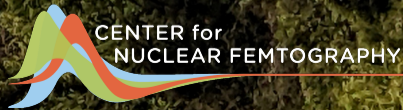


Mechanical Properties of Hadrons

Adam Freese
Center for Nuclear Femtography
June 1-3, 2026





- ☛ Center at Jefferson Lab
- ☛ Concerted effort in theory, experiment & computation
- ☛ Funded by Commonwealth of Virginia to “...to facilitate the application of modern developments in data science to the problem of imaging and visualization of sub-femtometer scale structure of protons, neutrons, and atomic nuclei.”
- ☛ Focused primary on **spatial structure of hadrons**:
 - ✦ Generalized parton distributions
 - ✦ High-energy electro- & photo-nuclear reactions
 - ✦ **Mechanical properties of hadrons**

Center for Nuclear Femtography: the people



- 🌿 **Latifa Elouadrhiri:** director
- 🌿 **David Richards:** associate director
- 🌿 **Adam Freese:** postdoctoral researcher
- 🌿 **Krishna Neupane:** postdoctoral researcher

<https://www.femtocenter.org>



🌿 Seminar series

- ✦ Usually in-person; always streamed on Zoom
- ✦ Sign up at [this link](#) or email afreese@jlab.org

🌿 Lecture series & schools

- ✦ **First International School of Hadron Femtography**
(jointly organized with Quark-Gluon Tomography collaboration)
- ✦ Second school planned for summer 2027
- ✦ Upcoming lecture series *this summer* by Cédric Lorcé

🌿 Partial support for CNUGS!

Upcoming lecture series: Cédric Lorcé

- Speaker: **Cédric Lorcé**
- Dates: July 13-17
- Location: Jefferson Lab

In this series of lectures, we propose an introduction to one of the modern approaches to exploring the internal structure of protons, and more generally hadrons. This approach is based on generalized parton distributions (GPDs) and allows one to construct tomographic representations of the interior of hadrons that extend the original partonic picture proposed by Feynman. GPDs constitute, in and of themselves, a vast topic encompassing a wide range of experimental, phenomenological and theoretical aspects. In these lectures, we will focus on the physical motivations and the reasons why we believe GPDs are so important that they constitute one of the fundamental pillars of the research program of the future Electron-Ion Collider. In particular, we will discuss to what extent they provide information about the spatial distributions of quarks and gluons inside hadrons, illustrate some of the intricacies associated with boosts and spin in quantum field theory, and present the connection to the energy-momentum tensor. The latter is of particular interest since, besides constituting the source of the gravitational field in general relativity, it encodes fundamental properties of the hadron such as mass, spin, and even mechanical properties, which we also plan to address. These lectures are primarily intended for PhD students and young postdocs who wish to understand the big picture behind this field. As recent developments will also be addressed, they may as well be of interest to more experienced researchers.



Pressure in the proton

🍃 Pressure in the proton has become a hot topic.

🍃 Empirical extractions happening at JLab!

- ✦ Burkert, Elouadrhiri & Girod, *Nature* (2018)
- ✦ Duran & al., *Nature* (2023)
- ✦ Joosten & al., 2602.14416
- ✦ CLAS collaboration, 2605.11690

🍃 Interpretation is still controversial

- ✦ What does the pressure mean?
- ✦ What do we learn from it?
- ✦ Is relativity properly incorporated?

🍃 **These lectures will be my own hot takes**

- ✦ Not everything here is settled science
- ✦ Reasonable people can disagree

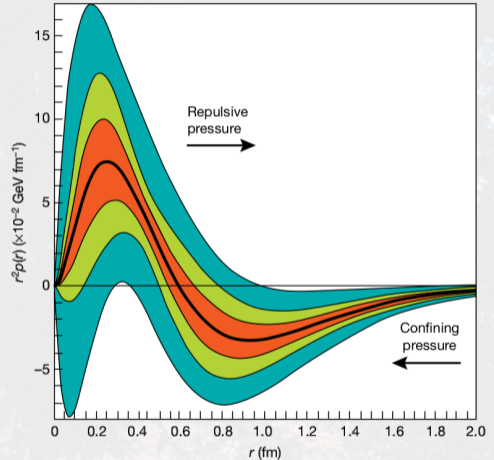


Figure: Burkert, Elouadrhiri & Girod, *Nature* (2018)

The questions

1 Proton mass

- ✧ How much of the proton mass is due to quark mass?
- ✧ Can any of the proton's mass be called "anomalous"?

2 Proton spin

- ✧ How much is carried by quark & gluon spin?
- ✧ How much is orbital motion?

3 Proton pressure

- ✧ Is it actually meaningful to talk about proton pressure?
- ✧ Do we actually learn anything about QCD forces?

4 Nuclear forces

- ✧ Does the stress tensor tell us anything about nuclear forces?

5 Proton densities

- ✧ How are relativistic densities defined?

The **energy-momentum tensor** is at the center of all these questions.

A close-up photograph of a large, dark rock covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and some as buds. Small green plants with broad leaves are growing from the moss. The background is a soft-focus view of a dense cherry blossom tree under a pale sky.

The Energy-Momentum Tensor

The energy-momentum tensor

- ☛ The energy-momentum tensor describes **density** and **flow** of energy & momentum.
- ☛ Also known as the stress-energy tensor.

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Energy density

Momentum densities

Energy fluxes

Stress tensor

The canonical energy tensor

- ✦ **Noether's theorems:** symmetries entail conservation laws
 - ✦ Spatial translation symmetry \implies momentum conservation
 - ✦ Time translation symmetry \implies energy conservation
- ✦ Can use this connection to *define* energy & momentum density!
 - ✦ Energy is what's conserved due to time translation symmetry
 - ✦ Momentum is what's conserved due to space translation symmetry
- ✦ Lagrangian depends on space & time only through the fields

$$\partial_\nu \mathcal{L} = \left\{ \frac{\partial \mathcal{L}}{\partial \psi} \partial_\nu \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\nu \partial_\mu \psi \right\}$$



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$$\partial_\nu \mathcal{L} = \left\{ \frac{\partial \mathcal{L}}{\partial \psi} \partial_\nu \psi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\nu \psi \right] - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right] \partial_\nu \psi \right\}$$



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$$\partial_\nu \psi \left\{ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] - \frac{\partial \mathcal{L}}{\partial \psi} \right\} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\nu \psi - \delta_\nu^\mu \mathcal{L} \right]$$



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$$\partial_\nu \psi \underbrace{\left\{ \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right] - \frac{\partial \mathcal{L}}{\partial \psi} \right\}}_{\text{Euler-Lagrange equations (= 0)}} = \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\nu \psi - \delta_\nu^\mu \mathcal{L} \right]}_{\text{canonical energy tensor}}$$

- ✦ Canonical EMT is conserved iff Euler-Lagrange equations satisfied



Densities and fluxes

Continuity equation:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Energy/momentum transmitted locally.

