6/10/2026

Studying the Unitary Fermi Gas using Discretized Pion-less EFT on the Lattice.

Hunter Duggin (UNC), Charles Kacir (UNC), Amy Nicholson (UNC, LBNL), Dmitra Pefkou (LBNL, UC Berkeley)

The University of North Carolina at Chapel Hill

Outline

Unitary Fermi Gas

Outline

Unitary Fermi Gas

Theoretical Methods

Outline

Unitary Fermi Gas

Theoretical Methods

Effective Mass & Current Matrix Elements

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Unitary Fermi Gas

Theoretical Methods

Effective Mass & Current Matrix Elements

Conclusion & Future Work

The Unitary Fermi Gas

Optical Theorem

- We know from constraints given by the unitarity of the S-matrix, the **S-wave scattering amplitude** is given by

$$\mathcal{A} = \frac{4\pi}{M} \frac{1}{p \cot \delta_0 - ip}$$

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$$\mathcal{A} = \frac{4\pi}{M}\frac{1}{p\cot\delta_0 - ip}$$

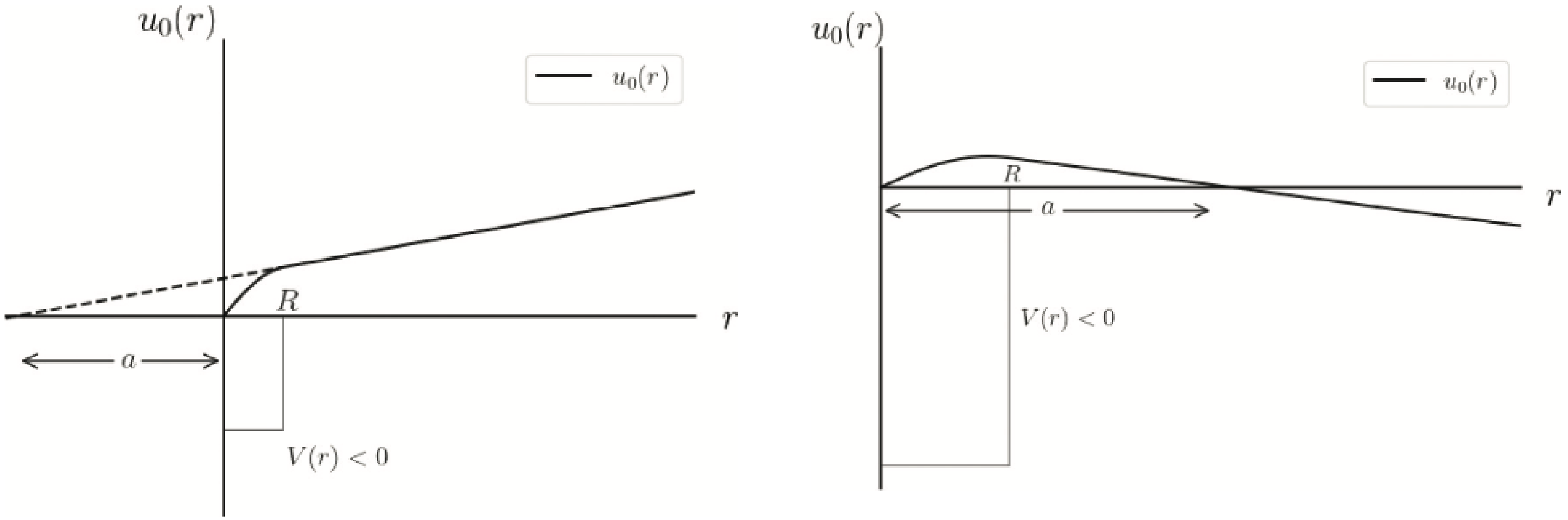
- At low energies it makes sense to use the **Effective range** expansion

$$p\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0\, p^2 \sum_{i=0}^{\infty} (r_i^2\, p^2)^i$$

- Where **a** is the **scattering length**, r_0 is known as the **effective range** and r_i are the **shape parameters**

Unitarity

- **Unitarity** is the special limit where the scattering length goes to infinity, and the effective range goes to zero.



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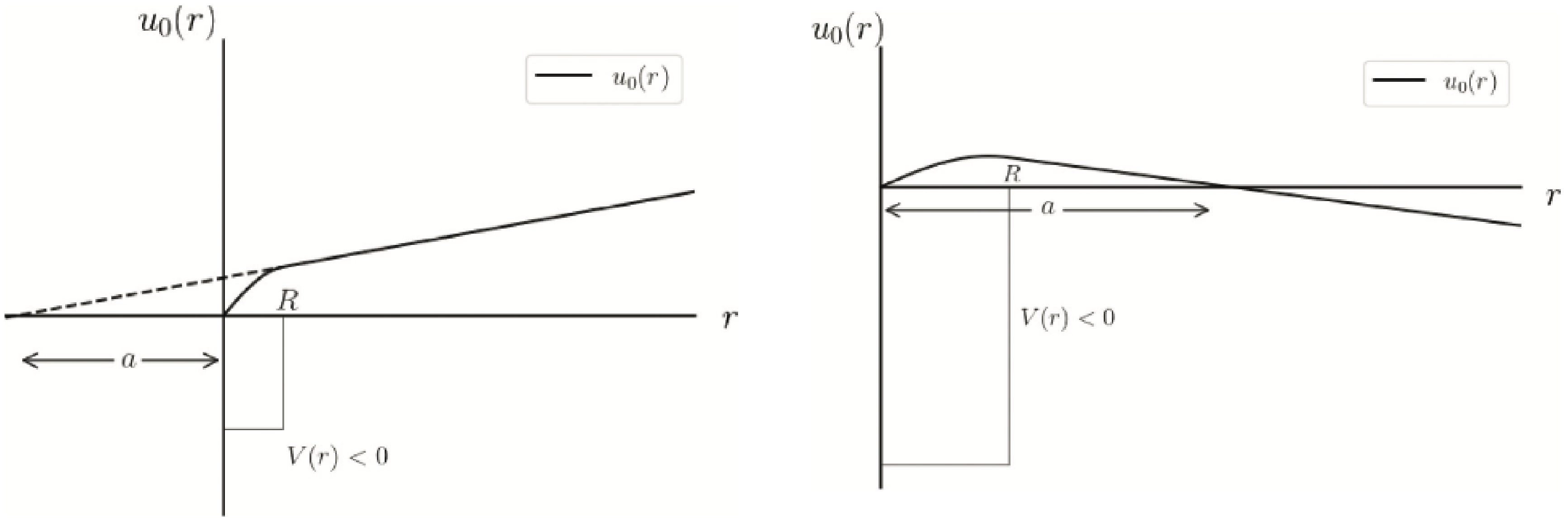

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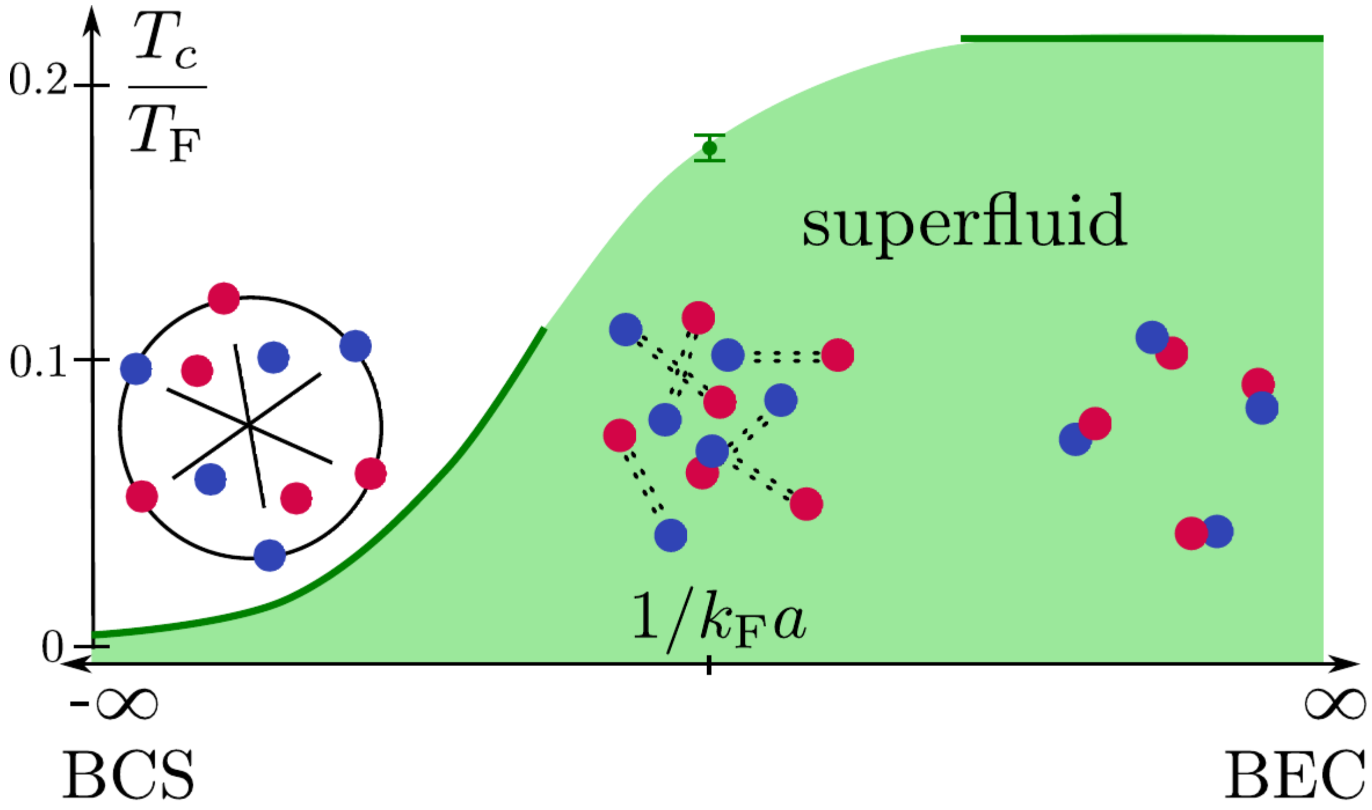
The Unitary Fermi Gas

- The **UFG** is a gas of Fermions tuned to the **unitary limit**
- Applicable to systems with large numbers of fermions (like neutrons!)



