$\eta \to 3\pi$ as a Precision Probe of Isospin Breaking and the Light-Quark Mass Ratio

Payton Arber

Department of Physics

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC

Outline

1. Why $\eta \rightarrow 3\pi$ is forbidden in the isospin limit
2. How the Standard Model makes it possible
3. What experiments measure: rates and Dalitz plots

The η meson is a particularly clean probe of fundamental symmetries

η $\quad I^G(J^{PC}) = 0^+(0^{-+})$

Mass $m = 547.862 \pm 0.017$ MeV
Full width $\Gamma = 1.31 \pm 0.05$ keV

- It's a flavor conserving neutral pseudoscalar meson
- Long-lived compared to many hadrons
- Strong and electromagnetic decays are often symmetry suppressed
- Sensitive to low-energy QCD and possible rare effects – which makes it accessible to precisions studies and BSM searches

The Decay Channels

$$\eta \to \pi^0\pi^+\pi^- \quad \text{and} \quad \eta \to 3\pi^0$$

Branching Fractions:

$3\pi^0$	(32.56 ± 0.21) %
$\pi^+\pi^-\pi^0$	(23.02 ± 0.25) %

Where does the rest of the decay go?

2γ	(39.36 ± 0.18) %

$\eta \rightarrow 3\pi$ is forbidden in the isospin limit

- Isospin symmetry treats the up and down quarks as a doublet

$$\begin{pmatrix} u \\ d \end{pmatrix} \text{ is an isospin doublet}$$

- In the exact isospin limit: $m_u = m_d$ and $e = 0$

For $\eta \rightarrow 3\pi$

- $I_\eta = 0,$ and $I_{3\pi} = 1$ in the allowed symmetric configuration
- Strong isospin breaking $m_u \neq m_d$
- Electromagnetic effects: quarks have different charges

What part of the Standard Model makes this decay possible?

$$\mathcal{L}_{mass} = -m_u \bar{u}u - m_d \bar{d}d - m_s \bar{s}s$$

$$\hat{m} = \frac{m_u + m_d}{2}, \qquad \delta = \frac{m_u - m_d}{2}$$

What part of the Standard Model makes this decay possible?

$$\mathcal{L}_{mass} = -m_u \bar{u}u - m_d \bar{d}d - m_s \bar{s}s$$

$$\hat{m} = \frac{m_u + m_d}{2}, \qquad \delta = \frac{m_u - m_d}{2}$$

$$\therefore \quad \hat{m} + \delta = m_u\,, \qquad \hat{m} - \delta = m_d$$

What part of the Standard Model makes this decay possible?

$$\mathcal{L}_{mass} = -m_u \bar{u}u - m_d \bar{d}d - m_s \bar{s}s$$

$$\hat{m} = \frac{m_u + m_d}{2}, \qquad \delta = \frac{m_u - m_d}{2}$$

$$\therefore \quad \hat{m} + \delta = m_u\,, \qquad \hat{m} - \delta = m_d$$

$$-m_u \bar{u}u - m_d \bar{d}d = -(\hat{m} + \delta)\bar{u}u - (\hat{m} - \delta)\bar{d}d = -\hat{m}\left(\bar{u}u + \bar{d}d\right) - \delta\left(\bar{u}u - \bar{d}d\right)$$

What part of the Standard Model makes this decay possible?

$$\mathcal{L}_{mass} = -m_u\bar{u}u - m_d\bar{d}d - m_s\bar{s}s$$

$$\hat{m} = \frac{m_u + m_d}{2}, \qquad \delta = \frac{m_u - m_d}{2}$$

$$\therefore \quad \hat{m} + \delta = m_u\,, \qquad \hat{m} - \delta = m_d$$

$$-m_u\bar{u}u - m_d\bar{d}d = -(\hat{m}+\delta)\bar{u}u - (\hat{m}-\delta)\bar{d}d = -\hat{m}\left(\bar{u}u + \bar{d}d\right) - \delta\left(\bar{u}u - \bar{d}d\right)$$

$$\mathcal{L}_{mass} = -\hat{m}\left(\bar{u}u + \bar{d}d\right) - m_s\bar{s}s - \frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)$$