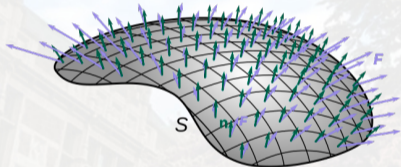


Image: Wikimedia

Integral form for spatial components:

$$\frac{d}{dt} \underbrace{\left[\int_V d^3x T^{0\nu}(\mathbf{x}, t) \right]}_{\text{Energy/momentum in region}} = - \underbrace{\oint_{\partial V} dS \hat{n}_i T^{i\nu}(\mathbf{x}, t)}_{\text{Flux out of region}}$$

Energy density

Momentum densities

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Energy fluxes

Stress tensor

Canonical energy tensor vs. gauge invariance

QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \sum_q \bar{q} \left(\frac{i}{2} \overleftrightarrow{\mathcal{D}}_{\mu} \gamma^{\mu} - m_q \right) q - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

$$\mathcal{D}_{\mu} \psi = \partial_{\mu} \psi - i g A_{\mu}^a T_a \psi$$

$$\mathcal{D}_{\mu} \bar{\psi} = \partial_{\mu} \bar{\psi} + i g A_{\mu}^a T_a \bar{\psi}$$

$$F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g f_{abc} A_{\mu}^b A_{\nu}^c$$

$$[T_a, T_b] = i f_{abc} T_c$$

The canonical energy tensor:

$$T_{\text{can}}^{\mu\nu} = \sum_q \frac{i}{2} \underbrace{\bar{q} \gamma^{\mu} \overleftrightarrow{\partial}^{\nu} q}_{\text{not gauge invariant!}} - F_a^{\mu\rho} \overbrace{(\partial^{\nu} A_{\rho}^a)}^{\text{not gauge invariant!}} - \frac{1}{4} \eta^{\mu\nu} \sum_q \bar{q} \left(\frac{i}{2} \overleftrightarrow{\mathcal{D}}_{\rho} \gamma^{\rho} - m_q \right) q + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta}$$

Lack of gauge invariance considered a fatal flaw!

Simpler example: Maxwell theory

Let's work out canonical EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\rho)} [F_{\alpha\beta} F^{\alpha\beta}]$$

Canonical EMT formula:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - \eta^{\mu\nu} \mathcal{L}$$

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Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} = -\frac{1}{2}F^{\alpha\beta}(\delta_\alpha^\mu \delta_\beta^\rho - \delta_\beta^\mu \delta_\alpha^\rho)$$

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$$T^{\mu\nu} = -F^{\mu\rho} \partial^\nu A_\rho + \eta^{\mu\nu} \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

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$$T^{\mu\nu} = -F^{\mu\rho} \underbrace{\partial^{\nu} A_{\rho}}_{\text{not gauge invariant}} + \eta^{\mu\nu} \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

Worth working this out for QCD too (pretend it's a homework problem)

Belinfante improvement

🍃 **Belinfante improvement:** a way to “fix” the canonical energy tensor

$$T_{\text{Bel}}^{\mu\nu} = T_{\text{can}}^{\mu\nu} + \partial_\rho \Lambda^{\mu\nu\rho}$$
$$\Lambda^{\mu\nu\rho} = \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} \sum_q \bar{q} \gamma_\lambda \gamma_5 q + F^{\mu\rho} A_a^\nu$$

- ✧ $\partial_\rho \Lambda^{\mu\nu\rho}$ is called a **superpotential**
- ✧ $\partial_\mu \partial_\rho \Lambda^{\mu\nu\rho} = 0$ is a *mathematical triviality*

🍃 Assuming Euler-Lagrange equations are true:

$$T_{\text{Bel}}^{\mu\nu} = \sum_q \frac{i}{4} \bar{q} \gamma^{\{\mu} \overleftrightarrow{\mathcal{D}}^{\nu\}} q + F_a^{\mu\rho} F_\rho^{a\nu} - \frac{1}{4} \eta^{\mu\nu} \sum_q \bar{q} \left(\frac{i}{2} \overleftrightarrow{\mathcal{D}}_\rho \gamma^\rho - m_q \right) q + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta}$$

🍃 Procedure works, but is ad hoc (and I find it unsatisfying)

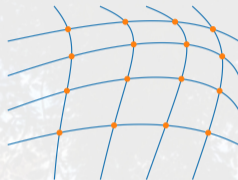
Local translations

🍃 **Noether's theorems:** symmetries entail conservation laws

✦ Try *local* spacetime translations instead?



$$\xrightarrow{x \rightarrow x - \xi(x)}$$



🍃 General coordinate transformations:

$$\phi'(x') = \phi(x) \quad A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \quad \text{etc.}$$

🍃 Local translations (move the fields) mix up field components

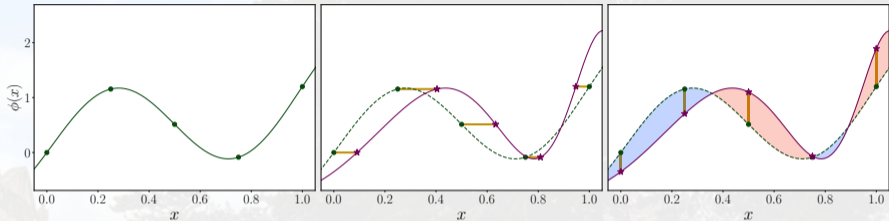
$$\delta_\xi \phi \equiv \phi'(x) - \phi(x) = \xi^\nu \partial_\nu \phi \quad \delta_\xi A_\mu = \xi^\nu \partial_\nu A_\mu + \underbrace{(\partial_\mu \xi^\nu) A_\nu}_{\text{new term!}} \quad \text{etc.}$$

🍃 Noether procedure now depends on tensor type of field

Local translations

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🍃 Noether procedure now depends on tensor type of field

Local translations: scalar fields

$$\phi'(x') = \phi(x)$$

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

Change in field

$$\delta_{\xi}\phi = \phi'(x) - \phi(x)$$

Change in derivative

$$\delta_{\xi}(\partial_{\mu}\phi) = \partial_{\mu}\phi'(x) - \partial_{\mu}\phi(x)$$

☛ Action changes only through changes to the fields

$$\delta_{\xi}S = \int d^4x \delta_{\xi}\mathcal{L}$$

Local translations: scalar fields

$$\phi'(x') = \phi(x)$$

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

Change in field

$$\delta_{\xi}\phi = \phi'(x' + \xi(x)) - \phi(x)$$

Change in derivative

$$\delta_{\xi}(\partial_{\mu}\phi) = \partial_{\mu}\phi'(x' + \xi(x)) - \partial_{\mu}\phi(x)$$

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Local translations: scalar fields

$$\phi'(x') = \phi(x)$$

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Change in field

$$\delta_{\xi}\phi \approx \phi'(x') + \xi^{\nu}(\partial_{\nu}\phi) - \phi(x)$$

Change in derivative

$$\delta_{\xi}(\partial_{\mu}\phi) \approx \partial_{\mu}\phi'(x') + \partial_{\mu}(\xi^{\nu}\partial_{\nu}\phi) - \partial_{\mu}\phi(x)$$

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- Action changes only through changes to the fields

$$\delta_{\xi}S = \int d^4x \left\{ -\xi^{\nu} \partial_{\mu} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} (\partial_{\nu}\phi) - \delta_{\nu}^{\mu} \mathcal{L} \right]}_{\text{energy-momentum tensor}} \right\}$$

- The same as canonical EMT!
- EMT conserved if action is invariant
- Action is invariant *iff Euler Lagrange equations satisfied* (see reference)