Olga Goulko

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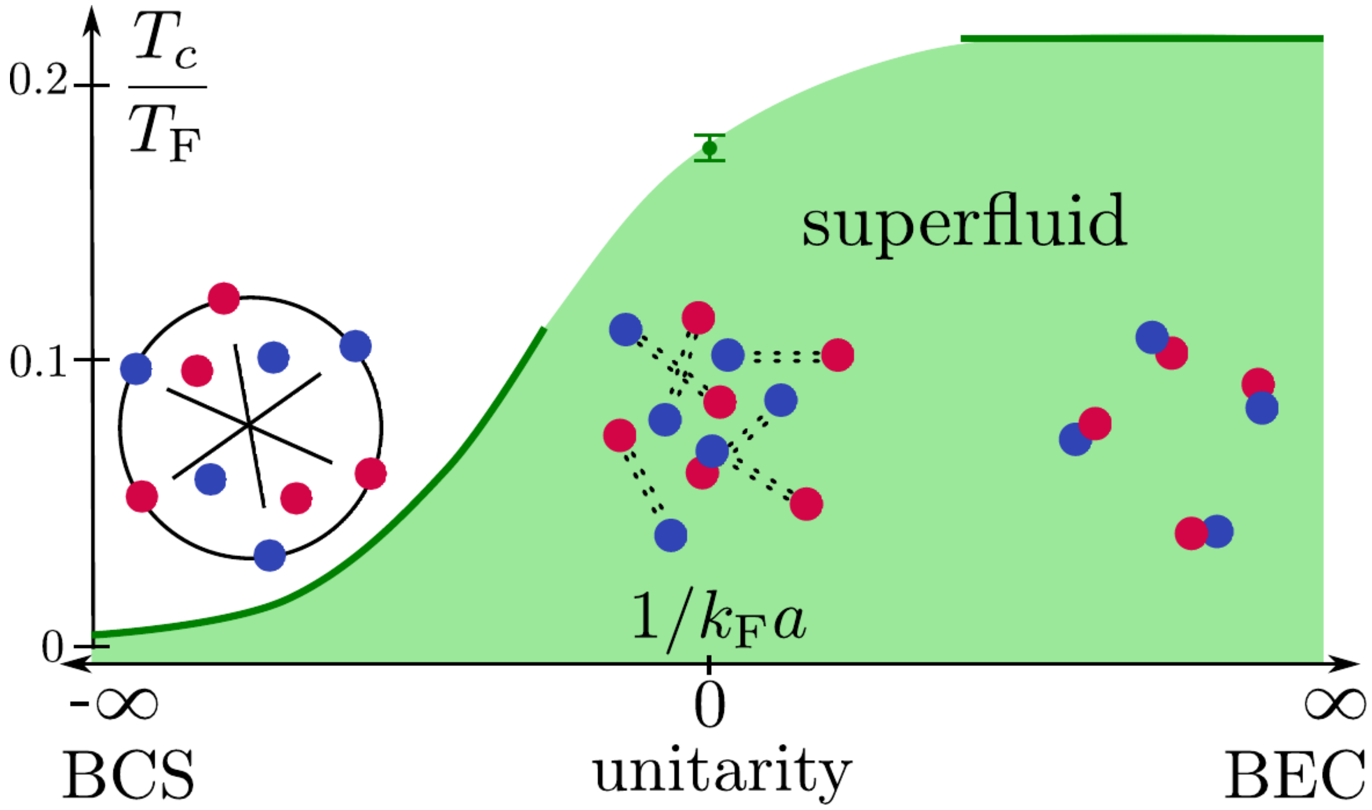
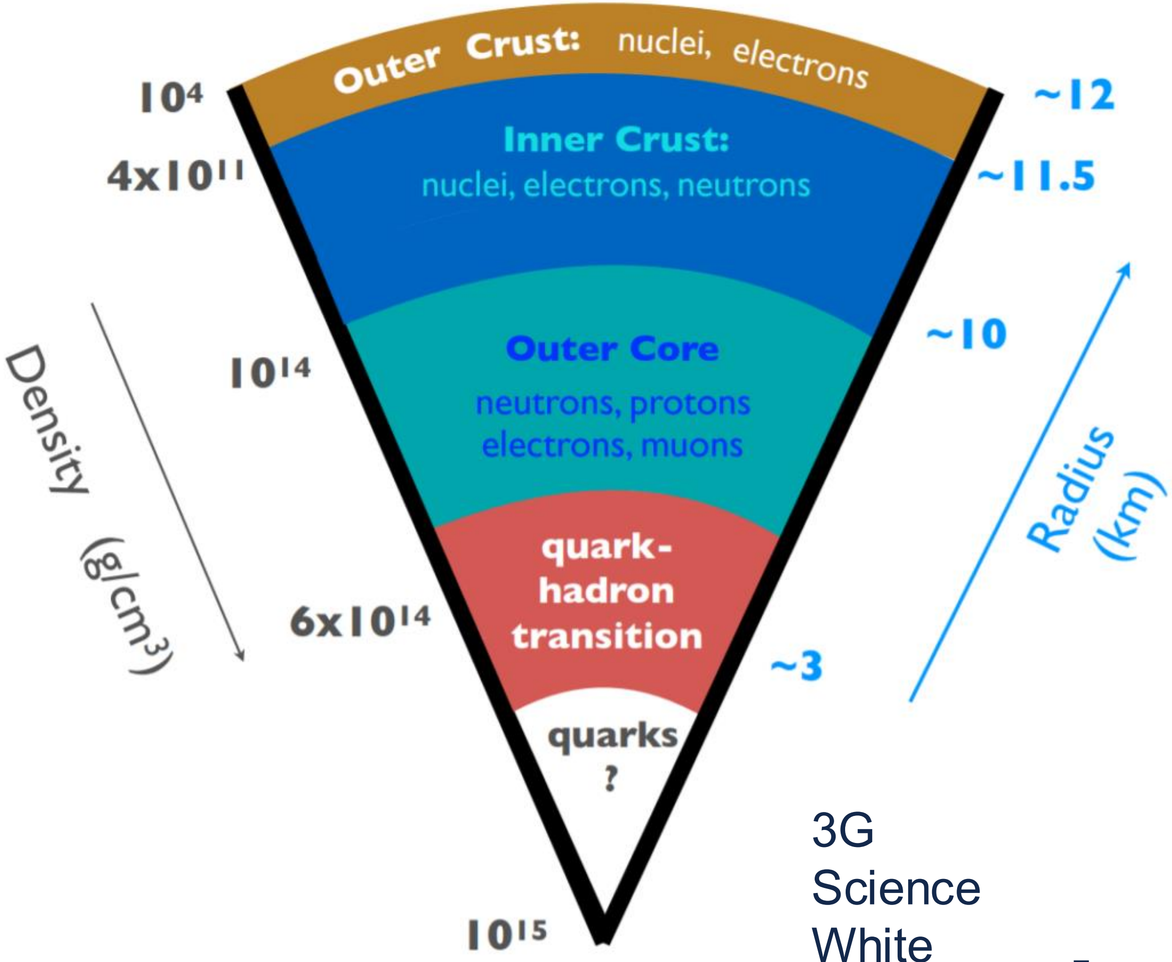


[Olga Goulko](https://)