What part of the Standard Model makes this decay possible?

$$\mathcal{L}_{mass} = -m_u\bar{u}u - m_d\bar{d}d - m_s\bar{s}s$$

$$\widehat{m} = \frac{m_u + m_d}{2}, \qquad \delta = \frac{m_u - m_d}{2}$$

$$\therefore \quad \widehat{m} + \delta = m_u\,, \qquad \widehat{m} - \delta = m_d$$

$$-m_u\bar{u}u - m_d\bar{d}d = -(\widehat{m} + \delta)\bar{u}u - (\widehat{m} - \delta)\bar{d}d = -\widehat{m}\left(\bar{u}u + \bar{d}d\right) - \delta\left(\bar{u}u - \bar{d}d\right)$$

$$\mathcal{L}_{mass} = -\widehat{m}\left(\bar{u}u + \bar{d}d\right) - m_s\bar{s}s - \frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)$$

$$\mathcal{L}_{IB} = -\frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)$$

Importance of quark mass ratio Q

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}, \qquad with\ \hat{m} = \frac{m_u + m_d}{2}$$

$$A_{\eta\to3\pi} \propto m_d - m_u \Longrightarrow A_{\eta\to3\pi} = \frac{1}{Q^2}\mathcal{M}(s,t,u)$$

$$\Gamma(\eta \to 3\pi) \propto |A|^2 \propto Q^{-4}$$

- Quantifies light-quark mass hierarchy info
- Links hadronic observables to SM input parameters
- Provides a precision test of low energy QCD

Observables

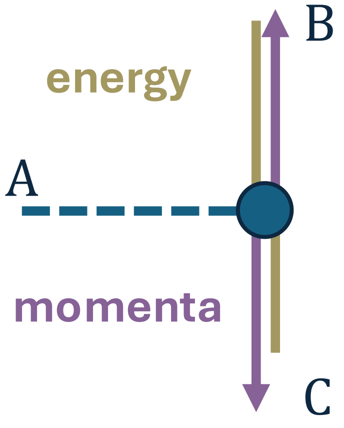
- Branching fractions & Partial widths
- Event distributions over phase space
- Dalitz plot parameters
- Comparisons between charged and neutral channels

Why Dalitz plots help describe three-body decays

- Two body decay:

$$A \to B + C$$

fixed final-state energy in parent rest frame



$$E_B + E_C = m_A$$

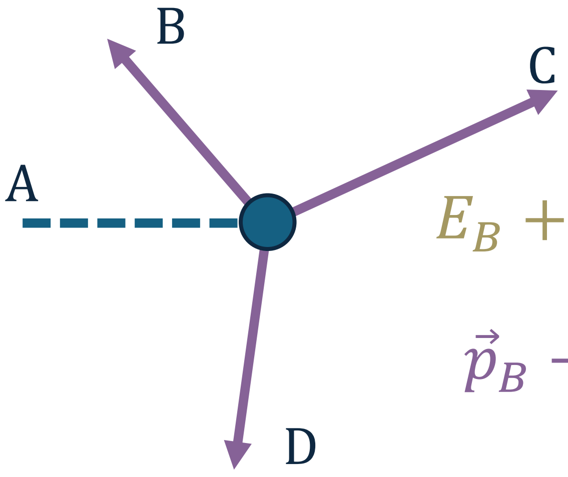
$$\vec{p}_B = -\vec{p}_c$$

- Three body decay:

