CNUGS

CHRISTOPHER NEWPORT UNIVERSITY
GRADUATE SUMMER WORKSHOP
AT JEFFERSON LAB 2026

formerly The HUGS Program

QCD factorization and global analysis

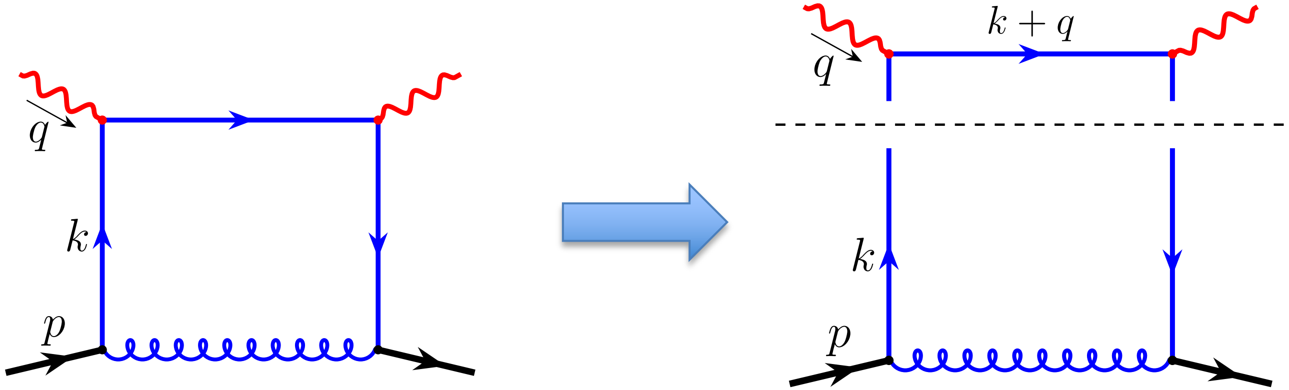
Zhite Yu (BNL, High Energy Physics)

- Introduction to QCD factorization
- Parton distribution functions and global analysis
- From inclusive to exclusive processes
- Generalized parton distributions and global analysis

CNUGS Summer Workshop, June 3, 2026

What is factorization?

- ❑ **Naively, $W = H \cdot F$ …**

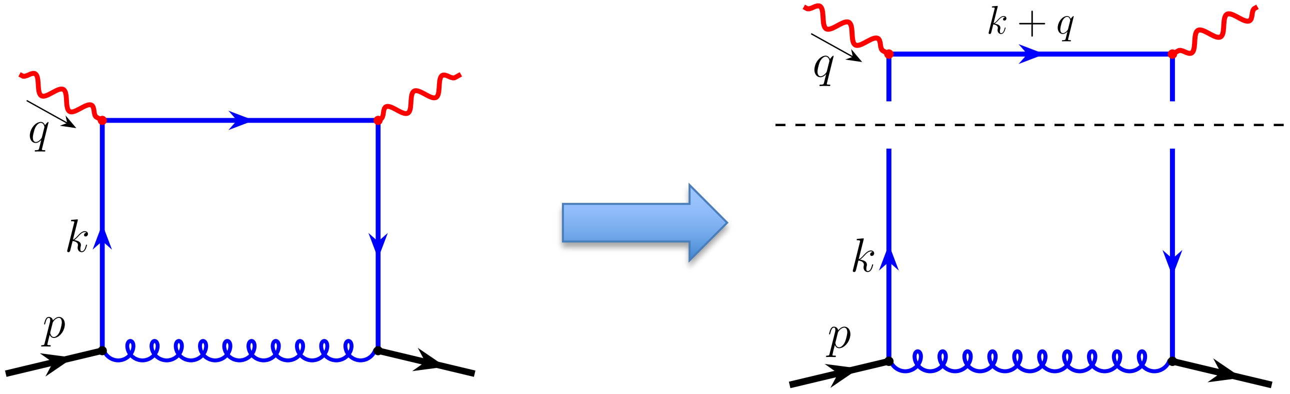


$$= \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\big[\, U({\color{red}k}, q)\, L({\color{red}k}, p)\,\big]$$

- **This separation is always doable!**
- **Not useful.**

What is factorization?

☐ **Naively,** $W = H \cdot F$ ...



$$= \int \frac{d^4 k}{(2\pi)^4}\, \mathrm{Tr}\big[\, U(k,q)\, L(k,p) \,\big]$$

- **This separation is always doable!**
- **Not useful.**

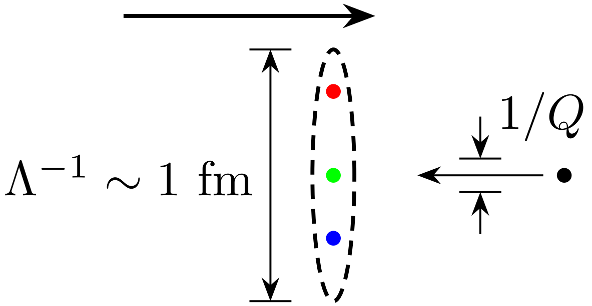
☐ **But, more crucially…**

We require:

- H **only depends on short-distance “hard” interaction**
- F **carries all long-distance “soft” dynamics**
- **The formalism applies to all orders**

QCD and factorization

❑ Feynman’s basic intuition

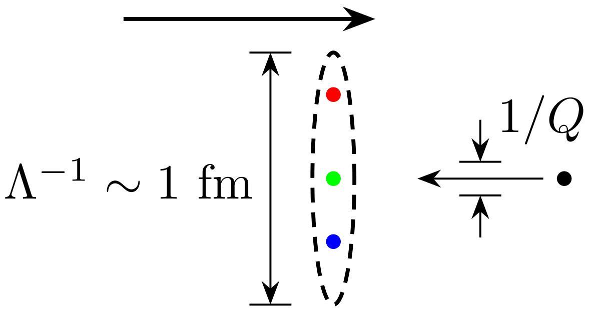


- Interaction happens locally: $\tau \sim 1/Q$
- Hadron has size: $1/\Lambda \sim \mathrm{fm}$
- Time dilation: Q/Λ

⇒ Short-distance interaction generally ***decouples*** from long-distance interaction.

QCD and factorization

Feynman's basic intuition



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Short-distance interaction generally ***decouples*** from long-distance interaction.

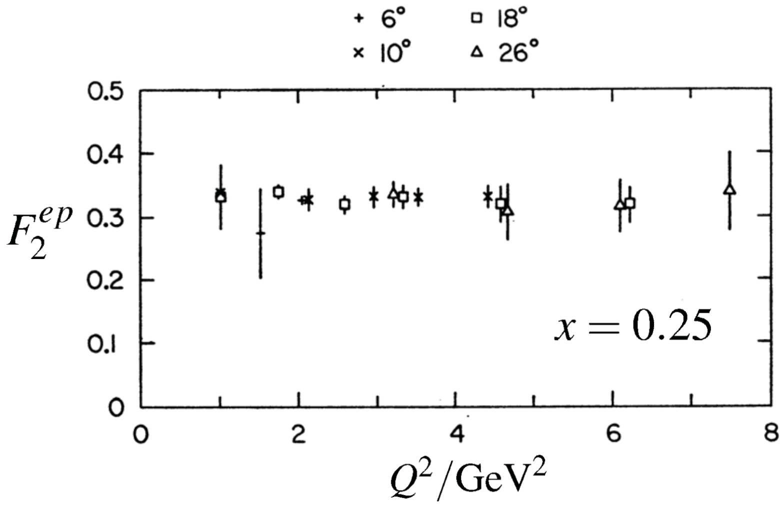
But, this is not sufficient!

Experimentally: **scaling** at large Q^2.

Proton constituents behave like
free massless particles at short distance!

“Partons”

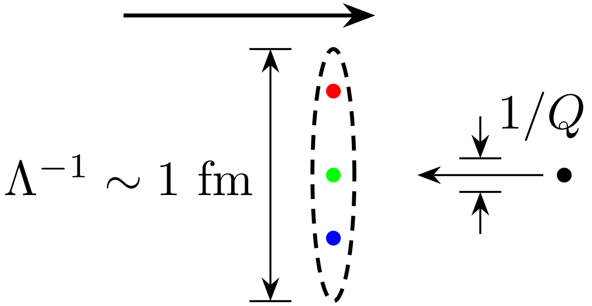
Interactions among partons are **asymptotically free**.



J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

QCD and factorization

❑ Parton model



<u>Parton distribution function</u> (PDF): given by long-distance dynamics

$$\sigma_{\rm DIS}(x_B, Q^2) \simeq \sum_i \int dx\, f_i(x) \cdot \hat{\sigma}_i(x_B/x, Q^2)$$

<u>Hard scattering</u> of on-shell parton: dominated by short-distance dynamics

QCD and factorization

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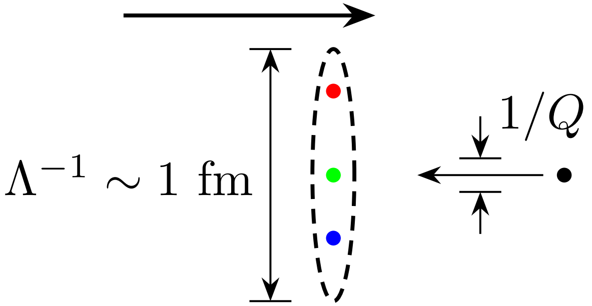

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Hard scattering of on-shell parton: dominated by short-distance dynamics

❑ More importantly, it makes a perturbative approach sensible

In reality:

- Partons are generally **off shell** due to their interactions.
- There are **a lot** of them.

But:

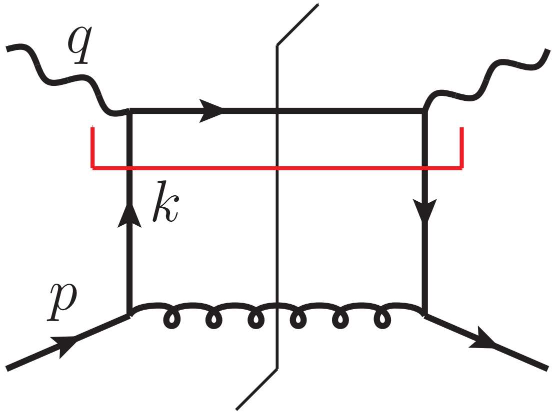
- ➢ Their interactions are time dilated.
- ➢ Less likely for a second parton to enter the local interaction.

