From Ioffe Time to the Light Cone

Lattice QCD Computations of PDFs and GPDs

CNUGS — Jefferson Lab, June 1-2, 2026

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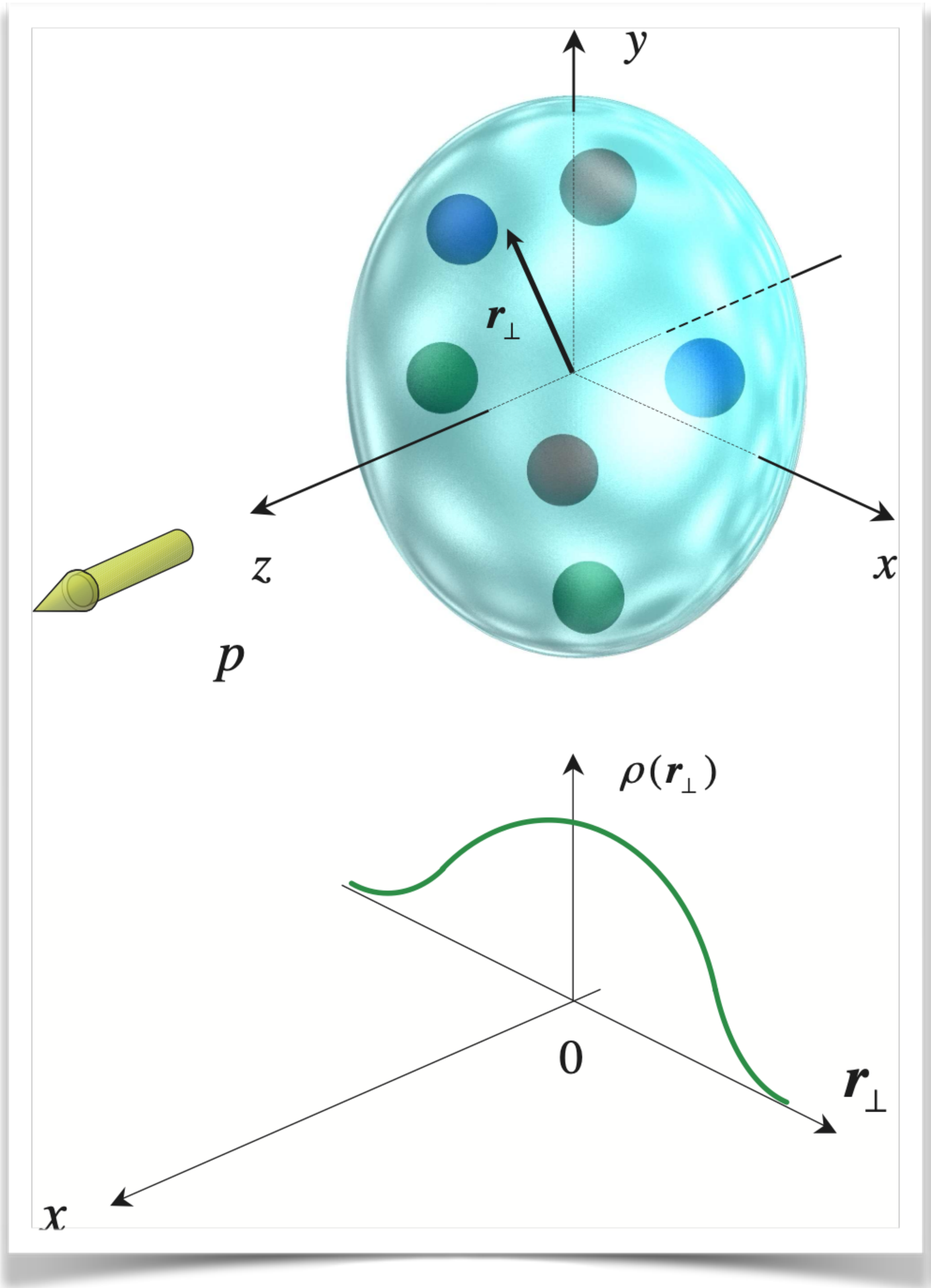
HadStruc

Introduction

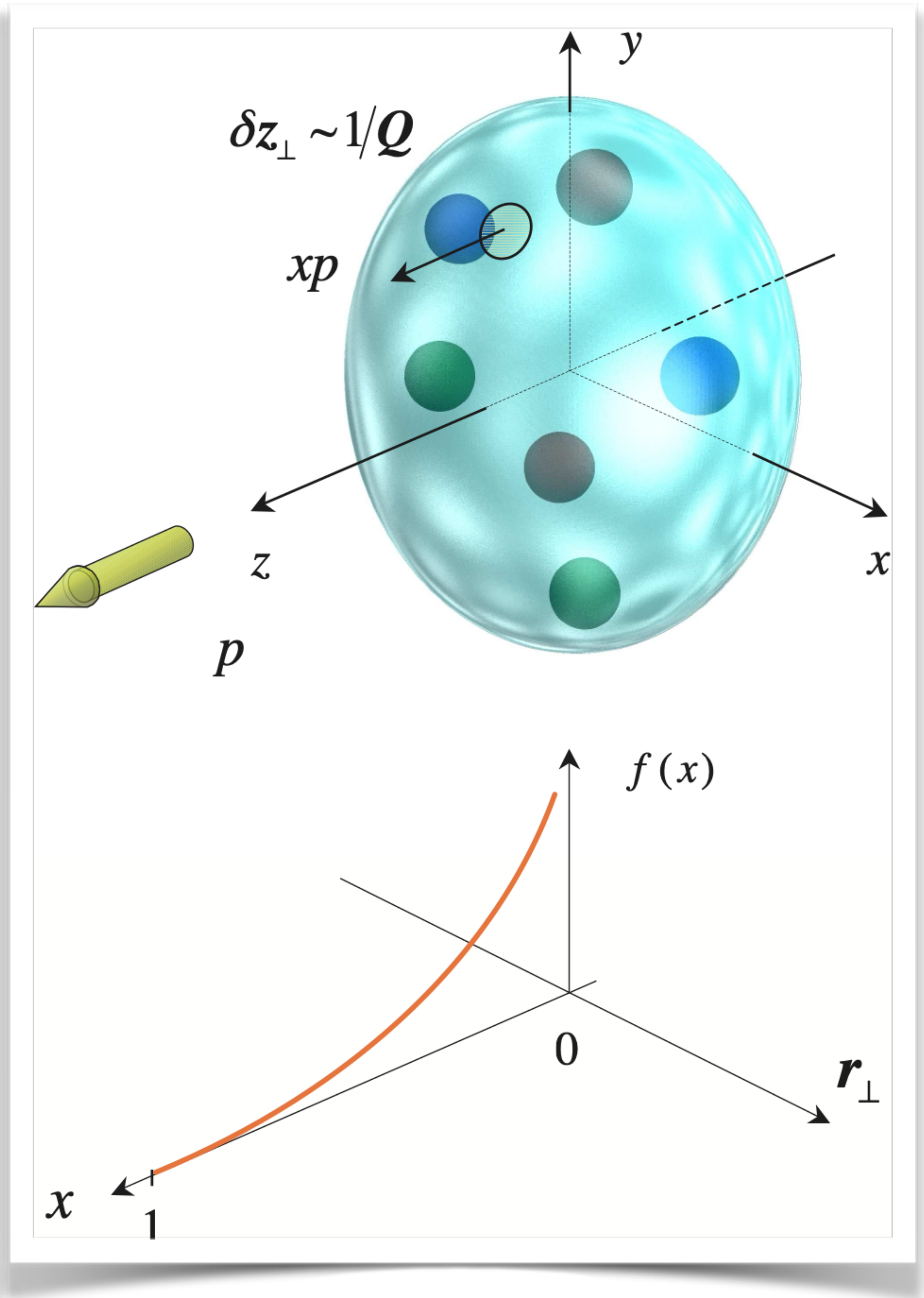
Objectives

- Compute the structure of Hadrons from first principles
  - Quantum ChromoDynamics (QCD)— Theory of strong interactions
  - Hadrons of interest: Proton, Neutron, pion, etc. (… resonances …)
  - Theory and Computation go hand in hand
- Why is this important?
  - Understand and interpret High Energy Scattering experiments (JLab 12 GeV, EIC)
    - Discover new physics
  - Fundamental questions in hadron physics:
    - Origin of mass, Spin, Mechanical properties, Confinement, etc.

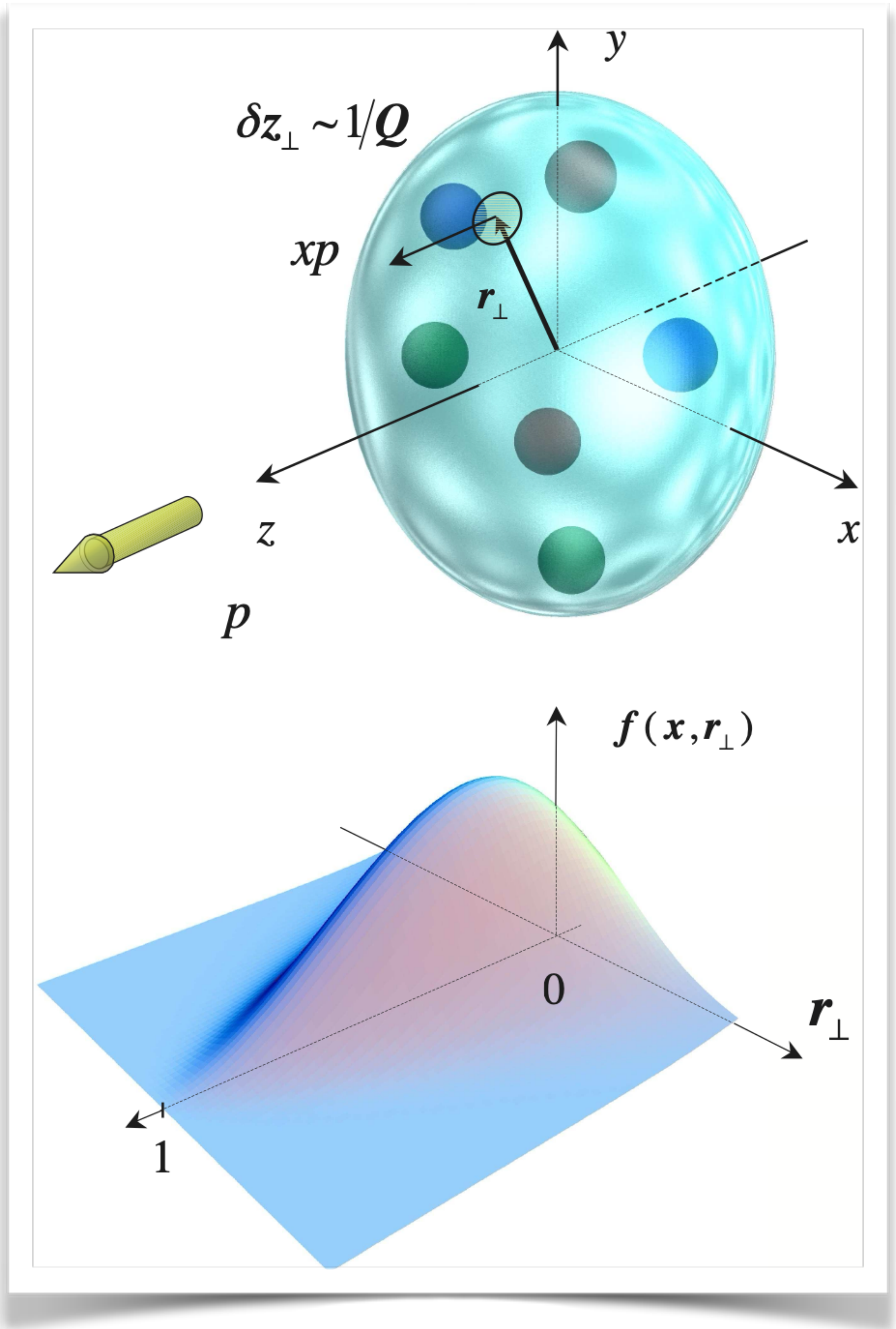
X. Ji, D. Muller, A. Radyushkin (1994-1997)



Form Factors

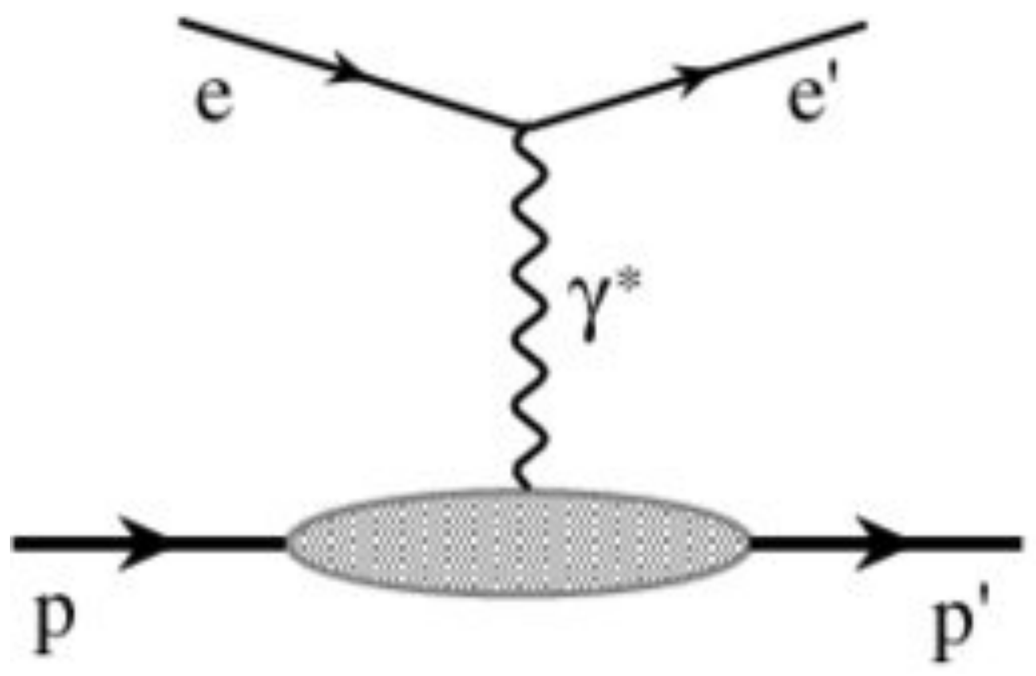


Parton Distribution functions

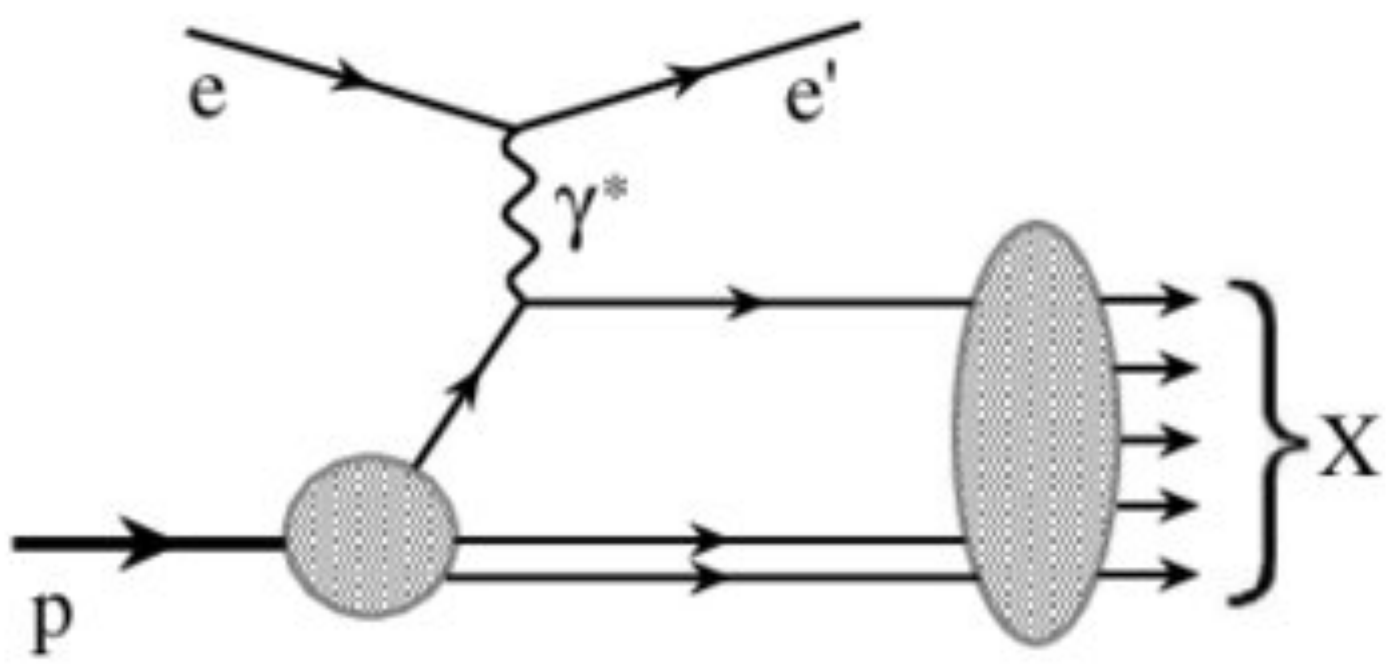


Generalized Parton Distribution functions

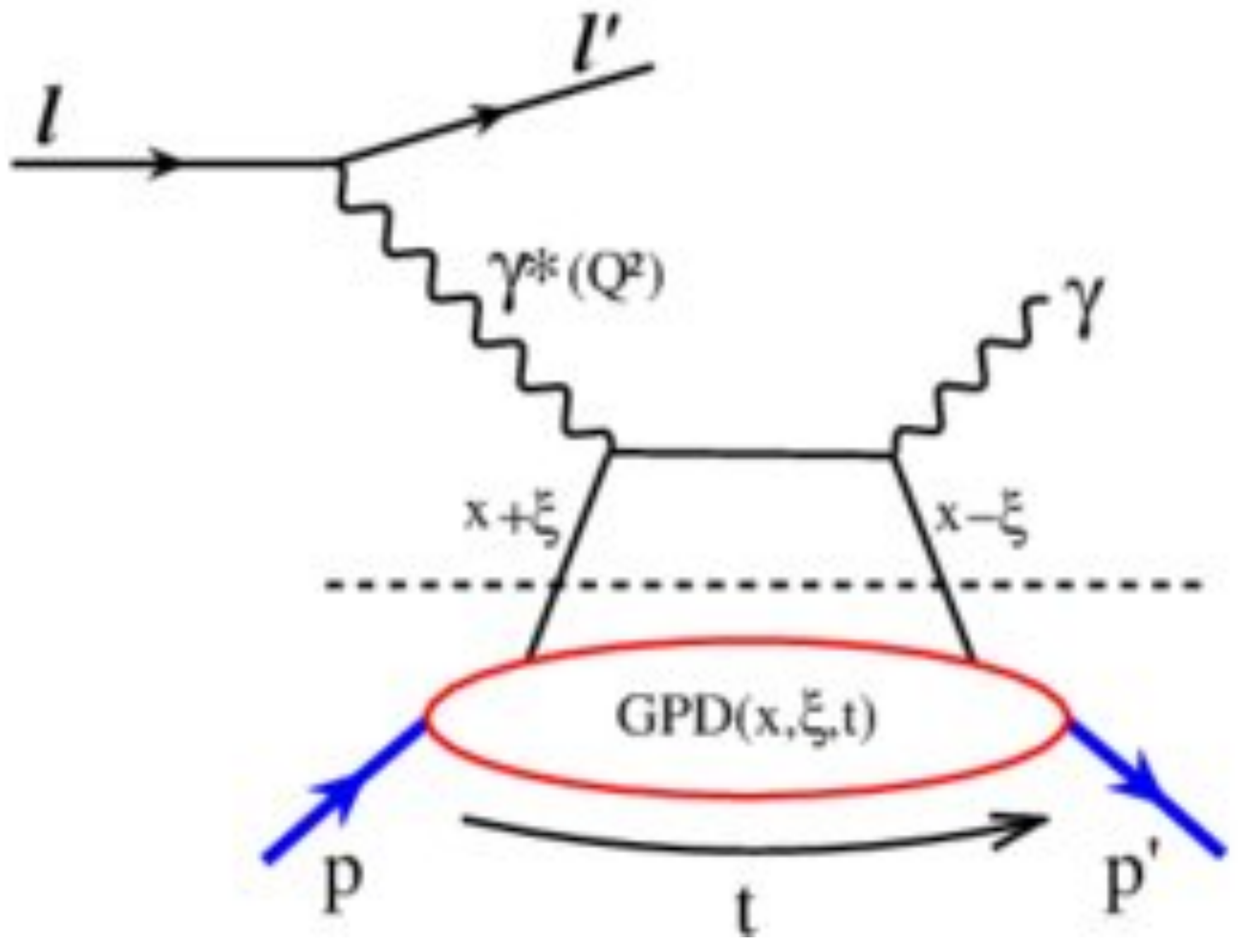
Factorization → non-perturbative structure

