volume norm
- smooth current kernel

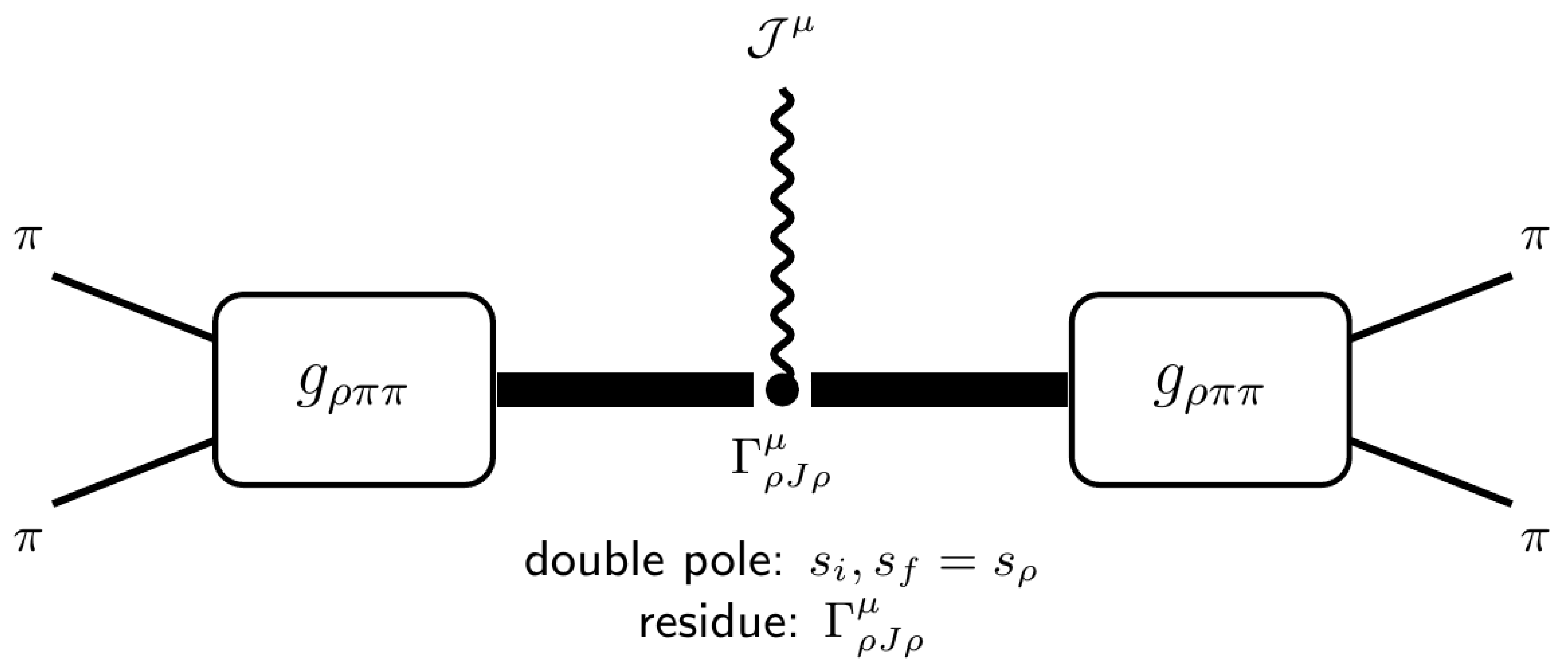


Refs: Briceno-Hansen 2016; Baroni et al. 2019; Briceno et al. 2021

Resonance form factor

$$\mathcal{W}^{\mu}_{\pi\pi\leftarrow\pi\pi} \sim \frac{g_{\rho\pi\pi}\,\Gamma^{\mu}_{\rho J\rho}(Q^2)\,g_{\rho\pi\pi}}{(s_f - s_\rho)(s_i - s_\rho)}\,, \qquad s_i\,, s_f \to s_\rho$$