➡ **Suppressed by Λ/Q**

➡ **Parton model is correct up to *power corrections.***

Example: “LO” factorization of DIS

❑ Consider the perturbative diagram with elementary target



Kinematics in terms of lightfront coordinates

$$p = (p^+, 0^-, \mathbf{0}_T), \quad q = \left(-x_B\, p^+, \frac{Q^2}{2x_B\, p^+}, \mathbf{0}_T\right)$$

Breit frame

$$V^\mu = (V^+, V^-, \boldsymbol{V}_T)\,, \quad V^\pm = \frac{V^0 \pm V^3}{\sqrt{2}}, \quad \boldsymbol{V}_T = (V^1, V^2)$$

1. The bottom cut line sets a condition on $\boldsymbol{k^-}$

$$\delta^+\left((p-k)^2\right) = \delta^+\left(2(p^+ - k^+)(-k^-) - k_T^2\right) = \frac{\theta(1-x)}{2(1-x)p^+}\delta\left(k^- + \frac{k_T^2}{2(1-x)p^+}\right)$$

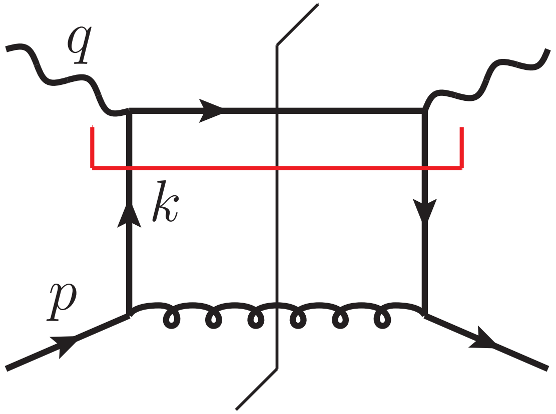
$$k^\mu = (k^+, k^-, \boldsymbol{k}_T), \quad k^+ = x\, p^+ < p^+$$

2. Quark propagators

$$k^2 = 2k^+k^- - k_T^2 = 2xp^+\left(-\frac{k_T^2}{2(1-x)p^+}\right) - k_T^2 = -\frac{k_T^2}{1-x}$$

goes *on-shell* as $\boldsymbol{k_T \to 0}$

Example: “LO” factorization of DIS



❑ Collinear singularity ⇔ Long-distance dynamics

k around this collinear singularity: **collinear region**

$$k^\mu = \left(xp^+, -\frac{k_T^2}{2(1-x)p^+}, \boldsymbol{k}_T\right) \sim \left(Q, \frac{k_T^2}{Q}, k_T\right) \qquad k_T \ll Q$$

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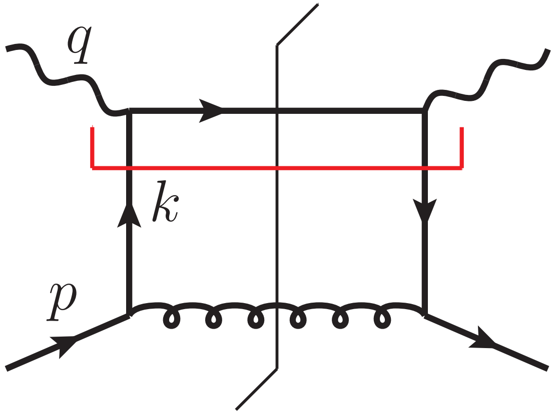


❑ How to extract contribution from this region?

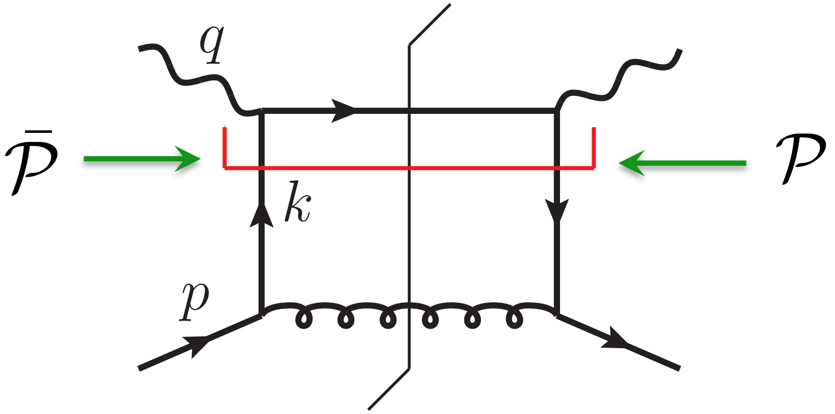
1. **Neglect $\boldsymbol{k_T}$ against $\boldsymbol{Q}$ in the "hard" interaction**

$$k' = k + q \simeq \left(k^+ + q^+, q^-, \mathbf{0}_T\right) \quad \text{or} \quad k \mapsto \hat{k} = (k^+, 0^-, \mathbf{0}_T)$$

$$\Rightarrow \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left[H(k,q)\, C(k,p)\right] \simeq \int \frac{dk^+}{k^+} \operatorname{Tr}\left[k^+ H(\hat{k},q) \int \frac{dk^- d^2\boldsymbol{k}_T}{(2\pi)^4} C(k,p)\right]$$

Collinear factor $C(k,p) \propto \dfrac{1}{(k^2)^2} \propto \dfrac{1}{(k_T^2)^2}$ carries low-virtuality lines along with $\int d^2k_T$

Example: "LO" factorization of DIS



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❑ How to extract contribution from this region?

2. How about quark propagator numerators?

$$\gamma \cdot k = \gamma^+ k^- + \gamma^- k^+ - \boldsymbol{\gamma}_T \cdot \boldsymbol{k}_T \simeq \gamma^- k^+$$

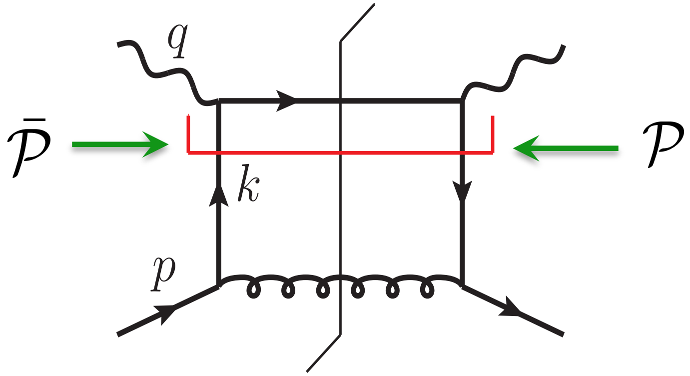
Extract this large component with spinor projectors

$$\bar{\mathcal{P}} = \frac{\gamma^-\gamma^+}{2}, \quad \mathcal{P} = \frac{\gamma^+\gamma^-}{2} \qquad \mathcal{P}^2 = \mathcal{P}, \quad \bar{\mathcal{P}}^2 = \bar{\mathcal{P}}, \quad \mathcal{P} + \bar{\mathcal{P}} = 1$$

$$\boxed{\begin{array}{c} C(k,p) \\ \Big\downarrow \\ \bar{\mathcal{P}}\, C(k,p)\, \mathcal{P} \end{array}}$$

$\bar{\mathcal{P}}$ acting on left keeps $\gamma^- k^+$ unchanged, but $\mathcal{P}$ doesn't;
(and conversely on the right).

Example: “LO” factorization of DIS



- ☐ **Putting the two approximations together**

$$\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\big[\,H(k,q)\,C(k,p)\,\big] \;\simeq\; \int \frac{dk^+}{k^+}\,\mathrm{Tr}\Big[\,k^+\,H(\hat{k},q)\,\underbrace{\bar{\mathcal{P}}\int \frac{dk^- d^2\boldsymbol{k}_T}{(2\pi)^4}\,C(k,p)\,\mathcal{P}}_{F(k^+/p^+)}\Big]$$

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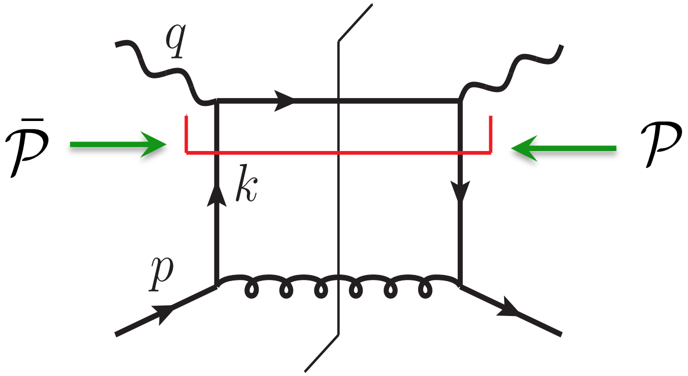

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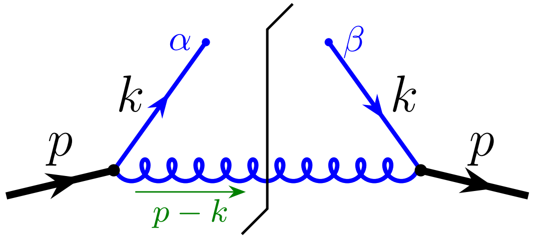
☐ **Collinear factor can be written as a Green function**



$$= C_{\alpha\beta}(k,p) = \int d^4z\, e^{-ik\cdot z}\,\langle p|\,\bar{\psi}_\beta(z)\,\psi_\alpha(0)\,|p\rangle\Big|_{\mathrm{LO}}$$

(unpolarized case)

$\bar{\mathcal{P}}$ and $\mathcal{P}$ project out a single Dirac structure $F(k^+/p^+) = \left[\bar{\mathcal{P}}\int \frac{dk^-\,d^2\boldsymbol{k}_T}{(2\pi)^4}\,C(k,p)\,\mathcal{P}\right]_{\alpha\beta} = f(x)\,\frac{\gamma^-_{\alpha\beta}}{2}$

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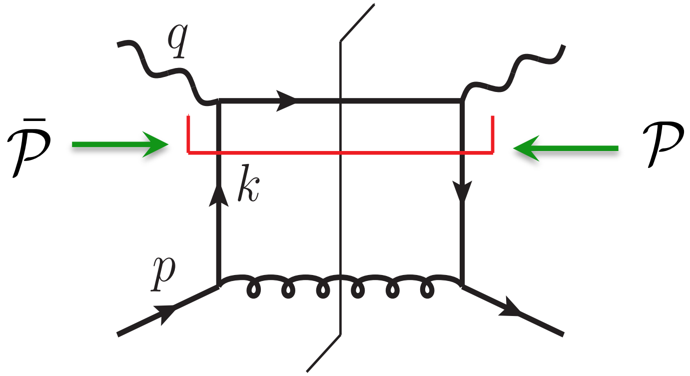

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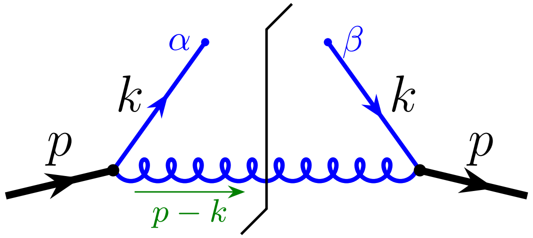

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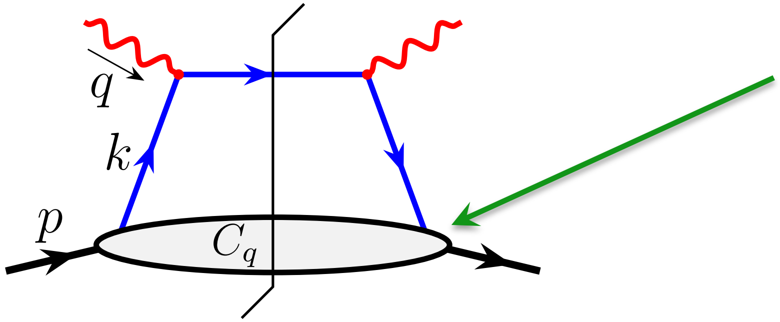
- ❑ **Factorization formula at LO**

$$\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\,H(k,q)\,C(k,p)\,\right] \simeq \int \frac{dx}{x}\,f(x)\,\underbrace{\mathrm{Tr}\left[\frac{\gamma\cdot\hat{k}}{2}\,H(\hat{k},q)\right]}_{\text{On-shell massless quark scattering}}$$