$$A \to B + C + D$$

energy can be shared continuously

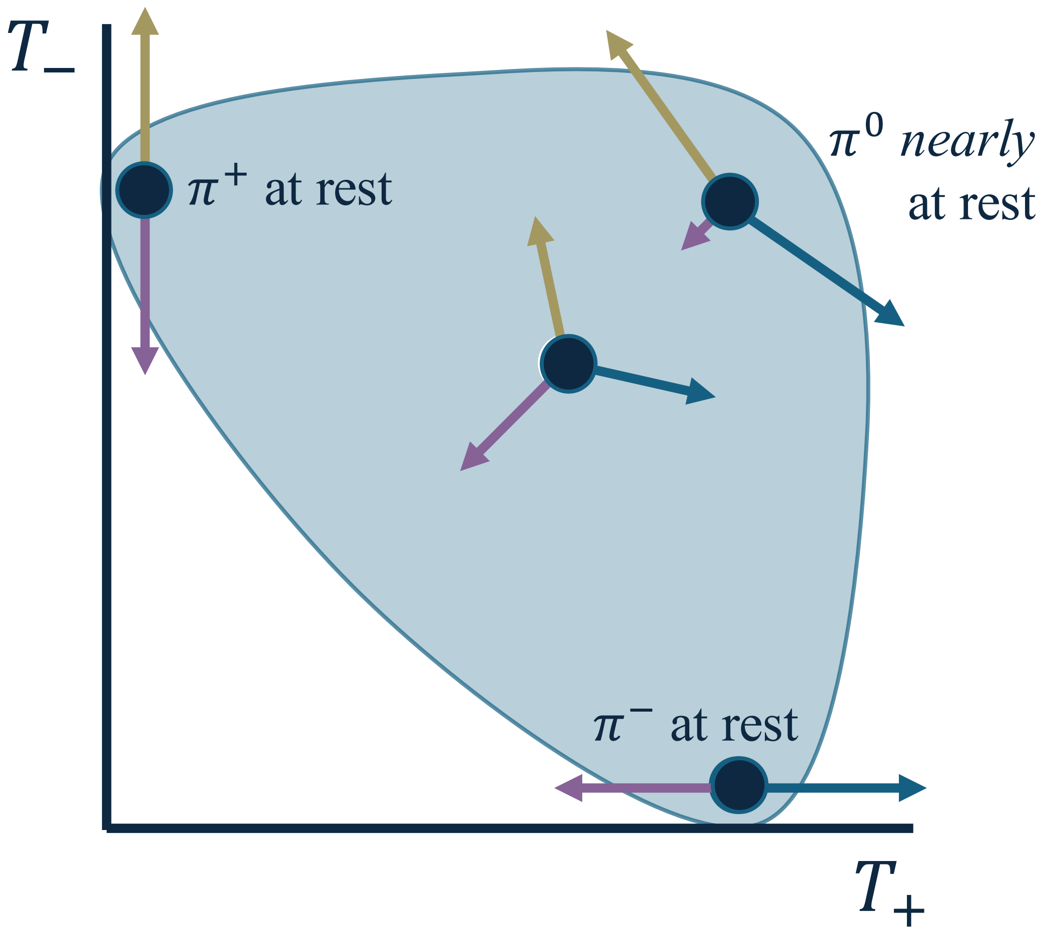
$$\frac{d^2\Gamma}{dXdY} \propto |A(X,Y)|^2$$



$$E_B + E_C + E_D = m_A$$

$$\vec{p}_B + \vec{p}_C + \vec{p}_D = 0$$

Charged π kinetic energy in the η rest frame



π^+ momentum

π^- momentum

π^0 momentum

Kinematic variables
→ Dimensionless variables X and Y

$$s = (P_\eta - p_0)^2 = (p_{\pi^+} + p_{\pi^-})^2$$

$$t = (P_\eta - p_{\pi^+})^2 = (p_{\pi^-} + p_0)^2$$

$$u = (P_\eta - p_{\pi^-})^2 = (p_{\pi^+} + p_0)^2$$

$$s + t + u = M_\eta^2 + 2M_{\pi^+}^2 + M_{\pi^0}^2$$

∴ there are only two degrees of freedom describing $\eta \to \pi^+\pi^-\pi^0$

Introducing X and Y:

- We want variables that
  - Are dimensionless
  - Put the center of phase space at the origin
  - Make the symmetry of $t \leftrightarrow u$ obvious
  - Are simple in the η rest frame

Constructing the Dalitz variables

Combining the Mandelstam variables and applying energy conservation, we obtain

$$T_0 = \frac{\left(M_\eta - M_{\pi^0}\right)^2 - s}{2M_\eta} \quad \text{and} \quad T_+ - T_- = \frac{u - t}{2M_\eta}$$