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu = A'_\mu(x) - A_\mu(x)$$

Change in field strength

$$\delta_\xi F_{\mu\nu} = F'_{\mu\nu}(x) - F_{\mu\nu}(x)$$

Local translations: vector fields

$$A'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} F_{\alpha\beta}(x)$$

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

Change in four-potential

$$\delta_{\xi} A_{\mu} = A'_{\mu}(x' + \xi(x)) - A_{\mu}(x)$$

Change in field strength

$$\delta_{\xi} F_{\mu\nu} = F'_{\mu\nu}(x' + \xi(x)) - F_{\mu\nu}(x)$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx A'_\mu(x') + \xi^\nu (\partial_\nu A_\mu) - A_\mu(x)$$

Change in field strength

$$\delta_\xi F_{\mu\nu} \approx F'_{\mu\nu}(x') + \xi^\rho (\partial_\rho F_{\mu\nu}) - F_{\mu\nu}(x)$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx (\delta_\mu^\nu + \partial_\mu \xi^\nu) A_\nu(x) + \xi^\nu (\partial_\nu A_\mu) - A_\mu(x)$$

Change in field strength

$$\delta_\xi F_{\mu\nu} \approx (\delta_\mu^\alpha \delta_\nu^\beta + (\partial_\mu \xi^\alpha) \delta_\nu^\beta + \delta_\mu^\alpha (\partial_\nu \xi^\beta)) F_{\alpha\beta}(x) + \xi^\rho (\partial_\rho F_{\mu\nu}) - F_{\mu\nu}(x)$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\nu} \approx (\partial_\mu \xi^\rho) F_{\rho\nu} + (\partial_\nu \xi^\rho) F_{\mu\rho} + \xi^\rho (\partial_\rho F_{\mu\nu})$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \delta_\xi \mathcal{L}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial A_\mu} \delta_\xi A_\mu + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} \delta_\xi F_{\mu\rho} \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial A_\mu} \left(\xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu \right) + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} \left(\xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu} \right) \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \xi^\nu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial A_\mu} (\partial_\nu A_\mu) + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} (\partial_\nu F_{\mu\rho}) \right)}_{=\partial_\nu \mathcal{L}} + \frac{\partial \mathcal{L}}{\partial A_\mu} (\partial_\mu \xi^\nu) A_\nu + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} \left((\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu} \right) \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \xi^\nu \delta^\mu_\nu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial A_\mu} (\partial_\mu \xi^\nu) A_\nu + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} \left((\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu} \right) \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

$$x'^\mu = x^\mu - \xi^\mu(x)$$

Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \xi^\nu \delta^\mu_\nu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial A_\mu} (\partial_\mu \xi^\nu) A_\nu + \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} (\partial_\mu \xi^\nu) F_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial F_{\rho\mu}} (\partial_\mu \xi^\nu) F_{\rho\nu} \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

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$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \xi^\nu \delta^\mu_\nu \partial_\mu \mathcal{L} + \frac{\partial \mathcal{L}}{\partial A_\mu} (\partial_\mu \xi^\nu) A_\nu + 2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} (\partial_\mu \xi^\nu) F_{\nu\rho} \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

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Change in four-potential

$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ \xi^\nu \delta^\mu_\nu \partial_\mu \mathcal{L} - \xi^\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial A_\mu} A_\nu \right] - \xi^\nu \partial_\mu \left[2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F_{\nu\rho} \right] \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

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$$\delta_\xi A_\mu \approx \xi^\nu (\partial_\nu A_\mu) + (\partial_\mu \xi^\nu) A_\nu$$

Change in field strength

$$\delta_\xi F_{\mu\rho} \approx \xi^\nu (\partial_\nu F_{\mu\rho}) + (\partial_\mu \xi^\nu) F_{\nu\rho} + (\partial_\rho \xi^\nu) F_{\mu\nu}$$

☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ -\xi^\nu \partial_\mu \left[2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial A_\mu} A_\nu - \delta_\nu^\mu \mathcal{L} \right] \right\}$$

Local translations: vector fields

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x)$$

$$F'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta}(x)$$

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☛ Action changes only through changes to the fields

$$\delta_\xi S = \int d^4x \left\{ -\xi^\nu \partial_\mu \underbrace{\left[2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial A_\mu} A_\nu - \delta^\mu_\nu \mathcal{L} \right]}_{\text{energy-momentum tensor}} \right\}$$

☛ **Not** the same as the canonical EMT!

☛ EMT conserved if action is invariant

☛ Action is invariant *iff* Euler Lagrange equations satisfied

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{4} \frac{\partial}{\partial F_{\mu\rho}} [F_{\alpha\beta}F^{\alpha\beta}]$$

New EMT formula:

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F^{\nu}_{\rho} - \eta^{\mu\nu} \mathcal{L}$$

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{2}F^{\alpha\beta} \frac{\partial}{\partial F_{\mu\rho}} [F_{\alpha\beta}]$$

New EMT formula:

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} F^{\nu}_{\rho} - \eta^{\mu\nu} \mathcal{L}$$

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{2}F^{\alpha\beta}\delta_{\alpha}^{\mu}\delta_{\beta}^{\rho}$$

New EMT formula:

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}}F^{\nu}_{\rho} - \eta^{\mu\nu}\mathcal{L}$$

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{2}F^{\mu\rho}$$

New EMT formula:

$$T^{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}}F^{\nu}_{\rho} - \eta^{\mu\nu}\mathcal{L}$$

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{2}F^{\mu\rho}$$

New EMT formula:

$$T^{\mu\nu} = -F^{\mu\rho}F^{\nu}_{\rho} + \eta^{\mu\nu}\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

Example: Maxwell theory

Let's work out this new EMT for the Maxwell Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \quad F_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$$

Relevant derivative:

$$\frac{\partial \mathcal{L}}{\partial F_{\mu\rho}} = -\frac{1}{2}F^{\mu\rho}$$

New EMT formula:

$$T^{\mu\nu} = F^{\mu\rho}F_{\rho}{}^{\nu} + \eta^{\mu\nu}\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

Gauge-invariant—and the same as the Belinfante EMT!

Again worth working this out for QCD too (pretend it's a homework problem)

Quick note about spinors

🌿 Spinors transform like scalars under local translations:

$$\psi'(x') = \psi(x)$$

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

🌿 Happens because there's no spinor representation of local translations

- ✦ More technically: no finite linear double cover of $GL(4, \mathbb{R})$
- ✦ There's a non-linear double cover (metilinear group, $ML(4, \mathbb{R})$) ...but it's messy and rarely used

🌿 Will result in **asymmetric EMT** for theories with spinors!