Neutron Stars

- The **Outer Core** region of a neutron star

ESA

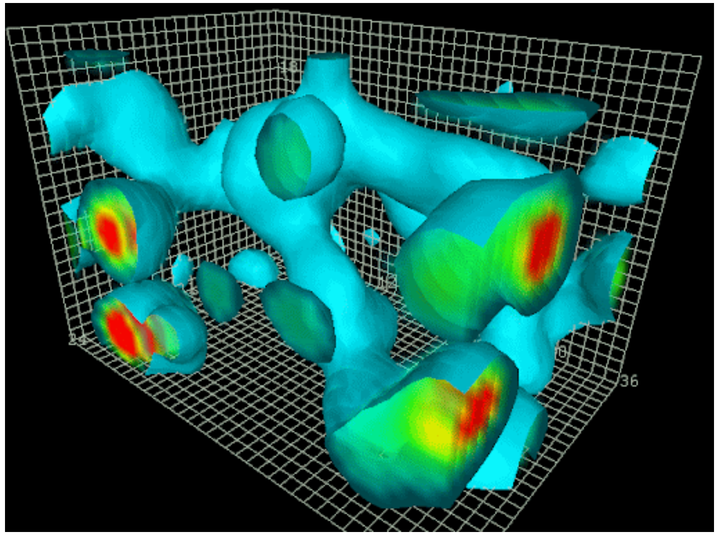


3G Science White

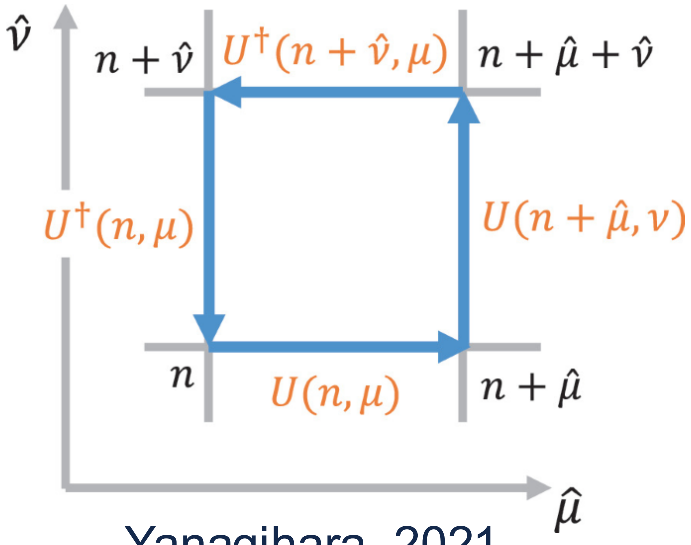
Theoretical Methods: Lattice EFT

Lattice Field Theory

- Lattice field theory is a **non-perturbative** tool used to study properties of **strongly interacting** field theories.
- Allows for **direct evaluation** of the path integral using numerical techniques.
- Has **well documented** systematic **errors**.
- Naturally **regulates** the theory.



Leinweber – QCD Vacuum



Yanagihara, 2021

Pion-less EFT

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) \psi + g_0 \left(\psi^\dagger \psi \right)^2 + \cdots$$

- Effective field theory involving two point-like nucleons interacting via contact interactions.

- The scattering amplitude is a geometric sum of **bubble diagrams**

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Geometric Series →

$$\frac{\sum C_{2n}(\mu)\, p^{2n}}{1 - I_0 \sum C_{2n}(\mu)\, p^{2n}} = \frac{4\pi}{M} \frac{1}{p \cot \delta - ip}$$

← **Optical Theorem**

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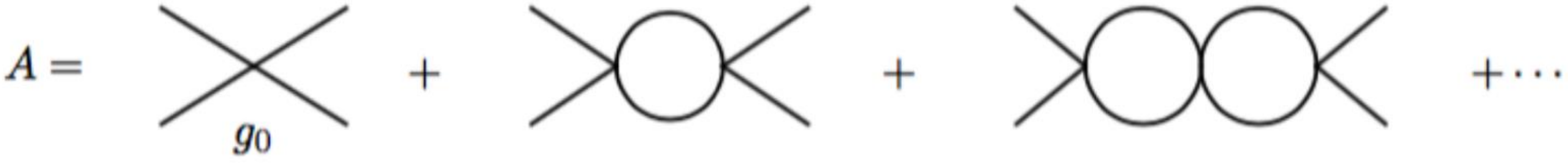

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Optical Theorem

$$I_0 = \left(\frac{\mu}{2}\right)^{4-D} \int \frac{d^{D-1}\mathbf{q}}{(2\pi)^{D-1}} \frac{1}{E - \frac{|\mathbf{q}|^2}{M} + i\epsilon}$$

Pion-less EFT: Finite to Infinite Volume

- Looking for poles in the amplitude, we see

$$\frac{1}{\sum C_{2n}(\mu)\; p^{2n}} - \mathrm{Re}(I_0^{(PDS)}(L)) = 0$$

$$I_0^{(PDS)}(L) = -\frac{M}{4\pi}\mu + \frac{1}{L^3}\sum_{\mathbf{k}}^{\Lambda}\frac{1}{E - \frac{|\mathbf{k}|^2}{M}} + M\int^{\Lambda}\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{|\mathbf{k}|^2}$$

- This relates the **discrete energy spectrum** to the **infinite volume scattering amplitude**
- Analogous to the **Lüscher** quantization condition
- We can solve this for the states to see the following eigenvalue problem

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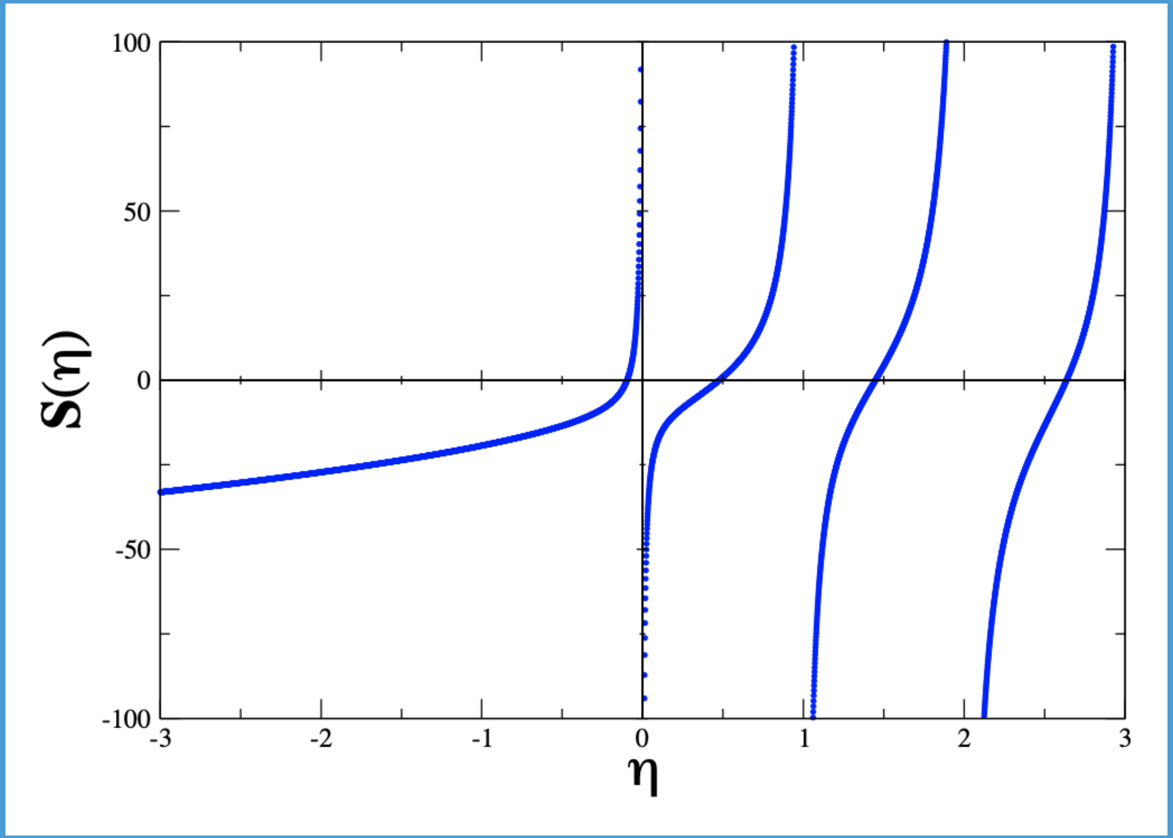
$$p\cot\delta(p) = \frac{1}{\pi L}\,\mathbf{S}\left(\left(\frac{Lp}{2\pi}\right)^2\right) \quad \text{where} \quad \mathbf{S}(\eta) \equiv \sum_{\mathbf{j}}^{\Lambda_j} \frac{1}{|\mathbf{j}|^2 - \eta} - 4\pi\Lambda_j$$