Constructing the Dalitz variables

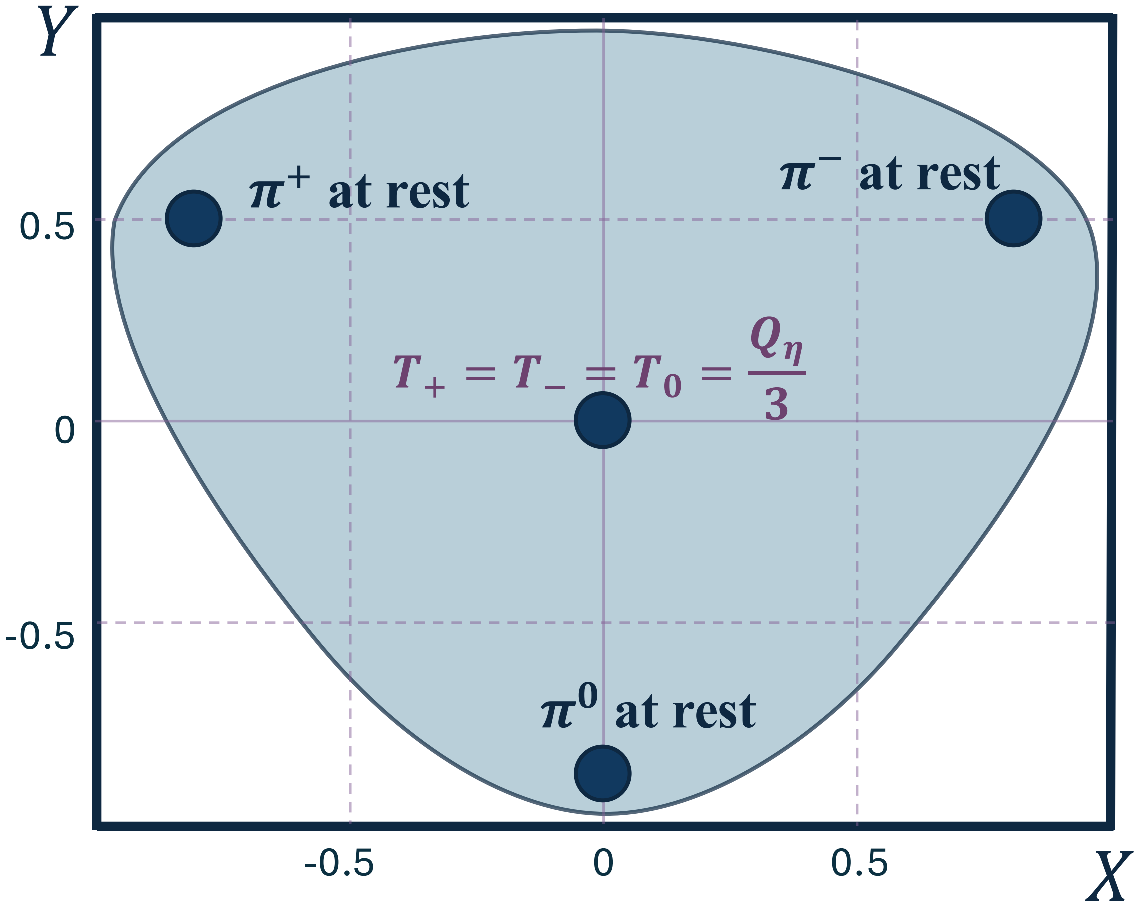
Combining the Mandelstam variables and applying energy conservation, we obtain

$$T_0 = \frac{\left(M_\eta - M_{\pi^0}\right)^2 - s}{2M_\eta} \quad \text{and} \quad T_+ - T_- = \frac{u - t}{2M_\eta}$$

Define X and Y such that when $T_+ = T_- = T_0 = \frac{Q_\eta}{3}$, $\mathrm{X} = \mathrm{Y} = 0$

$$Y \equiv \frac{3T_0}{Q_\eta} - 1 \quad \text{and} \quad X \equiv \frac{\sqrt{3}(T_+ - T_-)}{Q_\eta}$$