Cartan, *The Theory of Spinors* (original proof)

AF, *Phys Rev D* 113 (2026) 016011 (elementary proof in appendix)

Kinetic energy tensor

- Result from local translations called **kinetic EMT**:

$$T^{\mu\nu} = \sum_q \frac{i}{2} \underbrace{\bar{q} \gamma^\mu \overleftrightarrow{\mathcal{D}}^\nu q}_{\text{not symmetrized}} + F_a^{\mu\rho} F_\rho^{a\nu} - \frac{1}{4} \eta^{\mu\nu} \sum_q \bar{q} \left(\frac{i}{2} \overleftrightarrow{\mathcal{D}}_\rho \gamma^\rho - m_q \right) q + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta}$$

- Unlike Belinfante energy tensor, is not symmetric under $\mu \leftrightarrow \nu$
- Could have obtained from improvement procedure using different superpotential:

$$\Lambda^{\mu\nu\rho} = F^{\mu\rho} A_a^\nu$$

- Shows that superpotential is somewhat arbitrary
- Minimal choice needed to restore gauge invariance
- Meaning of symmetric vs. asymmetric can be sussed out using **mechanical form factors**

Leader & Lorcé, *Phys Rept* 541 (2014) 163

AF, *Phys Rev D* 113 (2026) 016011

Mechanical form factors

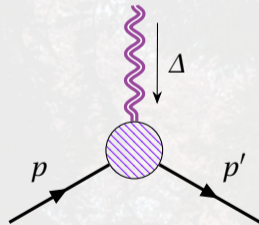
- Energy-momentum tensor parametrized using **mechanical form factors**

$$\langle \mathbf{p}', s' | \hat{T}_{q,g}^{\mu\nu}(0) | \mathbf{p}, s \rangle = \bar{u}(\mathbf{p}', s') \left\{ \frac{P^\mu P^\nu}{m} A_{q,g}(\Delta^2) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4m} D_{q,g}(\Delta^2) + m g^{\mu\nu} \bar{c}_{q,g}(\Delta^2) \right. \\ \left. + \frac{i P^{[\mu} \sigma^{\nu]\Delta}}{2m} J_{q,g}(\Delta^2) - \frac{i P^{[\mu} \sigma^{\nu]\Delta}}{2m} S_{q,g}(\Delta^2) \right\} u(\mathbf{p}, s)$$

- General form from Lorentz covariance
- MFFs encode hadron structure
- $S_{q,g}(\Delta^2)$ present for asymmetric tensor only

$$P^\mu = \frac{1}{2}(p + p')^\mu$$

$$\Delta^\mu = (p' - p)^\mu$$



Kobzarev & Okun, *Sov Phys JETP* 16 (1963) 1343

H. Pagels, *Phys Rev* 144 (1966) 1250

X. Ji, *PRL* 78 (1997) 610

Different nomenclatures

Mechanical form factors

- ✓ Describe mechanical properties—which are the main focus
- ✗ It's new jargon (not in widespread use)

EMT form factors

- ✓ Form factors of the energy-momentum tensor (EMT)
- ✗ Some literature uses “stress energy tensor” instead

Gravitational form factors

- ✓ Densities are gravitational sources
- ✗ We're not using gravitation to measure them or studying gravity

☞ All are acceptable nomenclature, with various pros and cons.

☞ **Mechanical form factors** is just my own preference

Mechanical form factors — sum rules

☛ Momentum sum rule:

$$\sum_{q,g} A_c(0) = 1$$

☛ Angular momentum sum rule (Ji sum rule):

$$\sum_{q,g} J_c(0) = \frac{1}{2}$$

☛ Local momentum conservation:

$$\sum_{q,g} \bar{c}_c(\Delta^2) = 0$$

- ✦ **Violated for open sub-systems**
- ✦ Non-zero $\bar{c}_c(\Delta^2)$ **quantifies forces**

$$\langle f_c^j(\mathbf{b}) \rangle = -m \nabla_j \int \frac{d^3 \Delta}{(2\pi)^3} \bar{c}_c(\Delta^2) e^{-i\Delta \cdot \mathbf{b}}$$

- ✦ Called **non-conserved form factor** — quantifies local conservation violation

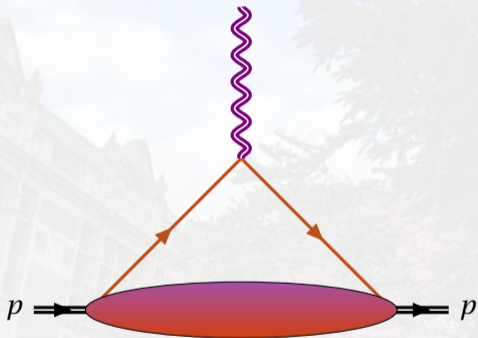
X. Ji, [PRL78 \(1997\) 610](#)

Polyakov & Son, [JHEP 09 \(2018\) 156](#)

How to get MFFs?

Lattice QCD

- Directly compute EMT matrix elements

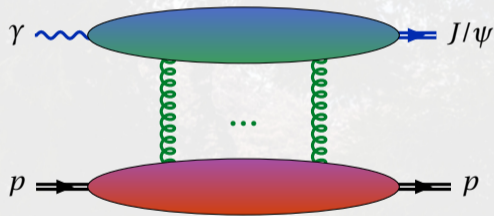


Hackett, Pefkou & Shanahan, *PRL* 132 (2024) 251904

- I'll just talk about what you learn once you've got them.

Exclusive reactions

- Deeply virtual Compton scattering
- Meson photo/electro-production
- Production amplitude formally related to QCD EMT operator



Hatta *et al.*, *PTEP* (2025) ptaf076

Chatagnon *et al.*, 2602.22128

1 Proton mass

- ✧ How much of the proton mass is due to quark mass?
- ✧ Can any of the proton's mass be called "anomalous"?

2 Proton spin

- ✧ How much is carried by quark & gluon spin?
- ✧ How much is orbital motion?

3 Proton pressure

- ✧ Is it actually meaningful to talk about proton pressure?
- ✧ Do we actually learn anything about QCD forces?

4 Proton densities

- ✧ How are relativistic densities defined?

Energy density

Momentum densities

$$\begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Stress tensor

A close-up photograph of a tree trunk covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and others as buds. The background is a soft-focus view of a large tree in full cherry blossom, creating a dreamy, ethereal atmosphere. The text 'Proton Mass' is written in a white, elegant cursive font, centered over the mossy trunk.

Proton Mass

The energy density

- Proton mass = proton energy in rest frame
- Energy density is $T^{00}(x)$

Energy density

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

- Belinfante & kinetic tensors give same result:

$$T^{00}(x) = \underbrace{\sum_q m_q \bar{q}q}_{\text{quark masses}} + \underbrace{\sum_q \frac{-i}{2} q^\dagger \boldsymbol{\alpha} \cdot \overleftrightarrow{\mathcal{D}} q}_{\text{quark kinetic energy}} + \underbrace{F_a^{0\rho} F_\rho^{a0} + \frac{1}{4} F_{\alpha\beta}^a F_a^{\alpha\beta}}_{\text{gluon energy}}$$

🍃 Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle = \int d^3x \langle \Psi | \hat{T}^{00}(x) | \Psi \rangle$$

Lorcé, *Eur Phys J C* 78 (2018) 120

Metz, Pasquini & Rodini, *Phys Rev D* 102 (2020) 114042

Lorcé, Metz, Pasquini & Rodini, *JHEP* 11 (2021) 121

☛ Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_{s,s'} \int d^3x \int \frac{d^3p}{2E_p (2\pi)^3} \int \frac{d^3p'}{2E_{p'} (2\pi)^3} \langle \Psi | \mathbf{p}', s' \rangle \langle \mathbf{p}', s' | \hat{T}^{00}(x) | \mathbf{p}, s \rangle \langle \mathbf{p}, s | \Psi \rangle$$