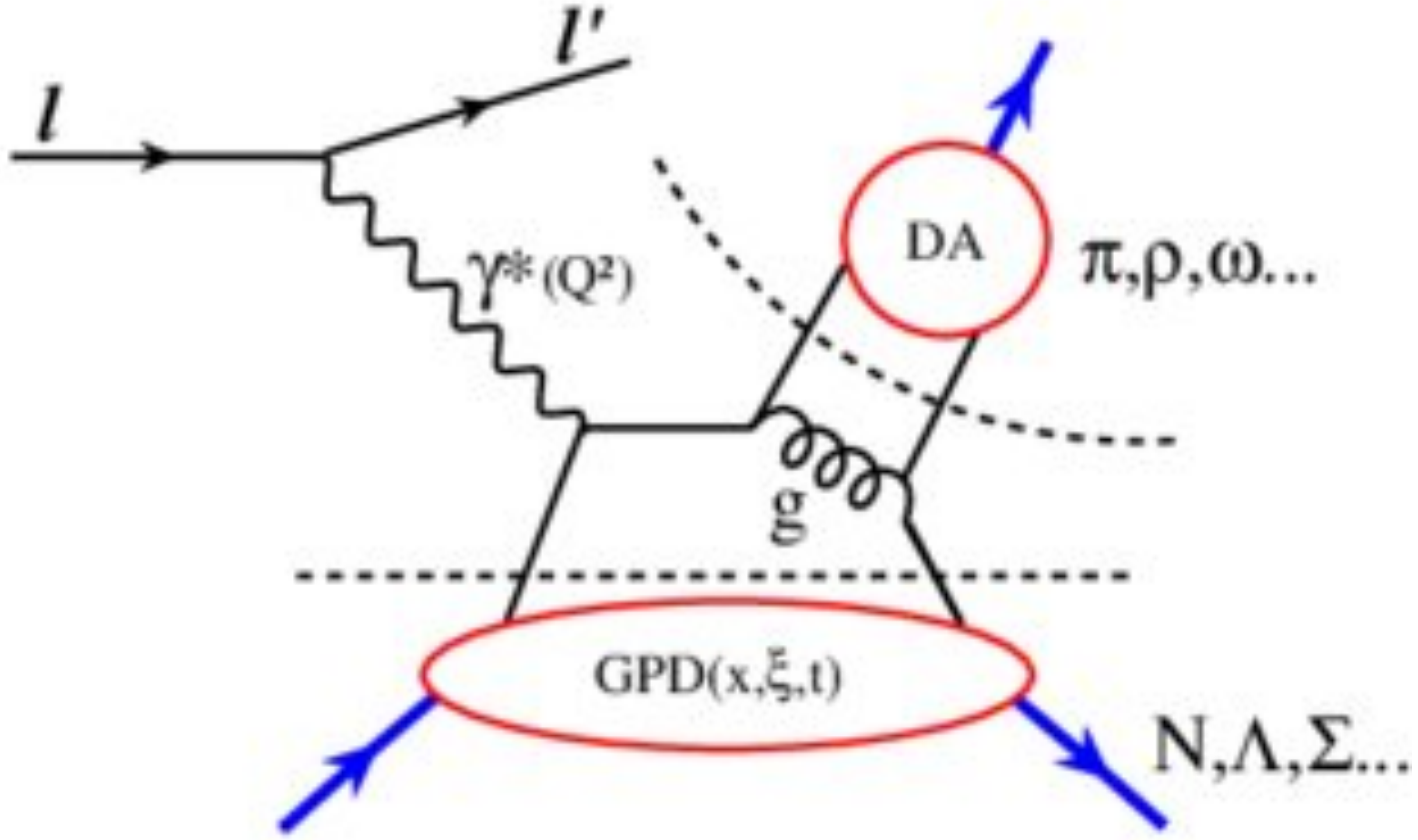


Elastic scattering: Form factor



DIS: Parton distributions

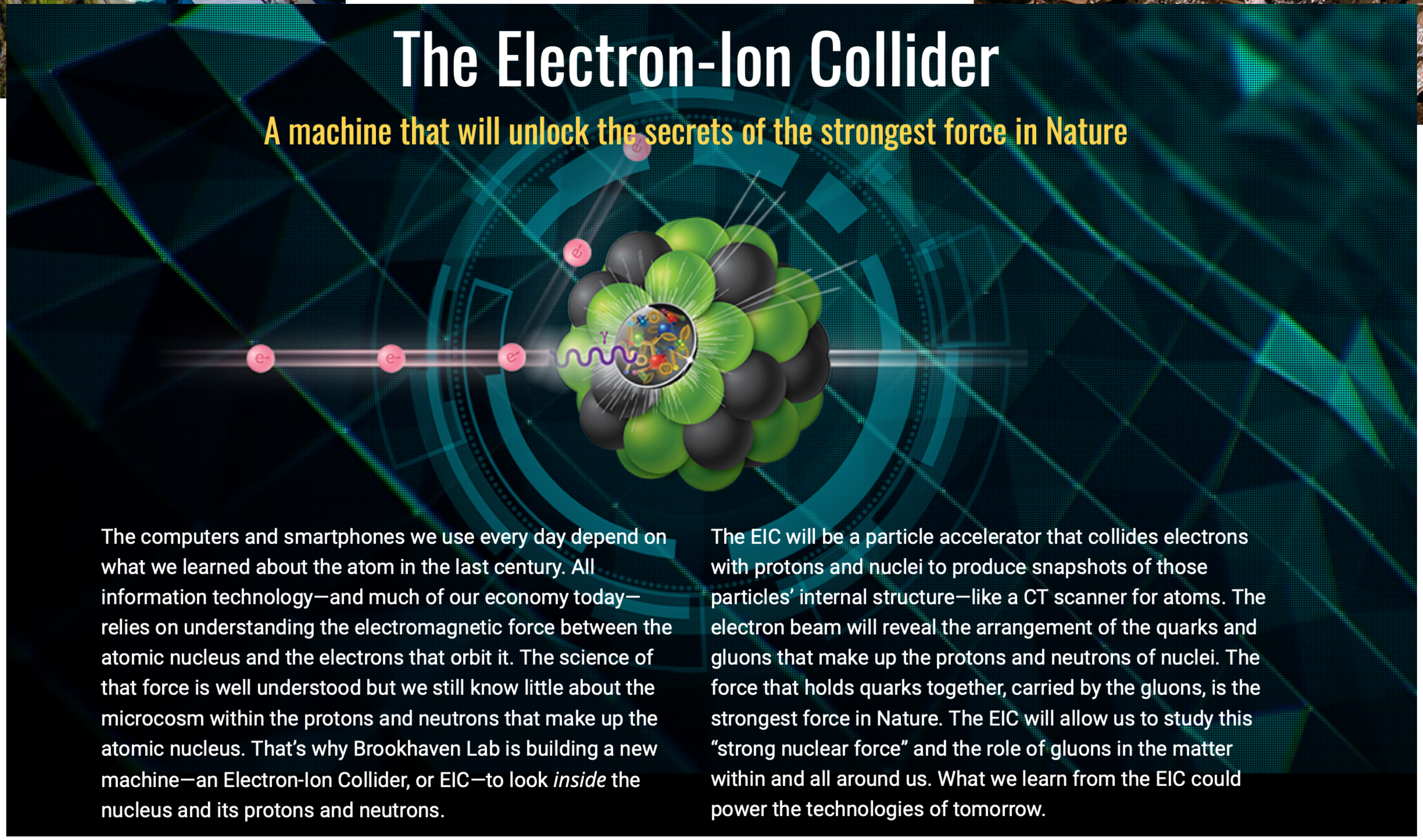




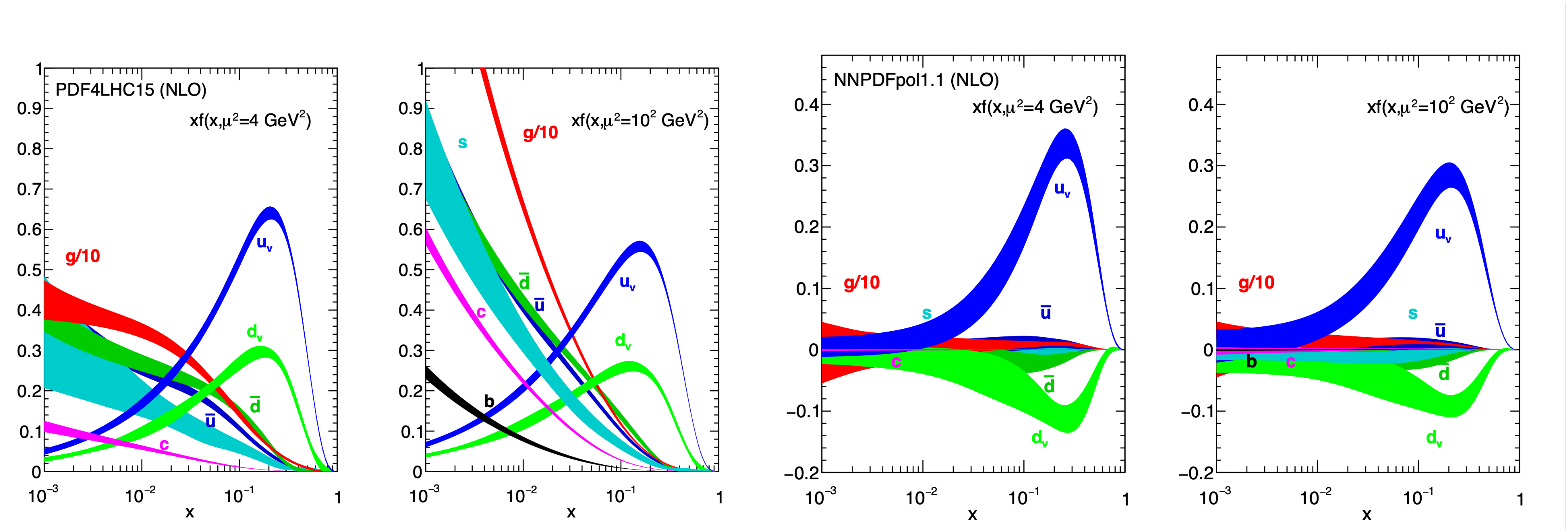
DVCS or DVMP: Generalized Parton distributions


JLab 12 GeV


Large Hadron Collider
CMS
p
p
LHCb
ATLAS
ALICE

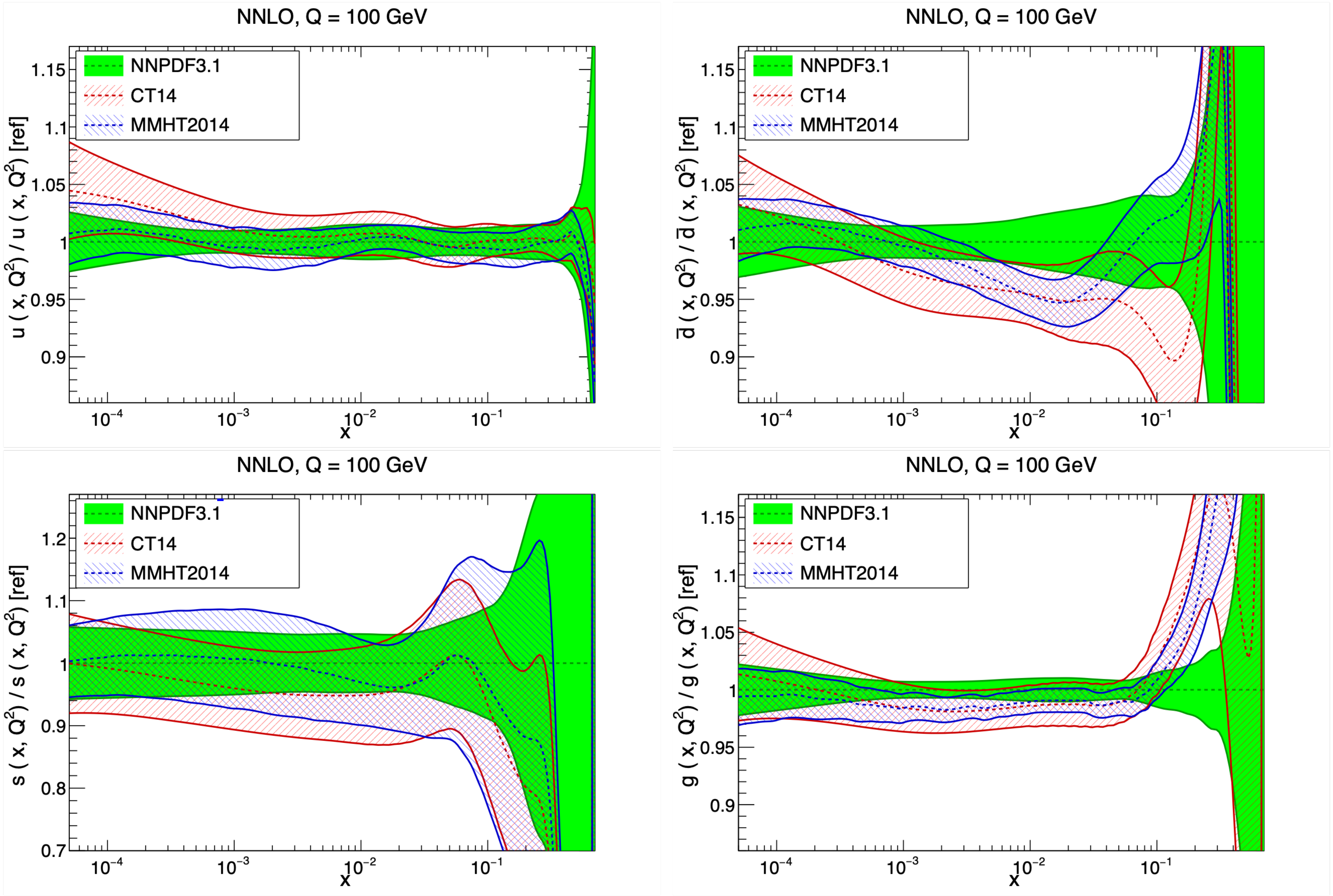

The Electron-Ion Collider
A machine that will unlock the secrets of the strongest force in Nature
The computers and smartphones we use every day depend on what we learned about the atom in the last century. All information technology—and much of our economy today—relies on understanding the electromagnetic force between the atomic nucleus and the electrons that orbit it. The science of that force is well understood but we still know little about the microcosm within the protons and neutrons that make up the atomic nucleus. That's why Brookhaven Lab is building a new machine—an Electron-Ion Collider, or EIC—to look *inside* the nucleus and its protons and neutrons.
The EIC will be a particle accelerator that collides electrons with protons and nuclei to produce snapshots of those particles' internal structure—like a CT scanner for atoms. The electron beam will reveal the arrangement of the quarks and gluons that make up the protons and neutrons of nuclei. The force that holds quarks together, carried by the gluons, is the strongest force in Nature. The EIC will allow us to study this "strong nuclear force" and the role of gluons in the matter within and all around us. What we learn from the EIC could power the technologies of tomorrow.

Determination of Parton distribution functions from Experiment



Fits to experimental cross section data

Determination of Parton distribution functions from Experiment

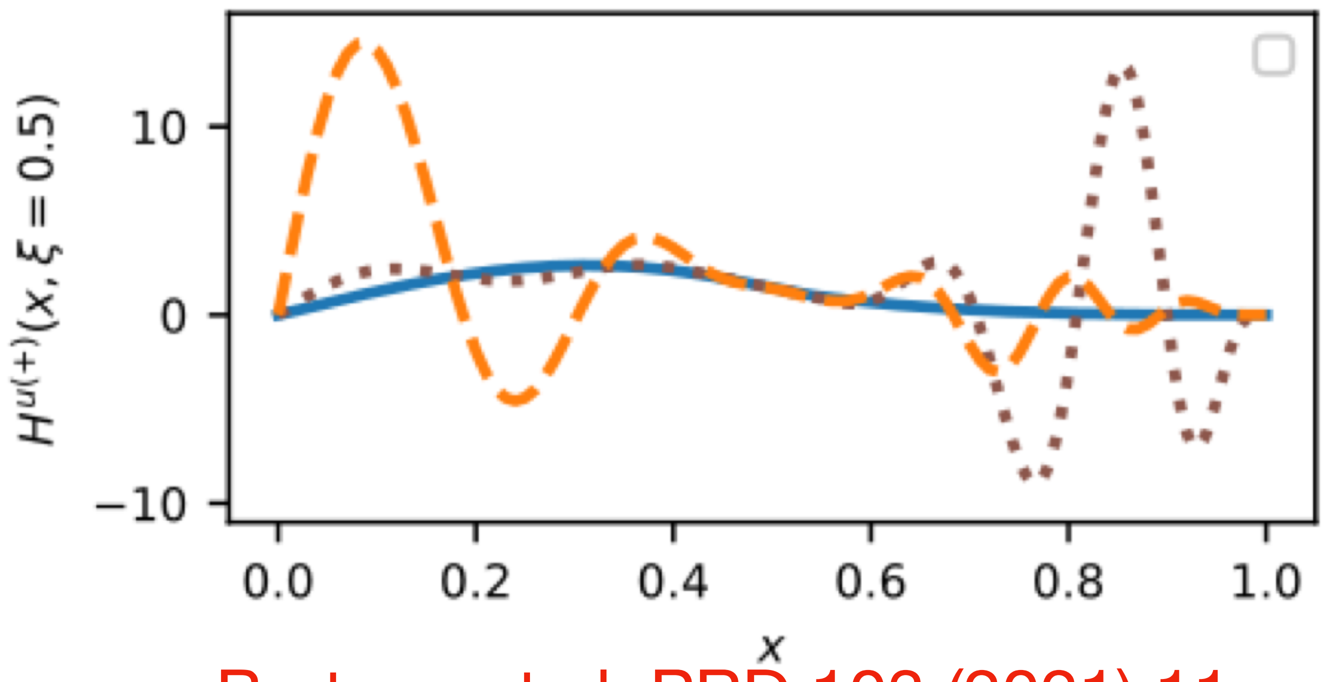


[Parton distributions and lattice QCD calculations: a community white paper]

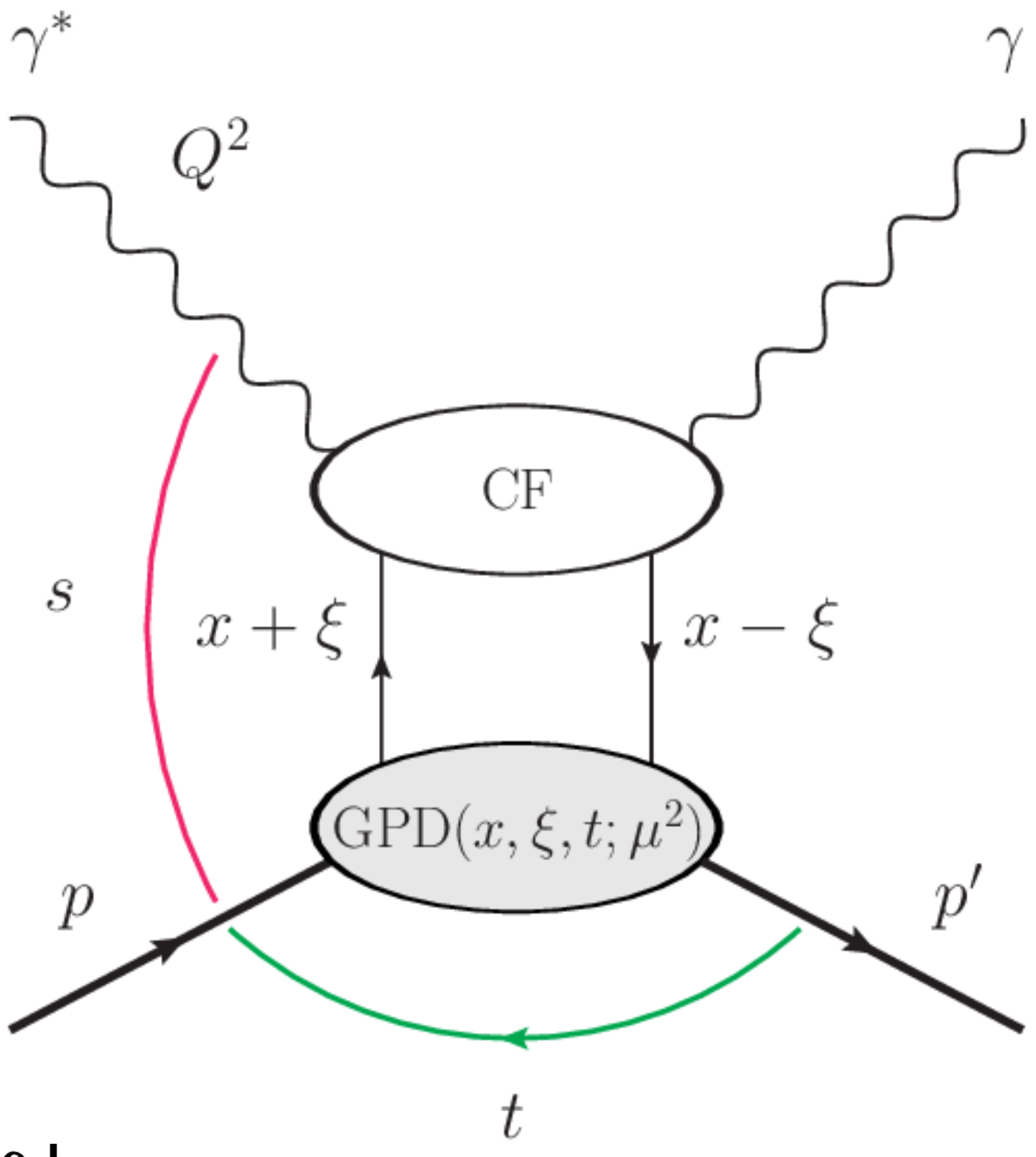
arXiv:1711.07916

DVCS factorization

Shadow GPDs



Bertone et al. PRD 103 (2021) 11



$$\mathcal{F}(\xi, t, Q^2) = \int_{-1}^{1} dx\, C(x, \xi, a_s(\mu), Q/\mu) G(x, \xi, t, \mu)$$

Ill-defined inverse problem — Lattice QCD computations are essential

Theoretical input on hadron structure is essential for the EIC physics program

Lattice QCD computations are uniquely poised to provide this information

GPDs: Definition

GPDs:

$$\bar{u}(P')\left(\gamma^+ H(x,\xi,t) + i\frac{\sigma^{+k}\Delta_k}{2m}E(x,\xi,t)\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega^-}{4\pi}e^{-i\xi P^+\omega^-}\left\langle P'\left|T\,\overline{\psi}(0,\omega^-,\mathbf{0}_\mathrm{T})W(\omega^-,0)\gamma^+\frac{\lambda^a}{2}\psi(0)\right|P\right\rangle_\mathrm{C}$$

$$W(\omega^-,0) = \mathcal{P}\exp\left[-ig_0\int_0^{\omega^-}\mathrm{d}y^- A_\alpha^+(0,y^-,\mathbf{0}_\mathrm{T})T_\alpha\right]$$

$$\langle P'|P\rangle = (2\pi)^3 2P^+\delta\left(P^+ - P'^+\right)\delta^{(2)}\left(\mathbf{P}_\mathrm{T} - \mathbf{P}'_\mathrm{T}\right)$$

$$\Delta = P' - P$$

$$t = \Delta^2$$

Moments:

$$\int_{-1}^{1} dx x^{n-1}\begin{bmatrix} H(x,\xi,t) \\ E(x,\xi,t)\end{bmatrix} = \sum_{k=0}^{[(n-1)/2]}(2\xi)^{2k}\begin{bmatrix} A_{n,2k}(t) \\ B_{n,2k}(t)\end{bmatrix} \pm \delta_{n,\mathrm{even}}(2\xi)^n C_n(t).$$

Matrix elements of twist-2 operators

$$\mathcal{O}_0^{\{\mu_1\cdots\mu_n\}} = i^{n-1}\overline{\psi}(0)\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_n\}}\frac{\lambda^a}{2}\psi(0) - \text{traces}$$

PDFs: Definition

Light-cone PDFs:

$$f^{(0)}(\xi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega^-}{4\pi} e^{-i\xi P^+ \omega^-} \left\langle P \middle| T\,\overline{\psi}(0, \omega^-, \mathbf{0}_\mathrm{T}) W(\omega^-, 0) \gamma^+ \frac{\lambda^a}{2} \psi(0) \middle| P \right\rangle_\mathrm{C} .$$

$$W(\omega^-, 0) = \mathcal{P} \exp\left[-ig_0 \int_0^{\omega^-} \mathrm{d}y^- A_\alpha^+(0, y^-, \mathbf{0}_\mathrm{T}) T_\alpha \right]$$

$$\langle P'|P\rangle = (2\pi)^3 2P^+ \delta\left(P^+ - P'^+\right) \delta^{(2)}\left(\mathbf{P}_\mathrm{T} - \mathbf{P}'_\mathrm{T}\right)$$

Moments:

$$a_0^{(n)} = \int_0^1 \mathrm{d}\xi\, \xi^{n-1} \left[f^{(0)}(\xi) + (-1)^n \overline{f}^{(0)}(\xi) \right] = \int_{-1}^1 \mathrm{d}\xi\, \xi^{n-1} f(\xi)$$

Local matrix elements:

$$\left\langle P | \mathcal{O}_0^{\{\mu_1 \dots \mu_n\}} | P \right\rangle = 2a_0^{(n)} \left(P^{\mu_1} \cdots P^{\mu_n} - \text{traces} \right)$$

$$\mathcal{O}_0^{\{\mu_1 \cdots \mu_n\}} = i^{n-1} \overline{\psi}(0) \gamma^{\{\mu_1} D^{\mu_2} \cdots D^{\mu_n\}} \frac{\lambda^a}{2} \psi(0) - \text{traces}$$

Beyond Moments

Compute the x-dependence of parton distributions

- Goal: Compute full x-dependence (generalized) parton distribution functions (GPDFs)
- Operator product: Mellin moments are local matrix elements that can be computed in Lattice QCD
  - Power divergent mixing limits us to few moments
- X. Ji suggested an approach for obtaining PDFs from Lattice QCD
- First calculations quickly became available
- Other approaches based on the hadronic tensor existed or appeared later

X. Ji, Phys.Rev.Lett. 110, (2013)

Y.-Q. Ma J.-W. Qiu (2014) 1404.6860

H.-W. Lin, J.-W. Chen, S. D. Cohen, and X. Ji, Phys.Rev. D91, 054510 (2015)

C. Alexandrou, et al, Phys. Rev. D92, 014502 (2015)

K-F Liu et al Phys. Rev. Lett. 72 (1994) , Phys. Rev. D62 (2000) 074501
Detmold and Lin 2005
M. T. Hansen et al arXiv:1704.08993.
UKQCD-QCDSF-CSSM Phys. Lett. B714 (2012), arXiv:1703.01153