Constructing a Dalitz Plot

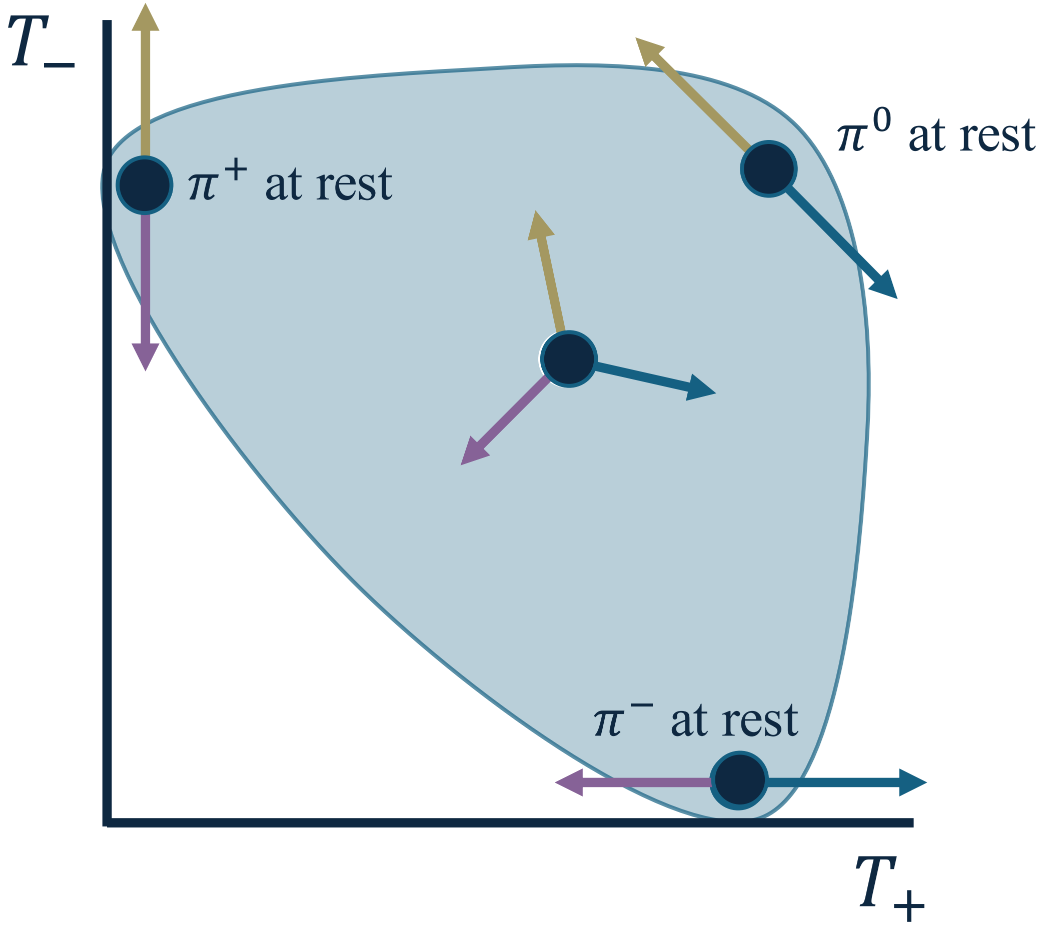


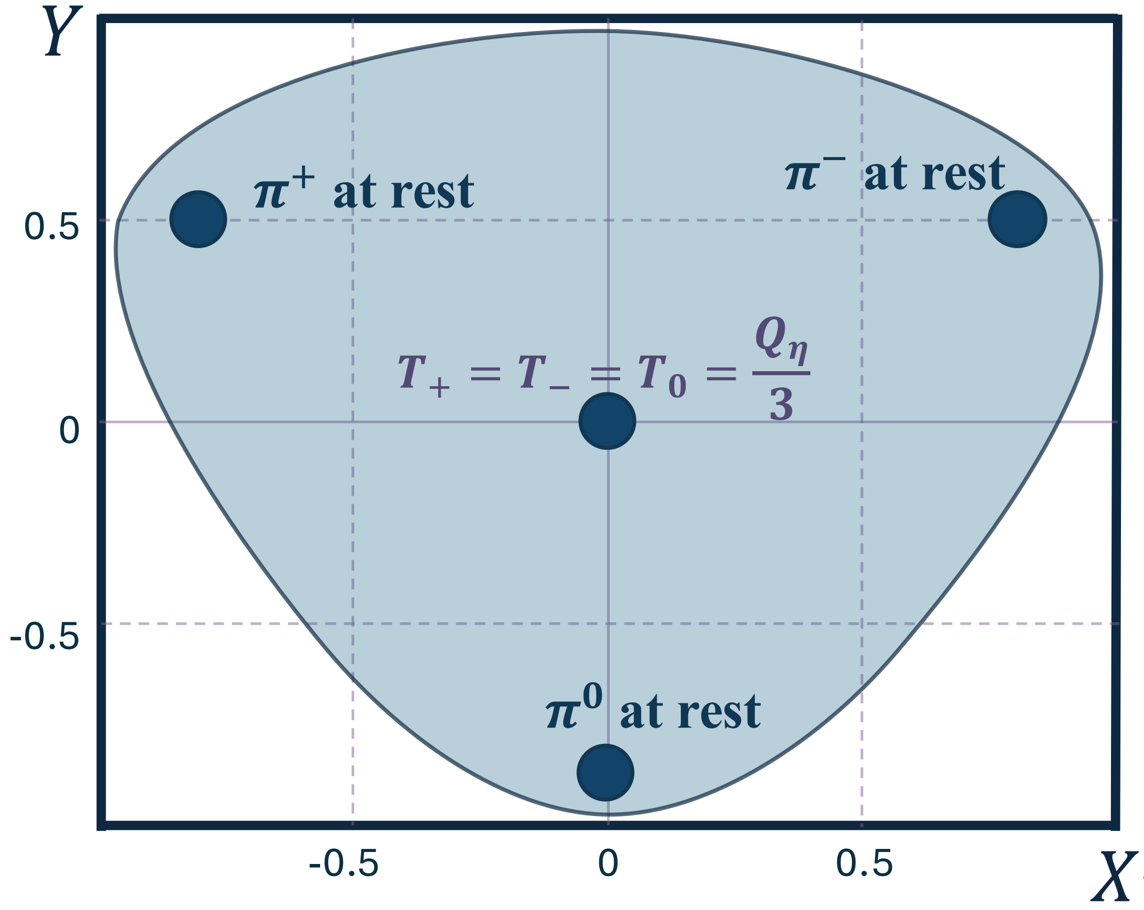
$$Y \equiv \frac{3T_0}{Q_\eta} - 1$$

$$= 2 - \frac{3(T_+ + T_-)}{Q_\eta}$$

$$X \equiv \frac{\sqrt{3}(T_+ - T_-)}{Q_\eta}$$

Comparing Dalitz Plots