- resonance pole in pi-pi amplitude
- double-pole limit in both channels
- residue: resonance form factor



Ref: Briceno-Jackura-Ortega-Gama-Sherman 2021

pp electroweak processes

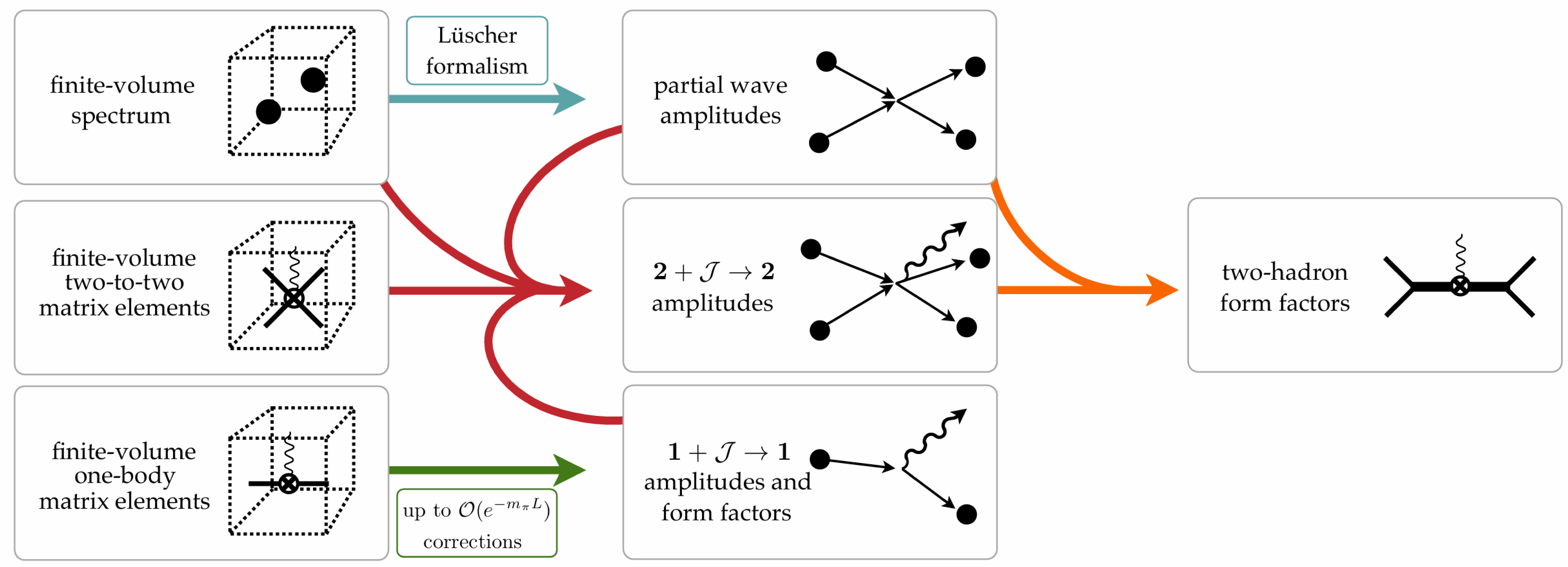
$$pp + \mathcal{J}_W^+ \to pn \quad \Longrightarrow \quad pp \to d\, e^+ \nu_e \quad (\text{final } pn \text{ pole})$$

$$pp + \mathcal{J}_{\text{em},\, Z} \to pp \quad \Longrightarrow \quad \text{two}-\text{proton electroweak response}$$

- two-proton scattering input
- charged and neutral currents
- deuteron pole from final proton and neutron
- two-nucleon current operators