- $\hat{k} = (xp^+, 0^-, \mathbf{0}_T)$ enters hart part
- $\gamma\cdot\hat{k}/2$ is its on-shell spin average

Slight generalization

☐ Suppose a general collinear blob



Blob = sum over all possible Feynman diagrams

$$C_{\alpha\beta}(k,p) = \int d^4z\, e^{-ik\cdot z} \langle p|\, \bar{\psi}_\beta(z)\, \psi_\alpha(0)\, |p\rangle \Big|_{\text{all orders}}$$

☐ Spinor decomposition

$$C_{\alpha\beta} = S\, \mathbf{1}_{\alpha\beta} + P\,(\gamma_5)_{\alpha\beta} + C^\mu(\gamma_\mu)_{\alpha\beta} + \Delta C^\mu(\gamma_\mu\gamma_5)_{\alpha\beta} + T^{\mu\nu}(\sigma_{\mu\nu})_{\alpha\beta}$$

$\bar{\mathcal{P}}$ and $\mathcal{P}$ project out three Dirac structures

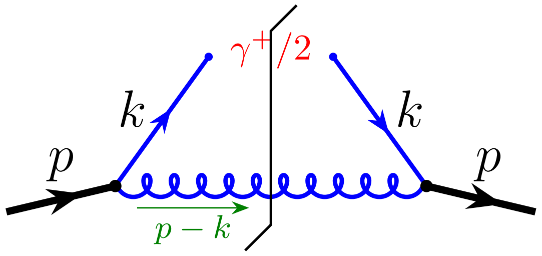
$$\bar{\mathcal{P}} C_{\alpha\beta} \mathcal{P} = C^+(\gamma^-)_{\alpha\beta} + \Delta C^+(\gamma^-\gamma_5)_{\alpha\beta} + \sum_{i=1,2} T^{+i}(\sigma^{-i})_{\alpha\beta}$$

For C^+ component, it gives the unpolarized PDF:

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{-i\,x\,p^+z^-} \left\langle p \,\middle|\, \bar{\psi}(z^-)\, \frac{\gamma^+}{2}\, \psi(0) \,\middle|\, p \right\rangle$$

UV divergence of PDF

❑ Once factorized, PDF contains UV divergence



$$= -g^2 C_F \int \frac{dk^- d^2\boldsymbol{k}_T}{(2\pi)^4} (2\pi)\delta^+\left((p-k)^2\right) \frac{1}{(k^2)^2} \, \mathrm{Tr}\left[\frac{\gamma^+}{2} \not{k} \gamma^\mu \frac{\not{p}}{2} \gamma_\mu \not{k}\right]$$

$$= \frac{\alpha_s C_F}{2\pi}(1-x)\int_0^\infty \frac{dk_T^2}{k_T^2}$$

- Collinear divergence at $k_T \to 0$ is expected.
- But what about UV divergence at $k_T \to \infty$?

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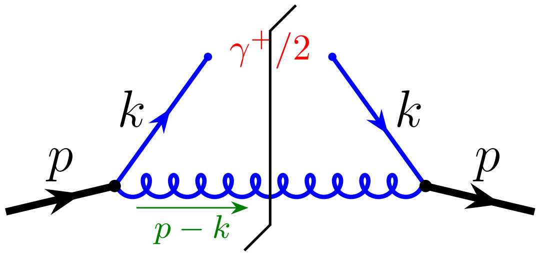

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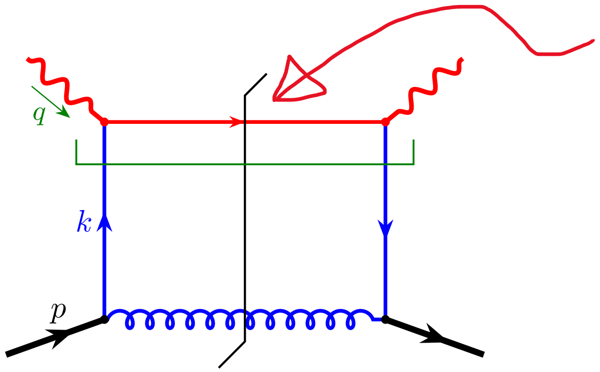
- Collinear divergence at $k_T \to 0$ is expected.
- But what about UV divergence at $k_T \to \infty$?

❑ Such UV divergence is absent in original diagram



"Hard" part contains a δ-function that constrains x to

$$x = 1 - \frac{1-x_B}{2}\left(1 - \sqrt{1 - \frac{4x_B}{1-x_B}\frac{k_T^2}{Q^2}}\right) \quad \text{or} \quad x = x_B + \frac{1-x_B}{2}\left(1 - \sqrt{1 - \frac{4x_B}{1-x_B}\frac{k_T^2}{Q^2}}\right)$$

This constrains k_T to

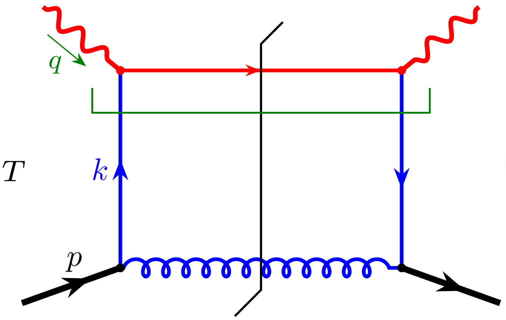
$$k_T^2 \leq \frac{1-x_B}{4x_B}Q^2$$

There is **NO** UV divergence before factorization.

Renormalization of PDF and factorization scale

❑ UV divergence is an “artifact” of factorization

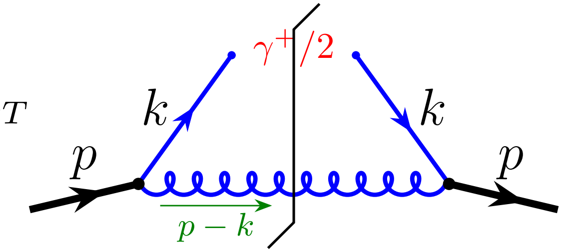
$$\int^{\sim Q} d^2\boldsymbol{k}_T$$



k_T is bounded

→

$$\int^{\infty} d^2\boldsymbol{k}_T$$



k_T extends to infinity

→ UV divergent

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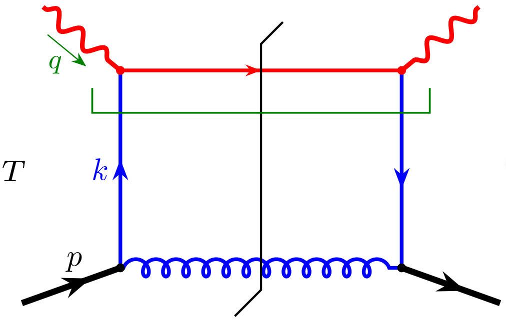
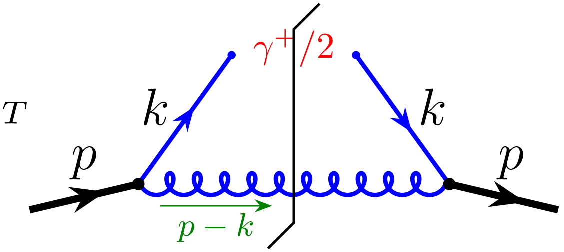


$$\longrightarrow \int^{\infty} d^2\boldsymbol{k}_T$$



→ UV divergent

k_T is bounded

k_T extends to infinity

☐ PDF needs extra renormalization

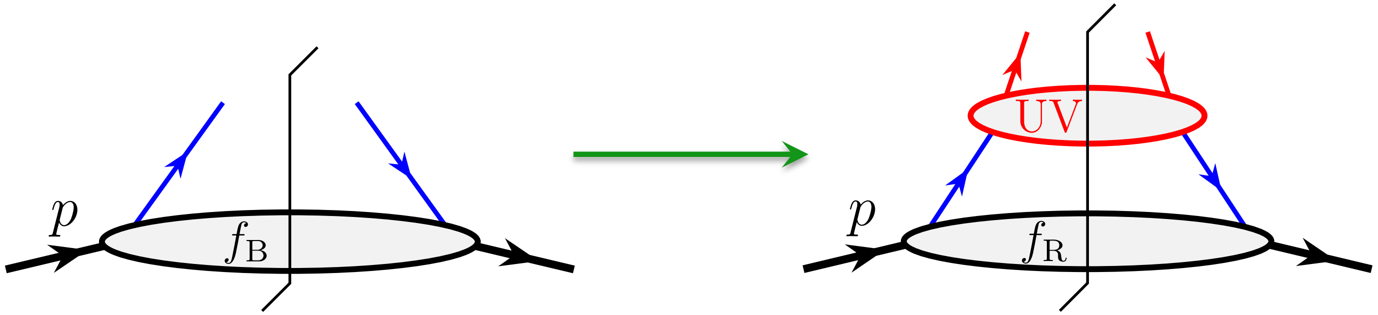
- Simplest way: introduce an upper cutoff μ^2 $\longrightarrow \int^{\mu^2} d^2\boldsymbol{k}_T$

  Interpretation: PDF carries away physics below μ^2. Hard part happens between μ^2 and Q^2.

- Realistically: dimensional regularization + $\overline{\mathrm{MS}}$
- Factorization separates dynamics at the **factorization scale μ^2**.

Renormalization of PDF and DGLAP evolution

Multiplicative renormalization



UV is like a hard part with $Q \to \Lambda = \infty$

Renormalized PDF, factorized from UV
in the same way as from a hard part.

Omitting flavor mixing for now, UV of PDF is "factorized":

$$f_B\left(x;\frac{1}{\epsilon}\right) = \int \frac{dx'}{x'}\, \underbrace{Z^{-1}\left(\frac{x}{x'}, \alpha_s(\mu^2); \frac{1}{\epsilon}\right)}_{\text{UV div.}} f_R(x', \mu^2)$$

Reversely, PDF is multiplicatively renormalized

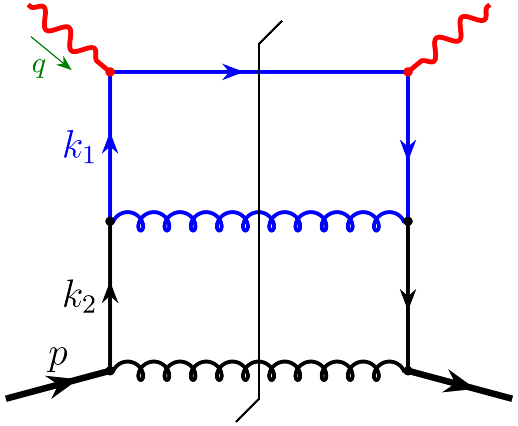
$$\boxed{f_R(x,\mu^2) = \int \frac{dx'}{x'}\, Z\left(\frac{x}{x'}, \alpha_s(\mu^2); \frac{1}{\epsilon}\right) f_B\left(x'; \frac{1}{\epsilon}\right)}$$

Evolution with respect to μ^2

DGLAP equation: $$\frac{d}{d\ln\mu^2} f_R(x,\mu^2) = \int \frac{dx'}{x'}\, P\left(\frac{x}{x'}, \alpha_s(\mu^2)\right) f_R(x', \mu^2)$$

"One-loop" factorization of DIS: Ladder diagram

❑ More than one region in any diagram



Assume $\boldsymbol{k_2}$ **is always collinear to** $\boldsymbol{p}$, as in "LO" case.