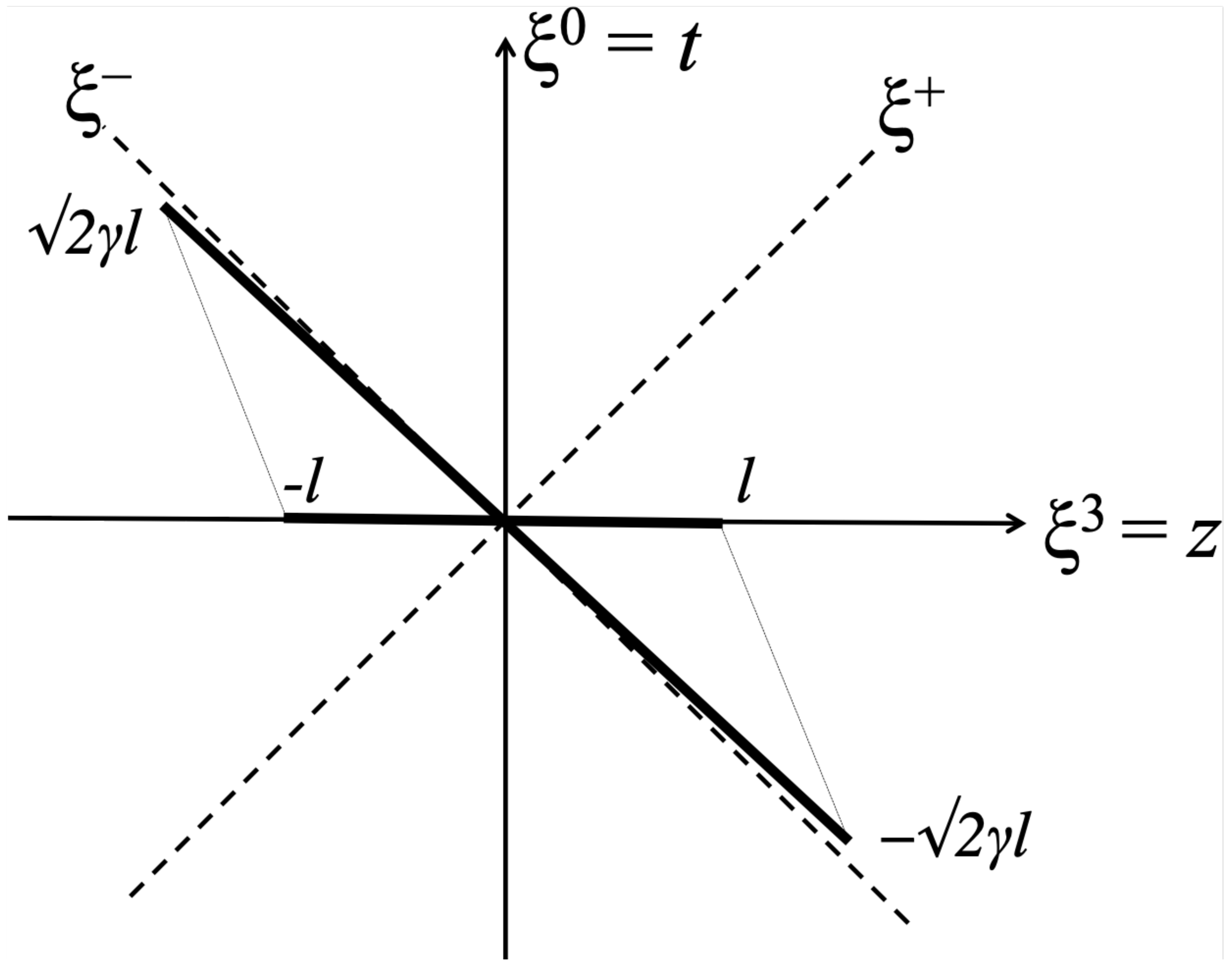
X. Ji, Phys.Rev.Lett. 110, (2013)

X. Ji (2014) Sci. China Phys. Mech. Astron. 57 arXiv:1404.6680

Quasi-PDF

X. Ji's Basic idea

- Lattice QCD computes equal time matrix elements
- Displace quarks in space-like interval
- Boost states to infinite momentum
- On the frame of the proton displacement becomes light-like
- Infinite momentum not possible on the lattice
    - Perturbative matching from finite momentum
    - LaMET



Renormalization of UV divergences is required

Good Lattice Cross sections

Current-Current Correlators

Y.-Q. Ma J.-W. Qiu (2014) arXiv:1404.6860
Y.-Q. Ma J.-W. Qiu (2017) arXiv:1709.03018

4-quark bi-local matrix elements:

$$\sigma_n(\nu, z^2) = \langle P \mid T\{\mathcal{O}_n(z)\} \mid P \rangle$$

equal time matrix element

Ex.

$$\begin{aligned} \mathcal{O}_S(z) &= (z^2)^2 Z_S^2 [\bar{\psi}_q \psi_q](z)[\bar{\psi}_q \psi](0) \\ \mathcal{O}_{V'}(z) &= z^2 Z_{V'}^2 [\bar{\psi}_q (z\cdot\gamma)\psi_{q'}](z)[\bar{\psi}_{q'} z\cdot\gamma\psi](0), \end{aligned}$$

Short distance factorization:

$$\sigma_n(\nu, z^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x, \mu^2)\, K_n^a(x\nu, z^2\mu^2) + \mathcal{O}(z^2\Lambda_{\rm QCD}^2),$$

PDFs can be obtained

Imitate scattering experiments: factorization

Renormalization of UV divergences of local operators is required

Pseudo-PDFs

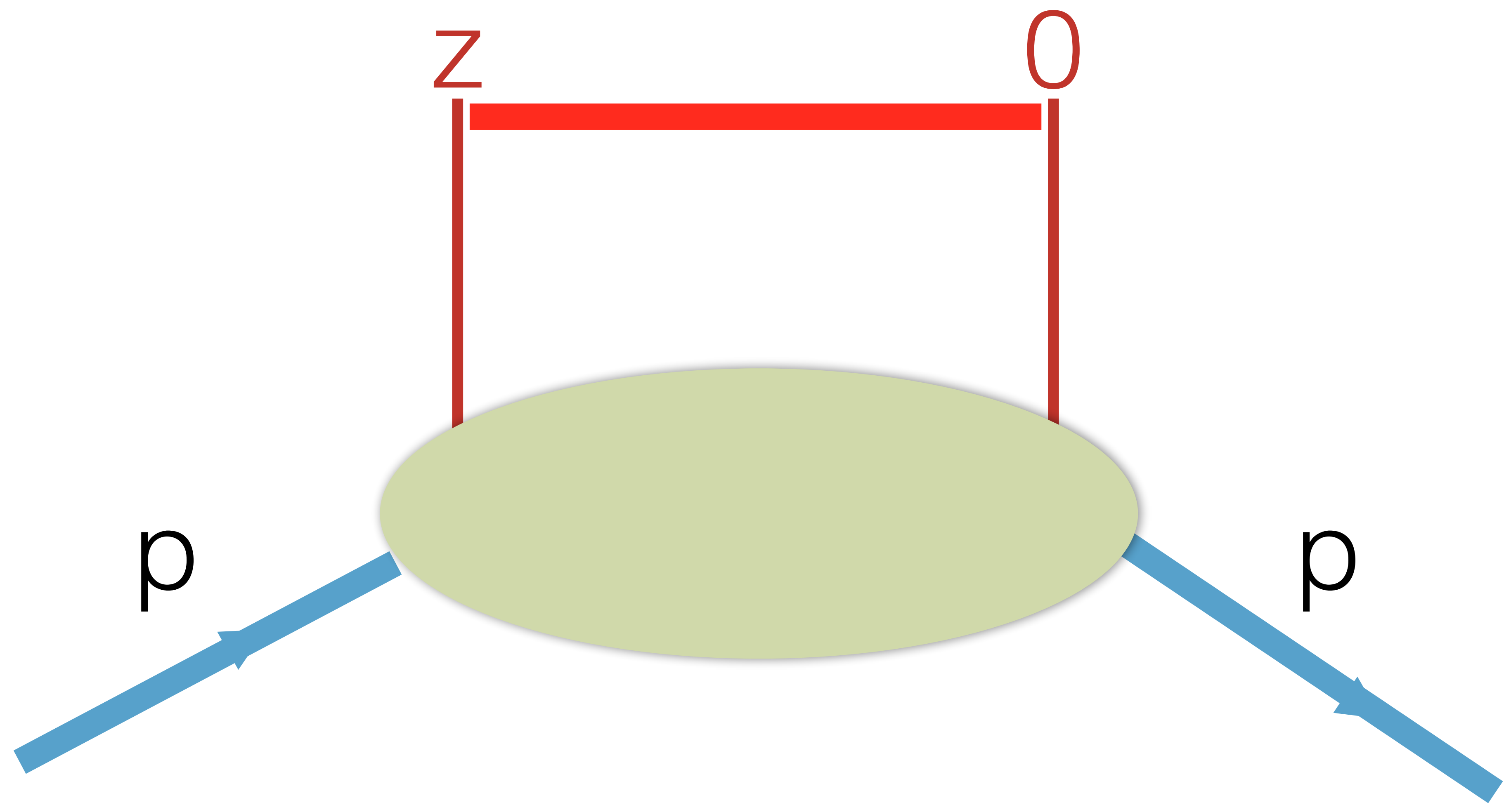
An alternative point of view

A. Radyushkin Phys.Lett. B767 (2017)

Unpolarized PDFs proton:

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^\alpha\,\hat{E}(0,z;A)\psi(z)|p\rangle$$

$$\hat{E}(0,z;A) = \mathcal{P}\exp\left[-ig\int_0^z \mathrm{d}z'_\mu\,A^\mu_\alpha(z')T_\alpha\right]$$



space-like separation of quarks

Lorentz decomposition:

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha\mathcal{M}_p(-(zp),-z^2) + z^\alpha\mathcal{M}_z(-(zp),-z^2)$$

A. Radyushkin Phys.Lett. B767 (2017)

Pseudo-PDFs

Connection to light-cone PDFs

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha \mathcal{M}_p(-(zp),-z^2) + z^\alpha \mathcal{M}_z(-(zp),-z^2)$$

Collinear PDFs: Choose

$$z = (0, z_-, 0)$$

$$p = (p_+, 0, 0)$$

$$\gamma^+$$

$$\mathcal{M}^+(z,p) = 2p^+ \mathcal{M}_p(-p_+ z_-, 0)$$

Definition of PDF:

$$\mathcal{M}_p(-p_+ z_-, 0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+ z_-}$$

Lorentz invariance allows for the computation of invariant form factors in any frame

Use equal time kinematics for LQCD

A. Radyushkin Phys.Lett. B767 (2017)

Lattice QCD calculation:

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar\psi(0)\,\gamma^\alpha\,\hat E(0,z;A)\psi(z)|p\rangle$$

Choose
$$p = (p_0, 0, 0, p_3)$$
$$z = (0, 0, 0, z_3)$$
$$\gamma^0$$

On shell equal time matrix element computable in Euclidean space

Briceno *et al* arXiv:1703.06072

Obtaining only the relevant

$$\mathcal{M}_p(\nu, z_3^2) = \frac{1}{2p_0}\mathcal{M}^0(z_3, p_3)$$

$$\mathcal{P}(x,-z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\,\mathcal{M}_p(\nu,-z^2)e^{-ix\nu}$$

the pseudo-PDF $x \in [-1, 1]$

Radyushkin Phys.Lett. B767 (2017) 314-320

Collinear singularity at $-z^2 \to 0$

Matching to $\overline{MS}$

$$\mathcal{M}_p(\nu, z^2) = \int_0^1 d\alpha\, \mathcal{C}(\alpha, z^2\mu^2, \alpha_s(\mu))\mathcal{Q}(\alpha\nu, \mu) \; + \mathcal{O}(z^2\Lambda_{qcd}^2)$$

$\mathcal{Q}(\nu, \mu)$ is called the Ioffe time PDF

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

$$\mathcal{Q}(\nu, \mu) = \int_{-1}^1 dx\, e^{-ix\nu} f(x, \mu)$$

Calculation of the matching Kernel

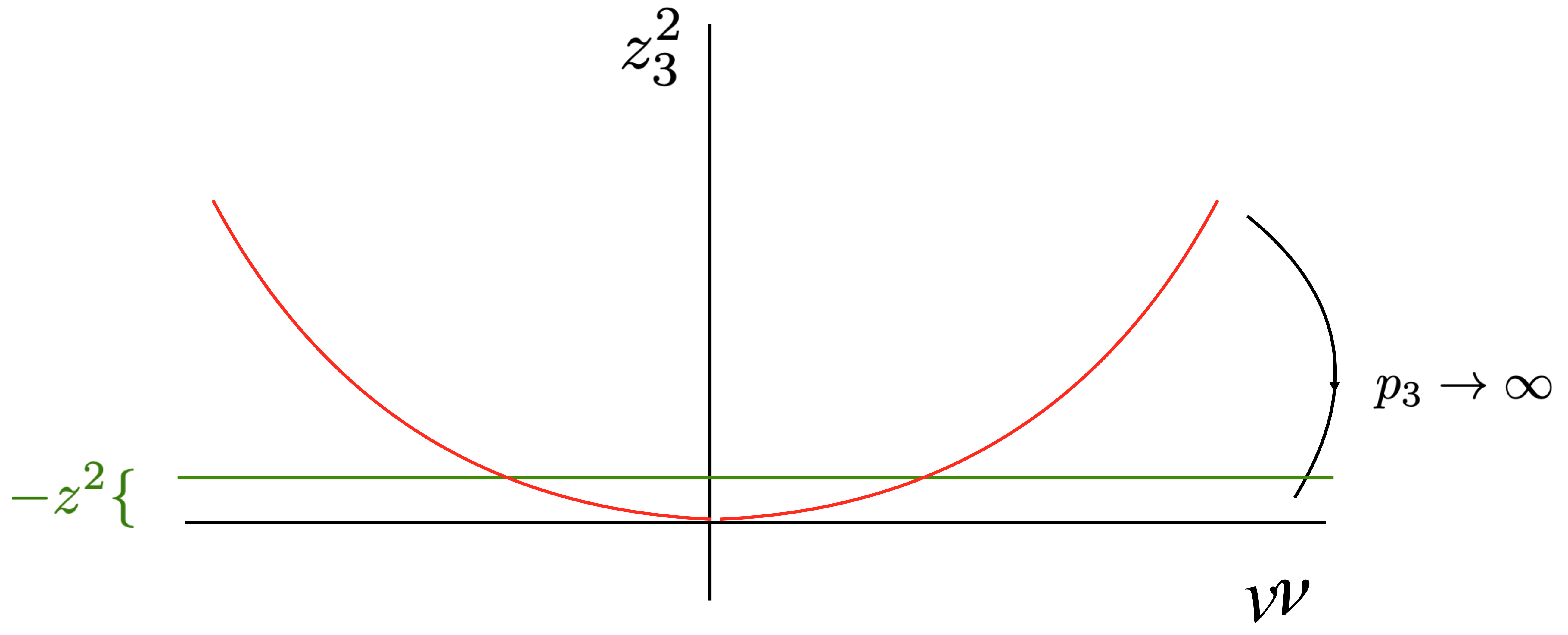
Radyushkin Phys.Rev. D98 (2018) no.1, 014019
Izubuchi et al. Phys.Rev. D98 (2018) no.5, 056004
Zhang et al. Phys.Rev. D97 (2018) no.7, 074508

Ioffe time $-z \cdot p = \nu$

$$Q(y, p_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \mathcal{M}_p(\nu, \nu^2/p_3^2) e^{-iy\nu}$$ Ji's quasi-PDF

Large values of $z_3 = \nu/p_3$ are problematic

Alternative approach to the light-cone:



$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \, \mathcal{M}_p(\nu, -z^2) e^{-ix\nu}$$

PDFs can be recovered $-z^2 \to 0$

Note that $x \in [-1, 1]$

Radyushkin Phys.Lett. B767 (2017) 314-320

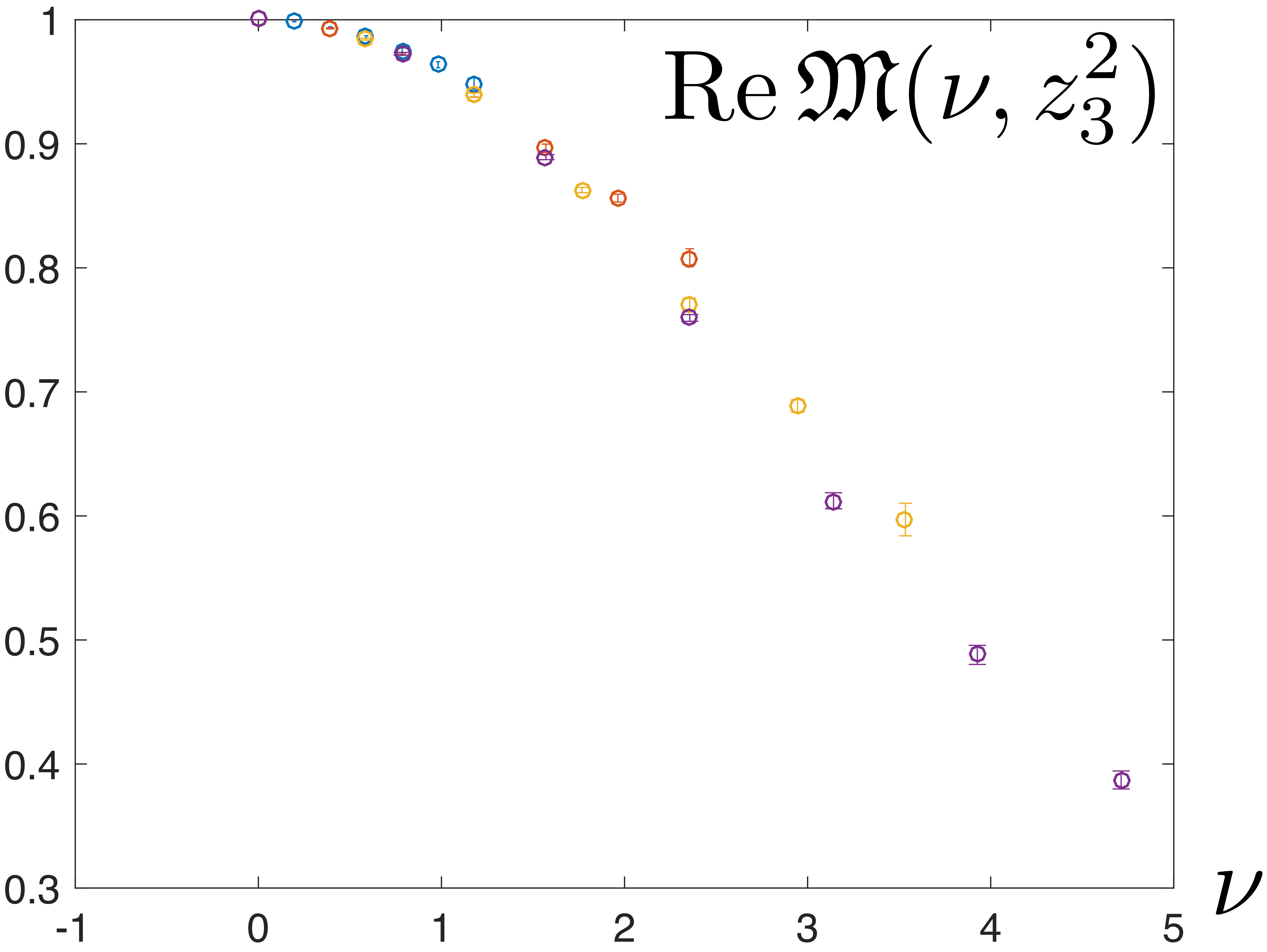
Evolution equation

Altarelli-Parisi Kernel (DGLAP)

$$\frac{d}{d\ln z_3^2}\,\mathfrak{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi}\,C_F \int_0^1 du\, B(u)\mathfrak{M}(u\nu, z_3^2),$$

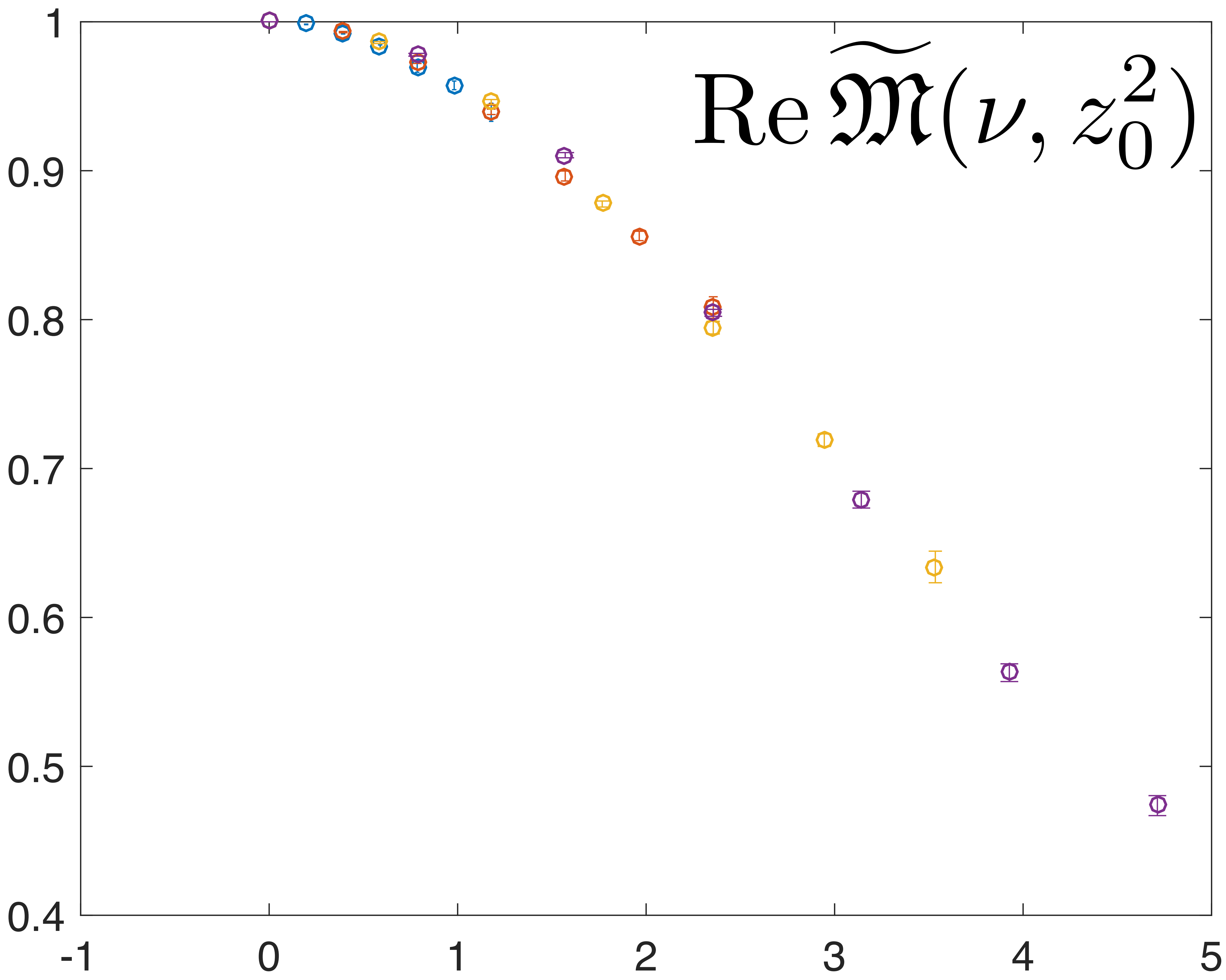
$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

Quenched QCD



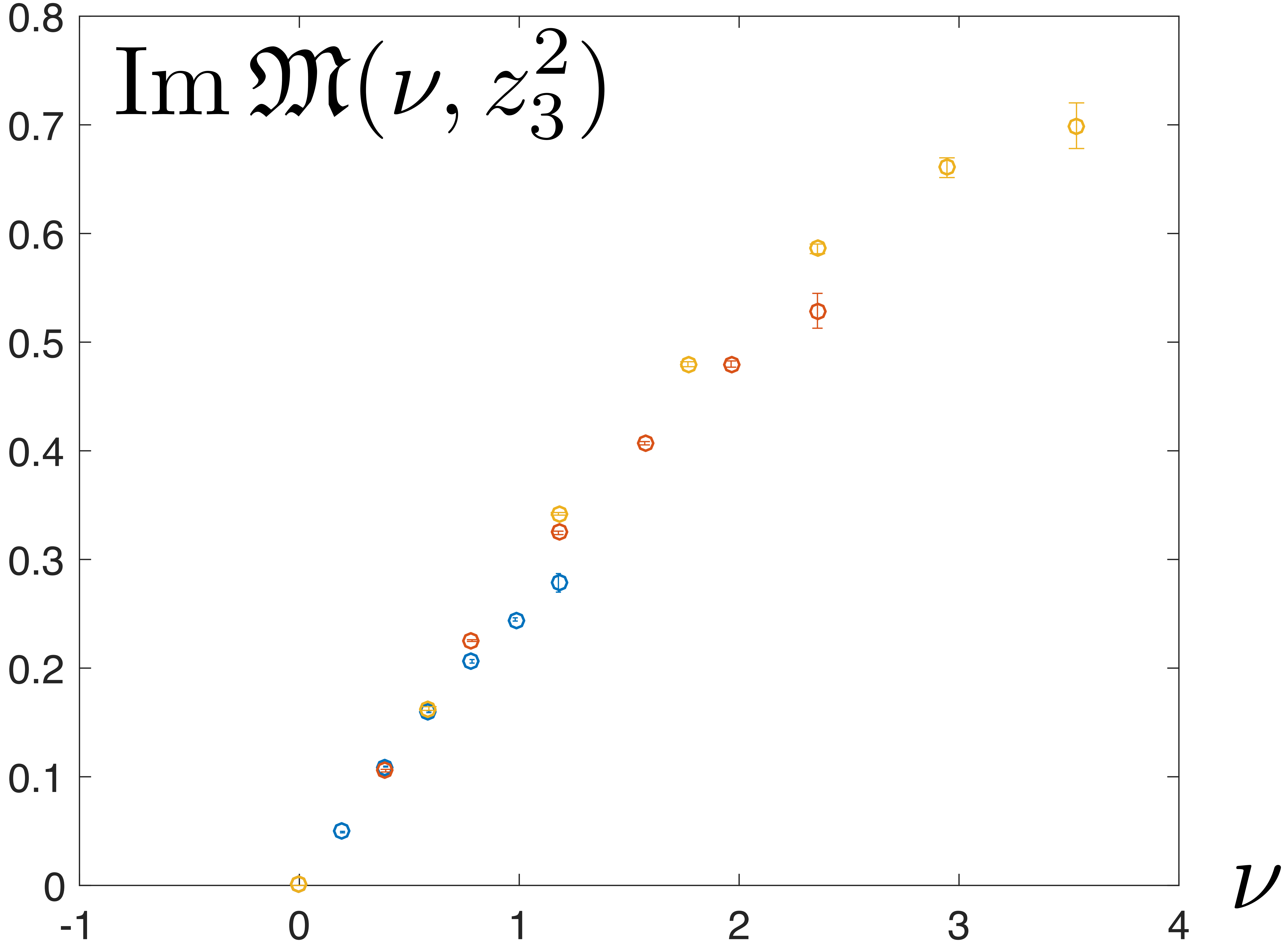
Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD



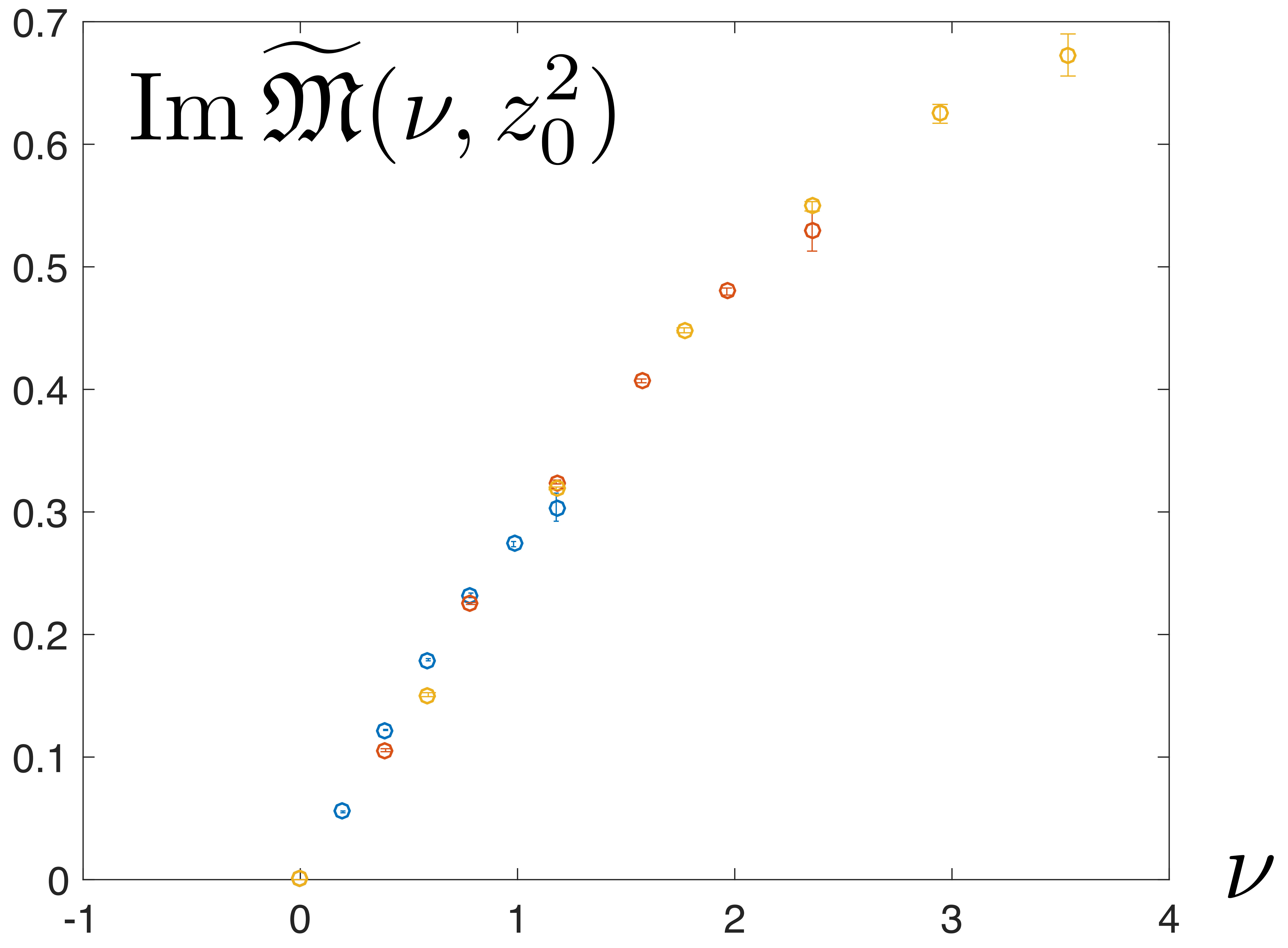
Evolved to 1GeV

Quenched QCD



Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD



Evolved to 1GeV

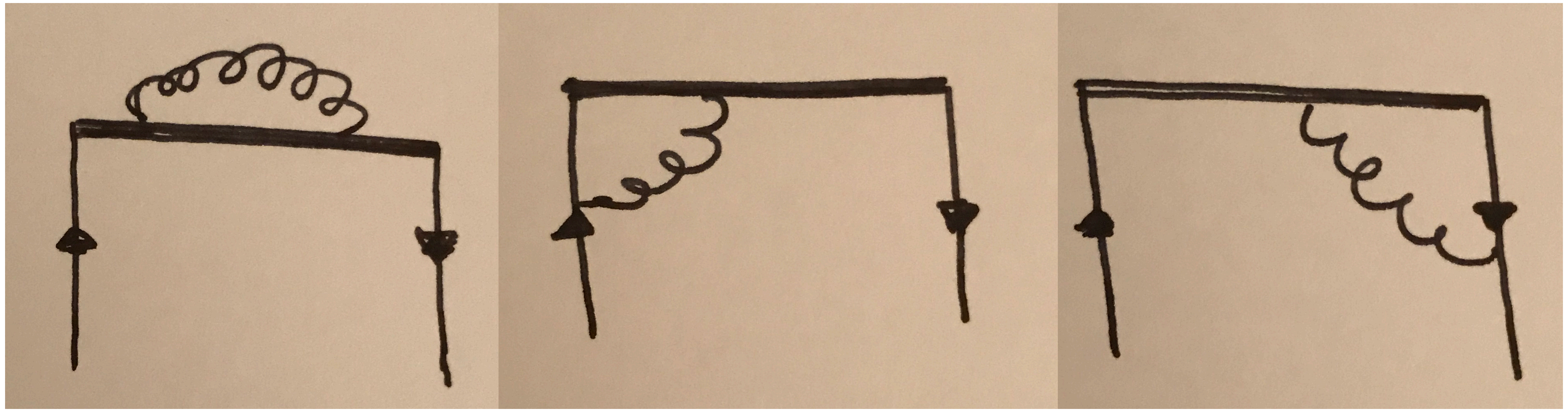
UV divergences

Introduction of a regulator

- Need to introduce a regulator of UV divergences:
    - Lattice cutoff needs to be removed and UV divergences renormalized

Rossi & Testa argue that in lattice computations 1/a divergences are still present in the moments of the computed PDFs

Rossi & Testa: PhysRev D 96, 014507 (2017), arXiv:1806.00808



Re-summation of one loop calculation of the UV divergences results in

$$\mathcal{M}^0(z,P,a) \sim e^{-m|z|/a}\left(\frac{a^2}{z^2}\right)^{2\gamma_{end}}$$

after re-summation of one loop result resulting exponentiation

- J.G.M.Gatheral,Phys.Lett.133B,90(1983)
- J.Frenkel, J.C.Taylor,Nucl.Phys.B246,231(1984),
- G.P.Korchemsky, A.V.Radyushkin,Nucl.Phys.B283,342(1987).

Multiplicatively renormalizable

Consider the ratio $$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}_p(\nu, z_3^2)}{\mathcal{M}_p(0, z_3^2)}$$

UV divergences will cancel in this ratio

The lattice regulator can now be removed

$\mathfrak{M}^{cont}(\nu, z_3^2)$ Universal independent of the lattice

$\mathcal{M}_p(0,0) = 1$ Isovector matrix element

$$\mathfrak{M}(\nu, z^2) = \int_0^1 d\alpha\, \mathfrak{C}(\alpha, z^2\mu^2, \alpha_s(\mu))\mathcal{Q}(\alpha\nu, \mu) \; + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu)(z^2)^k$$

$$\mathcal{B}_k(\nu)(z^2)^k \sim \mathcal{O}(\Lambda_{qcd}^{2k})$$

Polynomial corrections to the Ioffe time PDF may be suppressed

B. U. Musch, *et al* Phys. Rev. D 83, 094507 (2011)
M. Anselmino et al. 10.1007/JHEP04(2014)005

A. Radyushkin Phys.Lett. B767 (2017)

Rossi & Testa argument does not apply here if we use

$$\mathfrak{M}^{cont}(\nu, z_3^2)$$

Rossi & Testa: Phys Rev D 96, 014507 (2017), arXiv:1806.00808

The Moments

Karpie et al. arXiv:1807.10933

Using OPE:

$$\mathfrak{M}(\nu, z^2) = 1 + \frac{1}{2p^0}\sum_{k=1}^{\infty} i^k \frac{1}{k!} z_{\alpha_1}\cdots z_{\alpha_k} c_k(z^2\mu^2)\langle p|\mathcal{O}_{(k)}^{0\alpha_1\cdots\alpha_k}|p\rangle_\mu + \mathcal{O}(z^2)$$

$$\langle p|\mathcal{O}_{(k)}^{0\alpha_1\cdots\alpha_k}|p\rangle_\mu = 2[p^0 p^{\alpha_1}\cdots p^{\alpha_k} - \mathrm{traces}]_{\mathrm{sym}}\, a_{k+1}(\mu)\,,$$

Where $$a_n(\mu) = \int_{-1}^{1} dx\, x^{n-1}\, q(x,\mu)\,,$$

are the moments of the PDFs

The Moments

Karpie et al. arXiv:1807.10933

As a consequence:

$$(-i)^n \frac{\partial^n \mathfrak{M}(\nu, z^2)}{\partial \nu^n}\bigg|_{\nu=0} = c_n(z^2\mu^2) a_{n+1}(\mu) + \mathcal{O}(z^2)\,.$$

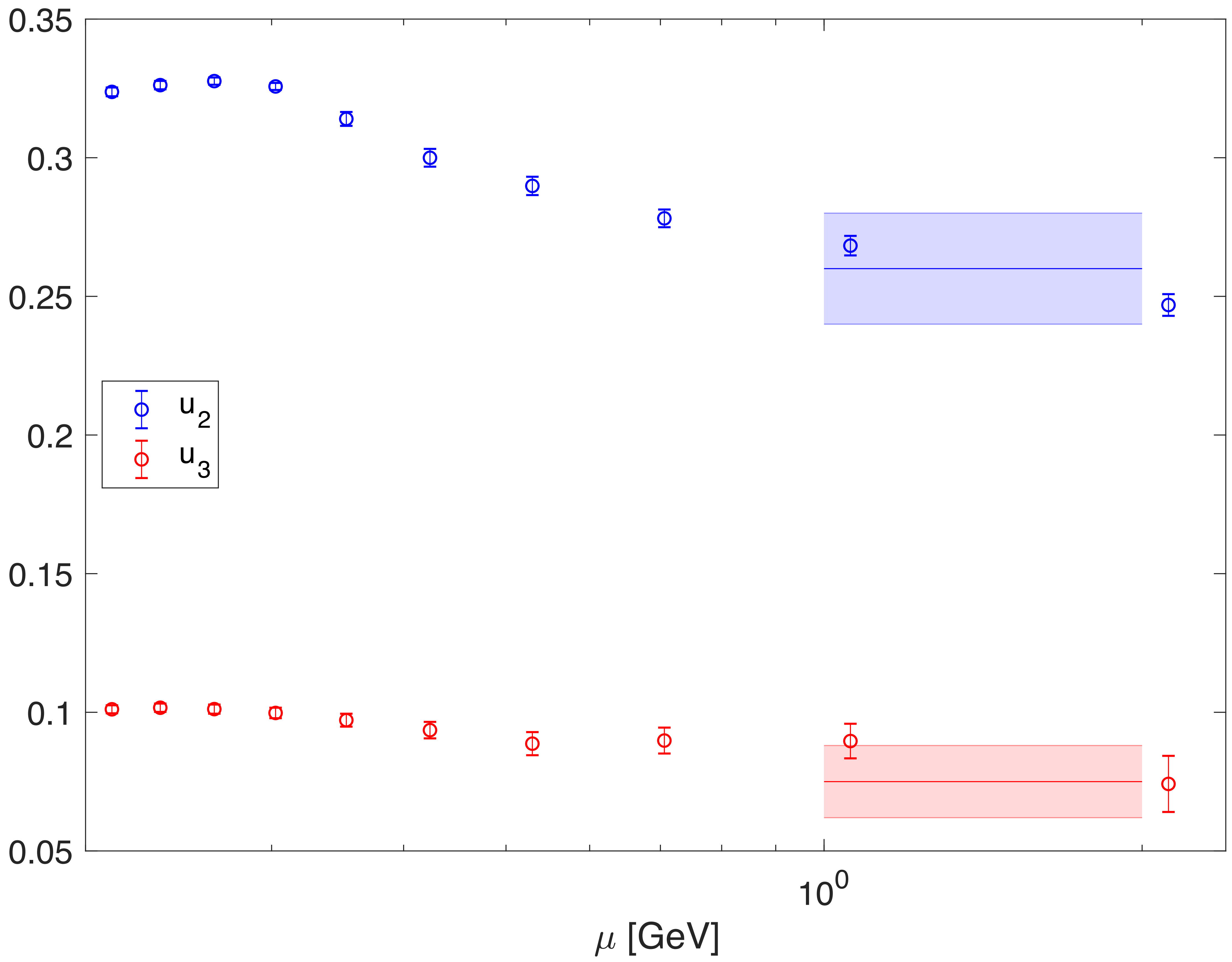
Where the Wilson coefficients are

$$c_n(z^2\mu^2) = \int_0^1 d\alpha\, \mathcal{C}(\alpha, z^2\mu^2, \alpha_s(\mu))\alpha^n\,.$$

By studying the small Ioffe time behavior moments free of any power divergences can be computed.

Low Ioffe time does not require large momentum.

Quenched QCD: Nucleon PDF moments



QCDSF: Phys.Rev. D53 (1996) 2317-2325 — shown as shaded patches at μ=2 GeV

$$\overline{MS} \quad \mu^2 = (2e^{-\gamma_E}/z_3)^2$$

Pseudo-PDF Framework

Basic features

- The continuum limit of the Ioffe-time reduced PDF can be obtained at fixed range of Ioffe time and fixed z^2.
- Small Ioffe-time allows for the extraction of moments avoiding the limitations that approaches based on local matrix elements had.
- A range of Ioffe time (which is limited primarily by the statistical precision of high momentum correlation functions) is possible. However, this range is not sufficient for a complete Fourier transform that is required for obtaining the x-dependent PDF.
- The x-dependent PDF/GPD can be inferred from the “data” obtained in these calculations.

Our inverse problem

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\rm cont}(\nu, z^2) + \sum_{n=1}\left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\rm QCD})^n R_n(\nu)\,.$$

$$\mathrm{Re}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\, \mathcal{K}_R(x\nu,\mu^2 z^2) q_-(x,\mu^2) + \mathcal{O}(z^2)$$

$$\mathrm{Im}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\, \mathcal{K}_I(x\nu,\mu^2 z^2) q_+(x,\mu^2) + \mathcal{O}(z^2)\,,$$

- Obtain the PDF from a limited set of matrix elements obtained from lattice QCD
- z^2 is a physical length scale sampled on discrete values
- z^2 needs to be sufficiently small so that higher twist effects are under control
- ν is dimensionless also sampled in discrete values
- the range of ν is dictated by the range of z and the range of momenta available and is typically limited
- Parametrization of unknown functions

Bayesian Inference

Optimize model parameters

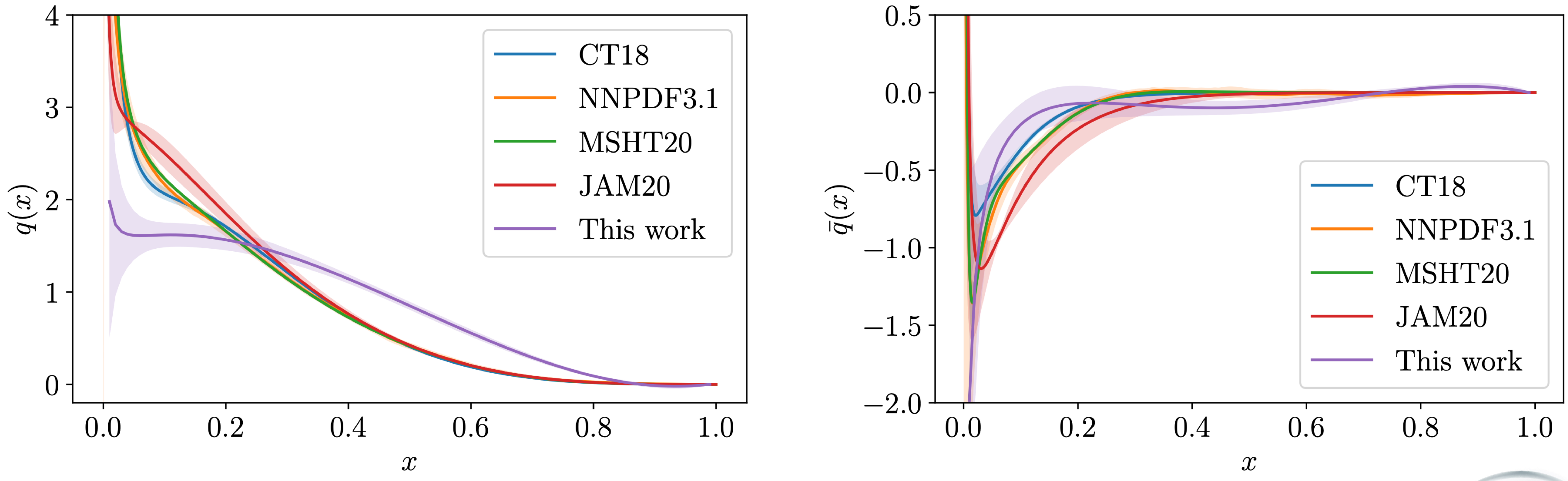
- Model the unknown functions with a set of reasonable models
- Use Bayesian Inference to optimize model parameters
- Average over models using AICc

$$P\left[\theta|\mathfrak{M}^L, I\right] = \frac{P\left[\mathfrak{M}^L|\theta\right] P\left[\theta|I\right]}{P\left[\mathfrak{M}^L|I\right]}\ .$$

Prior-knowledge is a required input for solving an ill-defined inverse problem

Isovector quark and anti-quark distributions

Comparison with phenomenology



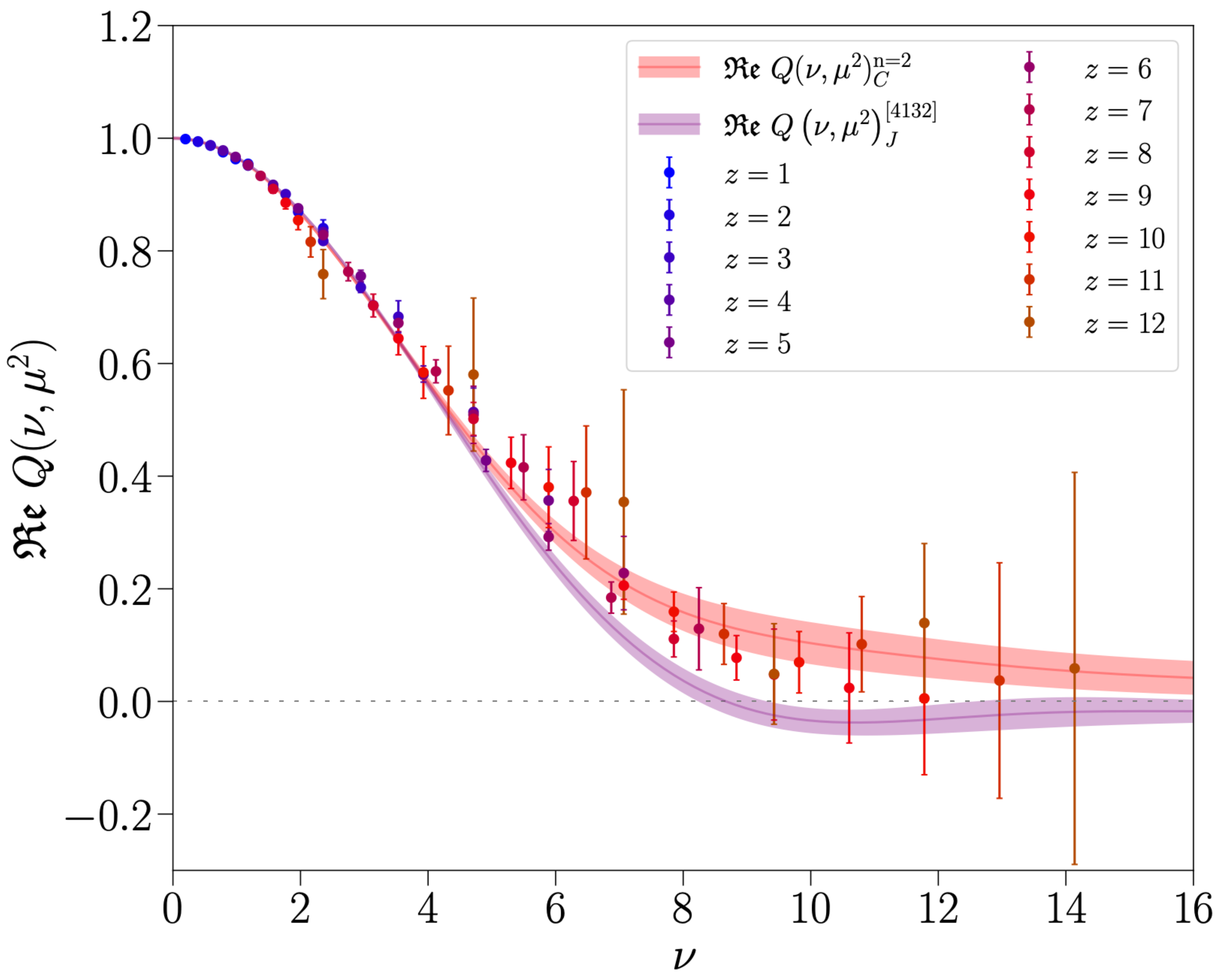
2 flavor QCD in the continuum limit with 450 MeV pions

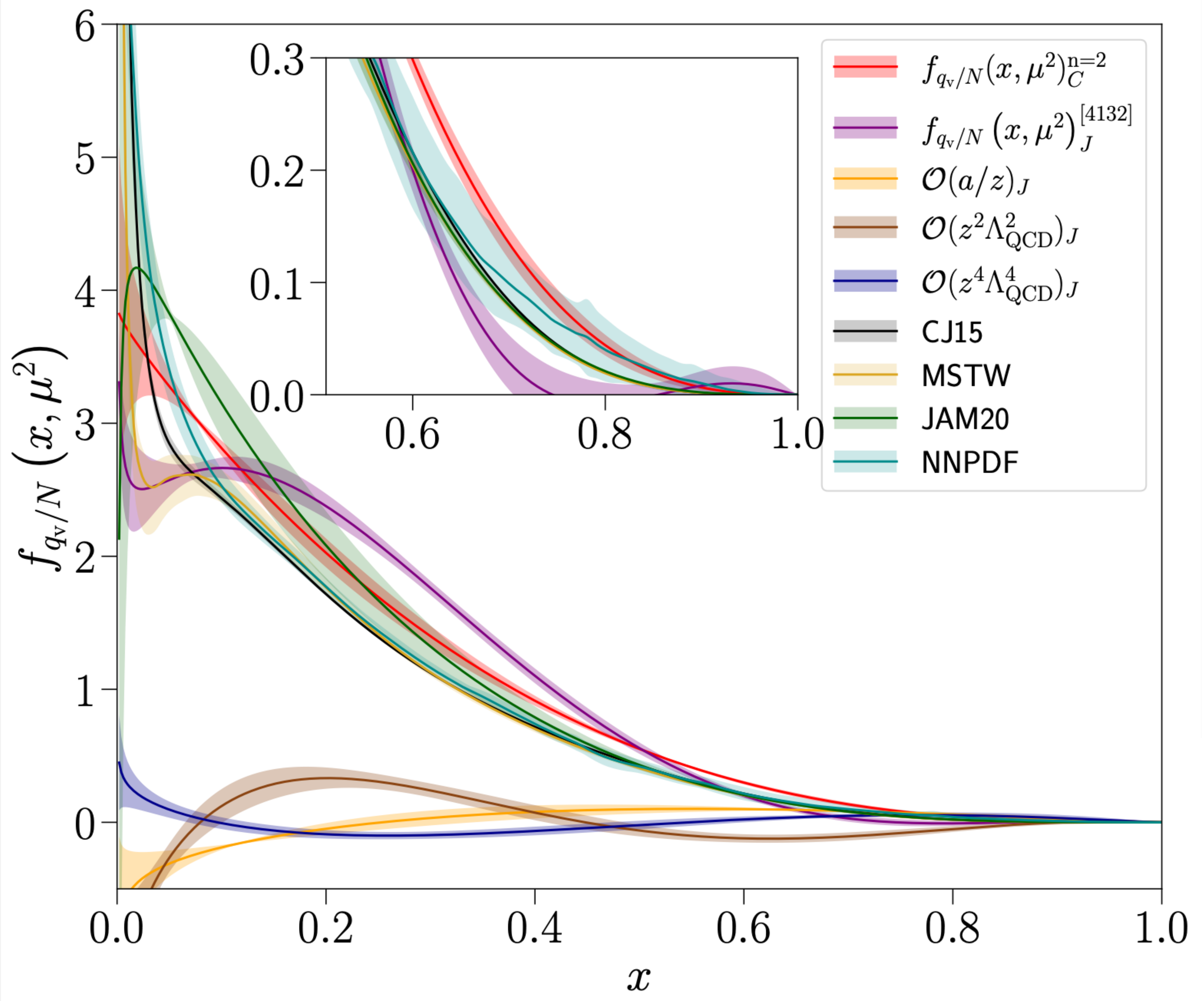
[arXiv:2105.13313](https://arxiv.org/abs/2105.13313) **[hep-lat] J. Karpie** *et al.*