The Dalitz Plot Parameterization for $\eta \rightarrow \pi^+\pi^-\pi^0$

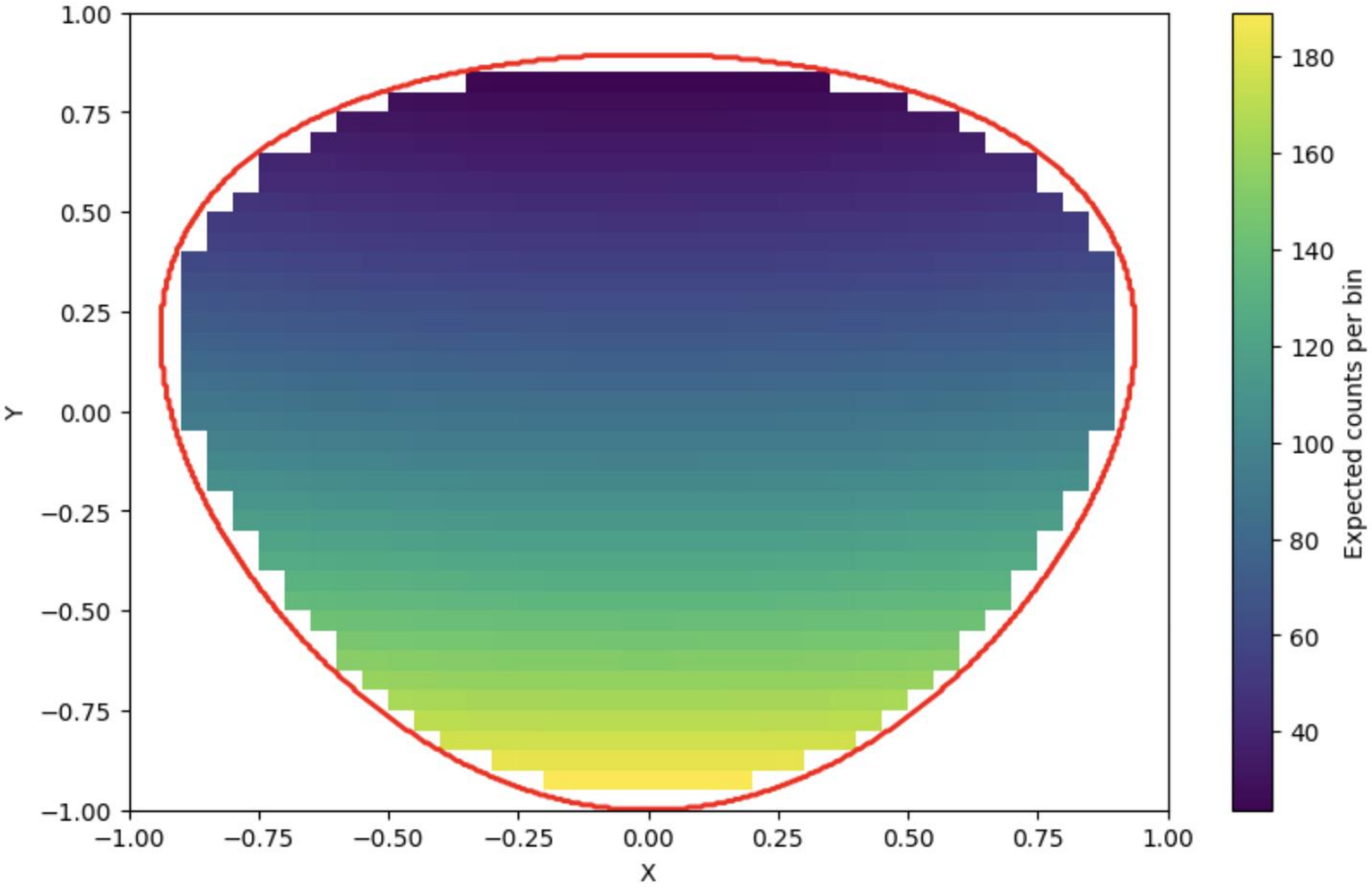
$$\Gamma(X,Y) = |A_c(s,t,u)|^2 \propto 1 + aY + bY^2 + cX + dX^2 + eXY + fY^3$$

a, b, c, d, e… are Dalitz Plot parameters

Charge conjugation symmetry requires terms odd in X to vanish

$$c = e = 0$$

Dalitz Plot for $\eta \to \pi^+\pi^-\pi^0$




```
Fit results:
a = -1.0607788730493255
b = 0.24178785950681664
d = 0.08516063431832312
f = -0.021113096379888747

Generated values:
a = -1.06
b =  0.24
d =  0.08
f = -0.03
```