There are two regions for k_1:

- $\boldsymbol{R_C}$: $\boldsymbol{k_{1T} \sim k_{2T} \ll Q} \implies$ **Both** $\boldsymbol{k_1}$ **and** $\boldsymbol{k_2}$ **are collinear**
- $\boldsymbol{R_H}$: $\boldsymbol{k_{1T} \sim Q} \implies$ $\boldsymbol{k_1}$ **is hard, and** $\boldsymbol{k_2}$ **is collinear**

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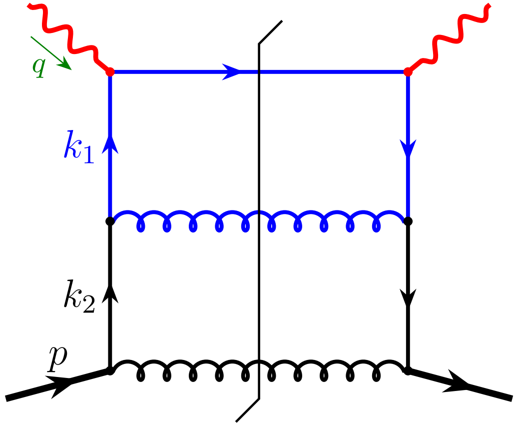

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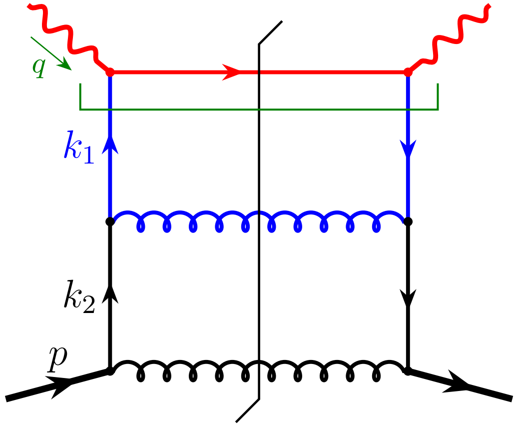
❑ Both regions need to be incorporated



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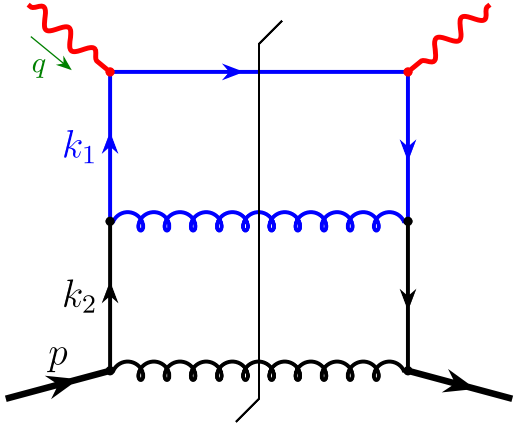

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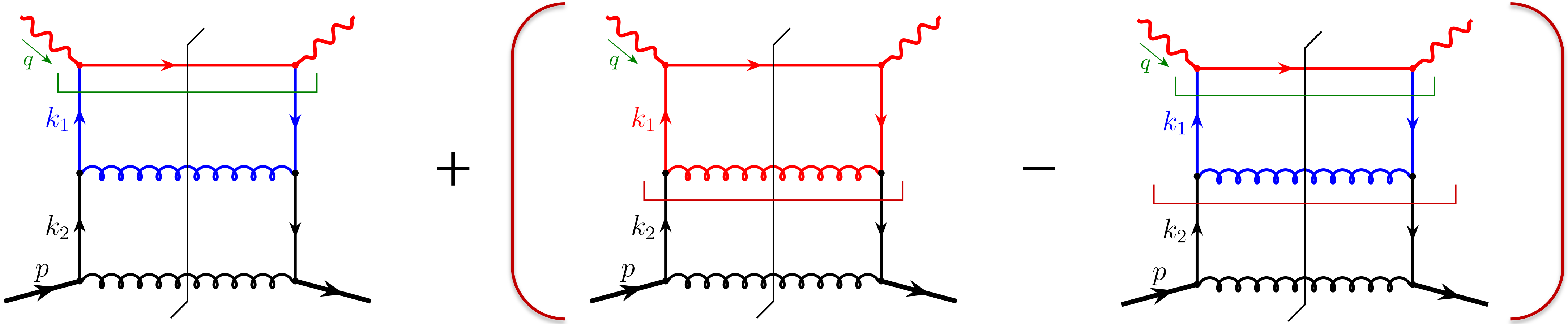
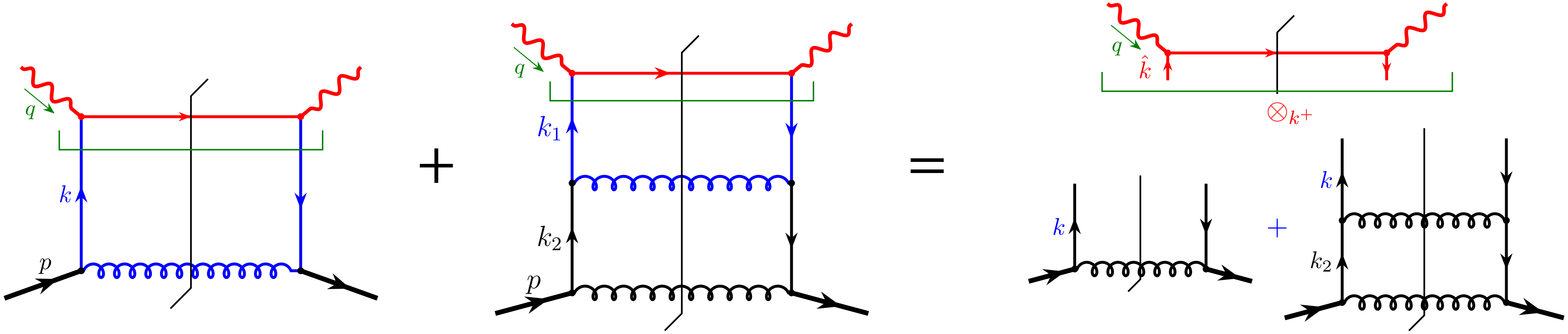
- $\boldsymbol{R_H}$: $\boldsymbol{k_{1T} \sim Q} \implies \boldsymbol{k_1}$ **is hard, and** $\boldsymbol{k_2}$ **is collinear**

❑ Both regions need to be incorporated

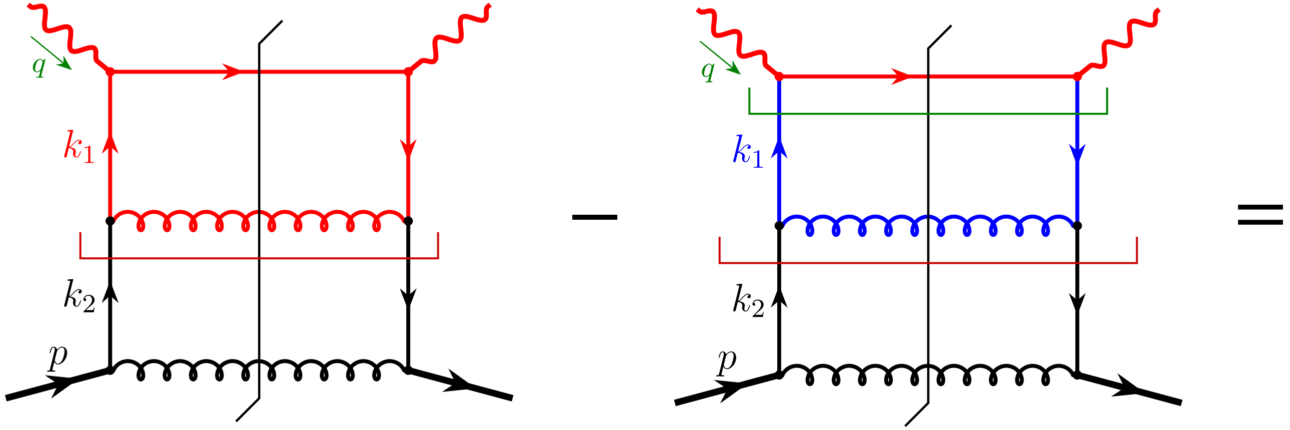


“One-loop” factorization of DIS: Combine with LO

- ❑ **Same hard parts combine**



- ❑ **High-order hard part contains collinear subtraction**



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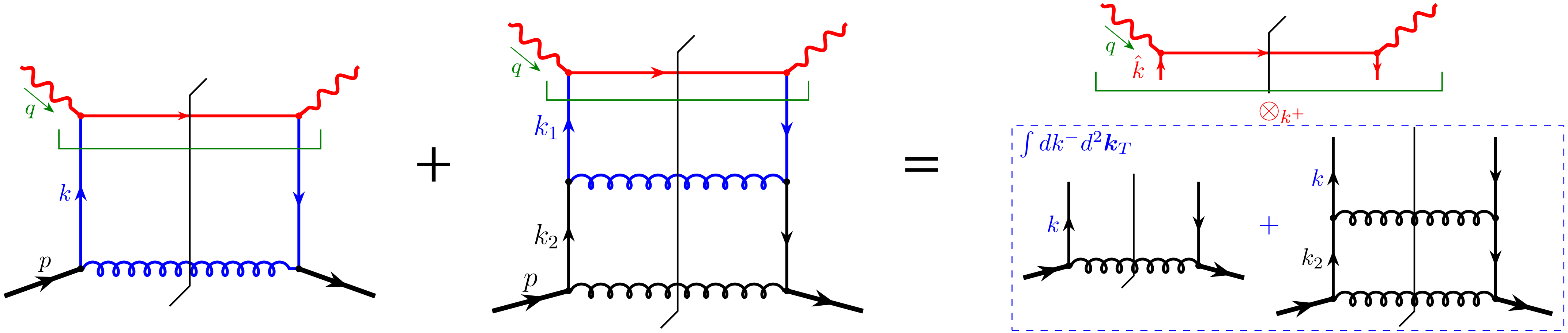