HadStruc

Unpolarized Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion



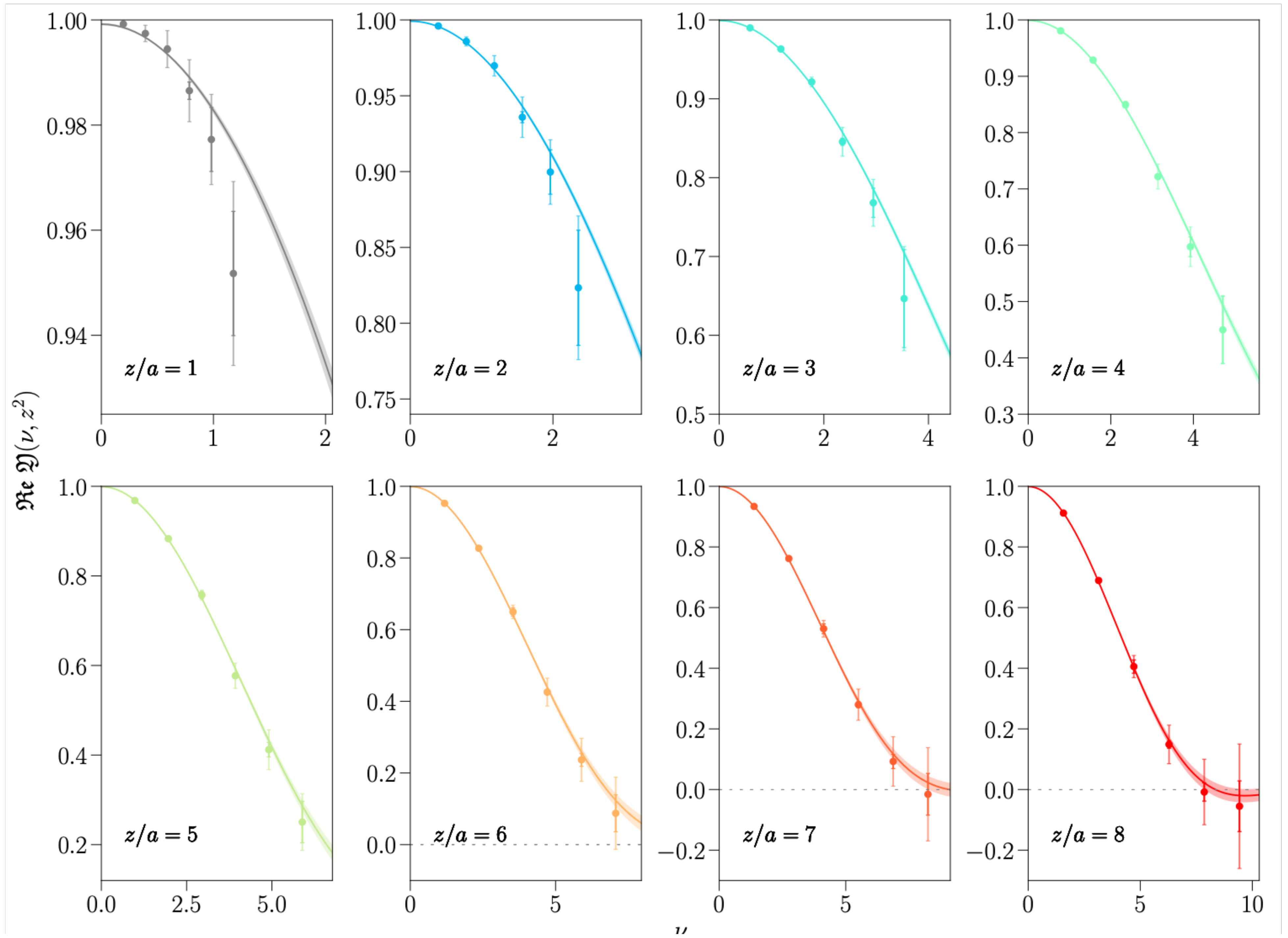


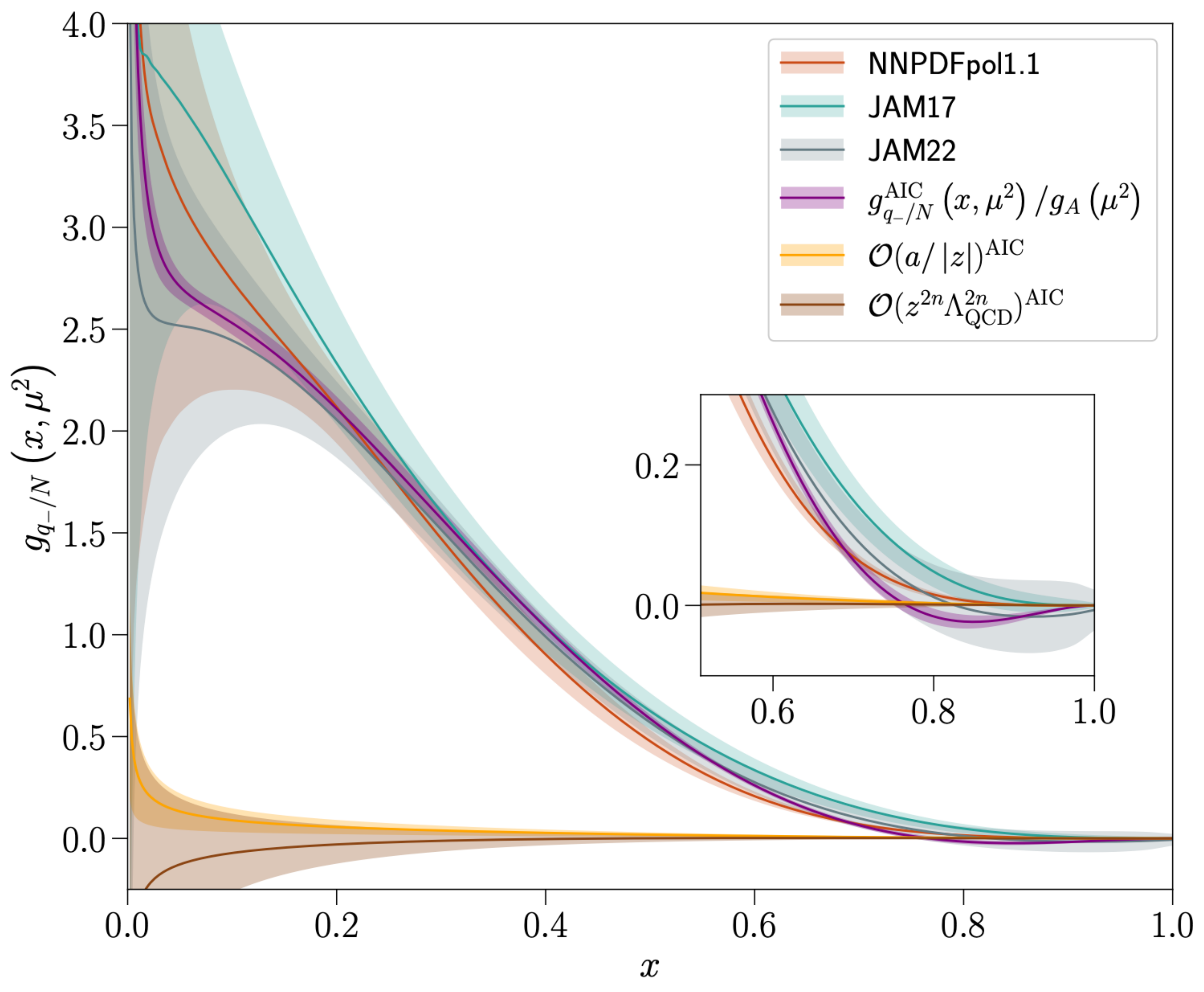
arXiv:2107.05199 [hep-lat] C. Egerer *et al.*

HadStruc

Helicity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion

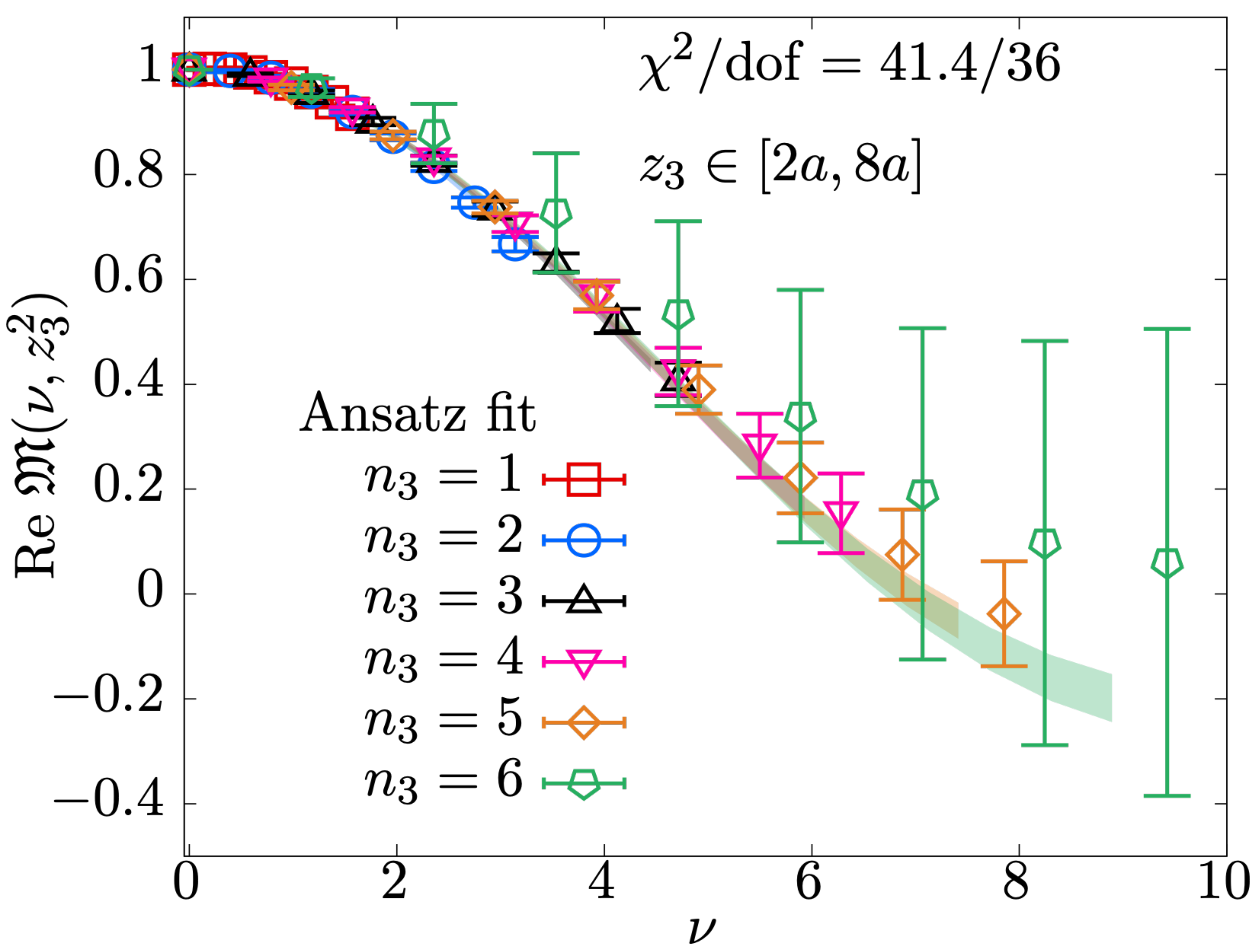


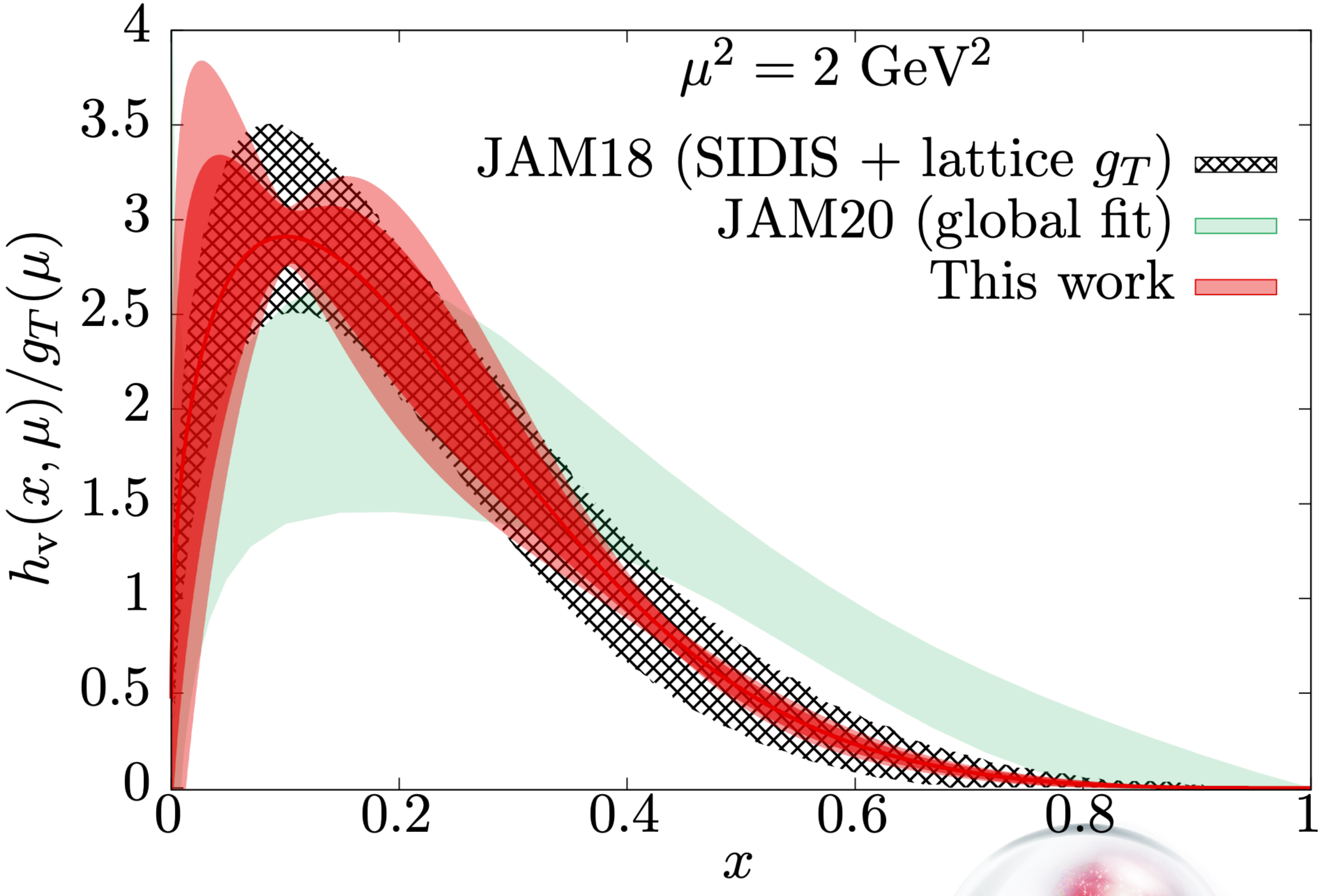


arXiv:2211.04434 **[hep-lat] C. Egerer** *et al.*

Transversity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





[arXiv:2111.01808](https://arxiv.org/abs/2111.01808) **[hep-lat] C. Egerer** *et al.*

Dutrieux et al. *Phys.Rev.D* 111 (2025) 3, 034515
Cahuana Medrano et al. arXiv:2510.21041

Gaussian Process

Non-parametric solution to the inverse problem

$$M(\nu) = B(\nu) \circ q$$

- ν takes values in a finite set
- B(ν) is a linear operator: $$B(\nu) \circ q = \int_0^1 dx \cos(\nu x) q(x)$$
- Bayesian inference: $$\mathbb{P}(f_q|M, I) = \frac{\mathbb{P}(M|f_q)\mathbb{P}(f_q|I)}{\mathbb{P}(M|I)}$$
- Prior : $$\mathbb{P}(f_q|I) = (\det(2\pi K))^{-1/2} \exp\left(-\frac{1}{2} \int \mathrm{d}x\mathrm{d}x' \; [q(x) - g(x)] K^{-1}(x, x')[q(x') - g(x')]\right)$$

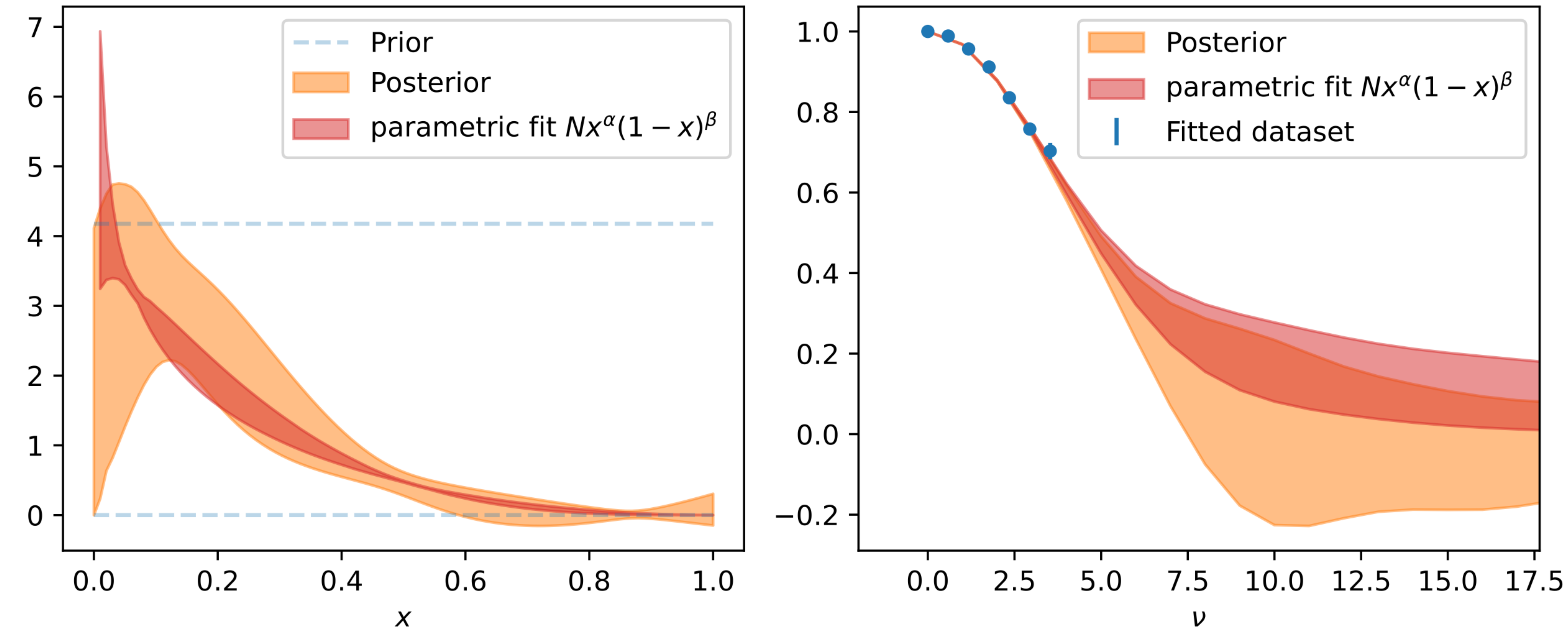
$$K(x, x') = \sigma^2 \exp\left(-\frac{(\ln(x) - \ln(x'))^2}{2l^2}\right)$$

- Gaussian nature—-> Solve for the posterior analytically
- Quantify information gain via KL divergence

Dutrieux et al. *Phys.Rev.D* 111 (2025) 3, 034515

Gaussian Process

Non-parametric solution to the inverse problem



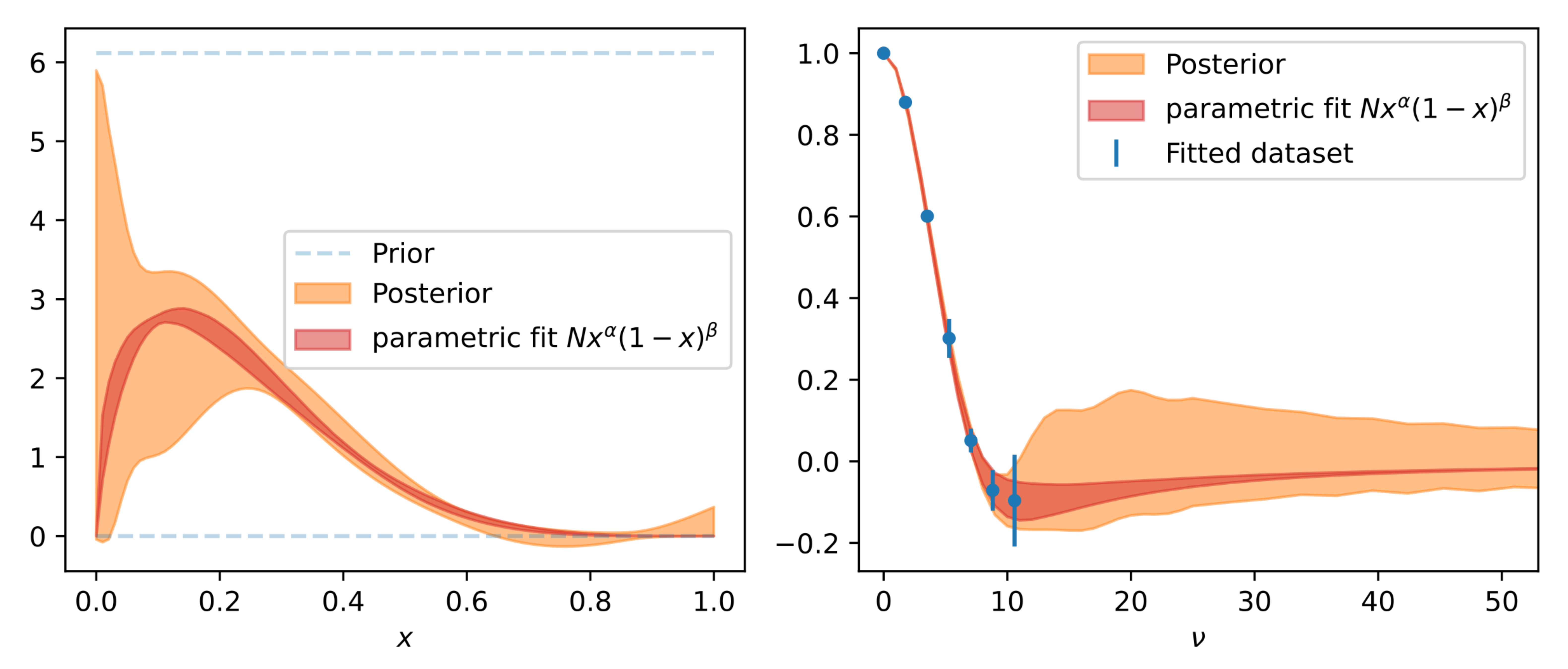
z=3a

Data from: [arXiv:2107.05199](https://arxiv.org/abs/2107.05199) **[hep-lat] C. Egerer** *et al.*