Refs: Moscoso et al. 2026; Wang et al. 2026

Lattice QCD input



Refs: Luscher 1986; Briceno-Hansen 2016; Baroni et al. 2019; Briceno et al. 2021

Lattice QCD input

$$C_2(t) = \langle 0|\mathcal{O}_{\pi\pi}(t)\mathcal{O}^\dagger_{\pi\pi}(0)|0\rangle \quad (t \gg 1) \sim \sum_n |Z_n|^2 e^{-E_n(L)t}$$

$$C_3^\mu(t_f,t,t_i) = \langle 0|\mathcal{O}_f(t_f)\mathcal{J}^\mu(t)\mathcal{O}_i^\dagger(t_i)|0\rangle \quad (t_f \gg t \gg t_i) \sim Z_f Z_i^* e^{-E_f(L)(t_f-t)} \langle E_f, L|\mathcal{J}^\mu|E_i, L\rangle e^{-E_i(L)(t-t_i)}$$

- two-point fits -> finite-volume energies
- three-point fits -> current matrix
- FV matrix -> matching input

Refs: Luscher 1986; Briceno-Hansen 2016; Baroni et al. 2019; Briceno et al. 2021

Lüscher: spectrum -> scattering

$$\det\left[\mathcal{K}^{-1}(E^{\star}) + F(E, \mathbf{P}, L)\right] = 0$$

- moving-frame kinematics
- Energy spectrum input
- K-matrix fit
- reconstructed amplitude

$$i\mathcal{M}(s) = i\mathcal{K}(s)\,\frac{1}{1 - i\rho\,\mathcal{K}(s)}$$

$$\rho_{\ell' m_{\ell'};\ell m_\ell} = \delta_{\ell'\ell}\,\delta_{m_{\ell'} m_\ell}\,\frac{\xi\, q^{\star}}{8\pi\sqrt{s}}$$

Ref: Luscher 1986

Finite-volume problem

2-2 Finite-volume matrix element

$$L^3 \langle P_f, L | \mathcal{J}^\mu(0) | P_i, L \rangle = \sqrt{\mathcal{R}(P_f, L)\mathcal{R}(P_i, L)}\,\mathcal{W}^\mu_{L,\text{df}}(P_f, P_i, L)$$

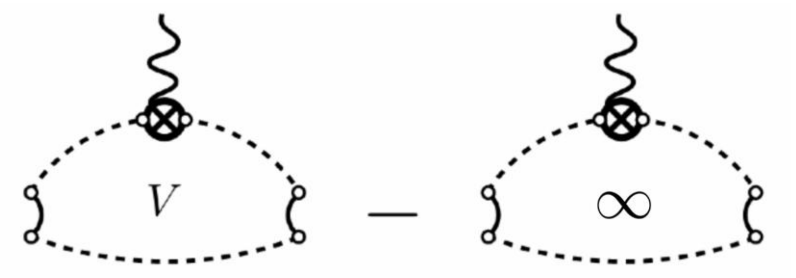
$$\mathcal{W}^\mu_{L,\text{df}}(P_f, P_i, L) = \mathcal{W}^\mu_{\text{df}}(P_f, P_i) + f(Q^2)\mathcal{M}(P_f^2)G^\mu(P_f, P_i, L)\mathcal{M}(P_i^2)$$

2-2 Infinite-volume divergence-free matrix element

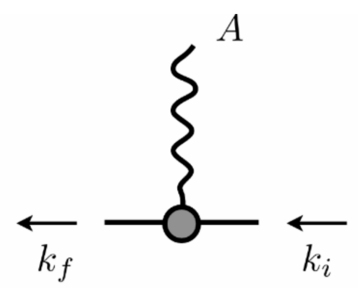
$$\mathcal{R}(E_n, \mathbf{P}) \equiv \lim_{E \to E_n} F(P, L)\left[\frac{(E - E_n)}{1 + \mathcal{M}(P^2)\, F(P, L)}\right]$$