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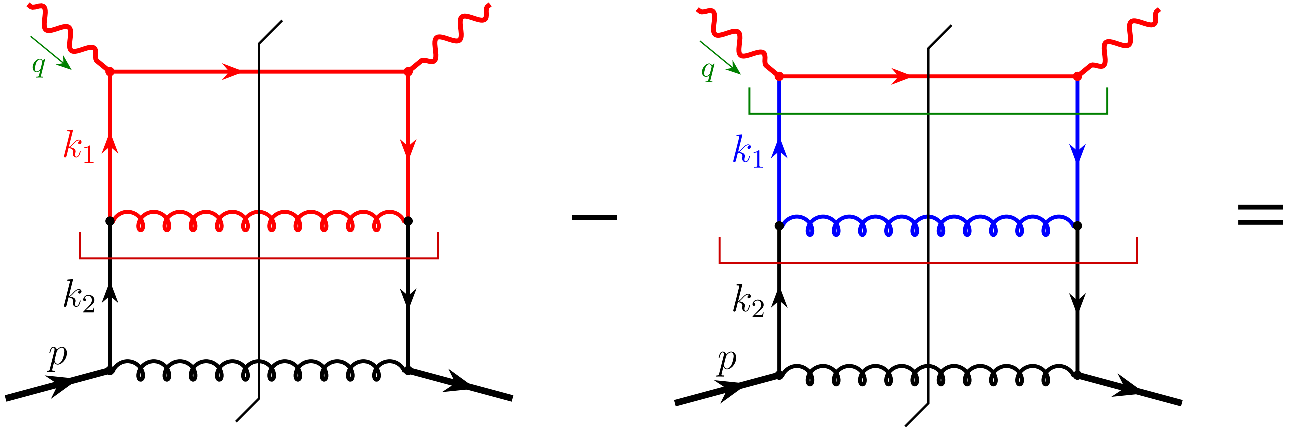

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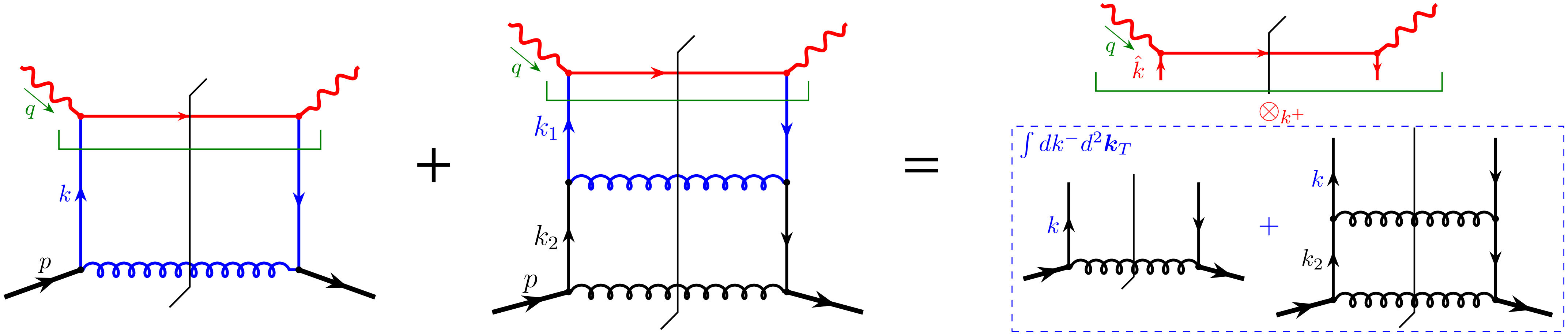

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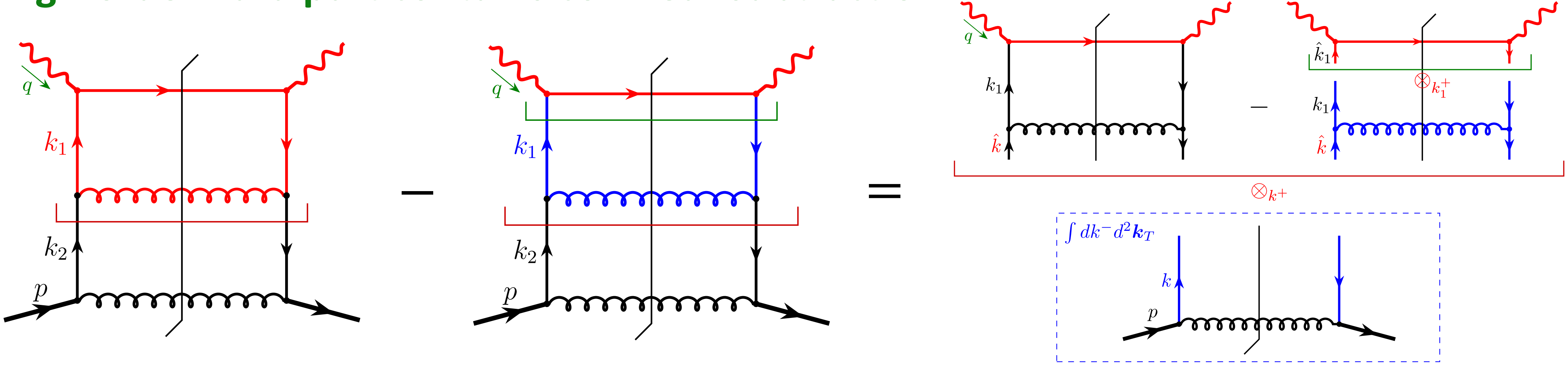
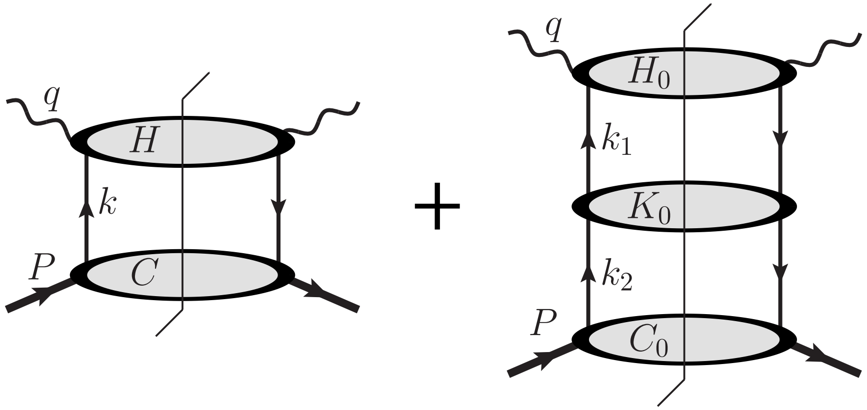


Non-ladder “one-loop” diagrams of DIS

❑ **DIS factorization could be generally obtained if only ladder diagrams are leading**

True in light-cone gauge!



$$+\cdots = \underbrace{H_0 \cdot \frac{1}{1-(1-\hat{T})K_0}}_{\text{hard part}}\, \hat{T}\, \underbrace{\frac{1}{1-K_0}\cdot C_0}_{\text{PDF}} + \mathcal{O}\left(\frac{m}{Q}\right)$$

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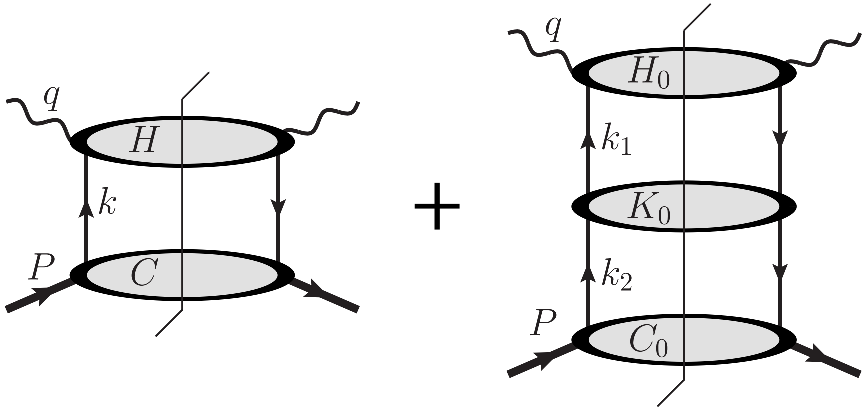

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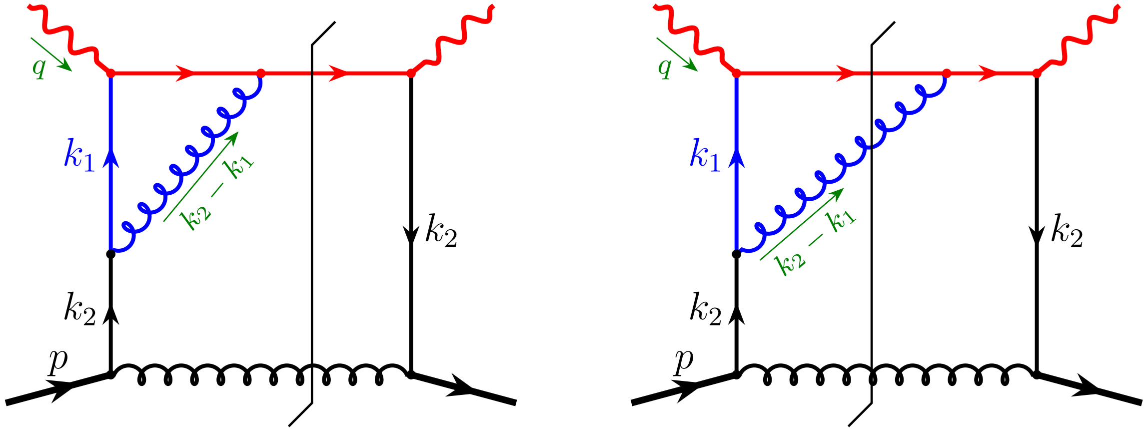
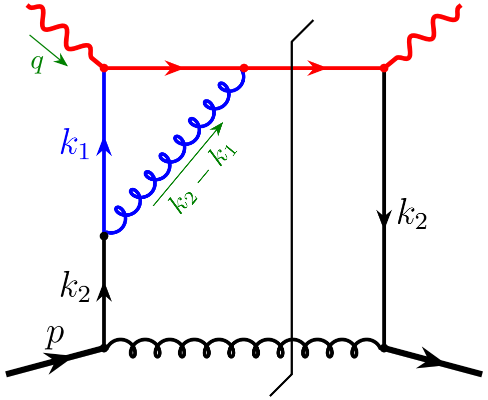
❑ **But there are other leading regions in Feynman gauge**



Consider: $\boldsymbol{k_2}$ **is always collinear.**

- If only **hard** k_1 region is leading, the loop reduces to the H_0 ladder.
- But if **collinear** k_1 can also be leading, we need further development!

Pinch singularity



Virtual diagram: k_1 is **loop** momentum.

$$\propto \int \frac{dk_1^+ dk_1^- d^2\boldsymbol{k}_{1T}}{(2\pi)^4} \frac{N(k_1; k_2, q)}{[(k_1+q)^2 + i\epsilon]\,(k_1^2 + i\epsilon)[(k_2 - k_1)^2 + i\epsilon]}$$

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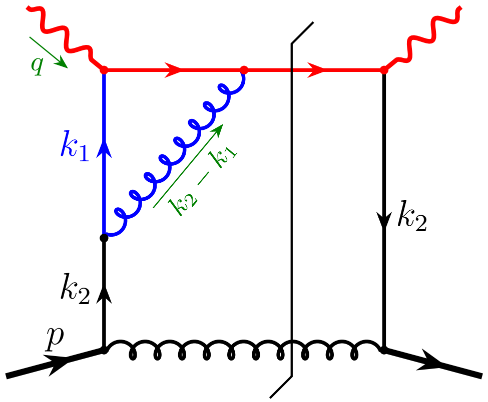

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Look at k_1^- integration first, in the **complex plane**!