Gaussian Process

Dutrieux et al. *Phys.Rev.D* 111 (2025) 3, 034515

Non-parametric solution to the inverse problem



z=9a

Data from: [arXiv:2107.05199](https://arxiv.org/abs/2107.05199) **[hep-lat] C. Egerer** *et al.*

Generalized Parton Distributions

Dutrieux et al. *JHEP* 08 (2024) 162 — arXiv:2405.10304

- Lorentz decomposition of non-local matrix elements for GPDs

$$
\begin{aligned}
\mathcal{M}^{\mu}\left(p_f, p_i, z\right) = & \langle\langle\gamma^{\mu}\rangle\rangle \mathcal{A}_1\left(\nu, \xi, t, z^2\right) + z^{\mu}\langle\langle\mathbb{1}\rangle\rangle \mathcal{A}_2\left(\nu, \xi, t, z^2\right) + i\langle\langle\sigma^{\mu z}\rangle\rangle \mathcal{A}_3\left(\nu, \xi, t, z^2\right) \\
& + \frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle \mathcal{A}_4\left(\nu, \xi, t, z^2\right) + \frac{q^{\mu}}{2m}\langle\langle\mathbb{1}\rangle\rangle \mathcal{A}_5\left(\nu, \xi, t, z^2\right) \\
& + \frac{i}{2m}\langle\langle\sigma^{zq}\rangle\rangle \left[P^{\mu}\mathcal{A}_6\left(\nu, \xi, t, z^2\right) + q^{\mu}\mathcal{A}_7\left(\nu, \xi, t, z^2\right) + z^{\mu}\mathcal{A}_8\left(\nu, \xi, t, z^2\right)\right]
\end{aligned}
$$

$$\nu = P \cdot z\,, \quad \bar{\nu} = -(q \cdot z)/2$$

$$\bar{\nu} \equiv \nu\xi$$

- Connection to GPDs

$$
\begin{aligned}
H^q(\nu, \xi, t) &= \lim_{z^2 \to 0}\left[\mathcal{A}_1 - \xi A_5\right], \\
E^q(\nu, \xi, t) &= \lim_{z^2 \to 0}\left[\mathcal{A}_4 + \nu\mathcal{A}_6 - 2\xi\nu\mathcal{A}_7 + \xi A_5\right],
\end{aligned}
$$

Matching to MSbar

$$F\left(\nu, \xi, t, z^2\right) = \overline{F}\left(\nu, \xi, t, \mu^2\right) - \frac{\alpha_s C_F}{2\pi}\left\{\ln\left(-\frac{e^{2\gamma_E+1}}{4} z^2\mu^2\right) B + L\right\} \otimes \overline{F} + \mathcal{O}\left(z^2\Lambda^2_{\mathrm{QCD}}\right) + \mathcal{O}\left(z^2 t\right)$$

Moments of GPDs

Dutrieux et al. *JHEP* 08 (2024) 162

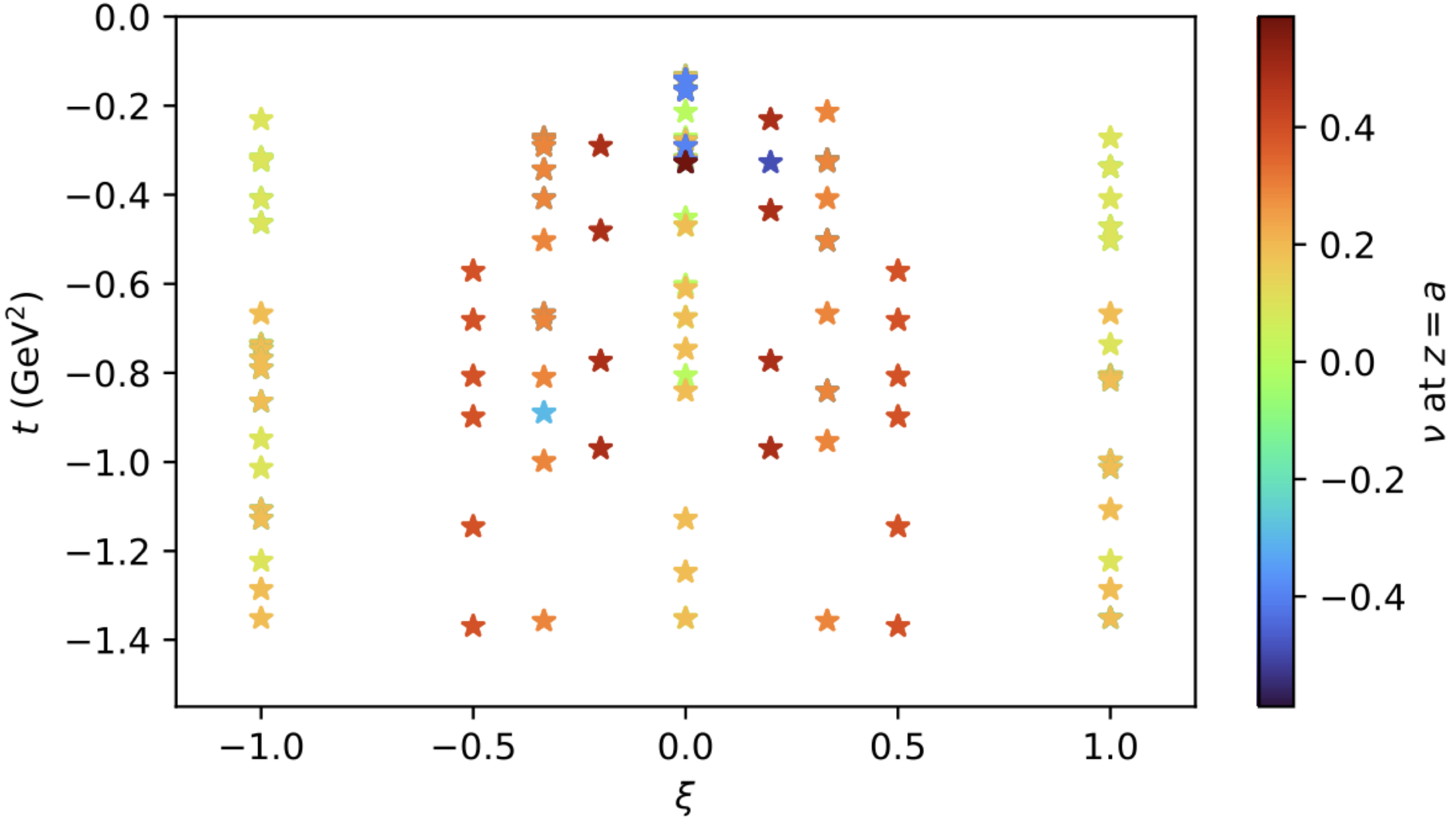
$$\frac{d^n}{d\nu^n} F(\nu, \xi, t, z^2)\Big|_{\nu=0} = (-i)^n \int_{-1}^{1} dx\, x^n F(x, \xi, t, z^2)\,.$$

$$F_n(\xi, t, z^2) \equiv \int_{-1}^{1} dx\, x^{n-1} F(x, \xi, t, z^2)\,,$$

- Focus on the small Ioffe-time limit
  - No need for large momentum
  - Statistically better resolved matrix elements and less prone to excited state contamination
- Moments of H and E GPDs up to order x3 including skewness dependence

Generalized Parton Distributions

Kinematic Coverage — Essential input for EIC DVCS program



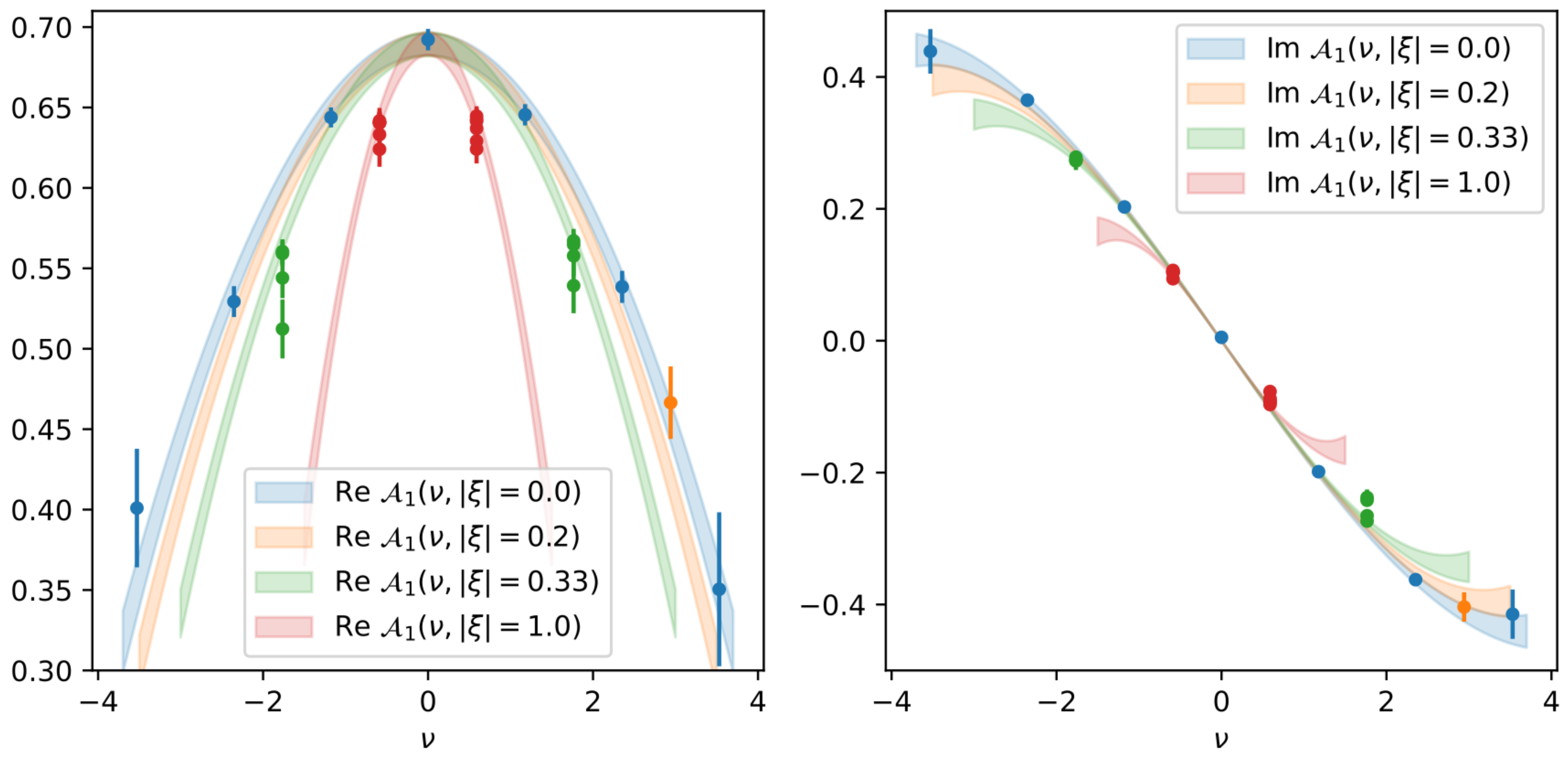
ID	a (fm)	m_π (MeV)	β	$m_\pi L$	$L^3 \times N_T$	N_{cfg}	N_{srcs}	rk ($\mathcal{D}$)
a094m358	0.094(1)	358(3)	6.3	5.4	$32^3 \times 64$	348	4	64

HadStruc

Extraction of moments

$$\mathrm{Re}\,\mathcal{A}_1(\nu,\xi,t,z^2) = F_1(t) - \frac{\nu^2}{2}\left(A_{3,0}(t,z^2) + \xi^2 A_{3,2}(t,z^2)\right) + \mathcal{O}(\nu^4) + \mathcal{O}(\Lambda^2_{\mathrm{QCD}} z^2, tz^2)\,,$$

$$\mathrm{Im}\,\mathcal{A}_1(\nu,\xi,t,z^2) = -\nu A_{2,0}(t,z^2) + \frac{\nu^3}{6}\left(A_{4,0}(t,z^2) + \xi^2 A_{4,2}(t,z^2)\right) + \mathcal{O}(\nu^5) + \mathcal{O}(\Lambda^2_{\mathrm{QCD}} z^2, tz^2)\,,$$



The z-expansion

Parametrizing the t-dependence of form factors and GPD moments

Conformal mapping

$$t_+ = 4\,m_\pi^2 \qquad \text{(two-pion production threshold)}$$

$$z(t,t_0) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_0}}$$

$$|z(t,t_0)| < 1 \quad \text{for} \quad t < t_+$$

- Maps the cut t-plane onto the unit disk
- t_0 is a free parameter chosen to minimize $|z|_{max}$ in the fit range
- Rapidly convergent expansion: $|z| \approx 0.1$–0.3 for typical lattice kinematics

Form factor expansion

$$F(t) = \sum_{k=0}^{k_{\max}} a_k\, z(t)^k$$

For generalized form factors (GPD moments):

$$A_{n,0}(t) = \sum_{k=0}^{k_{\max}} a_k^{(n)}\, z(t)^k$$

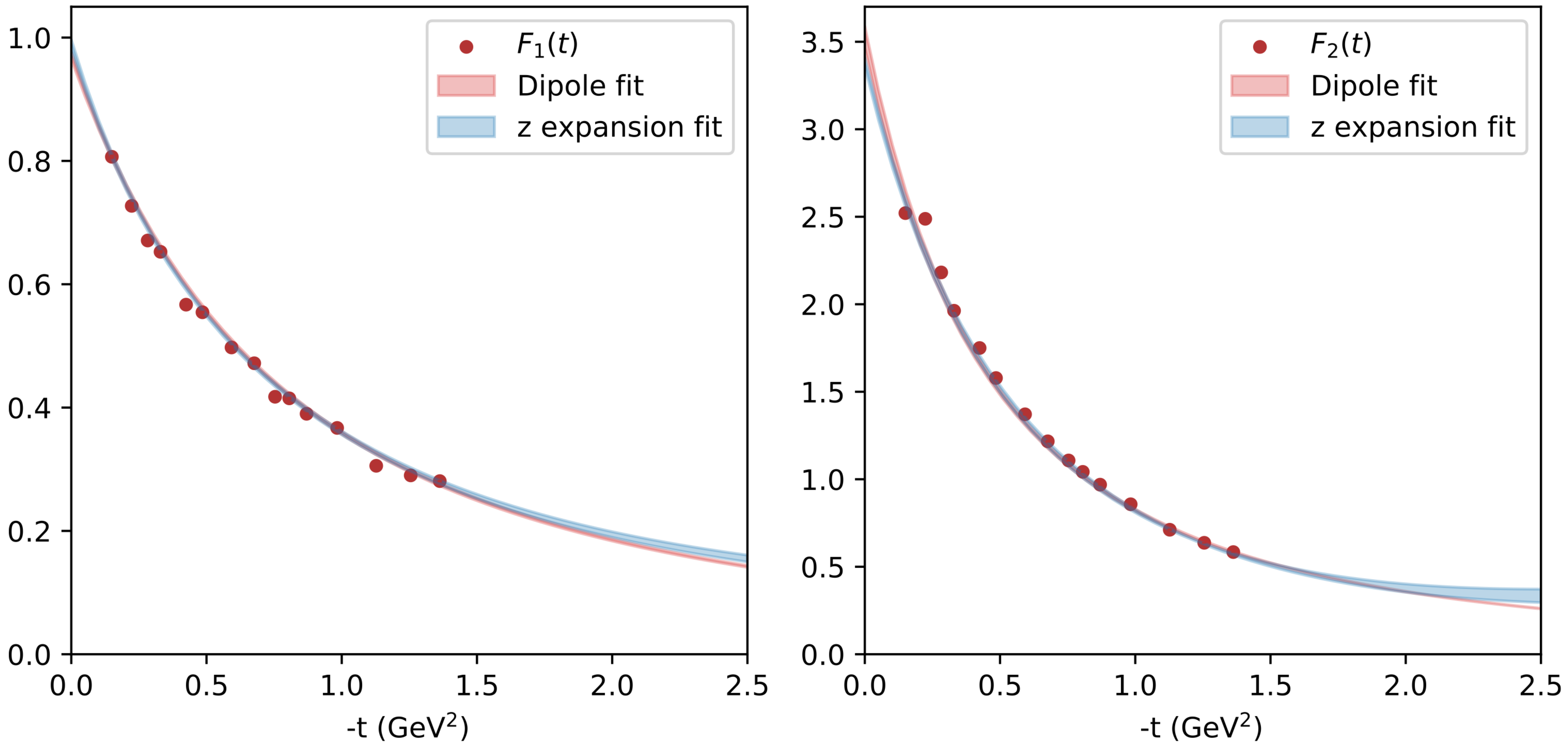
- Model-independent parametrization
- Coefficients a_k are bounded — unitarity constraints
- Truncation at k_{max} gives controlled systematic error
- Applied to extract $A_{n,0}(t)$, $B_{n,0}(t)$, $D_{n,0}(t)$ from lattice data

Optimal choice of t_0:

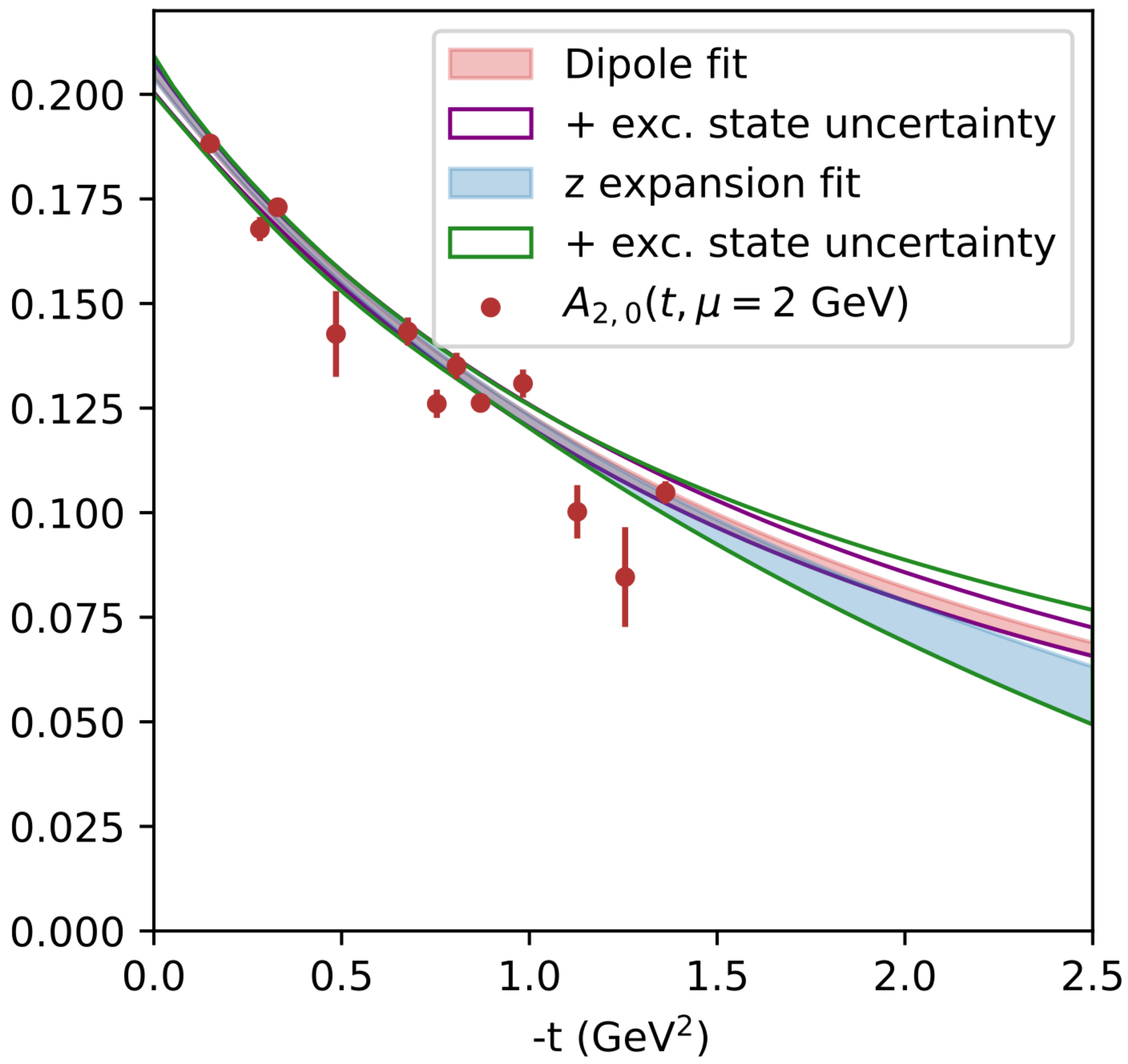
$$t_0 = t_+\left(1 - \sqrt{1 - t_{\min}/t_+}\right) \quad \text{minimizes } |z|_{\max}$$

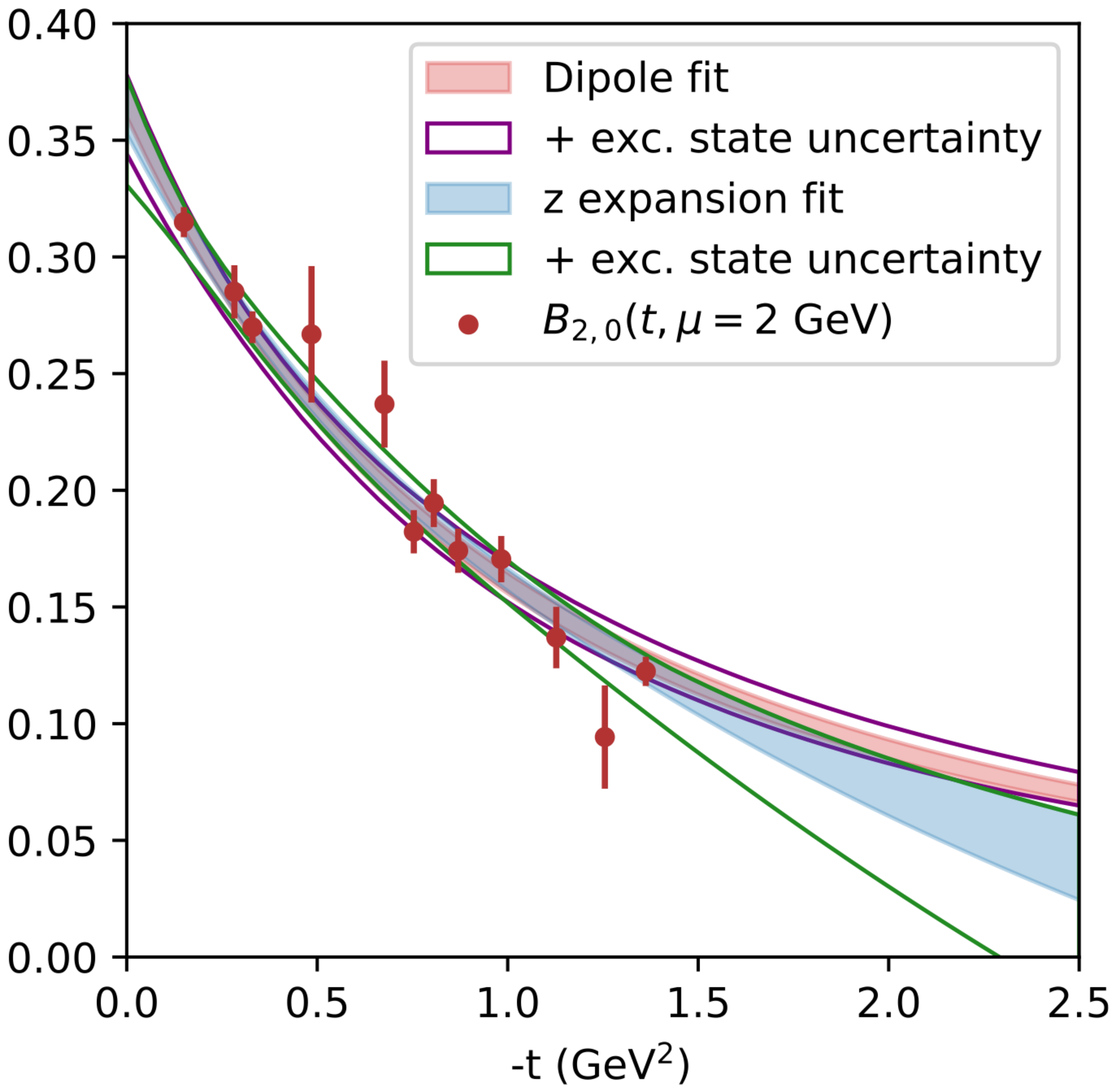
R.J. Hill, G. Paz, Phys.Rev. D82 (2010) 113005 · Bhattacharya et al. Phys.Rev. D92 (2015) · Applied to GPD moments: Dutrieux et al. JHEP 08 (2024) 162