Pion-less EFT: States



S.R. Beane, P.F.Bedaque, A. Parreno, M.J. Savage,
Two Nucleons on the Lattice

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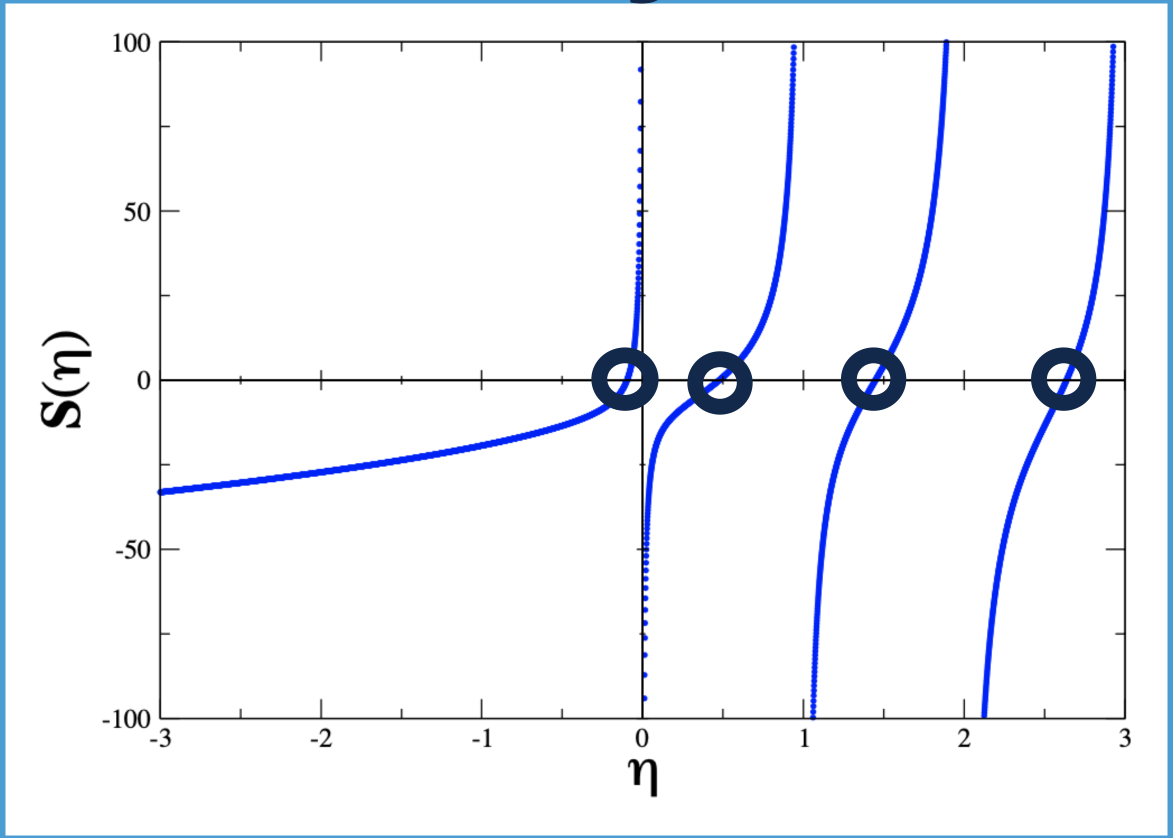

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Pion-less EFT: States

But Wait! There are Fermions!

As seen in Kostas Originos's lectures, **Fermions have issues on the lattice**

We introduce a bosonic auxiliary field that mediates the interaction

$$\mathcal{L}_{int} \rightarrow \sqrt{b_t g_0}\phi_{t,\vec{x}}\psi^\dagger_{t,\vec{x}}\psi_{t-1,\vec{x}}$$

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Common choices are

- $\mathbb{Z}_2$ Valued field ⇒evaluating the path integral amounts to summing over ±1
- Gaussian field*
- Compact continuous

$$Z = \int \mathcal{D}\phi\mathcal{D}\psi^\dagger\mathcal{D}\psi\rho[\phi]e^{-S[\phi,\psi^\dagger\psi]} \qquad \rho[\phi] = \begin{cases} \prod_n e^{-\phi_n^2/2} & \text{Gaussian} \\ \prod_n \frac{1}{2}\left(\delta_{\phi_{n,1}} + \delta_{\phi_{n,-1}}\right) & Z_2 \\ \prod_n \left(\theta(-\pi+\phi_n)\theta(\pi-\phi_n)\right) & \text{compact continuous} \end{cases}$$

Discretizing Pion-less EFT

We start by discretizing the pion-less EFT action in the following way

$$S = \frac{1}{b_t} \sum_{t,t'} \psi^\dagger_{t'} \left[K(\phi)\right]_{t't} \psi_t$$

$$K(\phi) \equiv \begin{pmatrix} D & -X(\phi_{T-1}) & 0 & 0 & . & . & . \\ 0 & D & -X(\phi_{T-2}) & 0 & . & . & . \\ . & . & . & . & & & \\ . & . & . & & . & & \\ . & . & . & & & D & X(\phi_0) \\ X(\phi_T) & . & . & & & 0 & D \end{pmatrix} .$$

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- K is a T x T block matrix, where each entry is an L^3 x L^3 matrix in itself
- D is a discretized kinetic operator $D \equiv 1 - \frac{b_s \nabla^2_L}{2})$
- $X(\phi_t)$is a rewriting of the Fermion interaction in terms of a **bosonic auxiliary field** that mediates the interaction

$$X(\phi_t) \equiv 1 - \sqrt{g_0}\phi_t$$

- We also choose ϕ_0 to only live on the **temporal links** to make our lives simpler

Tuning g_0 : Connecting to the Spectral Decomposition

From statistical mechanics:

$$\begin{aligned} C(\tau) &= \langle \Psi_{\mathrm{snk},2} | e^{-H\tau} | \Psi_{\mathrm{src},2} \rangle \\ &= \langle \Psi_{\mathrm{snk},2} | \left[e^{-H} \right]^{\tau} | \Psi_{\mathrm{src},2} \rangle \end{aligned}$$

Now we know that the logarithms of the eigenvalues of the transfer matrix give us the 2 body spectrum!

In principle, we could diagonalize $\mathcal{T}$, and its eigenvalues would be the 2-body spectrum.

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$$\langle pq | \mathcal{T} | p'q' \rangle = \frac{\delta_{pp'}\delta_{qq'} + \frac{g_0}{V}\delta_{p+q,p'+q'}}{\sqrt{\xi(p)\xi(q)\xi(q')\xi(p')}}$$

Where

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We can compare this result to the one discussed earlier using Lüscher methods to **tune the coupling**, g_0

Effective Mass & Current Insertion Matrix Elements

Effective Mass

In general, the transfer matrix with N ≥4 **cannot be solved** because the dimension increases with particle number

This is why we form **N-point** correlation functions in the following way

$$C_N(\tau) = \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-S[\psi^\dagger,\psi,\phi]}\Psi^{(b)}_{b_1\cdots b_N}(\tau)\Psi^{\dagger(a)}_{a_1\cdots a_N}(0) \ ,$$

where

$$\Psi^{(a)\dagger}_{a_1\cdots a_N}(\tau) = \int dx_1\cdots dx_N A^{(a)}(x_1\cdots x_N)\psi_{a_1}(x_1,\tau)\cdots\psi_{a_N}(x_N,\tau)$$

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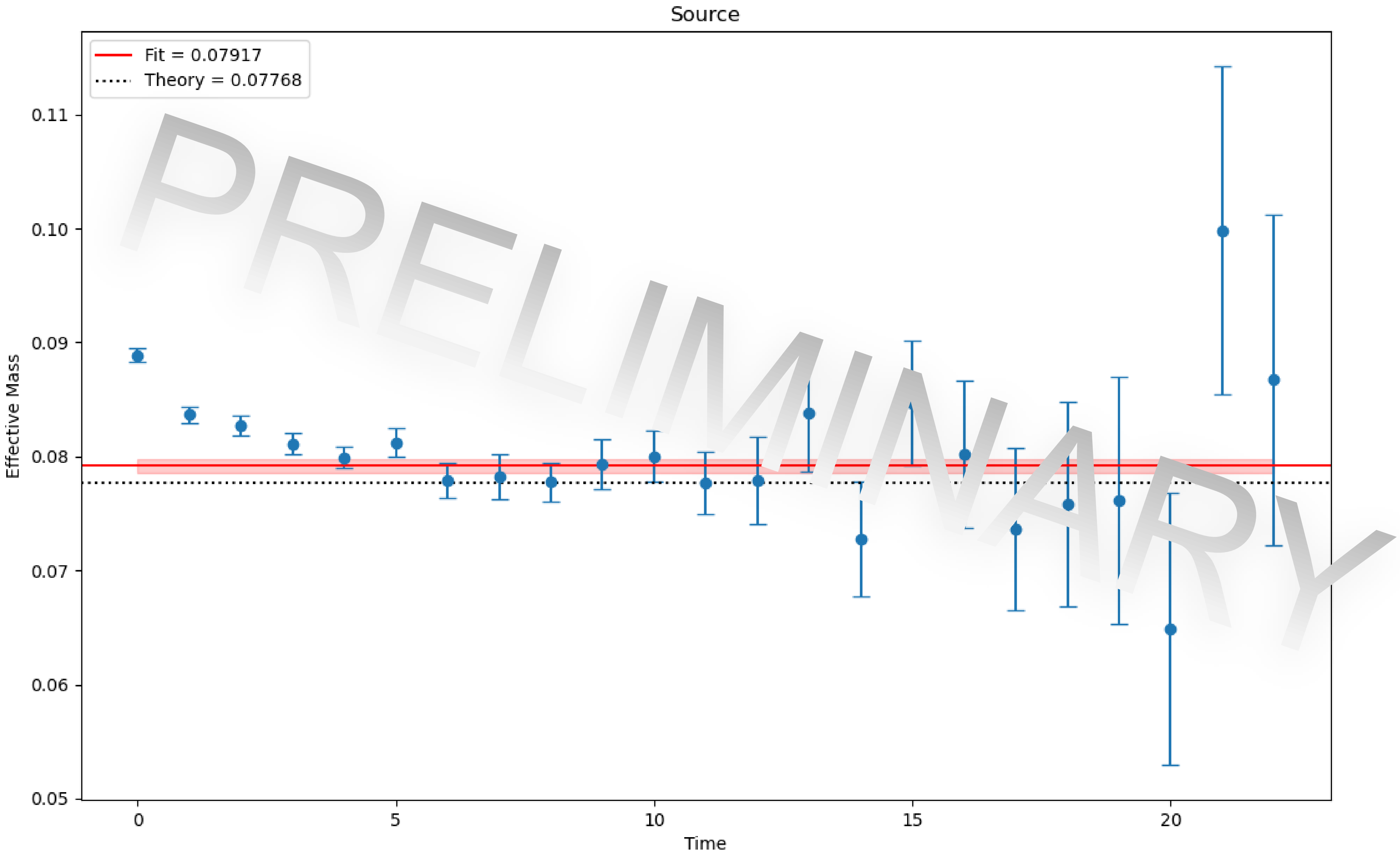
Using this correlation function, we can calculate the system's **Effective Mass**