Lorcé, *Eur Phys J C* 78 (2018) 120

Metz, Pasquini & Rodini, *Phys Rev D* 102 (2020) 114042

Lorcé, Metz, Pasquini & Rodini, *JHEP* 11 (2021) 121

☛ Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_{s,s'} \int d^3x \int \frac{d^3p}{2E_p (2\pi)^3} \int \frac{d^3p'}{2E_{p'} (2\pi)^3} \langle \Psi | \mathbf{p}', s' \rangle \langle \mathbf{p}', s' | e^{i\hat{P}\cdot x} \hat{T}^{00}(0) e^{-i\hat{P}\cdot x} | \mathbf{p}, s \rangle \langle \mathbf{p}, s | \Psi \rangle$$

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Static energy

Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_{s, s'} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \Delta}{(2\pi)^3} \int d^3 x e^{-i\Delta \cdot x} e^{i(E_{p'} - E_p)t} \frac{\langle \mathbf{p}', s' | \hat{T}^{00}(0) | \mathbf{p}, s \rangle}{4E_p E_{p'}} \Psi^*(\mathbf{p}', s') \Psi(\mathbf{p}, s)$$
$$P = \frac{1}{2}(p + p') \qquad \Delta = p' - p$$

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$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_{s,s'} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \Delta}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\Delta) e^{i(E_{p'} - E_p)t} \frac{\langle \mathbf{p}', s' | \hat{T}^{00}(0) | \mathbf{p}, s \rangle}{4E_p E_{p'}} \Psi^*(\mathbf{p}', s') \Psi(\mathbf{p}, s)$$

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🍃 Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle = \sum_s \int \frac{d^3 P}{2E_P (2\pi)^3} \frac{\langle \mathbf{P}, s | \hat{T}^{00}(0) | \mathbf{P}, s \rangle}{2E_P} |\Psi(\mathbf{P}, s)|^2$$

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Static energy

Static energy given by spatial integral of energy density:

$$\langle \Psi | \hat{H} | \Psi \rangle \xrightarrow{\text{zero-momentum plane wave}} \frac{\langle \mathbf{P}, s | \hat{T}^{00}(0) | \mathbf{P}, s \rangle}{2m} \equiv \langle\langle T^{00} \rangle\rangle$$

Gives the following proton mass decomposition:

$$\langle\langle T^{00} \rangle\rangle = \underbrace{\sum_q \langle\langle m_q \bar{q} q \rangle\rangle}_{\text{quark masses}} + \underbrace{\sum_q \frac{-i}{2} \langle\langle q^\dagger \boldsymbol{\alpha} \cdot \vec{\mathcal{D}} q \rangle\rangle}_{\text{quark kinetic energy}} + \underbrace{\left\langle\left\langle F_a^{0\rho} F_\rho^{a0} + \frac{1}{4} F_{\alpha\beta}^a F_a^{\alpha\beta} \right\rangle\right\rangle}_{\text{gluon energy}}$$

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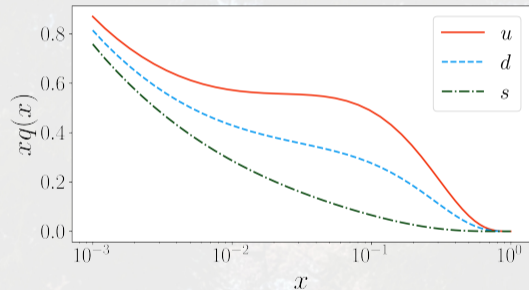
How much proton mass is quark mass?

Misconception: around 1%



Truth: 8-20%

$$T_{\text{mass}}^{00} = \sum_q m_q \bar{q}q$$



🍃 1% assumes proton is three quarks (uud)

✧ ...but we know that's not true.

✧ There's sea of quarks & anti-quarks

Gluon trace anomaly

☛ Gluon part of energy tensor is classically traceless:

$$\eta_{\mu\nu} \left\{ F_a^{\mu\rho} F_\rho^{a\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta} \right\} = -F_a^{\mu\nu} F_{\mu\nu}^a + \frac{4}{4} F_{\alpha\beta}^a F_a^{\alpha\beta} = 0$$

☛ Not true in the quantum theory!

- ✦ Quantities formally infinite without regularization & renormalization
- ✦ In **dimensional regularization** ($d = 4 - 2\epsilon$):

$$F_{\alpha\beta}^a F_a^{\alpha\beta} = \frac{C}{\epsilon} + \text{finite}$$

✦ Trace becomes:

$$\eta_{\mu\nu} \left\{ F_a^{\mu\rho} F_\rho^{a\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta}^a F_a^{\alpha\beta} \right\} = -F_a^{\mu\nu} F_{\mu\nu}^a + \frac{d}{4} F_{\alpha\beta}^a F_a^{\alpha\beta} = -\frac{\epsilon}{2} F_{\alpha\beta}^a F_a^{\alpha\beta} = -\frac{C}{2} + \mathcal{O}(\epsilon) \neq 0$$

✦ Called the **trace anomaly** or **conformal anomaly**

☛ Does this anomalous trace imply an anomalous energy?

Anomalous energy?

What an anomalous energy would look like:

$$[T_{\text{glue}}^{00}]_R \stackrel{?}{=} \sum_c \frac{1}{2} [\mathbf{E}_c^2 + \mathbf{B}_c^2]_R + E_{\text{anomaly}}$$

Complication: field operators have divergences that need to be renormalized

- Use dimreg! ($d = 4 - 2\epsilon$)
- Isolate and subtract $\frac{1}{\epsilon}$ divergences
- Caution:** not allowed to subtract $\frac{1}{\epsilon}$ or set $d = 4$ *until the very end*
- $[\dots]_R$ stands for **renormalized** quantity

Actual gluon energy:

$$T_{\text{glue}}^{00} = F_c^{0\rho} F_{c\rho}^0 + \frac{1}{4} F_{c\alpha\beta} F_c^{\alpha\beta}$$

- Can we rewrite in terms of chromoelectric and chromomagnetic fields?

Renormalized gluonic energy

Chromoelectric field

Vector field ($d - 1$ components)

$$E_c^i = F_c^{0i}$$

Chromomagnetic field

2-form field ($\frac{1}{2}(d - 1)(d - 2)$ components)

$$B_c^{ij} = F_c^{ij}$$

☛ Gluon energy (assume sum over c , even when not repeated):

$$T_{\text{glue}}^{00} = F_c^{0\rho} F_{c\rho}^0 + \frac{1}{4} F_{c\alpha\beta} F_c^{\alpha\beta}$$

☛ And now we renormalize:

$$[T_{\text{glue}}^{00}]_R = \frac{1}{2} [\mathbf{E}_c^2 + \mathbf{B}_c^2]_R$$

✧ That's it. No anomalous energy.