Form factor


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
-t (GeV^2)
$F_1(t)$
Dipole fit
z expansion fit
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2.5
-t (GeV^2)
$F_2(t)$
Dipole fit
z expansion fit

A generalized form factor





Generalized Parton Distributions

Moments of GPDs

Pion mass = 0.36 GeV - Proton mass = 1.12 GeV
No continuum limit - signs of discretization errors / light-cone uncertainty
Matching at 2 GeV with leading logarithmic accuracy

Value at t = 0

GPD H^{u-d}		GPD E^{u-d}	
$A_{1,0}$: 0.974^{+12}_{-5}		$B_{1,0}$: 3.40^{+7}_{-1}	
$A_{2,0}$: 0.206^{+2}_{-6}		$B_{2,0}$: 0.370^{+9}_{-24}	
$A_{3,0}$: 0.064^{+2}_{-6}	$A_{3,2}$: 0.39^{+11}_{-3}	$B_{3,0}$: 0.063^{+24}_{-8}	$B_{3,2}$: 1.1^{+4}_{-8}
$A_{4,0}$: 0.065^{+5}_{-19}	$A_{4,2}$: 0.5^{+3}_{-3}	$B_{4,0}$: 0.06^{+16}_{-2}	$B_{4,2}$: > 1.1

Dipole mass (GeV)

GPD H^{u-d}		GPD E^{u-d}	
$A_{1,0}$: 1.255^{+3}_{-29}		$B_{1,0}$: 0.987^{+2}_{-6}	
$A_{2,0}$: 1.83^{+9}_{-3}		$B_{2,0}$: 1.39^{+11}_{-5}	
$A_{3,0}$: 2.3^{+2}_{-5}	$A_{3,2}$: 1.10^{+7}_{-11}	$B_{3,0}$: 2.2^{+36}_{-5}	$B_{3,2}$: 0.78^{+77}_{-9}
$A_{4,0}$: > 3.5	$A_{4,2}$: > 0.9	$B_{4,0}$: > 0.6	$B_{4,2}$: 0.5^{+5}_{-2}

D-term^{u-d}

	Value at t = 0	Dipole mass (GeV)
C_2	0.025^{+8}_{-8}	> 2.2

HadStruc

- ***Dutrieux*** *et al.* *JHEP* 08 (2024) 162 • e-Print: 2405.10304

Reconstructing the full kinematic dependence of GPDs

from lattice pseudo-distributions

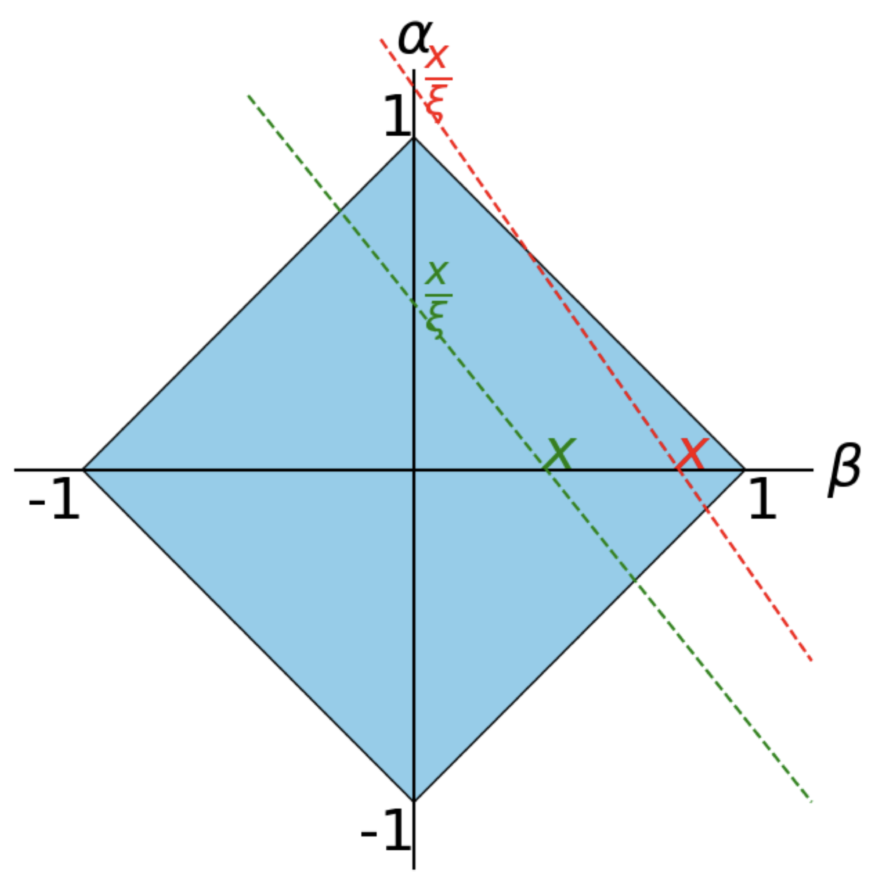
FIRST LATTICE DD EXTRACTION

FULL (x, ξ, t) GPDs

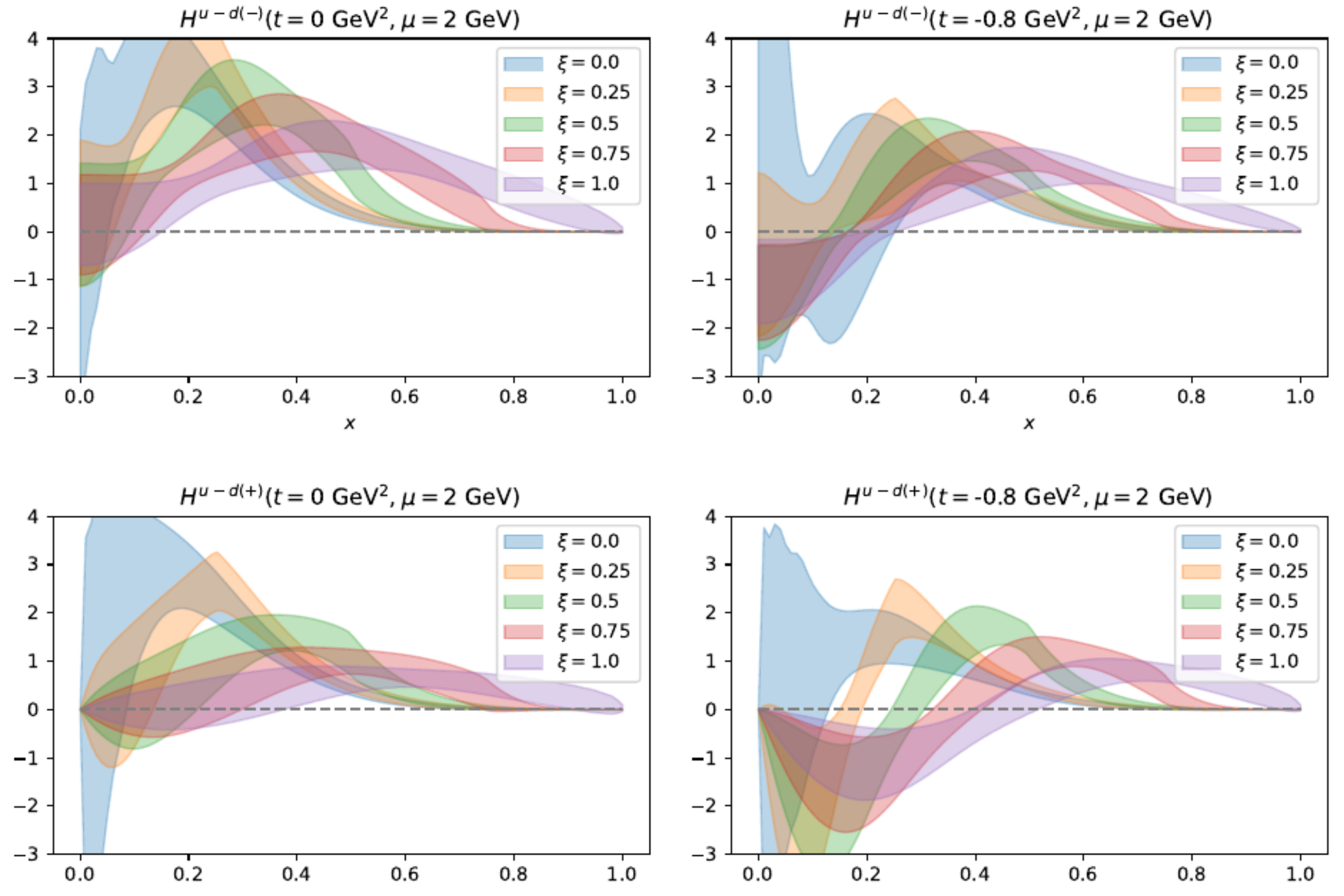
GPR INVERSE PROBLEM

Central idea

GPDs can be parametrized as double distributions, making polynomiality automatic and enabling a flexible reconstruction of proton isovector H and E GPDs.



Double-distribution support and Radon lines



Final H GPDs at μ = 2 GeV

Double Distributions

$$\mathcal{A}_i(\nu, \bar{\nu}, t, z^2) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \; e^{-i\beta\nu - i\alpha\bar{\nu}} a_i(\beta, \alpha, t, z^2)$$

$$|\alpha| + |\beta| \leq 1$$

$$\mathcal{A}_i(\nu, \xi\nu, t, z^2) = \int_{-1}^{1} \mathrm{d}x \, e^{-ix\nu} A_i(x, \xi, t, z^2)$$

$$A_i(x, \xi, t, z^2) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \, \delta(x - \beta - \alpha\xi) \, a_i(\beta, \alpha, t, z^2)$$

Polynomiality:

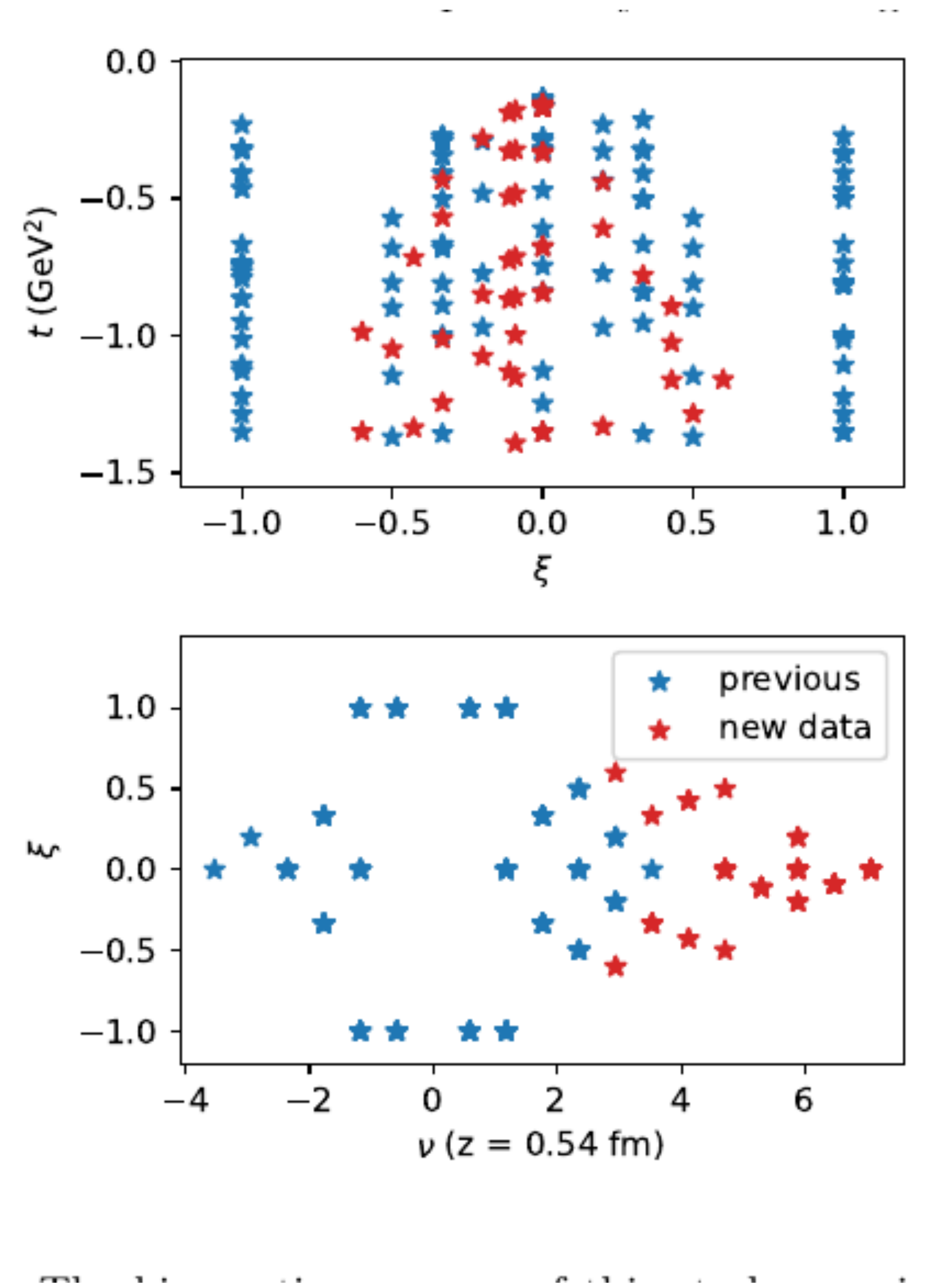
$$\int_{-1}^{1} \mathrm{d}x \, x^n A_i(x, \xi, t, z^2) = \sum_{k=0}^{n} (A_i)_{n+1,k}(t, z^2) \xi^k$$

$$
\begin{aligned}
\mathcal{M}^\mu\left(p_f, p_i, z\right) = \langle\langle\gamma^\mu\rangle\rangle\mathcal{A}_1\left(\nu, \xi, t, z^2\right) + z^\mu\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_2\left(\nu, \xi, t, z^2\right) + i\langle\langle\sigma^{\mu z}\rangle\rangle\mathcal{A}_3\left(\nu, \xi, t, z^2\right) \\
+ \frac{i}{2m}\langle\langle\sigma^{\mu q}\rangle\rangle\mathcal{A}_4\left(\nu, \xi, t, z^2\right) + \frac{q^\mu}{2m}\langle\langle\mathbb{1}\rangle\rangle\mathcal{A}_5\left(\nu, \xi, t, z^2\right) \\
+ \frac{i}{2m}\langle\langle\sigma^{zq}\rangle\rangle\left[P^\mu\mathcal{A}_6\left(\nu, \xi, t, z^2\right) + q^\mu\mathcal{A}_7\left(\nu, \xi, t, z^2\right) + z^\mu\mathcal{A}_8\left(\nu, \xi, t, z^2\right)\right]
\end{aligned}
$$

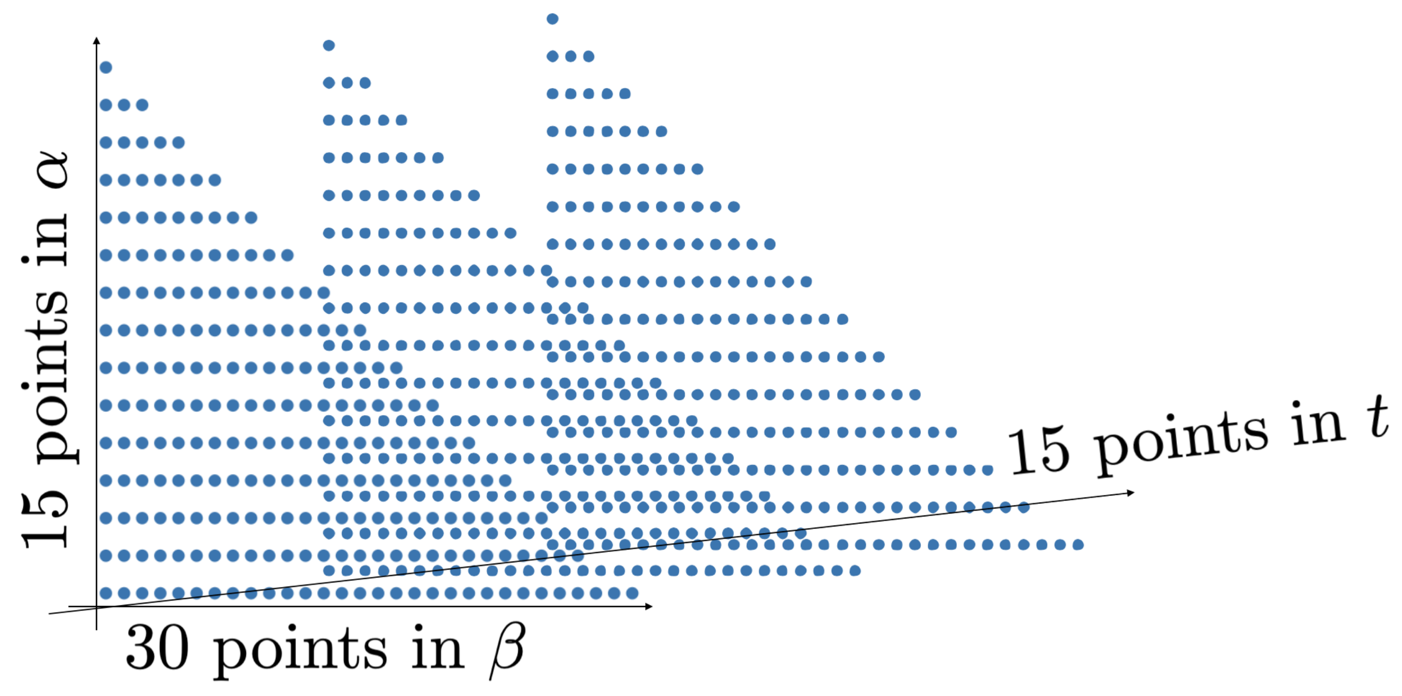
$$
\begin{aligned}
\mathcal{A}_1(\nu, \bar{\nu}, t, z^2) &= \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\, e^{-i(\beta\nu+\alpha\bar{\nu})} h^q(\beta, \alpha, t, z^2)\,, \\
[\mathcal{A}_4 + \nu\mathcal{A}_6 - 2\bar{\nu}\mathcal{A}_7](\nu, \bar{\nu}, t, z^2) &= \int_\Omega \mathrm{d}\beta\mathrm{d}\alpha\, e^{-i(\beta\nu+\alpha\bar{\nu})} e^q(\beta, \alpha, t, z^2)\,, \\
\mathcal{A}_5(\nu, \bar{\nu}, t, z^2) &= -\int_{-1}^{1} \mathrm{d}\alpha\, e^{-i\alpha\bar{\nu}} D^q(\alpha, t, z^2)\,.
\end{aligned}
$$

Central idea: DDs → full GPDs

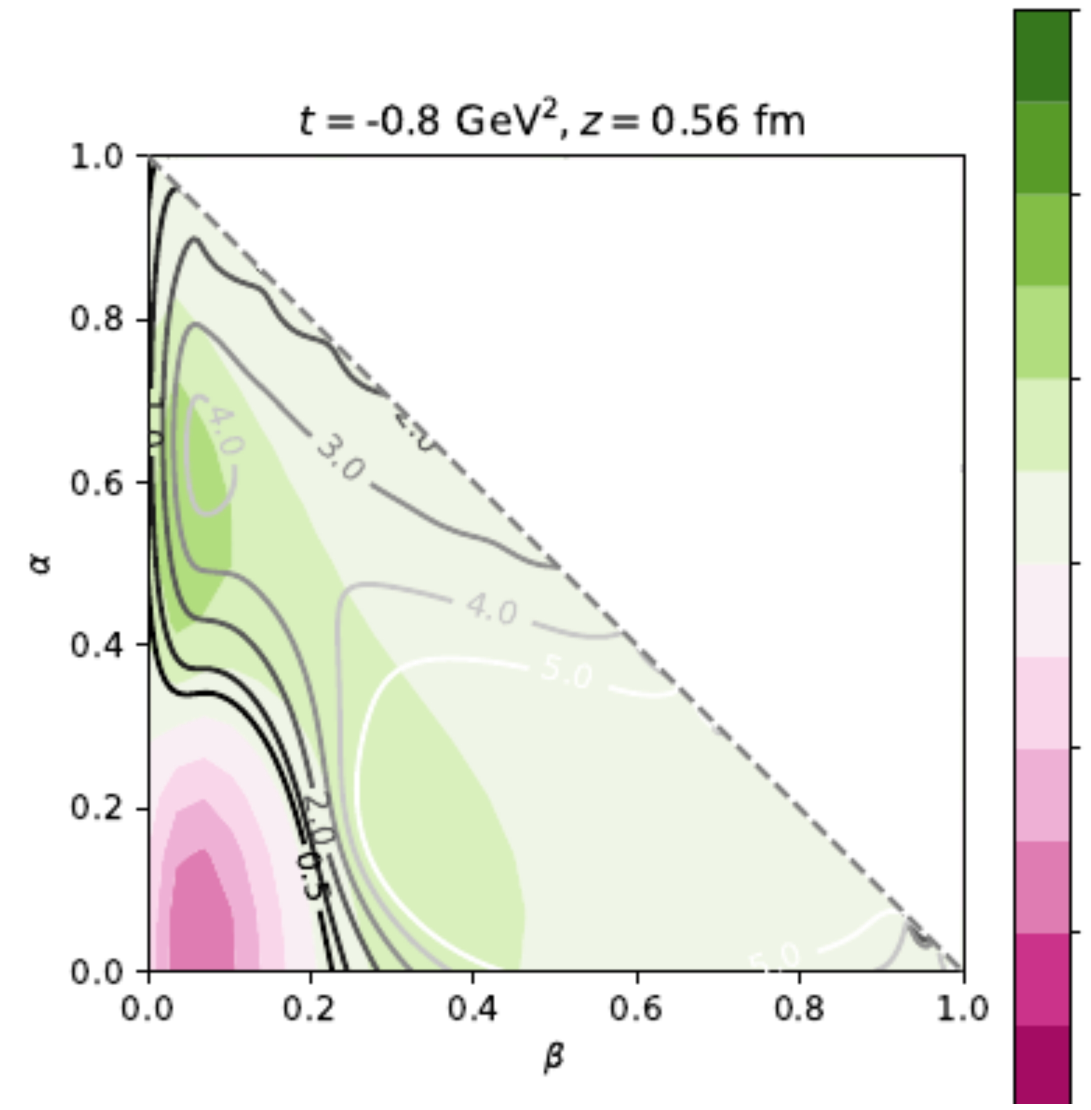
Double distributions encode the x–ξ correlation required by Lorentz symmetry.