- Lellouch Lushcer normalization factor
- current matrix-element matching

$G_\mu(P_f, P_i, L) =$



$f(Q^2) =$

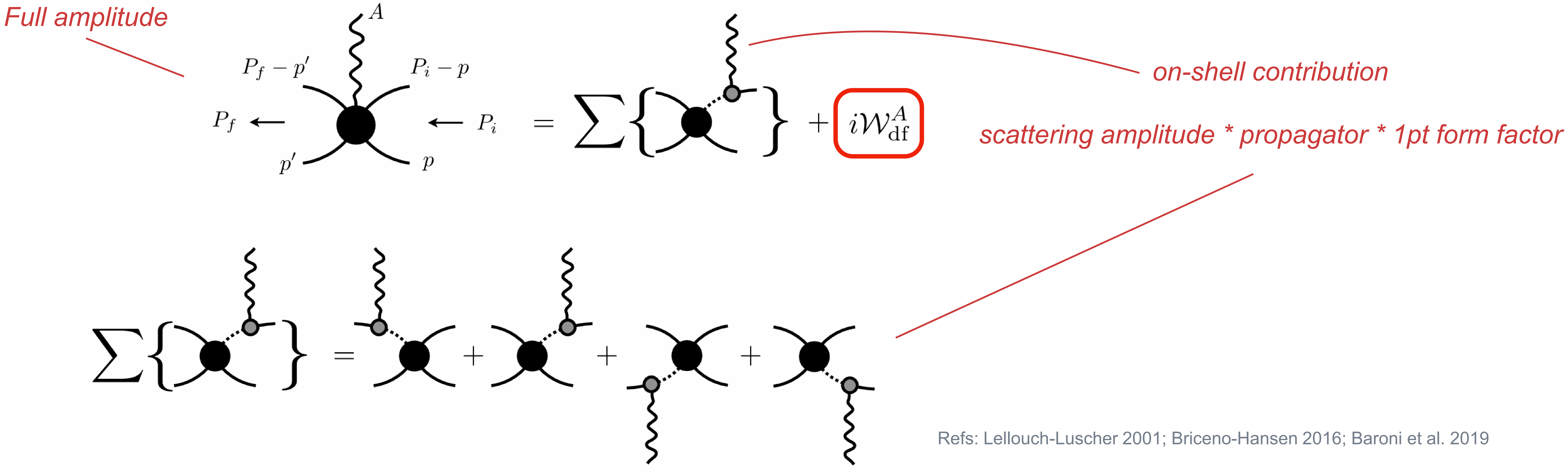


Refs: Lellouch-Luscher 2001; Briceno-Hansen 2016; Baroni et al. 2019

Finite-volume problem

$$L^3\langle P_f, L|\mathcal{J}^\mu(0)|P_i, L\rangle = \sqrt{\mathcal{R}(P_f, L)\mathcal{R}(P_i, L)}\mathcal{W}^\mu_{L,\mathrm{df}}(P_f, P_i, L)$$

$$\mathcal{W}^\mu_{L,\mathrm{df}}(P_f, P_i, L) = \mathcal{W}^\mu_{\mathrm{df}}(P_f, P_i) + f(Q^2)\mathcal{M}(P_f^2)G^\mu(P_f, P_i, L)\mathcal{M}(P_i^2)$$