Renormalized gluonic energy

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Vector field ($d - 1$ components)

$$E_c^i = F_c^{0i}$$

Chromomagnetic field

2-form field ($\frac{1}{2}(d - 1)(d - 2)$ components)

$$B_c^{ij} = F_c^{ij}$$

☛ Gluon energy (assume sum over c , even when not repeated):

$$T_{\text{glue}}^{00} = F_c^{0i} F_{ci}{}^0 + \frac{1}{2} F_{c0i} F_c^{0i} + \frac{1}{4} F_{cij} F_c^{ij}$$

☛ And now we renormalize:

$$[T_{\text{glue}}^{00}]_R = \frac{1}{2} [\mathbf{E}_c^2 + \mathbf{B}_c^2]_R$$

✧ That's it. No anomalous energy.

Renormalized gluonic energy

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2-form field ($\frac{1}{2}(d - 1)(d - 2)$ components)

$$B_c^{ij} = F_c^{ij}$$

☛ Gluon energy (assume sum over c , even when not repeated):

$$T_{\text{glue}}^{00} = \mathbf{E}_c^2 - \frac{1}{2}\mathbf{E}_c^2 + \frac{1}{2}\mathbf{B}_c^2$$

☛ And now we renormalize:

$$[T_{\text{glue}}^{00}]_R = \frac{1}{2}[\mathbf{E}_c^2 + \mathbf{B}_c^2]_R$$

✧ That's it. No anomalous energy.

Renormalized gluonic energy

Chromoelectric field

Vector field ($d - 1$ components)

$$E_c^i = F_c^{0i}$$

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$$B_c^{ij} = F_c^{ij}$$

✦ Gluon energy (assume sum over c , even when not repeated):

$$T_{\text{glue}}^{00} = \frac{1}{2}(\mathbf{E}_c^2 + \mathbf{B}_c^2)$$

✦ And now we renormalize:

$$[T_{\text{glue}}^{00}]_R = \frac{1}{2}[\mathbf{E}_c^2 + \mathbf{B}_c^2]_R$$

✦ That's it. No anomalous energy.

Origin of the myth of anomalous energy

- There's a common myth in the field that there's an anomalous gluon energy.
- Arises from inconsistently applying renormalization rules—here's how:

$$T_{\text{glue}}^{00} = F_c^{0\rho} F_{c\rho}^0 + \frac{1}{4} F_{c\alpha\beta} F_c^{\alpha\beta}$$

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$$T_{\text{glue}}^{00} = F_c^{0\rho} F_{c\rho}{}^0 + \frac{1}{d} F_{c\alpha\beta} F_c^{\alpha\beta} + \left(\frac{d-4}{4d} \right) F_{c\alpha\beta} F_c^{\alpha\beta}$$

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$$T_{\text{glue}}^{00} = \underbrace{F_c^{0\rho} F_{c\rho}{}^0 + \frac{1}{d} F_{c\alpha\beta} F_c^{\alpha\beta}}_{\text{traceless part}} + \underbrace{\left(\frac{d-4}{4d} \right) F_{c\alpha\beta} F_c^{\alpha\beta}}_{\text{traceful part}}$$

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$$T_{\text{glue}}^{00} = F_c^{0i} F_{ci}{}^0 + \frac{2}{d} F_{c0i} F_c^{0i} + \frac{1}{d} F_{cij} F_c^{ij} + \left(\frac{d-4}{4d} \right) \left(\frac{C}{\epsilon} + \text{finite} \right)$$

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$$T_{\text{glue}}^{00} = \left(1 - \frac{2}{d}\right) E_c^2 + \frac{2}{d} B_c^2 + \left(\frac{-\epsilon}{2d}\right) \left(\frac{C}{\epsilon} + \text{finite}\right)$$

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$$T_{\text{glue}}^{00} = \left(\frac{1}{2} - \frac{\epsilon}{4} + \mathcal{O}(\epsilon^2) \right) \mathbf{E}_c^2 + \left(\frac{1}{2} + \frac{\epsilon}{4} + \mathcal{O}(\epsilon^2) \right) \mathbf{B}_c^2 - \frac{C}{8} + \mathcal{O}(\epsilon)$$

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- The mistake:** drop apparent $\mathcal{O}(\epsilon)$ factors:

$$T_{\text{glue}}^{00} = \frac{1}{2} (\mathbf{E}_c^2 + \mathbf{B}_c^2) - \frac{C}{8}$$

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- Correct procedure:** recall \mathbf{E}_c^2 and \mathbf{B}_c^2 have $\frac{1}{\epsilon}$ divergences:

$$T_{\text{glue}}^{00} = \frac{1}{2} (\mathbf{E}_c^2 + \mathbf{B}_c^2) + \frac{\epsilon}{4} (\mathbf{B}_c^2 - \mathbf{E}_c^2) - \frac{C}{8} + \mathcal{O}(\epsilon)$$

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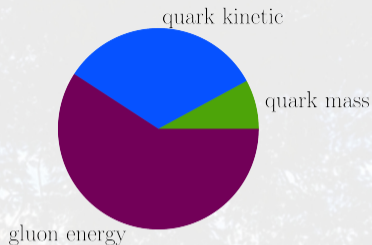
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Proton mass decomposition

☛ The most sensible proton mass breakdown has three terms:

$$T^{00} = \underbrace{\sum_q m_q \bar{q}q}_{\text{quark masses}} + \underbrace{\sum_q \frac{-i}{2} q^\dagger \boldsymbol{\alpha} \cdot \vec{\mathcal{D}} q}_{\text{quark kinetic energy}} + \underbrace{F_a^{0\rho} F_\rho^{a0} + \frac{1}{4} F_{\alpha\beta}^a F_a^{\alpha\beta}}_{\text{gluon energy}}$$



Scenario A, MS from MPR PRD

☛ In terms of mechanical form factors ...

$$\langle T_{\text{quark}}^{00} \rangle = m \left(A_q(0) + \bar{c}_q(0) \right) \quad \langle T_{\text{glue}}^{00} \rangle = m \left(A_g(0) + \bar{c}_g(0) \right)$$

✦ Mechanical form factors insufficient to separate quark mass & kinetic energy

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A close-up photograph of a tree trunk covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and others as buds. The background is a soft-focus sea of more cherry blossoms. The text "Proton Spin" is written in a white, elegant cursive font across the center of the mossy trunk.

Proton Spin

Momentum densities

- Momentum densities are the top row of the energy-momentum tensor

Momentum densities

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

- Classically, (orbital) angular momentum given by:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

- Define (orbital) angular momentum density via:

$$L_i(x) = \epsilon_{ijk} x^j T^{0k}(x)$$

Beware of twisted protons

☛ **Caution:** proton wave packet can carry angular momentum!

- ✦ Need to avoid this when building spin sum rules
- ✦ Use spherical wave packets, zero average momentum

$$\langle \Psi | \hat{L}_i | \Psi \rangle = \epsilon_{ijk} \int d^3x x^j \langle \Psi | \hat{T}^{0k}(x) | \Psi \rangle$$

☛ Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

$$\langle \Psi | \hat{L}_{q,g} | \Psi \rangle = J_{q,g}(0) - S_{q,g}(0)$$

- ✦ Different energy tensors imply different proton spin breakdowns
- ✦ **Caution:** what's meant by $J_{q,g}(0)$ and $S_{q,g}(0)$ can also differ between energy tensors!