Lattice Fourier harmonics



3D finite-element DD mesh



DD posterior

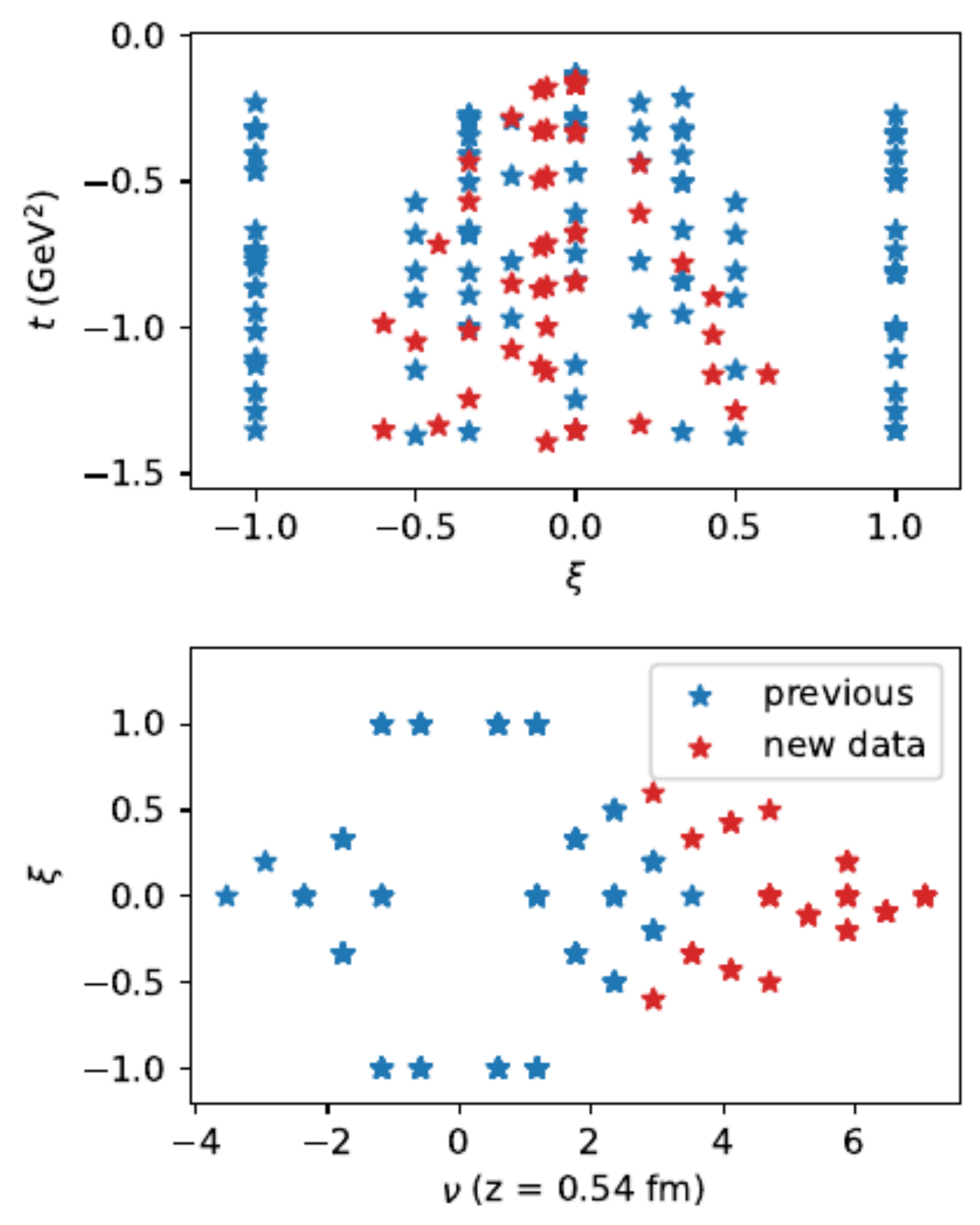
$$\mathcal{A}_i(\nu,\bar{\nu},t,z^2) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\; e^{-i\beta\nu - i\alpha\bar{\nu}} a_i(\beta,\alpha,t,z^2)$$

$$\begin{pmatrix} H^q \\ E^q \end{pmatrix}(x,\xi,t) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha\, \delta(x-\beta-\alpha\xi) \left[\begin{pmatrix} h^q \\ e^q \end{pmatrix}(\beta,\alpha,t) \pm \xi\delta(\beta) D^q(\alpha,t) \right]$$

GPR regularizes the inverse problem with physically motivated priors: small-β flexibility, large-β falloff, edge behavior, and smoothness in t.

Data reach: higher momentum, broader ξ

A substantially enlarged dataset gives access to longer Ioffe times and denser ξ coverage.



New red points extend kinematic coverage

mπ	a	Volume	Ncfg
358 MeV	0.094 fm	$32^3 \times 64$	1490 new

Iplmax	zmax	vmax	Pairs
2.7 GeV	0.56 fm	≈ 7	186 + 63

- ~4× more configurations than the previous GPD study.
- Momentum smearing improves high-momentum signals.
- Extended kinematics support full DD/GPD reconstruction.

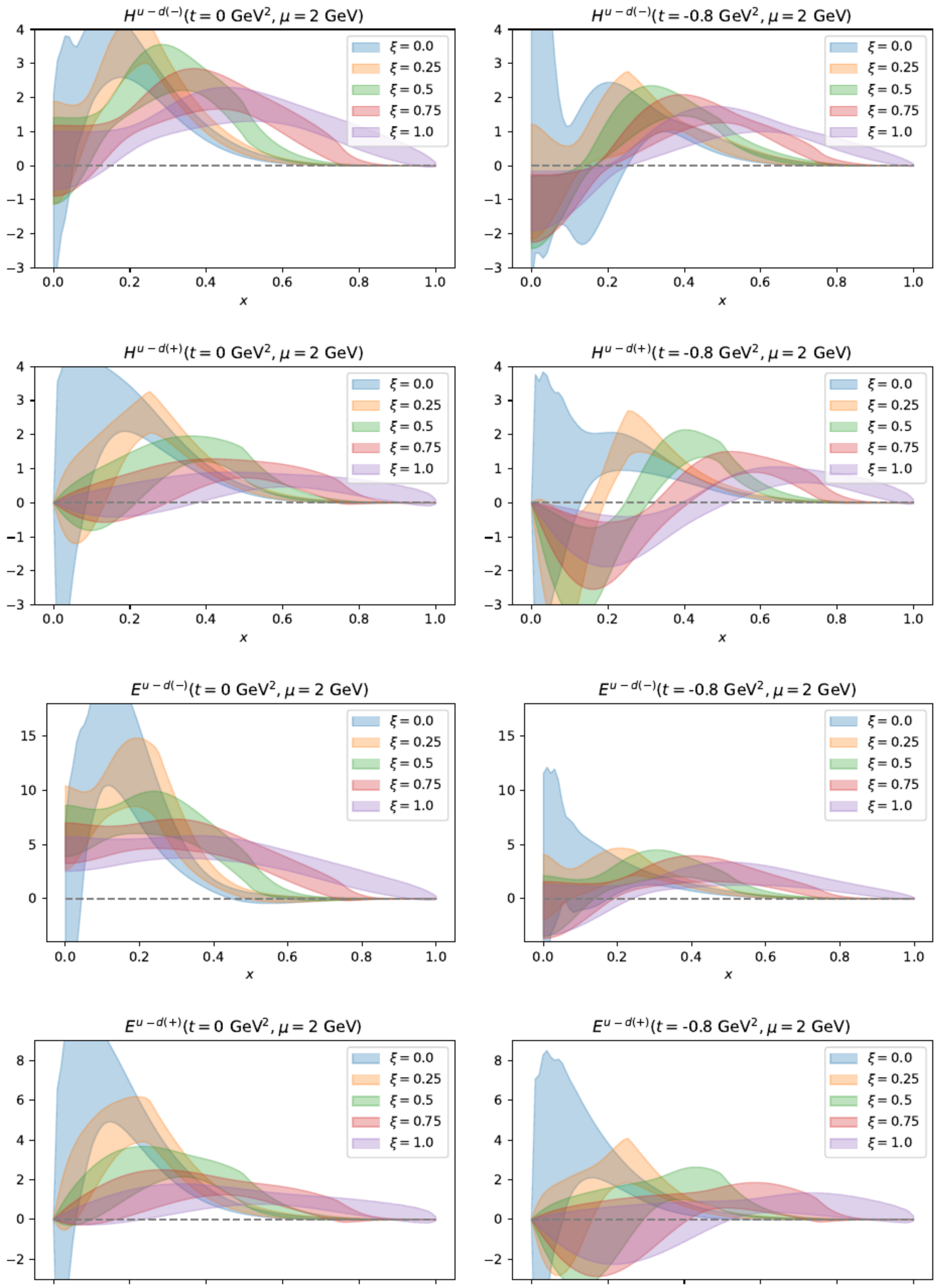
Main result: full H and E GPDs at μ = 2 GeV

The final reconstruction provides x, ξ, and t dependence for $H^{u\text{-}d}$ and $E^{u\text{-}d}$.

What is new?

- Direct lattice extraction of double distributions.
- Polynomiality is automatic by construction.
- Uncertainties: statistics, excited states, and GPR model dependence.
- Final baseline: z = 5a ≈ 0.47 fm, cross-checked with z = 6a and time cuts.

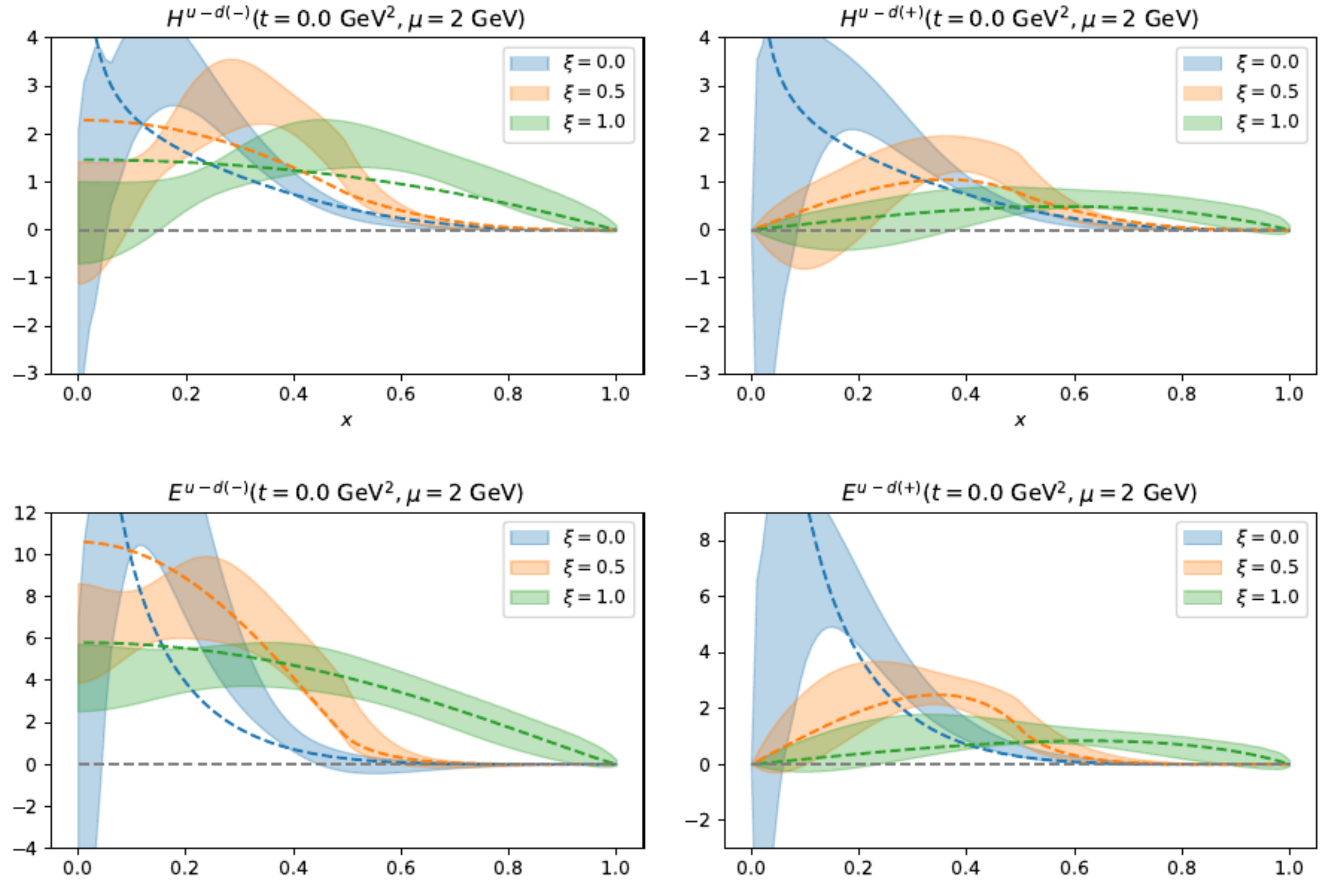
Interpretation: the result is a first-principles map of the proton's longitudinal momentum structure with momentum transfer and skewness dependence.



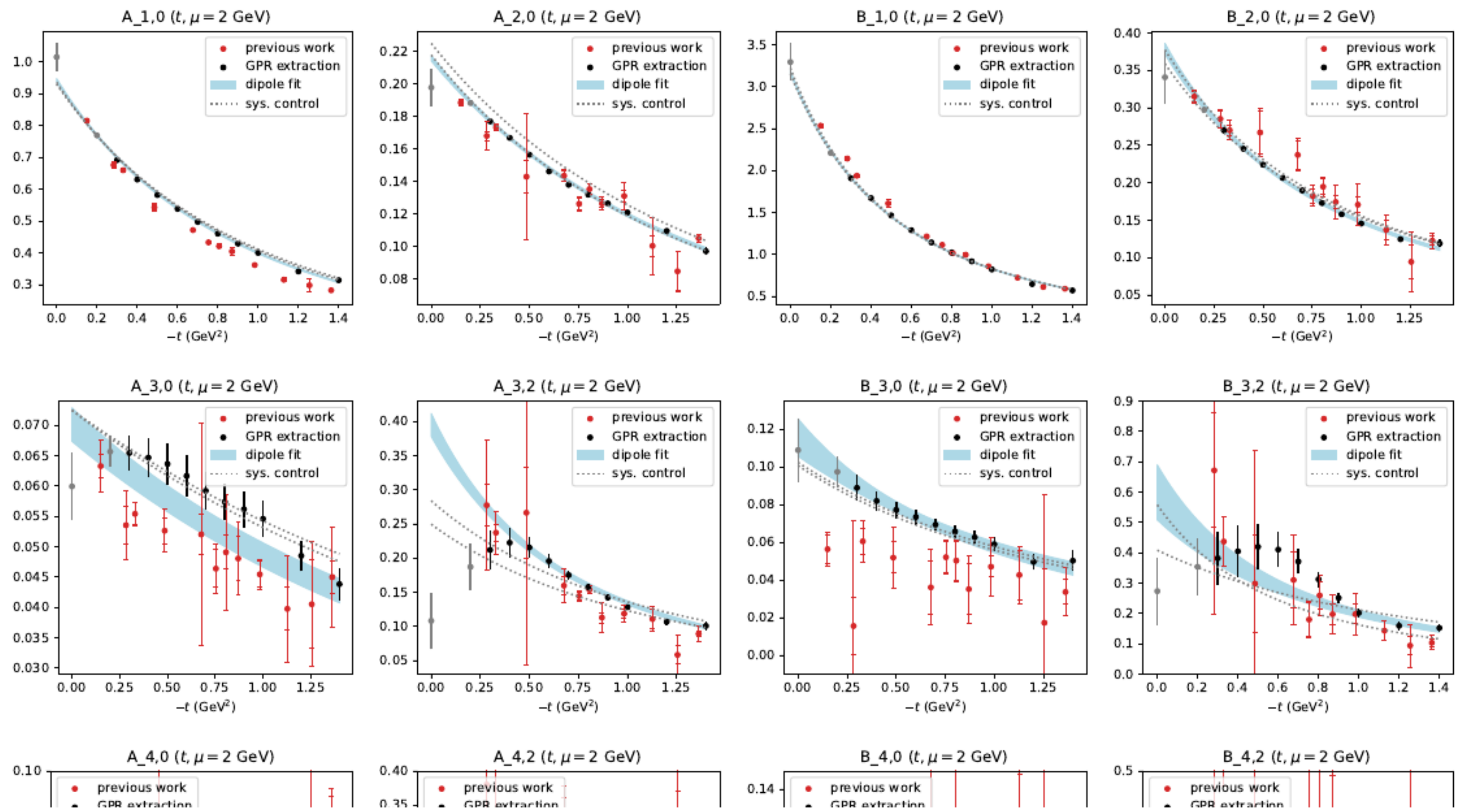
Final extraction of unpolarized isovector GPDs

Checks and implications

The reconstruction begins to connect lattice GPDs to phenomenological models and moments.



Comparison with Goloskokov–Kroll model at t = 0



Generalized form factors from Mellin moments

- Broad qualitative agreement with GK at t = 0.
- Positivity-bound checks are generally satisfied.
- Improved access to skewness-dependent moments and GFFs.
- Remaining systematics: heavy pion, one lattice spacing, finite-volume/discretization, and higher twist.

Takeaway: this is a proof-of-principle route toward computing exclusive-process amplitudes from lattice + perturbative QCD inputs.

Computational requirements

Statistical Errors —- Approach to the continuum Limit

- Inverse problem
  - Need large hadron momentum
  - Large momentum implies larger statistical errors which grow exponentially with momentum
- Extracting phenomenologically interesting information requires small lattice spacing (limit z^2 -> 0)
  - Critical slowing down
  - Large lattices are needed with small lattice spacing
- New algorithm development is needed
  - Improved sampling methods
  - Improved estimators for enhanced statistical precision

"Δός μοι πᾷ στῶ, και τάν γᾶν κινήσω" Αρχιμήδης

- Small lattice spacing for both continuum limit and small z^2
- Large momentum to extend the range of ν

  large ν $\longleftrightarrow$ small x

- Scaling $1/a^7$ (?)
- Large momentum $\longrightarrow$ bad signal to noise ratio

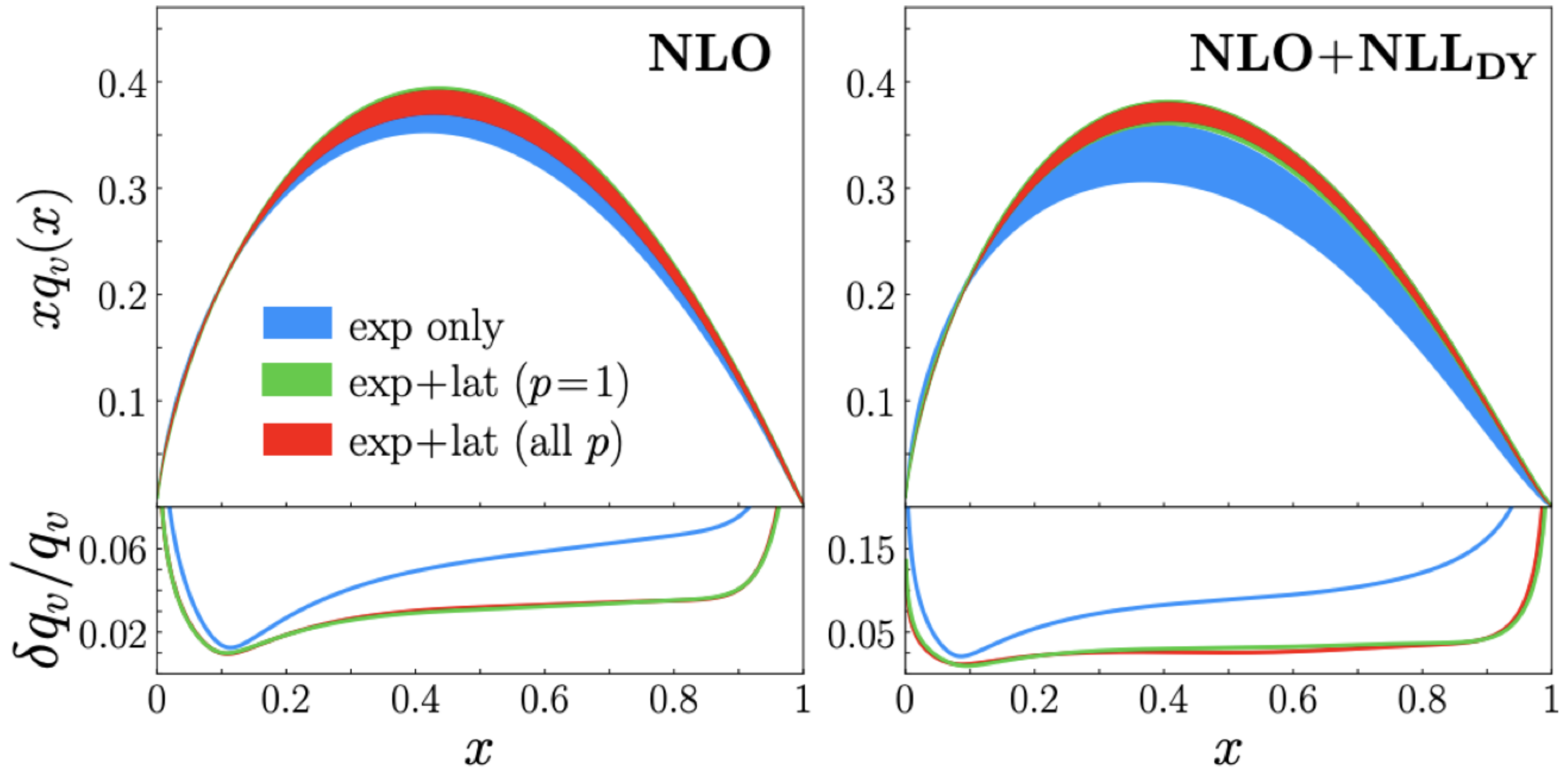
"Give me a big computer...."

Conclusions
Outlook

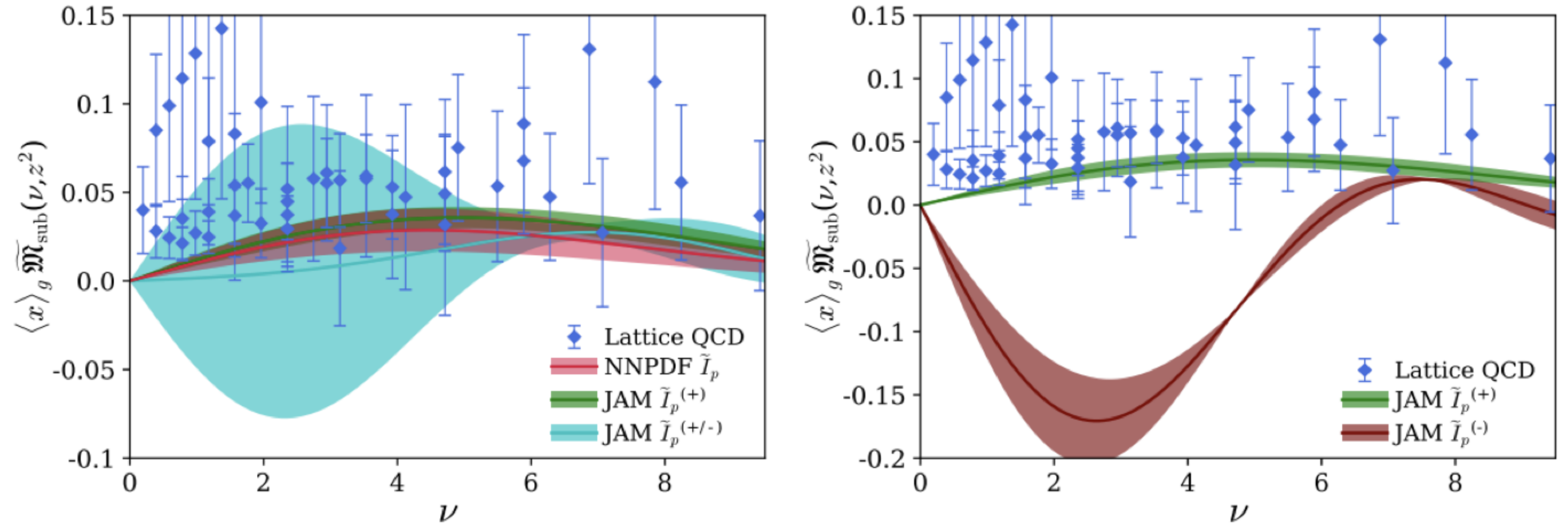
- Understanding hadronic structure is a major goal in nuclear physics
  - Large experimental effort: JLab 12 GeV and the upcoming EIC
- Lattice QCD calculations can in principle compute (Generalized) Parton distribution functions from first principles
- Controlling all systematics of the calculation is important and that complicates the solution of the inverse problem at hand
  - Both lattice spacing and higher twist effects need to be controlled
- New ideas are needed for pushing to higher momentum and improved sampling of the Ioffe time
  - The range of Ioffe time is essential for obtaining the x-dependence of distribution functions
- The synergy between lattice QCD and experiment (especially the EIC) will be essential in providing precision estimates of (Generalized) Parton distribution functions

Complementarity of LQCD and Phenomenology

Pion parton distribution — — Gluon helicity in the nucleon



Valence quark distributions (**top**) when extracted from experimental data alone (blue), combined with the $p = 1$ lattice data (green), and combined with all the lattice data (red) for the NLO (**left**) and NLO+NLL$_{\mathrm{DY}}$ (**right**) cases, along with the relative uncertainties (**bottom**). The bands represent a 1σ uncertainty level. Barry, Egerer, Karpie, Melnitchouk, Orginos, Richards, Sato, Sufian, Qiu, S.Z. Phys.Rev.D 105 (2022) 11, 114051



Egerer, Joo, Karpie, Karthik, Khan, Monahan, Morris, Orginos, Radyushkin, Richards, Romero, Sufian and S.Z, Phys.Rev.D 106 (2022) 9, 094511 De Florian, Forte and Vogelsang Phys. Rev. D 109, 074007 JAM Collaboration N.T. Hunt-Smith et al 2403.08117 [hep-ph] The lattice reduced pITD and the gluon helicity ITD constructed from global fits. In the LHS the red band denotes the ITD constructed from the gluon helicity distribution by the NNPDF collaboration. The green band and the cyan band represent the gluon helicity ITD determined by the JAM collaboration with and without the positivity constraint. The green band and the maroon band represent the gluon helicity ITD determined by the JAM collaboration associated with the positive and negative gluon helicity PDF solutions.

HadStruc