Refs: Lellouch-Luscher 2001; Briceno-Hansen 2016; Baroni et al. 2019

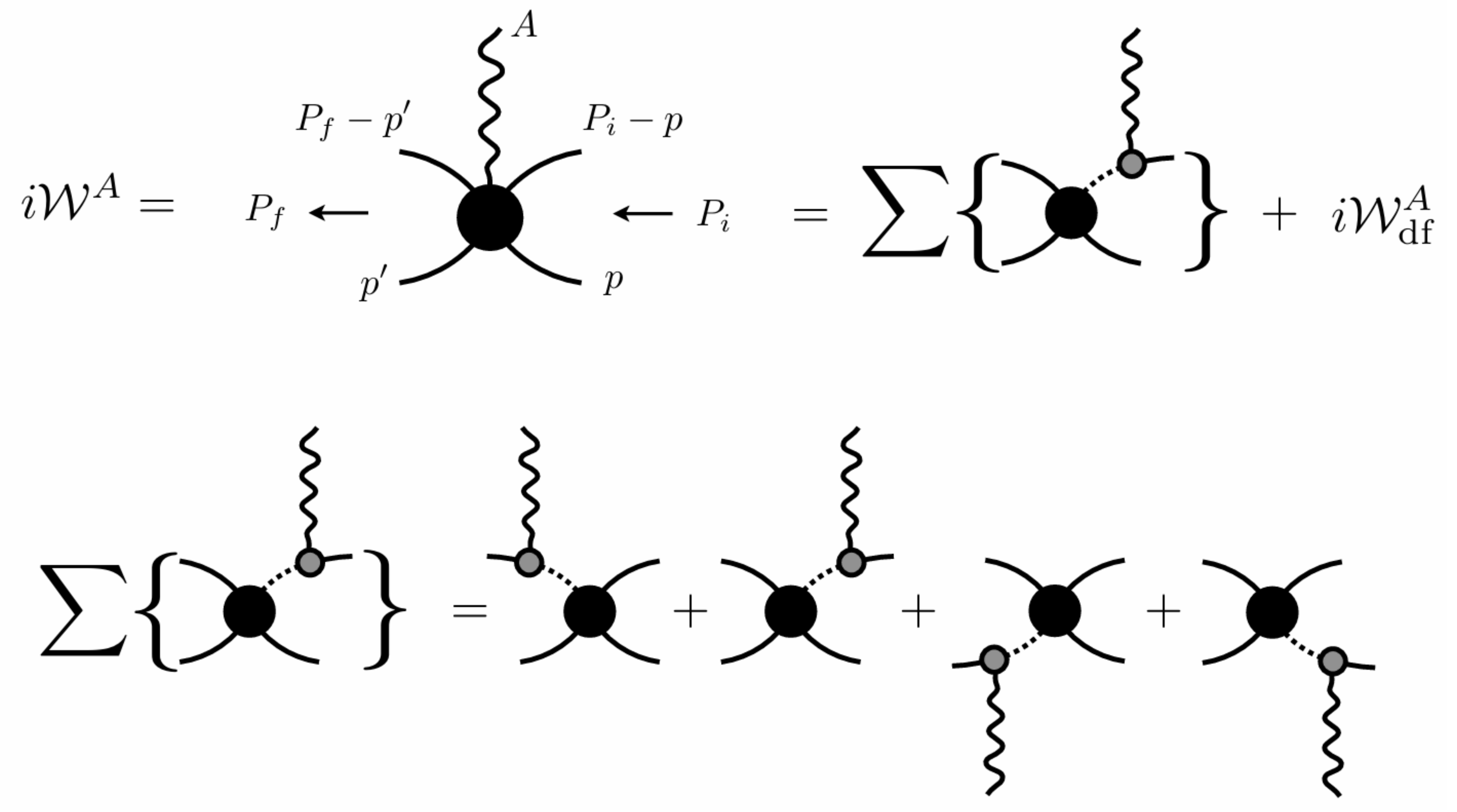
Finite-volume problem

$$i\mathcal{W}^{\mu}_{\rm df}(P_f;P_i) = i\mathcal{M}(s_f)\left[\mathcal{A}^{\mu}(P_f,P_i) + f(Q^2)\mathcal{G}^{\mu}(P_f,P_i)\right]\mathcal{M}(s_i),$$

Meromorphic!

non-analytic

- How to parametrize?
  Hint from EFT.



$$L^3\langle P_f, L|\mathcal{J}^{\mu}(0)|P_i, L\rangle = \sqrt{\mathcal{R}(P_f,L)\mathcal{R}(P_i,L)}\,\mathcal{W}^{\mu}_{L,\rm df}(P_f,P_i,L)$$

$$\mathcal{W}^{\mu}_{L,\rm df}(P_f,P_i,L) = \mathcal{W}^{\mu}_{\rm df}(P_f,P_i) + f(Q^2)\mathcal{M}(P_f^2)G^{\mu}(P_f,P_i,L)\mathcal{M}(P_i^2)$$