Three propagators give three poles to k_1^-

$$(k_1+q)^2 \quad \rightarrow \quad k_{1H}^- = -q^- + \frac{k_{1T}^2 - i\epsilon}{2(x_1 - x_B)p^+}$$

$$k_1^2 \quad \rightarrow \quad k_{1q}^- = \frac{k_{1T}^2 - i\epsilon}{2x_1 p^+}$$

$$(k_2 - k_1)^2 \quad \rightarrow \quad k_{1g}^- = k_2^- - \frac{(\boldsymbol{k}_{2T} - \boldsymbol{k}_{1T})^2 - i\epsilon}{2(x_2 - x_1)p^+}$$

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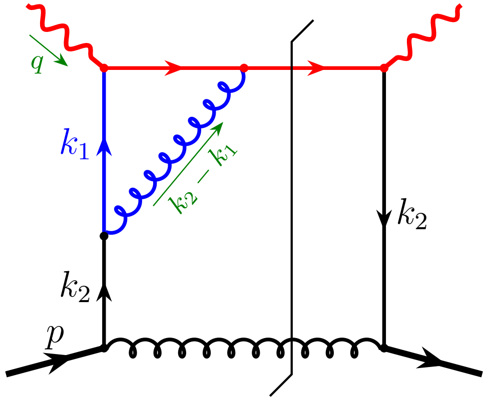

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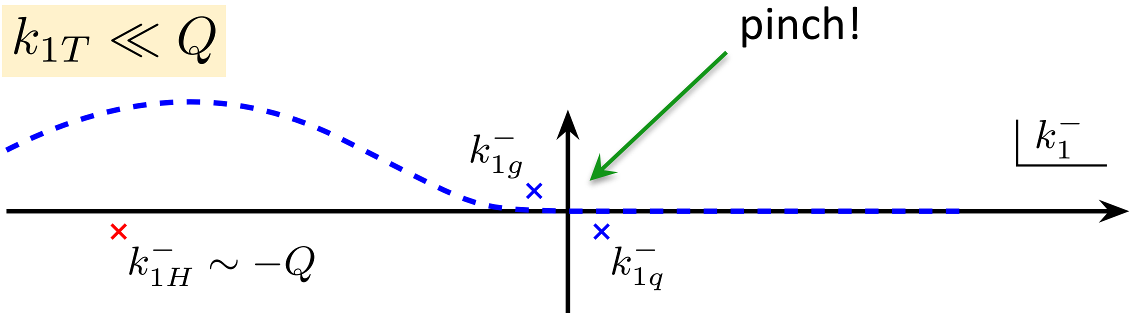

Consider $x_2 > x_1$, and x_1, x_2, $x_2 - x_1 \sim 1$.

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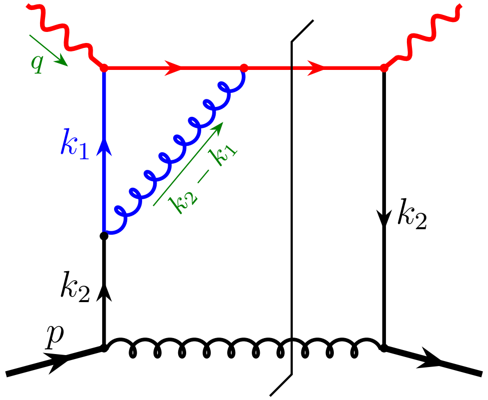

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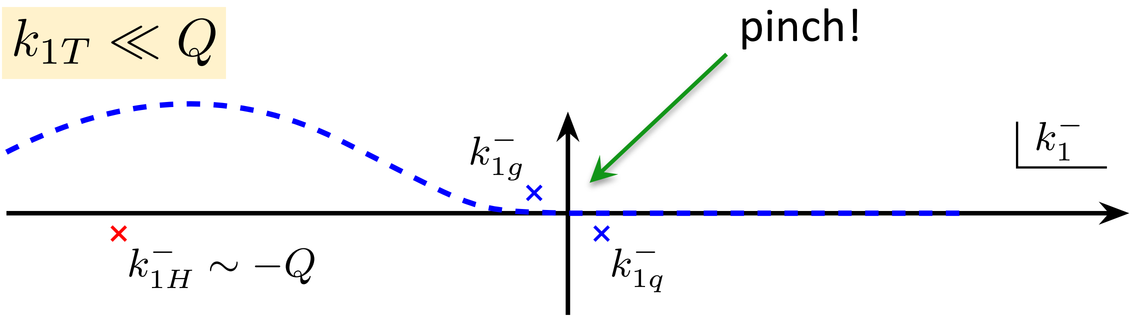

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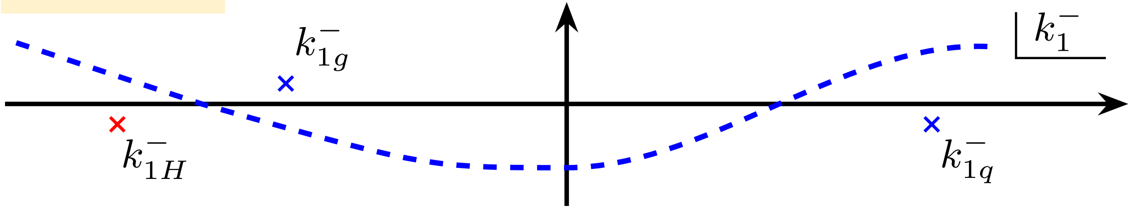

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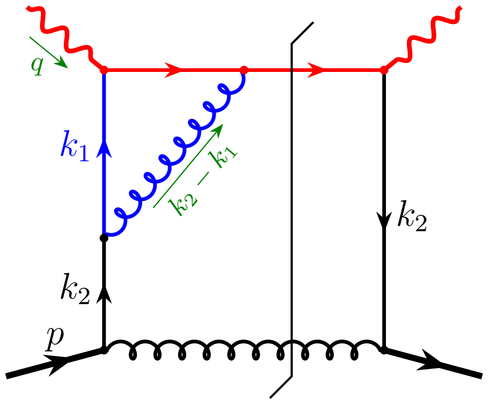

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A loop momentum is deformed to be "hard" unless its contour is pinched by on-shell poles!

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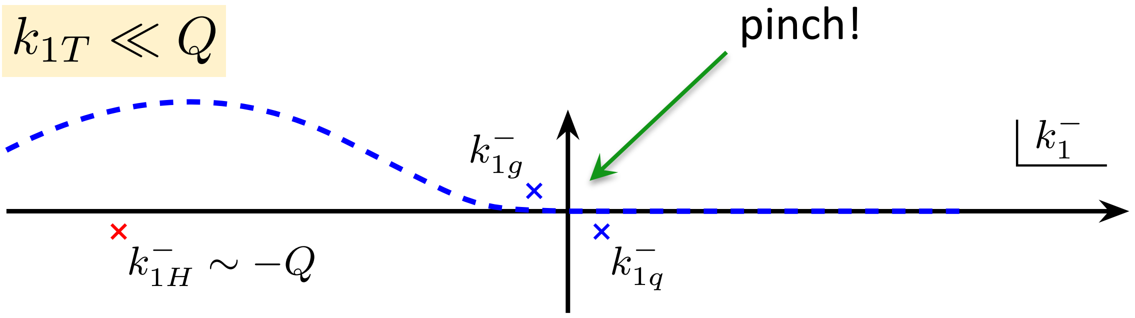
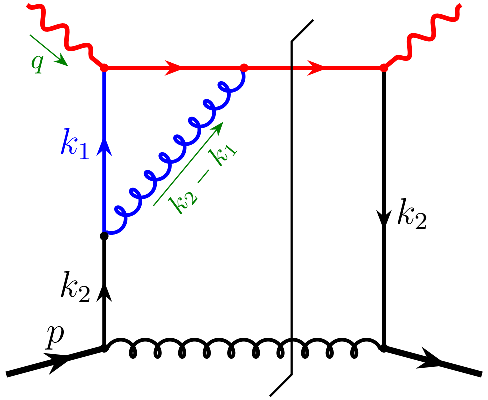


We only need to consider

- **Hard region**
- **Pinched region**

Pinch singularity



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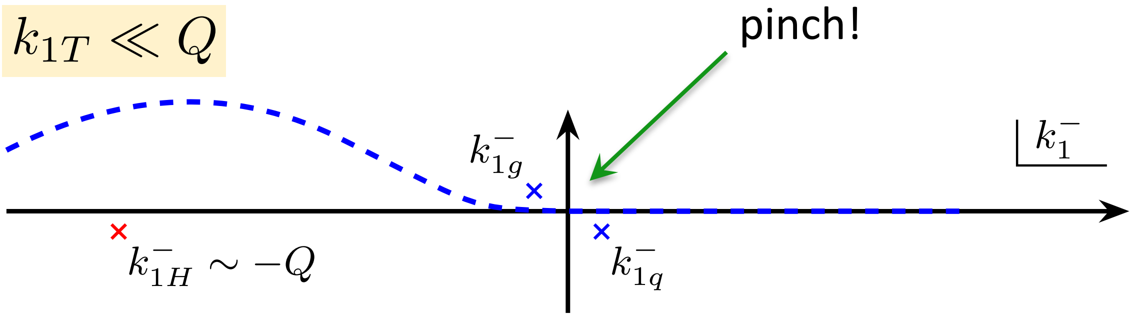

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$$k_2^\mu \sim \left(Q, \frac{k_{2T}^2}{Q}, k_{2T}\right)$$

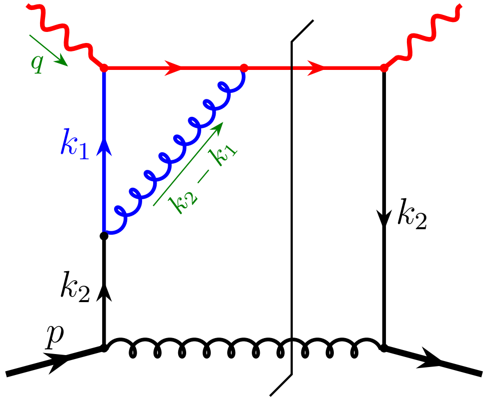
$$k_1^\mu \sim \left(Q, \frac{k_{1T}^2}{Q}, k_{1T}\right)$$

$k_{2T} \sim k_{1T} \ll Q$

In this case, pinch singularity is reached as $k_{1T} \to 0$

("Region" = momentum region *around* this pinch singularity)

Pinch singularity and power counting



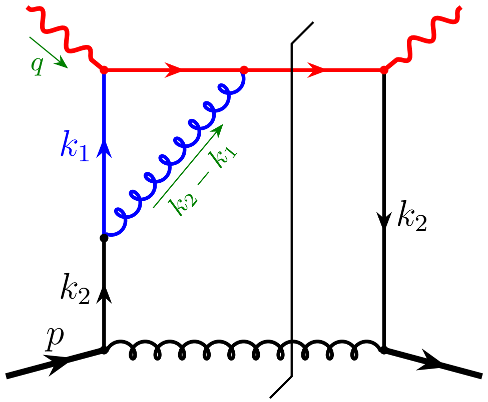
What about numerator factor?

$$\propto \int \frac{dk_1^+ dk_1^- d^2\boldsymbol{k}_{1T}}{(2\pi)^4} \frac{N(k_1;k_2,q)}{[(k_1+q)^2+i\epsilon]\,(k_1^2+i\epsilon)[(k_2-k_1)^2+i\epsilon]}$$

$$N(k_1;k_2,q) \propto \gamma_\rho\,\gamma\cdot(k_1+q)\,\gamma^\mu\,\underbrace{\gamma\cdot k_1}_{\gamma^- k_1^+}\,\gamma^\rho\,\underbrace{\gamma\cdot k_2}_{\gamma^- k_2^+}$$

$\rho = +$

Pinch singularity and power counting



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$\rho = +$

Power counting separates the gluon polarization.