$$M_{\text{eff}}(\tau) \equiv \ln \frac{C(\tau)}{C(\tau+1)} \xrightarrow[\tau\to\infty]{} E_0$$


Source
Fit = 0.07917
Theory = 0.07768
0.11
0.10
0.09
0.08
0.07
0.06
0.05
Effective Mass
0
5
10
15
20
Time
PRELIMINARY

Gravitational (Mechanical) Form Factors

- Related to the **Lorentz decomposition** of **EMT current matrix** elements
- GFFs can be related to **GPD** physics (through computing GPD moments)

$$J^q(t) = \frac{1}{2}\int_{-1}^{1} dx x[H^q(x,\xi,t) + E^q(x,\xi,t)]$$

$$M_2(t) + \frac{4}{5}d_1(t)\xi^2 = \frac{1}{2}\int_{-1}^{1} dx x H^q(x,\xi,t)$$

e.g. Proton form factors related to GPD moments (Girod, 2018)

- Following the methods discussed in Kostas Originos's lectures, we can extract matrix elements relevant to GFFs for the **unitary fermi gas** by taking **constrained ratios** of 3-pt and 2-pt functions!

Future Work

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- Change **wavefunction parameters** to make the effective mass plateau

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- **Lorentz decompose** the EMT and interpret the meaning of the GFFs

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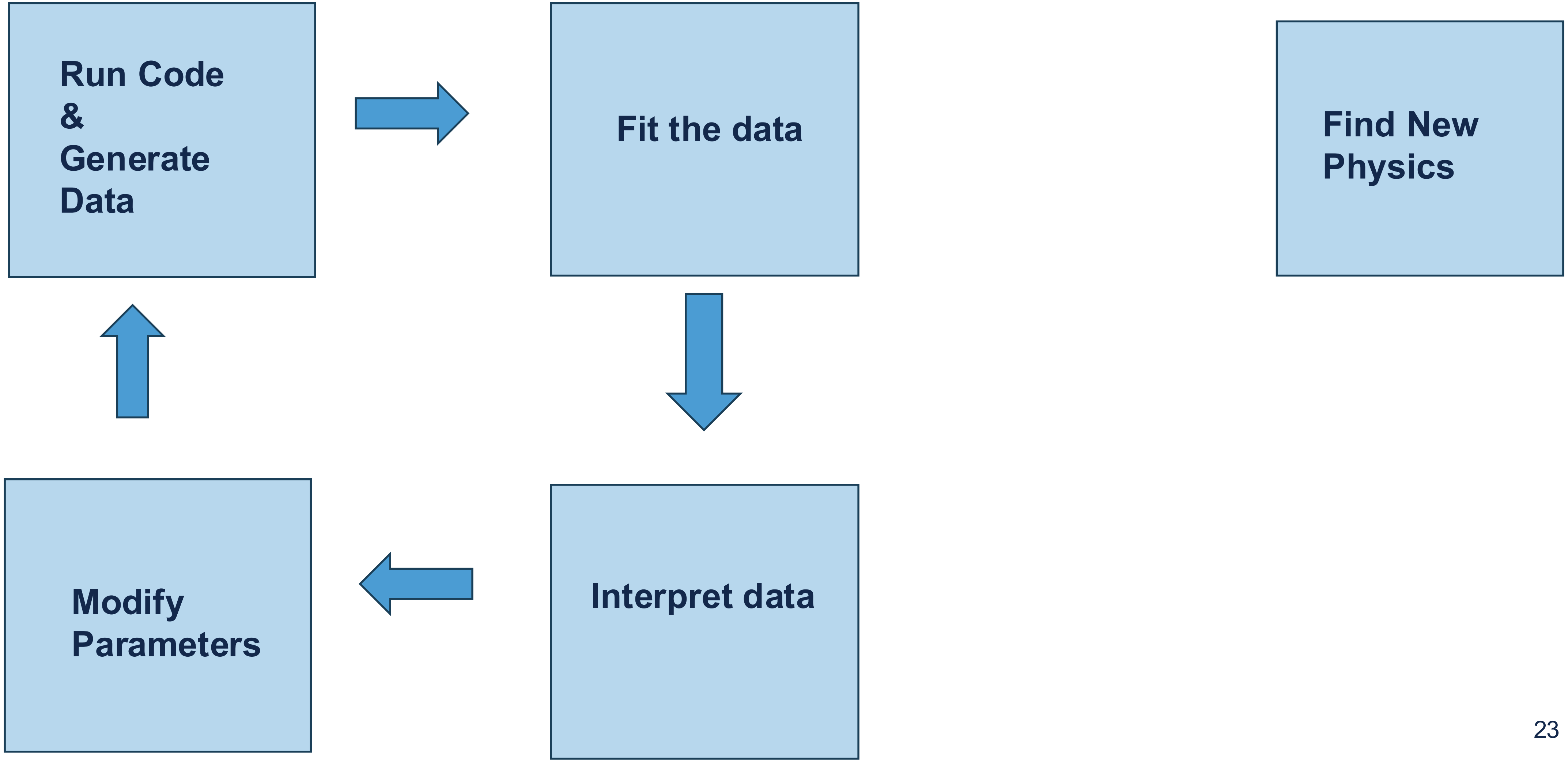
- Change **wavefunction parameters** to make the effective mass plateau

- **Lorentz decompose** the EMT and interpret the meaning of the GFFs

- Extract GFFs using the **ratio of 3pt and 2pt** functions

Where are we at now?

Where we are actually at



Thank You! Questions?

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BONUS SLIDES

Review: Field Theory

- Field theory concerns itself with calculating probability amplitudes for relativistic quantum systems
- The probability amplitude for a particle to propagate from one point (x) to another point (y) is called the **2-point function.**

$$G_{(2)}(x,y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

- In a free theory, the 2-point function is known as a **propagator**
- You can find propagators by direct integration, or by finding the Green function for the equations of motion.
- If we have an **Interacting theory** the calculation of 2-point functions becomes more complicated

$$_H\langle 0|T\{\phi_H(x)\phi_H(y)\}|0\rangle_H = \frac{_I\langle 0|T\{\phi_I(x)\phi_I(y)e^{iS_{int}}\}|0\rangle_I}{_I\langle 0|T\{e^{iS_{int}}\}|0\rangle_I}$$

- Typically, we expand the exponential in the above expression and apply **Wick's Theorem** to relate the time ordered vacuum expectation values (later time to the left) to normal ordered expectation values (all creation operators are to the left of annihilation operators).
- This process gives us a diagrammatic expansion of the amplitude we are interested in
- This can be generalized for **n-point functions** as well

Review: Path integrals

- Another way to calculate these amplitudes is to use the **path integral formulation** of QFT
- We define the generating free functional Z_0 [J] as
- $$Z_0[J(x)] \equiv < 0|T\{e^{i\int d^4x J(x)\phi(x)}\}|0 > \rightarrow N \int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}_{free})+J(x)\phi(x)}$$

Such that $$\frac{(-i)^2\delta^2 Z[J]}{\delta J(z_1)\delta J(z_2)}|_{J=0} = \langle 0|T\{\phi(z_1)\phi(z_2)\}|0\rangle$$

If we are in an interacting theory, the generating functional is written as

$$Z[J] = N' \int \mathcal{D}\phi e^{i\int d^4x \mathcal{L}_{int}(\frac{(-i)\delta}{\delta J})} Z_0[J]$$

In analogy with statistical mechanics, Z is sometimes known as the **partition function**. We can the expand the exponentials in powers of the coupling and take derivatives of Z w.r.t. J to get the interacting n-point functions in terms of free propagators.