Refs: Lellouch-Luscher 2001; Briceno-Hansen 2016; Baroni et al. 2019

Chiral perturbation theory

- Chiral Symmetry Spontaneous Breaking $SU(2)_L \times SU(2)_R \to SU(2)_V$
- Low energy effective field theory for pions

$$\mathcal{L}_2 = \frac{f^2}{4} \mathrm{tr}\left[D_\mu U^\dagger D^\mu U + \chi^\dagger U + U^\dagger \chi \right]$$

Definition

$$U \in SU(2) \quad \chi = M^2 \mathbb{1}_{2\times 2}$$

$$D_\mu U = \partial_\mu U + ieA_\mu [Q, U] \quad Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

Chiral perturbation theory

- Chiral Symmetry Spontaneous Breaking $SU(2)_L \times SU(2)_R \to SU(2)_V$
- Low energy effective field theory for pions

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}\left[D_\mu U^\dagger D^\mu U + \chi^\dagger U + U^\dagger \chi\right]$$

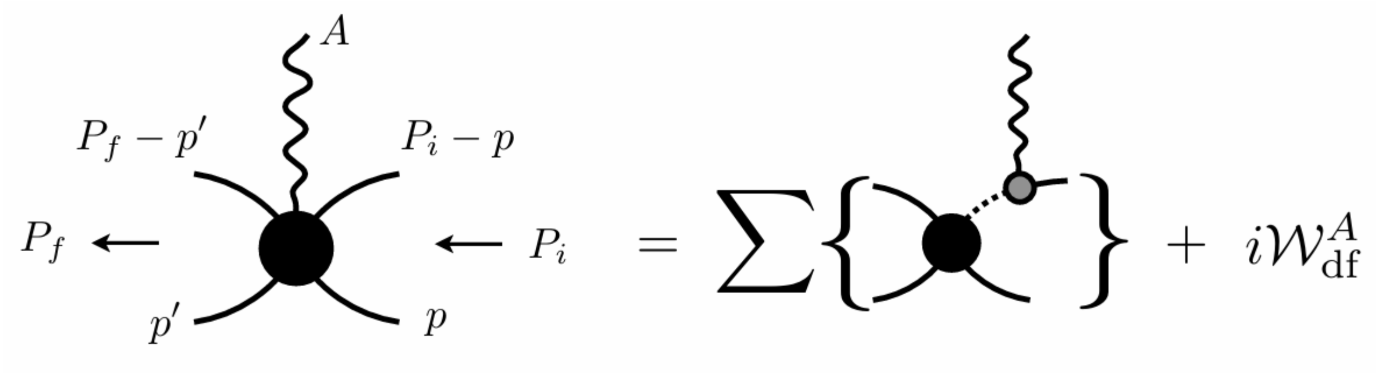
Definition

$$U \in SU(2) \quad \chi = M^2 \mathbb{1}_{2\times 2}$$

$$D_\mu U = \partial_\mu U + ieA_\mu [Q, U] \quad Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

Exponential parametrization: $U(x) = \exp\left(\frac{i\Phi(x)}{f}\right), \quad \Phi(x) = \sum_{a=1}^{3} \tau^a \pi^a = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$

Square root parametrization: $U(x) = \sqrt{1 - \frac{\Phi^2}{f^2}} + i\frac{\Phi}{f}, \quad \Phi = \sum_{a=1}^{3} \tau^a \pi^a$