Only “longitudinal” or “scalar” polarization is leading power.

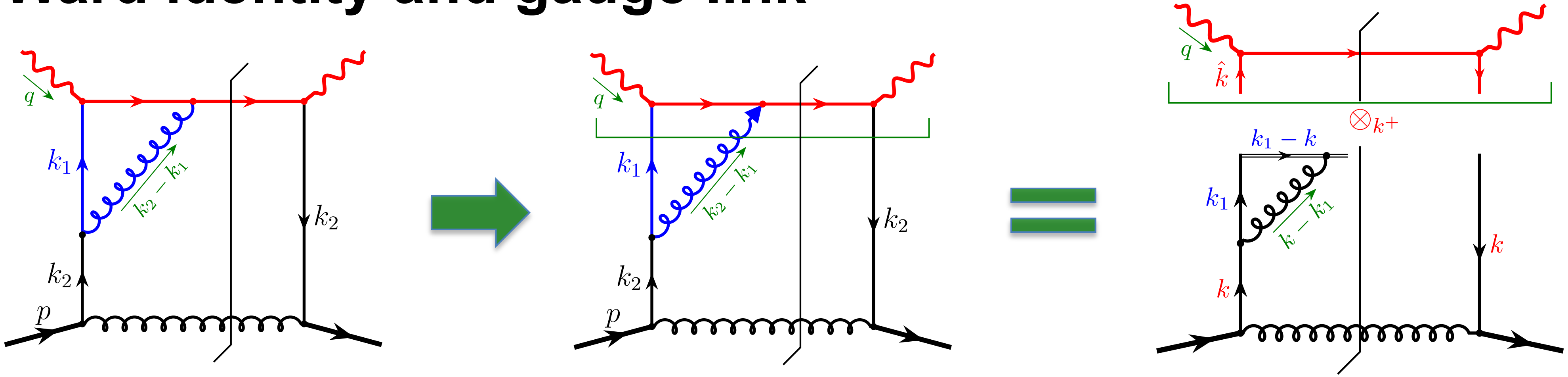
$$n^\rho = (0^+, 1^-, \mathbf{0}_T)$$

$$H_\rho(\hat{k}_1)\, J^\rho(k_1) \simeq H_+(\hat{k}_1)\, J^+(k_1) = H_+(\hat{k}_1)\, \frac{k_1^+ n^-}{k_1\cdot n}\, J^+(k_1) = \left[H_\rho(\hat{k}_1)\,\hat{k}_1^\rho\right]\left[\frac{n_\sigma}{k_1\cdot n} J^\sigma(k_1)\right]$$

Ward identity!

Eikonal factor!

Ward identity and gauge link



$$H_\rho(\hat{k}_1)\,J^\rho(k_1) \simeq \left[H_\rho(\hat{k}_1)\,\hat{k}_1^\rho\right]\left[\frac{n_\sigma}{k_1 \cdot n}\,J^\sigma(k_1)\right]$$

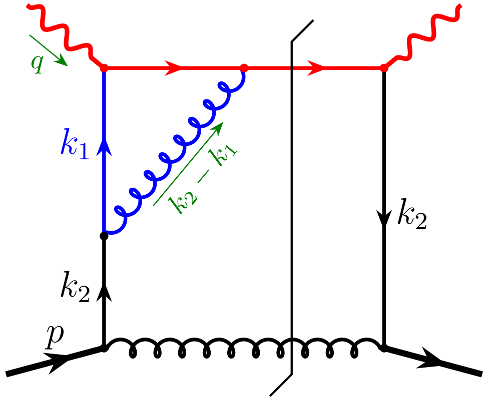
Collinear factor (PDF) now has a modified operator definition: **dressed by a Wilson line**

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{-i\,x\,p^+ z^-} \left\langle p \,\middle|\, \bar{\psi}(z^-)\,\frac{\gamma^+}{2}\,\left[W_{(1)}(\infty, 0; n)\,\psi(0)\right] \middle|\, p \right\rangle$$

$$W(\infty, x; n) = P\exp\left\{-ig\int_0^\infty d\lambda\, n^\rho A^a_\rho(x + \lambda n)t^a\right\}$$

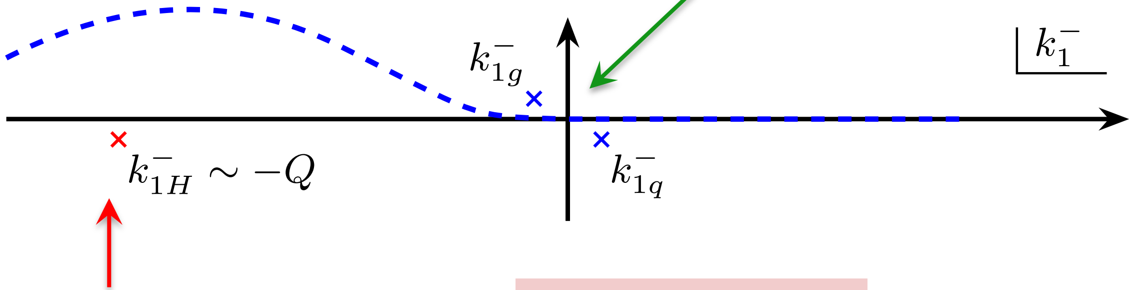
Expanded to $\mathcal{O}(g)$

Hard region with subtraction



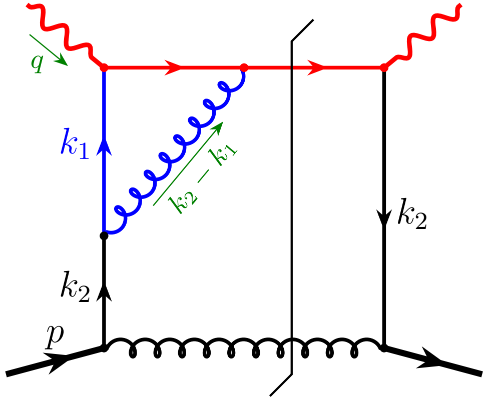
$k_{1T} \ll Q$

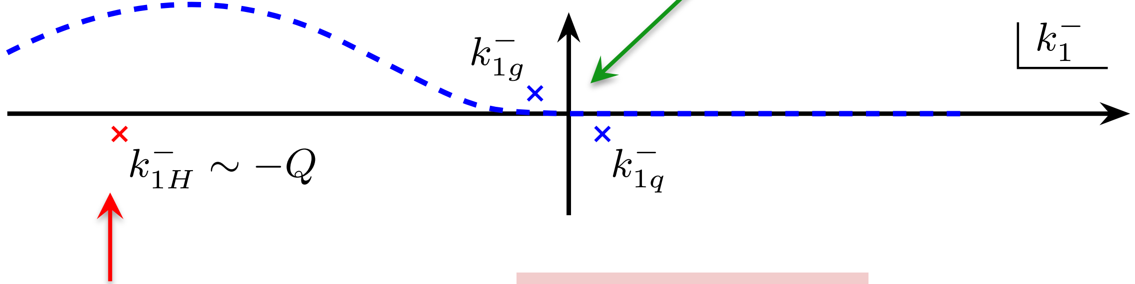
Collinear region



Hard region: No pinch. $k_1 \sim (Q, Q, Q)$

Hard region with subtraction



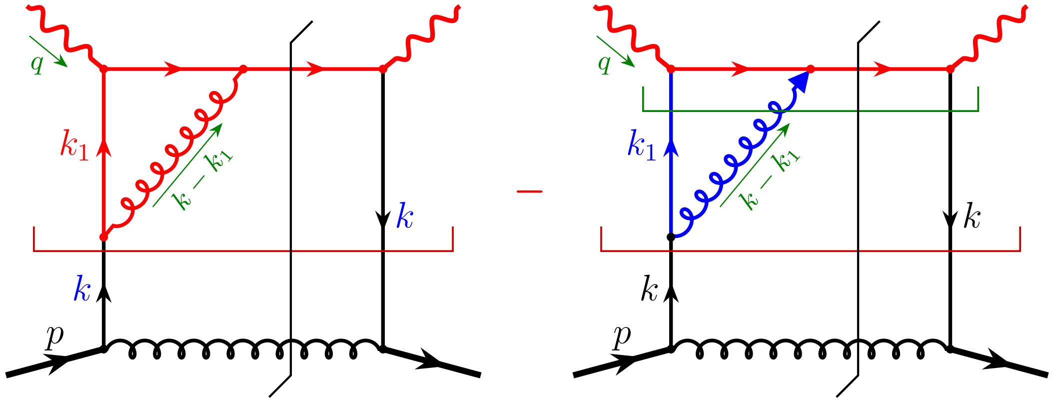


$k_{1T} \ll Q$

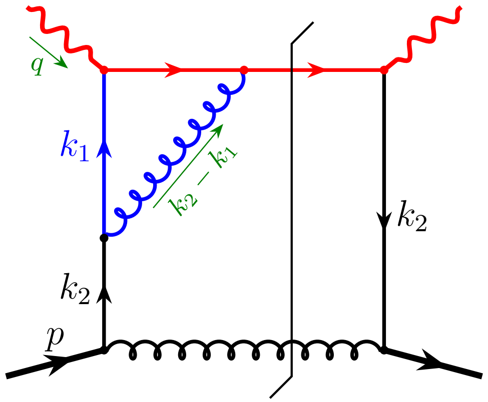
Collinear region

Hard region: No pinch. $k_1 \sim (Q, Q, Q)$

Contribution from hard region involves subtraction of collinear subregion.