Beware of twisted protons

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$$\langle \Psi | \hat{L}_i | \Psi \rangle = \epsilon_{ijk} \int d^3x x^j \int \frac{d^3p}{2E_p (2\pi)^3} \int \frac{d^3p'}{2E_{p'} (2\pi)^3} \langle \Psi | \mathbf{p}', s' \rangle \langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle \langle \mathbf{p}, s | \Psi \rangle$$

☛ Use a spin-up (z-axis) state; in terms of **mechanical form factors**:

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$$\langle \Psi | \hat{L}_i | \Psi \rangle = \sum_{s,s'} \epsilon_{ijk} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \Delta}{(2\pi)^3} \int d^3 x x^j e^{-i\Delta \cdot x} e^{i\Delta_E t} \frac{\langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle}{4E_{\mathbf{p}} E_{\mathbf{p}'}} \Psi^*(\mathbf{p}', s') \Psi(\mathbf{p}, s)$$

☛ Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

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Beware of twisted protons

☛ **Caution:** proton wave packet can carry angular momentum!

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- ✦ Use spherical wave packets, zero average momentum

$$\langle \Psi | \hat{L}_i | \Psi \rangle = -i \sum_{s,s'} \epsilon_{ijk} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \Delta}{(2\pi)^3} \int d^3 x e^{-i\Delta \cdot x} \frac{\partial}{\partial \Delta^j} \left[e^{i\Delta E t} \frac{\langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle}{4E_p E_{p'}} \Psi'^* \Psi \right]$$

☛ Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

$$\langle \Psi | \hat{L}_{q,g} | \Psi \rangle = J_{q,g}(0) - S_{q,g}(0)$$

- ✦ Different energy tensors imply different proton spin breakdowns
- ✦ **Caution:** what's meant by $J_{q,g}(0)$ and $S_{q,g}(0)$ can also differ between energy tensors!

Beware of twisted protons

☛ **Caution:** proton wave packet can carry angular momentum!

- ✦ Need to avoid this when building spin sum rules
- ✦ Use spherical wave packets, zero average momentum

$$\langle \Psi | \hat{L}_i | \Psi \rangle = -i \sum_{s,s'} \epsilon_{ijk} \int \frac{d^3 P}{(2\pi)^3} \int \frac{d^3 \Delta}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\Delta) \frac{\partial}{\partial \Delta^j} \left[e^{i\Delta_E t} \frac{\langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle}{4E_{\mathbf{p}} E_{\mathbf{p}'}} \Psi'^* \Psi \right]$$

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🍃 Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

$$\langle \Psi | \hat{L}_{q,g} | \Psi \rangle = J_{q,g}(0) - S_{q,g}(0)$$

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$$\langle \Psi | \hat{L}_i | \Psi \rangle = -i \sum_{s, s'} \epsilon_{ijk} \int \frac{d^3 P}{(2\pi)^3} \frac{\partial}{\partial \Delta^j} \left[\frac{\langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle}{2E_{\mathbf{P}}} \right] \Big|_{\Delta=0} \Psi^*(\mathbf{P}, s') \Psi(\mathbf{P}, s)$$

☛ Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

$$\langle \Psi | \hat{L}_{q,g} | \Psi \rangle = J_{q,g}(0) - S_{q,g}(0)$$

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$$\langle \Psi | \hat{L}_i | \Psi \rangle = -i \sum_{s,s'} \epsilon_{ijk} \frac{\partial}{\partial \Delta^j} \left[\frac{\langle \mathbf{p}', s' | \hat{T}^{0k}(0) | \mathbf{p}, s \rangle}{2m} \right] \Big|_{\Delta=0}$$

☛ Use a spin-up (z -axis) state; in terms of **mechanical form factors**:

$$\langle \Psi | \hat{L}_{q,g} | \Psi \rangle = J_{q,g}(0) - S_{q,g}(0)$$

- ✦ Different energy tensors imply different proton spin breakdowns
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A tale of two sum rules

1 Jaffe-Manohar sum rule

$$J = \sum_q \underbrace{\int d^3x \mathbf{x} \times q^\dagger (-i\nabla) q}_{\mathcal{L}_q} + \sum_q \underbrace{\int d^3x q^\dagger \frac{\boldsymbol{\sigma}}{4} q}_{S_q} + \underbrace{\int d^3x \mathbf{E}_a \times \mathbf{A}_a}_{\mathcal{L}_g} + \underbrace{\int d^3x E_a^i (\mathbf{x} \times \nabla) A_a^i}_{S_g}$$

- ✦ Motivated by **canonical energy tensor**
- ✦ Not gauge-invariant, but correct rotation generators
 - + **Chen decomposition** picks out Coulomb gauge

2 Ji sum rule

$$J = \sum_q \underbrace{\int d^3x \mathbf{x} \times q^\dagger (-i\mathcal{D}) q}_{L_q} + \sum_q \underbrace{\int d^3x q^\dagger \frac{\boldsymbol{\sigma}}{4} q}_{S_q} + \underbrace{\int d^3x \mathbf{x} \times (\mathbf{E}_a \times \mathbf{B}_a)}_{J_g}$$

- ✦ Motivated by **kinetic energy tensor**
- ✦ Gauge-invariant, but not rotation generators

Jaffe & Manohar, *Nucl Phys B* 337 (1990) 509

Ji, *Phys Rev Lett* 78 (1997) 610

Leader & Lorcé, *Phys Rept* 541 (2014) 163

Jaffe-Manohar decomposition

$$J = \underbrace{\sum_q \int d^3x \mathbf{x} \times q^\dagger (-i \nabla) q}_{\mathcal{L}_q} + \underbrace{\sum_q \int d^3x q^\dagger \frac{\boldsymbol{\sigma}}{4} q}_{S_q} + \underbrace{\int d^3x \mathbf{E}_a \times \mathbf{A}_a}_{\mathcal{L}_g} + \underbrace{\int d^3x E_a^i (\mathbf{x} \times \nabla) A_a^i}_{S_g}$$

- 🍃 **Orbital pieces** come from $\epsilon_{ijk} x^j T_{\text{can}}^{0k}(x)$
- 🍃 **Spin pieces** come from (missing) Belinfante improvement terms
- 🍃 All pieces obey $[J_i, J_j] = i\epsilon_{ijk} J_k$ individually
- 🍃 Pieces act like rotation generators:

$$[\mathcal{L}_{q,i}, q] = i(\mathbf{x} \times \nabla)_i q$$

$$[S_{q,i}, q] = -\frac{1}{4} \sigma_i q$$

$$[\mathcal{L}_{g,i}, A_j^a] = i(\mathbf{x} \times \nabla)_i A_j^a$$

$$[S_{g,i}, A_j^a] = i\epsilon_{ijk} A_k^a$$

Jaffe & Manohar, *Nucl Phys B* 337 (1990) 509

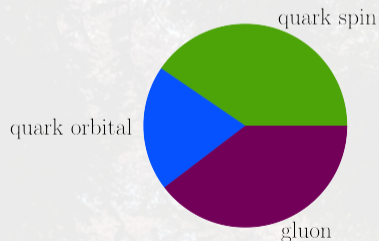
Leader & Lorcé, *Phys Rept* 541 (2014) 163

I didn't make a pie chart because the numbers are basically unknown.