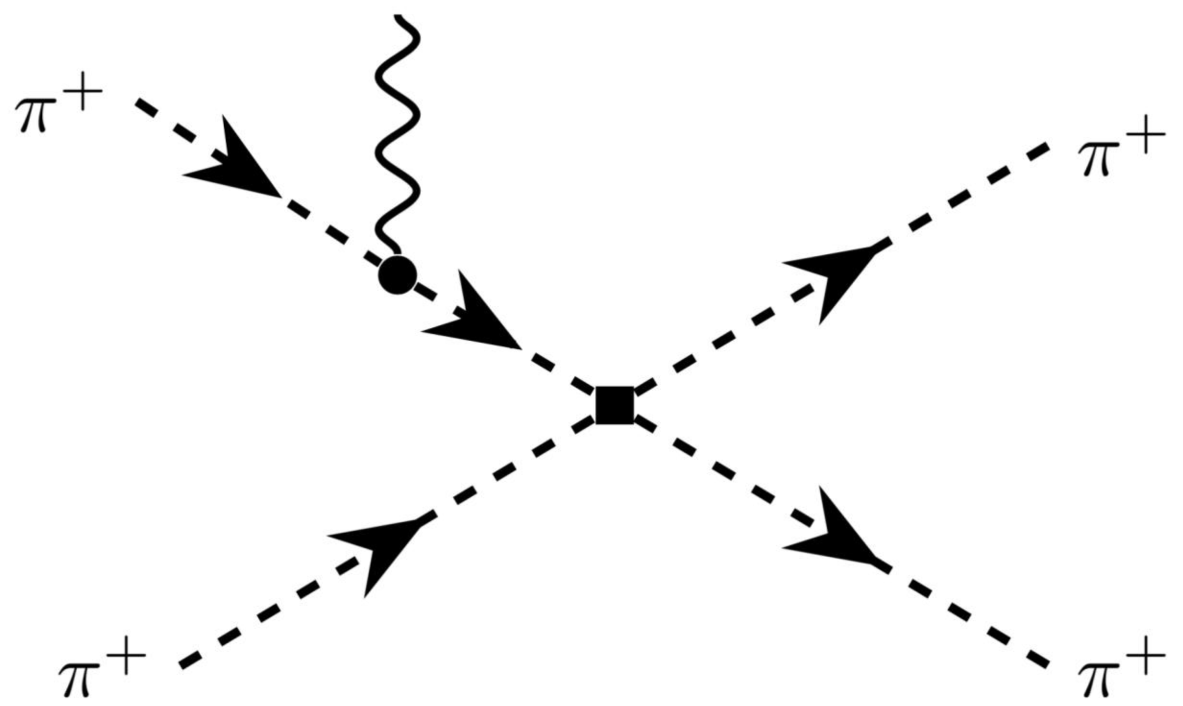


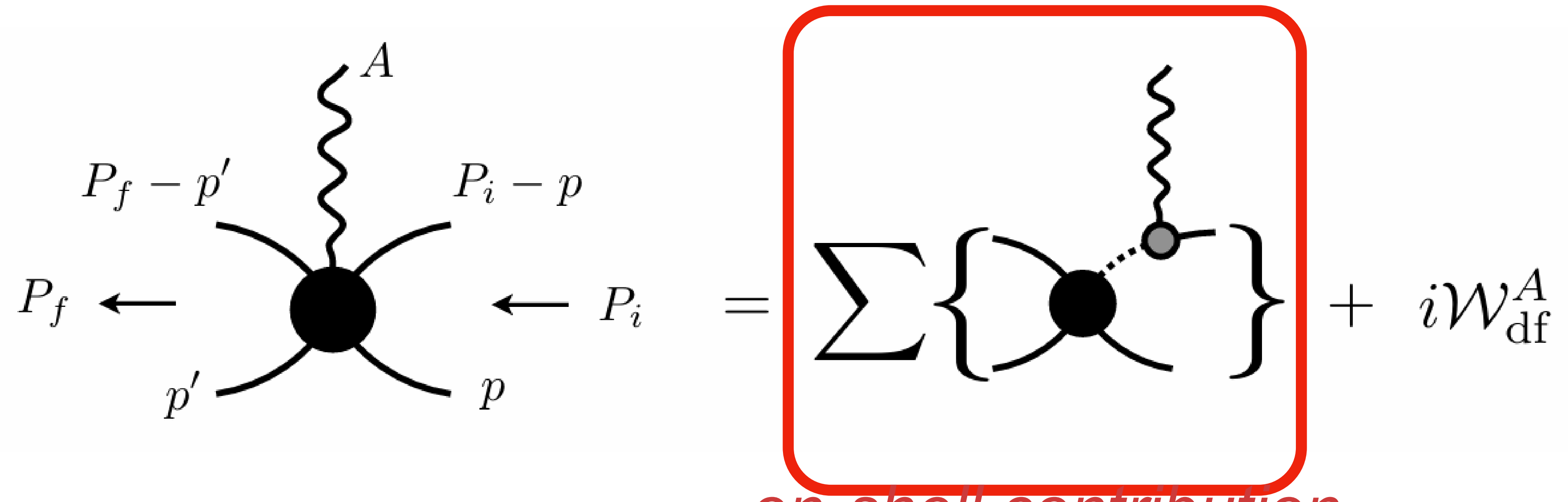
Physics observables are independent of parametrization!