Why the Dalitz Plots are Essential

Total rate:

$$\Gamma(\eta \to 3\pi) = \int d\Phi_3 |A(s,t,u)|^2$$

- Fixes overall strength
- Carries Q dependence
- $\Gamma \propto Q^{-4}$

Why the Dalitz Plots are Essential

Total rate:

$$\Gamma(\eta \to 3\pi) = \int d\Phi_3 |A(s,t,u)|^2$$

- Fixes overall strength
- Carries Q dependence
- $\Gamma \propto Q^{-4}$

Dalitz shape:

$$|A(X,Y)|^2 = N(1 + aY + bY^2 + dX^2 + \ldots)$$

- Measures momentum dependence
- Reveals $\pi\,\pi$ final-state interactions
- Constrains dispersive amplitudes

Why the Dalitz Plots are Essential

Total rate:

$$\Gamma(\eta \to 3\pi) = \int d\Phi_3 |A(s,t,u)|^2$$

- Fixes overall strength
- Carries Q dependence
- $\Gamma \propto Q^{-4}$

Dalitz shape:

$$|A(X,Y)|^2 = N(1 + aY + bY^2 + dX^2 + \dots)$$

- Measures momentum dependence
- Reveals $\pi\,\pi$ final-state interactions
- Constrains dispersive amplitudes

Rate constrains Q, while shape constrains the amplitude used to extract Q

Summary

- $\eta \rightarrow 3\pi$ is forbidden in exact isospin symmetry
- The decay is primarily driven by $m_d - m_u$
- It provides access to the light-quark mass ratio Q
- Dalitz plots contain essential dynamical information

Thank You ☺

Backup Slides

Motivation

What the $\eta \to 3\pi$ decay can teach us about QCD

- $\eta \to 3\pi$ is suppressed by isospin symmetry
- The decay occurs because isospin is not exact
- It's sensitive to the up-down quark mass difference
- Using precision measurements, we can constrain the light-quark mass ratio

Why the Dalitz Plots are Essential

Total rate:

$$\Gamma(\eta \to 3\pi) = \int d\Phi_3 |A(s,t,u)|^2$$

- Fixes overall strength
- Carries Q dependence
- $\Gamma \propto Q^{-4}$

Dalitz shape:

$$|A(X,Y)|^2 = N(1 + aY + bY^2 + dX^2 + \dots)$$

- Measures momentum dependence
- Reveals $\pi\,\pi$ final-state interactions
- Constrains dispersive amplitudes

Rate constrains Q, while shape constrains the amplitude used to extract Q

Constructing the Dalitz variables

In the η rest frame $P_\eta = (M_\eta, \mathbf{0})$

Final state energies $E_+ = M_{\pi^+} + T_+ \,, \quad E_- = M_{\pi^+} + T_- \,, \quad E_0 = M_{\pi^0} + T_0$

Energy conservation $E_+ + E_- + E_0 = M_\eta$

results in $T_+ + T_- + T_0 = M_\eta - 2M_{\pi^+} - M_{\pi^0} \equiv Q_\eta$

$$s = (P_\eta - p_0)^2 = (M_\eta - M_{\pi^0})^2 - 2M_\eta T_0 \qquad \therefore\ T_0 = \frac{(M_\eta - M_{\pi^0})^2 - s}{2M_\eta}$$

$$u - t = (P_\eta - p_{\pi^+})^2 - (P_\eta - p_{\pi^-})^2 = 2M_\eta(T_+ - T_-) \qquad \therefore\ T_+ - T_- = \frac{u - t}{2M_\eta}$$

Constructing the Dalitz variables

Define X and Y such that when $T_+ = T_- = T_0 = \frac{Q_\eta}{3}$, $X = Y = 0$

From $T_0 = \frac{(M_\eta - M_{\pi^0})^2 - s}{2M_\eta}$ we obtain

$$Y \equiv \frac{3T_0}{Q_\eta} - 1 = \frac{3}{2M_\eta Q_\eta}\left[(M_\eta - M_{\pi^0})^2 - s\right] - 1$$

And from $T_+ - T_- = \frac{u-t}{2M_\eta}$ we see

$$X \equiv \frac{\sqrt{3}(T_+ - T_-)}{Q_\eta} = \frac{\sqrt{3}}{2M_\eta Q_\eta}(u - t)$$

With

$$Q_\eta = M_\eta - 2M_{\pi^+} - M_{\pi^0}$$