Review: Divergences

- When performing such calculations, either using Wick's theorem or path integrals, **divergences** arise in the calculation.
- We can **regulate** these divergences however we like to make the integrals finite.

e.g. ϕ^4 scalar theory

$$\int \frac{d^4p}{(2\pi)^4} G_2(p) \to \infty$$

$$\int_0^\Lambda \frac{d^4p}{(2\pi)^4} G_2(p) \propto \Lambda^2 + m^2 \log\left(\frac{m^2}{m^2+\Lambda^2}\right)$$

Momentum (UV) cutoff regulator

$$\int_0^\infty \frac{\mu^\epsilon d^{4-\epsilon}p}{(2\pi)^{4-\epsilon}} G_2(p) \propto (m^2)^{1-\frac{\epsilon}{2}} \pi \csc\left(\frac{\pi\epsilon}{2}\right)$$

***Dimensional Regularization** ($\mathrm{d} = 4 - \epsilon$)

When does perturbation theory fail?

- PT fails when the coupling is > 1.
- Terms higher order in the coupling become more important.
- e.g. Low energy QCD, high energy QED, various many body systems

What non-perturbative techniques can we use?

- Functional RG
- Instantons
- Large N expansions
- Lattice field theory

The Lattice

- The lattice turns derivatives into differences and integrals into sums.

$$\partial_{\hat{k}}^{(L)} f_j = \frac{1}{b_s}\left[f_{j+\hat{k}} - f_j\right] \qquad \nabla_L^2 f_j = \sum_k \frac{1}{b_s^2}\left[f_{j+\hat{k}} + f_{j-\hat{k}} - 2f_j\right] \qquad S_{\text{free}} = \sum_{\tau,\tau'} \frac{1}{b_\tau}\psi_{\tau'}^\dagger \left[K_0\right]_{\tau,\tau'} \psi_\tau$$

Nicholson 2016

- Finite Euclidian volume (L).
  - Quantizes the momentum
- Lattice simulations give the **n-point functions** by directly evaluating the path integral using monte-carlo sampling.

$$G_{c,2pt}(x,y) = \frac{\delta^2 \log(Z[J])}{\delta J(x)\delta J(y)}\Big|_{J=0}$$

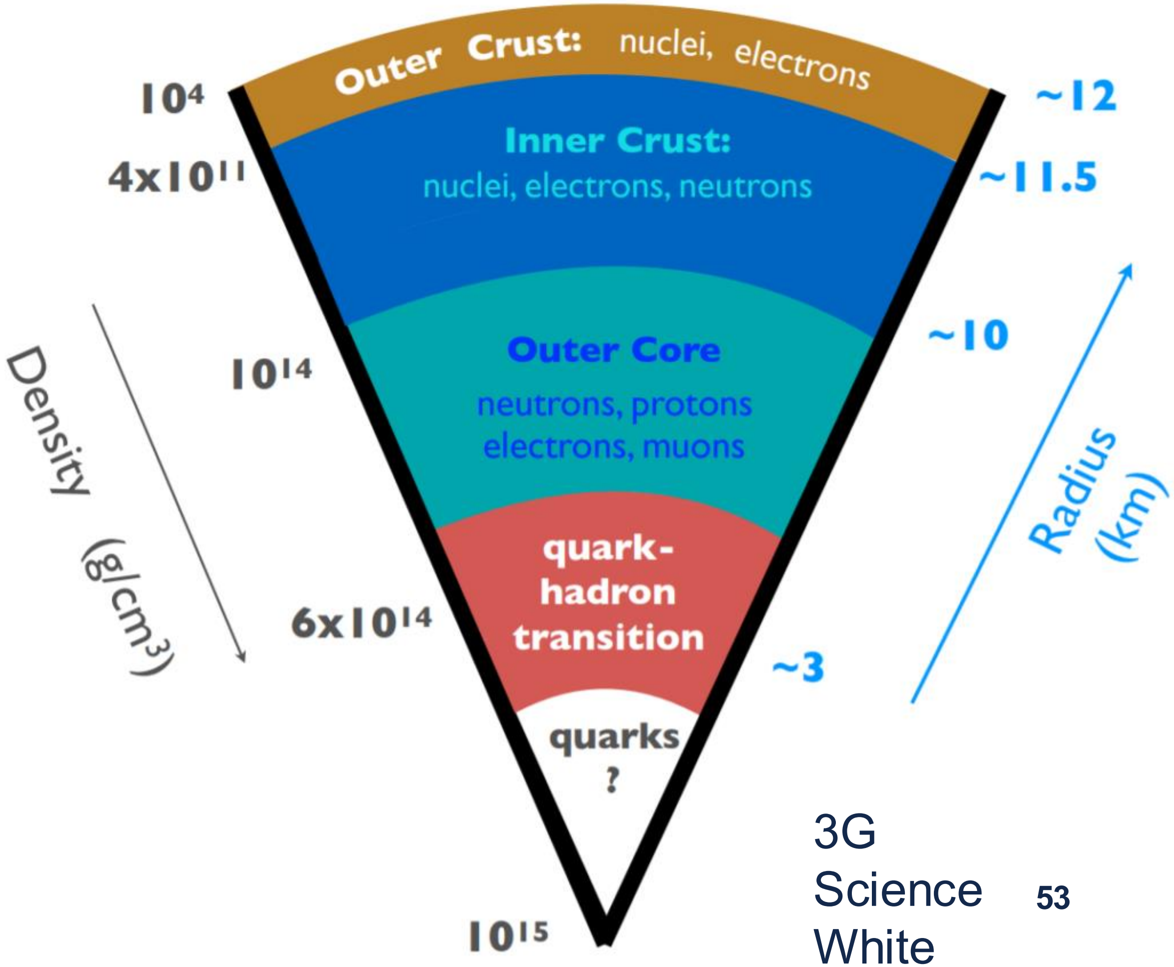
Problems on the Lattice

- Euclidian space
    - ⇒ Only **time independent** quantities can be calculated on the lattice.
    - Integrals go from oscillating to exponentially decaying
- Signal to noise problem
- Sign problem
- Symmetries continuous -> discrete

The Unitary Fermi Gas

- The system of interest is a gas of Fermions tuned to the **unitary limit**
- Applicable to systems with large numbers of fermions

ESA



3G Science White

Pion-less EFT

-
- Effective field theory involving two point-like nucleons interacting via contact interactions.

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) \psi + g_0 \left(\psi^\dagger \psi \right)^2 + \cdots$$

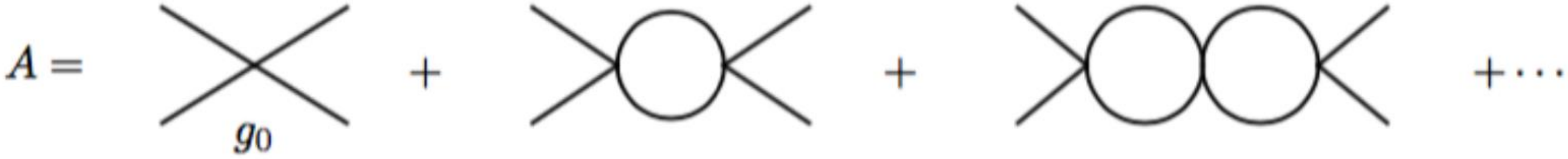
- Higher order terms add derivatives of fields, and therefore, different couplings to different spin states. Thus, we ignore the spin of the nucleons.

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-\int d\tau d^3x \left[\mathcal{L}(\psi^\dagger, \psi)\right]}$$

- Z is the Euclidian path integral for this theory.

Pion-less EFT: 2 body amplitude calculation (small g_0)

- Scattering amplitude is a sum of bubble diagrams



- But, we know from the optical theorem that

$$\mathcal{A} = \frac{4\pi}{M} \frac{1}{p\cot\delta - ip}$$

- Therefore, we know that if the coupling is small

$$\frac{\sum C_{2n}(\mu)\ p^{2n}}{1 - I_0 \sum C_{2n}(\mu)\ p^{2n}} = \frac{4\pi}{M} \frac{1}{p\cot\delta - ip}$$

$$I_0 = \left(\frac{\mu}{2}\right)^{4-D} \int \frac{d^{D-1}\mathbf{q}}{(2\pi)^{D-1}} \frac{1}{E - \frac{|\mathbf{q}|^2}{M} + i\epsilon}$$

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- K is a T x T block matrix, where each entry is an L^3 x L^3 matrix in itself
- D is a discretized kinetic operator $D \equiv 1 - \frac{b_s \nabla^2_L}{2})$
- $X(\phi_t)$is a rewriting of the Fermion interaction in terms of a **bosonic auxiliary field** that mediates the interaction $X(\phi_t) \equiv 1 - \sqrt{g_0}\phi_t$

- We also choose ϕ_0 to only live on the **temporal links** to make our lives simpler