Chiral perturbation theory

- Typical diagram for $\pi^+\pi^+ \xrightarrow{\mathcal{J}^{\mu}_{\mathrm{EM}}} \pi^+\pi^+$

Similar but not equal.
Kinematic poles cancel





on-shell contribution

*scattering amplitude * propagator * 1pt form factor*

Vertices generated via FeynRules.

Diagrams generated by FeynArts.

Amplitudes simplified with FeynCalc.

Loop Integrals calculated by PackageX.

Results

$$\mathcal{A}_{22}^{\mu} = \mathcal{A}_{22,\,\mathrm{LO}}^{\mu} + \mathcal{A}_{22,\,\mathrm{NLO}}^{\mu,\,\mathrm{loop}} + \mathcal{A}_{22,\,\mathrm{NLO}}^{\mu,\,\mathrm{ct}}$$

$$J_{\mathrm{em}}^{\mu} \propto \tau_3\,, \qquad I_J = 1$$

$$\pi^0\pi^0 + \gamma^* \to \pi^0\pi^0: \quad \mathcal{A}_{\mathrm{LO}}^{\mu} = 0, \quad \mathcal{A}_{\mathrm{NLO}}^{\mu} \neq 0$$

$$\mathrm{LO}:\ \ell_i = \ell_f = 0, \qquad \mathrm{NLO}:\ [\mathcal{A}_{22}^{\mu}]_{\ell_f I_f;\,\ell_i I_i}$$

- isospin channels coupled by current
- neutral channel generated beyond leading order
- higher partial waves enter through loops
- near threshold hierarchy: S wave largest

Summary

- two-body current amplitude target
- lattice data -> current kernels
- 1loop ChPT results, to be compared with lattice data

References

Luscher, Commun. Math. Phys. 105, 153 (1986).
Lellouch and Luscher, Commun. Math. Phys. 219, 31 (2001), hep-lat/0003023.
Briceno, Hansen, Walker-Loud, Phys. Rev. D 91, 034501 (2015), arXiv:1406.5965.
Briceno and Hansen, Phys. Rev. D 94, 013008 (2016), arXiv:1509.08507.
Baroni, Briceno, Hansen, Ortega-Gama, Phys. Rev. D 100, 034511 (2019), arXiv:1812.10504.
Briceno, Jackura, Ortega-Gama, Sherman, Phys. Rev. D 103, 114512 (2021), arXiv:2012.13338.
Moscoso, Ortega-Gama, Briceno, Jackura, Kacir, Nicholson, arXiv:2602.20373.
Wang, Feng, Jian, Jin, Liu, arXiv:2603.09554.
Ren et al., 1+1D O(3) sigma-model two-particle transition amplitude, in prep.
Gasser and Leutwyler, Annals Phys. 158, 142 (1984).

Thanks!