Hard region with subtraction



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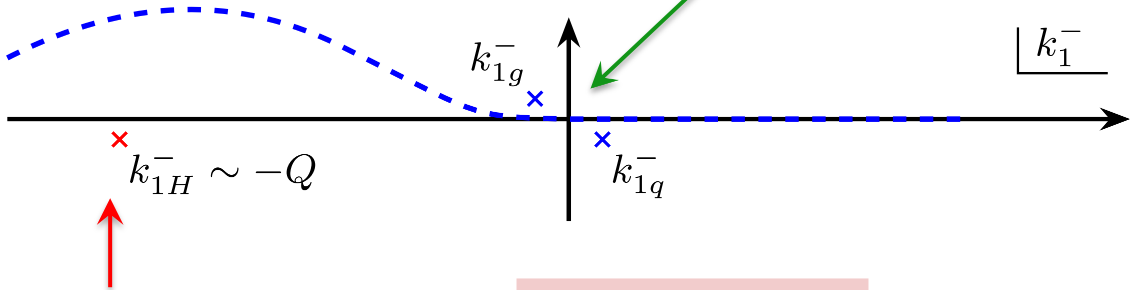

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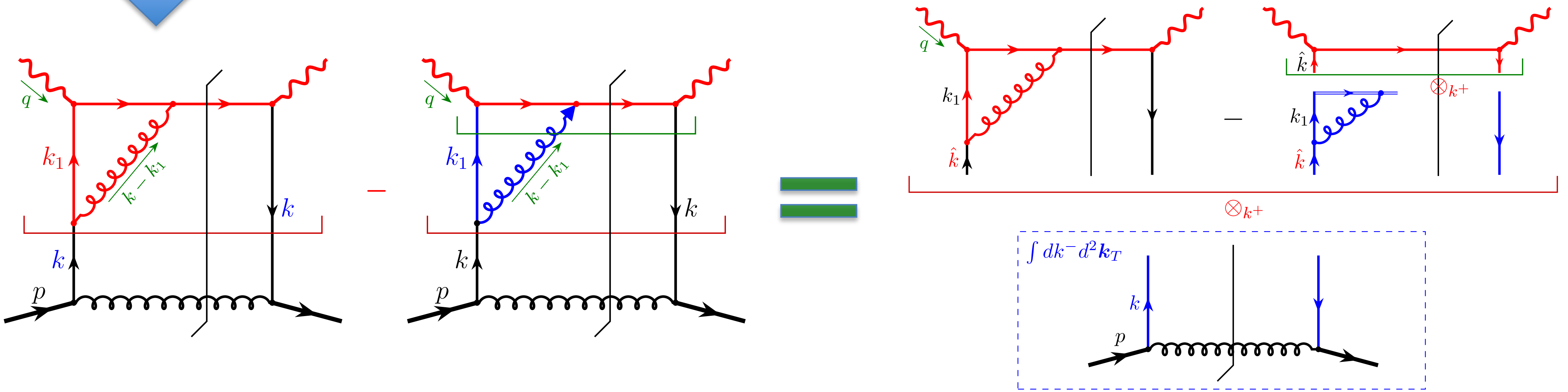
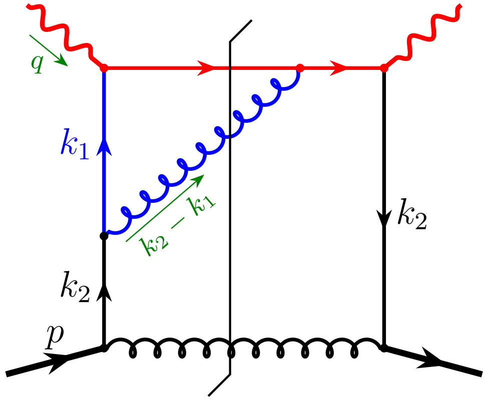


Similarly for a real gluon radiation



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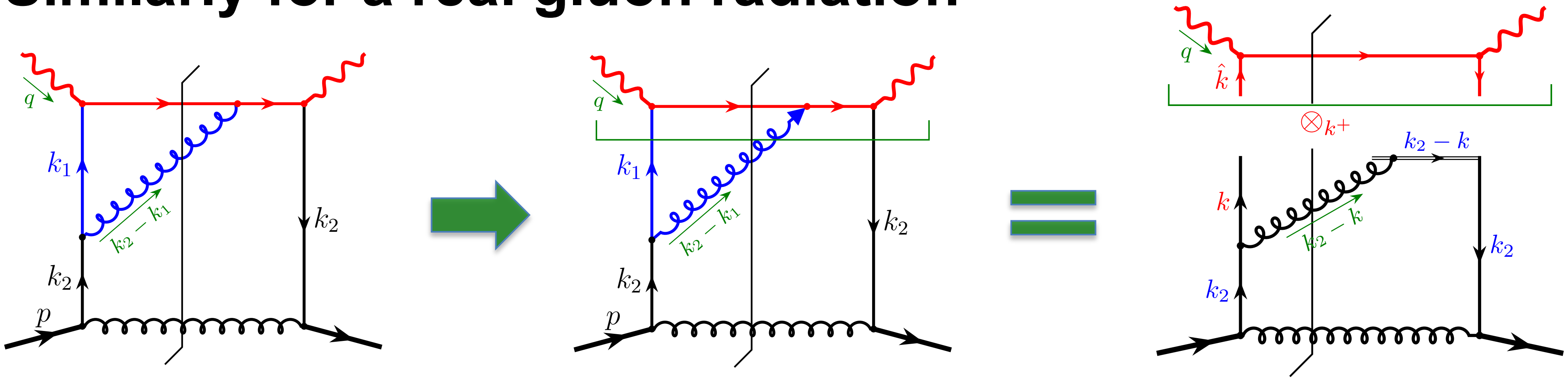

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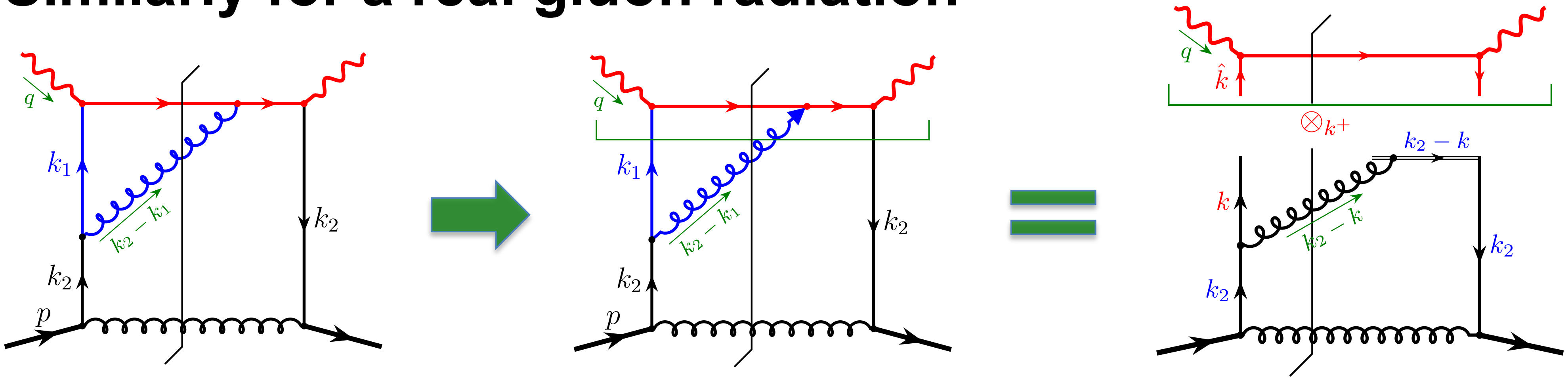

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$$f(x) = \int \frac{dz^-}{2\pi}\, e^{-i\,x\,p^+ z^-} \left\langle p \left| \left[\bar{\psi}(z^-)\, W^{\dagger}_{(1)}(\infty, z^-; n) \right] \frac{\gamma^+}{2}\, \psi(0) \right| p \right\rangle$$

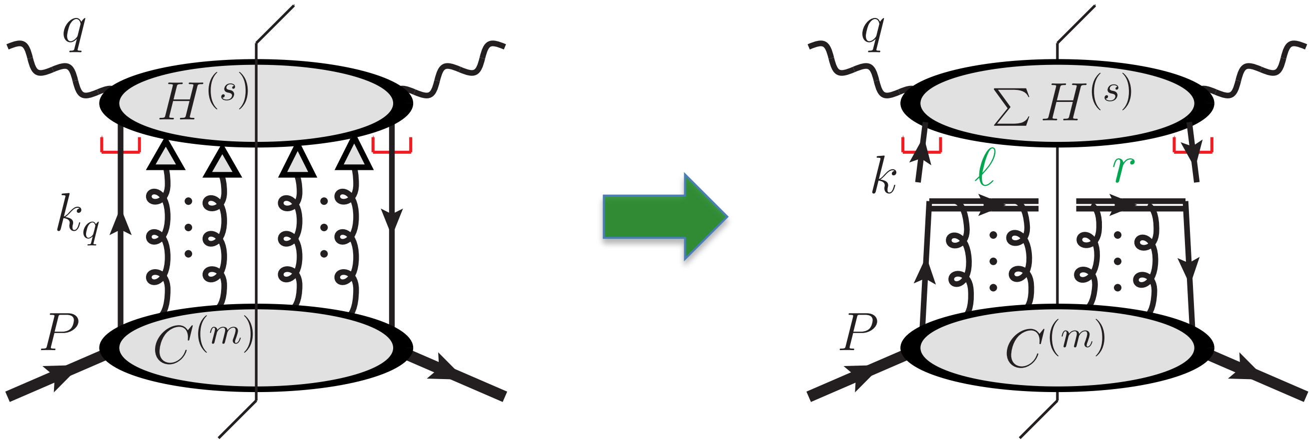
Expanded to $\mathcal{O}(g)$

Exercise:

1. Justify this approximation for using Ward identity.
2. Write down the appropriate subtraction structure for the hard region contribution.

All-order factorization of DIS

❑ An arbitrary leading region



❑ Include subtraction and sum over all diagrams $\Longrightarrow$ factorization

$$\sigma_{\rm DIS}(x_B,Q)=\sum_i\int_{x_B}^1\frac{dx}{x}\,f_i(x,\mu)\,C_i\!\left(\frac{x_B}{x},\frac{Q}{\mu};\alpha_s(\mu)\right)+\text{polarized}+\mathcal{O}(\Lambda_{\rm QCD}/Q)$$

$$f(x,\mu)=\int\frac{dz^-}{2\pi}\,e^{-i\,x\,p^+z^-}\left\langle p\,\right|\left[\bar\psi(z^-)\,W^\dagger(\infty,z^-;n)\right]\frac{\gamma^+}{2}\left[W(\infty,0;n)\,\psi(0)\right]\left|\,p\right\rangle\Big|_{\rm UV\ ren.}$$

$$=\int\frac{dz^-}{2\pi}\,e^{-i\,x\,p^+z^-}\left\langle p\,\right|\bar\psi(z^-)\,W(z^-,0;n)\,\frac{\gamma^+}{2}\,\psi(0)\left|\,p\right\rangle\Big|_{\rm UV\ ren.}$$

Hard part C_i contains subtraction of all collinear divergences

Infrared finite!

Summary: Remarks on factorization

❑ What have we done?

- Factorized pinch singularity from hard part for **perturbative** diagrams to **all orders**.
- Identified **perturbative** pinches at low virtuality with sensitivity to **nonperturbative** dynamics.
- Assumed hard coefficients obtained from perturbative analysis still apply to **hadron** scattering.

❑ How are they justified?

- Not really possible from first principle without knowledge of nonperturbative QCD.
- Justified from ***comparison to experiments.***
- Factorization reduces “**unknowns**” to “**unknowns**”. Predictive power comes from universality.
- Justified by (1) **scaling** phenomena, and (2) success of **global analyses**.