Calculating the spectrum of the 2 body system

- The two body correlation function becomes $C_2(\tau) = \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-S[\psi^\dagger,\psi,\phi]}\Psi^\dagger_{\text{src},2}\Psi_{\text{snk},2}$
- We can integrate out the fermion fields to see

$$Z_\phi = \int \mathcal{D}\phi P[\phi] \qquad P[\phi] \equiv \rho[\phi]\det K[\phi]$$

$$\begin{aligned} C_2(\tau) &= \frac{1}{Z_\phi}\int \mathcal{D}\phi P[\phi]\langle\Psi_{\text{snk},2}|K^{-1}(\tau,0)\otimes K^{-1}(\tau,0)|\Psi_{\text{src},2}\rangle \\ &= \frac{1}{4\tau}\sum_{\phi=\pm1}\langle\Psi_{\text{snk},2}|D^{-1}\otimes D^{-1}X(\phi_\tau)\otimes X(\phi_\tau)D^{-1}\otimes D^{-1}X(\phi_{\tau-1})\otimes X(\phi_{\tau-1})\cdots|\Psi_{\text{src},2}\rangle \end{aligned}$$

- After a bunch of simplification $\langle x_1x_1'|\mathcal{D}^{-1}|x_2x_2'\rangle \equiv D^{-1}_{x_1x_1'}D^{-1}_{x_2x_2'}\ ,\qquad \langle x_1x_2|\mathcal{V}|x_1'x_2'\rangle \equiv g_0\delta_{x_1x_1'}\delta_{x_2x_2'}\delta_{x_1x_2}$

$$\begin{aligned} C_2(\tau) &= \langle\Psi_{\text{snk},2}|\mathcal{D}^{-1}(1+\mathcal{V})\mathcal{D}^{-1}(1+\mathcal{V})\cdots\mathcal{D}^{-1}(1+\mathcal{V})\mathcal{D}^{-1}|\Psi_{\text{src}}\rangle \\ &= \langle\Psi_{\text{snk}}|\mathcal{D}^{-1/2}\mathcal{T}\mathcal{D}^{-1/2}|\Psi_{\text{src},2}\rangle\ , \end{aligned}$$

$$\mathcal{T} \equiv \mathcal{D}^{-1/2}(1+\mathcal{V})\mathcal{D}^{-1/2}$$

Transfer Matrix

Calculating the spectrum of the 2 body system

From statistical mechanics:

$$C(\tau) = \langle \Psi_{\mathrm{snk},2} | e^{-H\tau} | \Psi_{\mathrm{src},2} \rangle = \langle \Psi_{\mathrm{snk},2} | \left[e^{-H} \right]^{\tau} | \Psi_{\mathrm{src},2} \rangle$$

Now we know that the logarithms of the eigenvalues of the transfer matrix give us the 2 body spectrum!

In principle, we could diagonalize $\mathcal{T}$ and its eigenvalues would be the 2-body spectrum.

$$\langle pq | \mathcal{T} | p'q' \rangle = \frac{\delta_{pp'}\delta_{qq'} + \frac{g_0}{V}\delta_{p+q,p'+q'}}{\sqrt{\xi(p)\xi(q)\xi(q')\xi(p')}}$$

Where

$$\xi(p) \equiv 1 + \frac{\Delta(q)}{M} \;,\; \Delta(q) \equiv -\frac{1}{2}\langle \mathbf{q} | \Delta_L^2 | \mathbf{q} \rangle$$

We can compare this result to the one discussed earlier using Lüscher methods to tune the coupling

Effective Mass

In general, the transfer matrix with N ≥4 cannot be solved because the dimension increases with particle number

This is why we form N-point correlation functions

$$C_N(\tau) = \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-S[\psi^\dagger,\psi,\phi]}\Psi^{(b)}_{b_1\cdots b_N}(\tau)\Psi^{\dagger(a)}_{a_1\cdots a_N}(0) \ ,$$

where

$$\Psi^{(a)\dagger}_{a_1\cdots a_N}(\tau) = \int dx_1\cdots dx_N A^{(a)}(x_1\cdots x_N)\psi_{a_1}(x_1,\tau)\cdots\psi_{a_N}(x_N,\tau)$$

Effective Mass

We can expand $C_N(\tau)$

$$C_N(\tau) = \frac{1}{Z}\langle\tilde{\Psi}^{(a)}_{a_1\cdots a_N}|e^{-H\tau}|\tilde{\Psi}^{(b)}_{b_1\cdots b_N}\rangle = \frac{1}{Z}\sum_{m,n}\langle\tilde{\Psi}^{(a)}_{a_1\cdots a_N}|m\rangle\langle m|e^{-H\tau}|n\rangle\langle n|\tilde{\Psi}^{(b)}_{b_1\cdots b_N}\rangle$$

$$= \sum_m Z^{(a)}_m Z^{*(b)}_m e^{-E_n\tau} \; ,$$

$Z^{(a)}_m$ is the overlap of the wavefunction a with the eigenstate m

In the zero-temperature limit (large Euclidian time) we expect

$$C_N(\tau) \underset{\tau\to\infty}{\longrightarrow} Z^{(a)}_0 Z^{*(b)}_0 e^{-E_0\tau}$$

Effective Mass

$$C_N(\tau) \xrightarrow[\tau\to\infty]{} Z_0^{(a)} Z_0^{*(b)} e^{-E_0\tau}$$

We can expect excited states to be exponentially suppressed by $e^{-\Delta E\tau}$

A common way to extract the ground state is to construct the so called "**effective mass function**" and let Euclidian time approach infinity

$$M_{\text{eff}}(\tau) \equiv \ln \frac{C(\tau)}{C(\tau+1)} \xrightarrow[\tau\to\infty]{} E_0$$

Approach to calculating GFF’s on the lattice

Gravitational Form Factors (GFFs)

- A form factors are functions that describe the shape, size, and distributions of various different types of charges
- They can be calculated from scattering cross sections (observed experimentally)
- Related to the decomposition of current matrix elements
- GFFs can be related to GPD physics (through GPD moments)
- The gravitational form factors (GFFs) describe the angular momentum, mass, and force distributions in your system

$$J^q(t) = \frac{1}{2}\int_{-1}^{1} dx x[H^q(x,\xi,t) + E^q(x,\xi,t)]$$

$$M_2(t) + \frac{4}{5}d_1(t)\xi^2 = \frac{1}{2}\int_{-1}^{1} dx x H^q(x,\xi,t)$$

e.g. Proton form factors related to GPD moments (Girod, 2018)

Current Insertion

Proton GFF matrix element decomposition

$$\langle N(\mathbf{p}', s')| \hat{T}^{\mu\nu} |N(\mathbf{p}, s)\rangle = \frac{1}{m}\bar{u}(\mathbf{p}', s')\Big[P^\mu P^\nu A(t) + iP^{\{\mu}\sigma^{\nu\}\rho}\Delta_\rho J(t) + \frac{1}{4}(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)D(t)\Big]u(\mathbf{p}, s),$$

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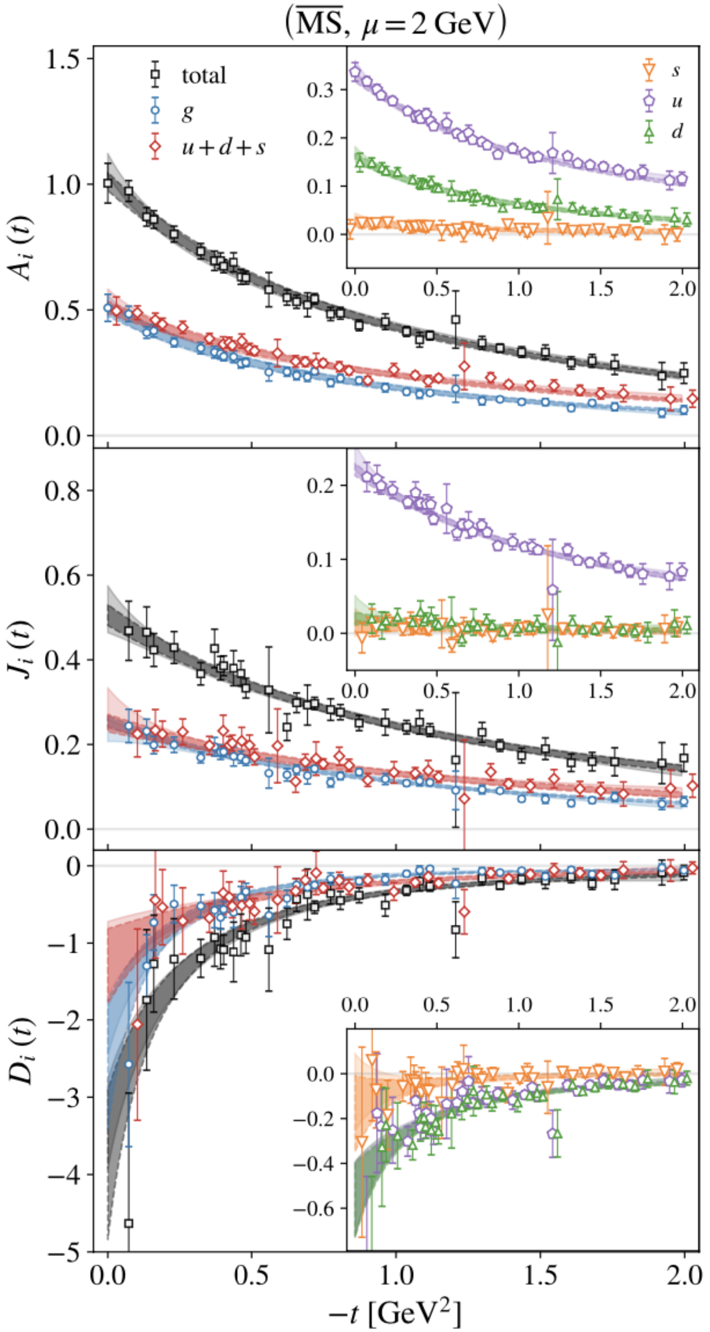
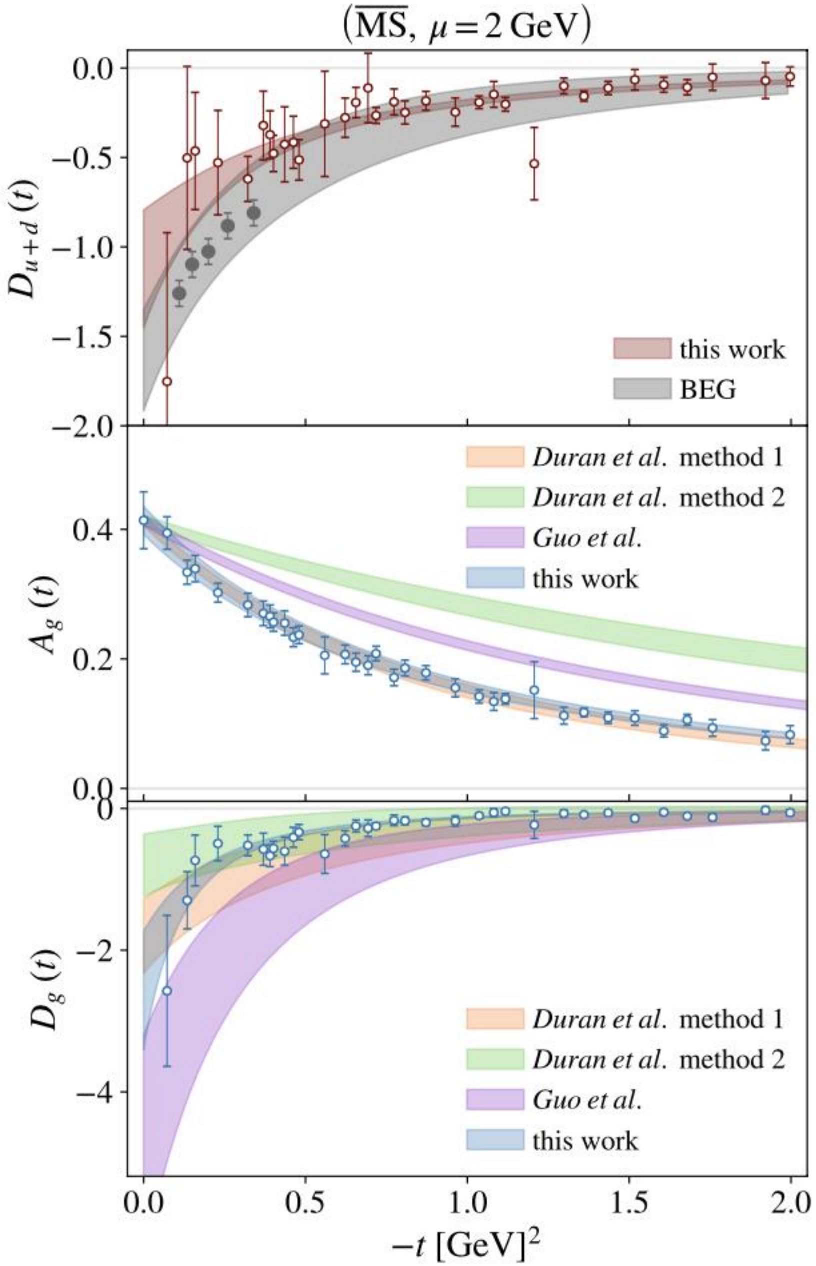
$$\hat{T}_g^{\mu\nu} = 2\ \mathrm{Tr}\left[-F^{\mu\alpha}F^\nu_{\ \alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

$$\hat{T}_q^{\mu\nu} = \sum_f \left[i\bar{\psi}_f D^{\{\mu}\gamma^{\nu\}}\psi_f\right],$$

Following the methods discussed in this paper, we can extract GFFs for the **unitary fermi gas** by taking **constrained ratios** of 3-pt and 2-pt functions!





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How is this done for the proton?

We define the momentum projected proton 2-pt correlation function

$$C^{2\mathrm{pt}}_{ss'}(\mathbf{p}, t_s; \mathbf{x}_0, t_0) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \mathrm{tr}\left[\Gamma_{s's} \langle \chi(\mathbf{x}, t_s + t_0)\bar{\chi}(\mathbf{x}_0, t_0)\rangle\right]$$

Where

$$\chi(x) = \epsilon^{abc}\left[\psi_u^{T,b}(x) C\gamma_5 \psi_d^c(x)\right]\psi_u^a(x)$$

Is fit to the usual

$$C^{2\mathrm{pt}}(\mathbf{p}, t_s) \sim \sum_{n=0} |Z^n_{\mathbf{p}}|^2 e^{-E^n_{\mathbf{p}} t_s}$$

How is this done for the proton?

- Next, we write the Euclidian EMT contributions as

$$\hat{T}_{f,\mu\nu}(x) = \bar{\psi}_f(x)\overleftrightarrow{D}_{\{\mu}\gamma_{\nu\}}\psi_f(x),$$
$$\hat{T}_{g,\mu\nu}(x) = 2\,\mathrm{Tr}\left[F_{\mu\rho}(x)F_{\nu\rho}(x) - \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}(x)F_{\alpha\beta}(x)\right]$$

- The elements of which transform like two different irreps of the hypercubic group that can be mapped back to Minkowski space

- We can then write the 3-pt function used to extract the matrix elements as

$$C^{\mathrm{3pt}}_{i\mathcal{R}\ell ss'}(\mathbf{p}', t_s; \mathbf{\Delta}, \tau; \mathbf{x_0}, t_0) = \sum_{\mathbf{x},\mathbf{y}} e^{-i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{x}_0)} e^{i\mathbf{\Delta}\cdot(\mathbf{y}-\mathbf{x}_0)} \mathrm{tr}\left[\langle \Gamma_{s's}\chi(\mathbf{x}, t_s + t_0)\hat{T}_{i\mathcal{R}\ell}\bar{\chi}(\mathbf{x}_0, t_0)\rangle\right]$$

How is this done for the proton?

In the long time limit

$$C^{3\text{pt}}_{i\mathcal{R}\ell ss'}(\mathbf{p}',t_s;\boldsymbol{\Delta},\tau;\mathbf{x_0},t_0) \xrightarrow[(t_s-t_0)\to\infty]{(t_s-\tau)\to\infty} Z^*_{\mathbf{p}}Z_{\mathbf{p}'}\frac{e^{-E_{\mathbf{p}'}(t_s-t_0)}e^{-(E_{\mathbf{p}}-E_{\mathbf{p}'})(\tau-t_0)}}{4E_{\mathbf{p}'}E_{\mathbf{p}}}\langle N(\mathbf{p}',s')|\,\hat{T}_{i\mathcal{R}\ell}\,|N(\mathbf{p},s)\rangle$$

To cancel out the overlap factors, we form the ratio

$$R_{i\mathcal{R}\ell ss'}(\mathbf{p}',t_s;\boldsymbol{\Delta},\tau) = \frac{C^{3\text{pt}}_{i\mathcal{R}\ell ss'}(\mathbf{p}',t_s;\boldsymbol{\Delta},\tau)}{C^{2\text{pt}}_{s's'}(\mathbf{p}',t_s)}\sqrt{\frac{C^{2\text{pt}}_{ss}(\mathbf{p},t_s-\tau)C^{2\text{pt}}_{s's'}(\mathbf{p}',t_s)C^{2\text{pt}}_{s's'}(\mathbf{p}',\tau)}{C^{2\text{pt}}_{s's'}(\mathbf{p}',t_s-\tau)C^{2\text{pt}}_{ss}(\mathbf{p},t_s)C^{2\text{pt}}_{ss}(\mathbf{p},\tau)}}$$

$$\xrightarrow[t_s\to\infty]{(t_s-\tau)\to\infty} \frac{\operatorname{tr}\left[\Gamma_{s's}(\not{p}'+m)\langle N(\mathbf{p}',s')|\,\hat{T}_{i\mathcal{R}\ell}\,|N(\mathbf{p},s)\rangle(\not{p}+m)\right]}{4\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}(E_{\mathbf{p}}+m)(E_{\mathbf{p}'}+m)}}.$$

Which is a linear combination of the form factors with kinematically known coefficients. (don’t ask me how, this feels like magic to me).

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