Ji decomposition

$$J = \underbrace{\sum_q \int d^3x \mathbf{x} \times q^\dagger (-i\mathcal{D}) q}_{L_q} + \underbrace{\sum_q \int d^3x q^\dagger \frac{\boldsymbol{\sigma}}{4} q}_{S_q} + \underbrace{\int d^3x \mathbf{x} \times (\mathbf{E}_a \times \mathbf{B}_a)}_{J_g}$$

- 🍃 **Gluon piece** and **quark orbital piece** come from $\epsilon_{ijk} x^j T_{\text{can}}^{0k}(x)$
- 🍃 **Quark spin piece** comes from (missing) Belinfante improvement term
- 🍃 Gauge-invariant breakdown!
- 🍃 ...but pieces are not rotation generators



Ji, *Phys Rev Lett* 78 (1997) 610

Leader & Lorcé, *Phys Rept* 541 (2014) 163

Alexandrou *et al.*, *Phys Rev D* 101 (2020) 094513 (numbers for pie chart)

A close-up photograph of a tree trunk covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and others as buds. The background is a soft-focus sea of more cherry blossoms. The text 'Proton Pressure' is written in a white, elegant cursive font across the center of the mossy trunk.

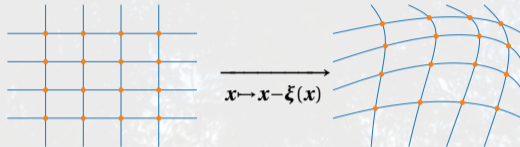
Proton Pressure

The stress tensor

✦ 3×3 sub-matrix of the energy-momentum tensor.

$$\begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

↑
Stress tensor



✦ Work needed to deform a system:

$$W[\mathbf{x} \mapsto \mathbf{x} - \boldsymbol{\xi}(\mathbf{x})] = - \int d^3x \overbrace{T_{\text{QCD}}^{ij}(\mathbf{x})}^{\text{stress tensor}} \left(\overbrace{\frac{1}{2}(\partial_i \xi^j(\mathbf{x}) + \partial_j \xi^i(\mathbf{x}))}^{\text{strain}} + \overbrace{\frac{1}{2}(\partial_i \xi^j(\mathbf{x}) - \partial_j \xi^i(\mathbf{x}))}^{\text{torsion}} \right)$$

- ✦ **Stress:** internal forces resisting *spatial* deformation
- ✦ Deformations are a virtual pressure gauge

Lorcé, Metz, Pasquini & Rodini, *JHEP* 11 (2021) 121

AF, *PRD* 113 (2026) 016011

Stress & momentum flux density

Continuity equation (integral form):

$$\frac{d}{dt} \left[\underbrace{\int_V d^3x T^{0j}(\mathbf{x}, t)}_{\text{Momentum in region}} \right] = - \underbrace{\oint_{\partial V} dS \hat{n}_i T^{ij}(\mathbf{x}, t)}_{\text{Flux out of region}}$$

$$T^{\mu\nu}(\mathbf{x}) = \begin{bmatrix} T^{00}(\mathbf{x}) & T^{01}(\mathbf{x}) & T^{02}(\mathbf{x}) & T^{03}(\mathbf{x}) \\ T^{10}(\mathbf{x}) & T^{11}(\mathbf{x}) & T^{12}(\mathbf{x}) & T^{13}(\mathbf{x}) \\ T^{20}(\mathbf{x}) & T^{21}(\mathbf{x}) & T^{22}(\mathbf{x}) & T^{23}(\mathbf{x}) \\ T^{30}(\mathbf{x}) & T^{31}(\mathbf{x}) & T^{32}(\mathbf{x}) & T^{33}(\mathbf{x}) \end{bmatrix}$$

Stresses are **momentum flux densities**

✦ Include **particle fluxes & forces**



Particle flux



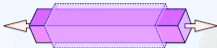
Force

Kinetic pressure is pressure

- ✦ It's the microscopic origin of gas & fluid pressure
- ✦ Photon flux is the origin of radiation pressure

Positive or negative stress?

Tension



Compression



Image: MikeRun (Wikimedia), very modified

Hanging monkeys

- 🍃 **Negative stress** / negative “pressure”
- 🍃 **Tension** / pulling / stretching
- 🍃 Highest monkey under most tension

$$\nabla_i T^{ij} = -\rho g \hat{z}^j = f_{\text{grav}}^j$$

Monkey image by Angie Freese (Instagram)

Stacked Turtles

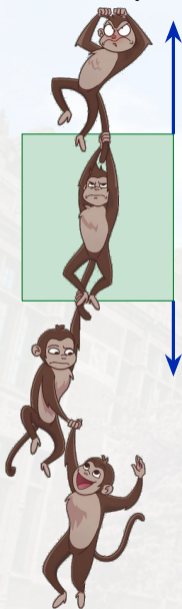
- 🍃 **Positive stress** / positive “pressure”
- 🍃 **Compression** / pushing / squishing
- 🍃 More compression at bottom of pile

$$\nabla_i T^{ij} = -\rho g \hat{z}^j = f_{\text{grav}}^j$$

Turtle image by FoxKids1302 (DeviantArt)



Cauchy momentum equation



$$F_{\text{up}} = +3Mg$$

$$F_g = -Mg$$

$$F_{\text{down}} = -2Mg$$

Cauchy momentum equation for an *open, static* system:

$$f_{\text{net}}^j(\mathbf{x}, t) = f_{\text{external}}^j(\mathbf{x}, t) - \nabla_i T^{ij}(\mathbf{x}, t) = 0$$

- ✦ Balance between **external forces** & **internal forces**
- ✦ Can infer external forces from stress tensor!

Works for expectation values in quantum systems!

$$\langle f_{\text{external}}^j(\mathbf{x}, t) \rangle = \nabla_i \langle T^{ij}(\mathbf{x}, t) \rangle$$

- ✦ Schrödinger, *Ann. Phys.* 387 (1927) 265
- ✦ Nielsen & Martin, *PRB* 32 (1985) 3780
- ✦ Polyakov & Son, *JHEP* 09 (2018) 156

Stress tensor in atom returns Coulomb force law!

- ✦ AF, *PRD* 111 (2025) 034047
- ✦ Could map out strong nuclear forces via stress tensor!

About that divergence-free part



✦ Both donkeys have the same $\nabla_i T^{ij}$

$$\nabla_i T^{ij} = -\rho g \hat{z}^j$$

- ✦ Only their *D*-term (donkey term?) differs.
- ✦ Divergence-free part describes (potentially large) mutually-cancelling internal stresses.
- ✦ **This is physical and can be felt.**

Define a **static stress tensor**:

$$t_{q,g}^{ij}(\mathbf{r}) \equiv \int \frac{d^3\Delta}{2E_{\Delta/2}(2\pi)^3} \left\langle \frac{\Delta}{2}, s \left| \hat{T}_{q,g}^{ij}(0) \right| -\frac{\Delta}{2}, s \right\rangle e^{-i\Delta \cdot \mathbf{r}}$$

- ✧ This is a *definition*—not derived
- ✧ The definition is controversial—we'll get to that next lecture
- ✧ Let's roll with it for now

In terms of **mechanical form factors**:

$$t_{q,g}^{ij}(\mathbf{r}) = \int \frac{d^3\Delta}{(2\pi)^3} \left(\frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4m} D_{q,g}(\Delta^2) - m \delta^{ij} \bar{c}_{q,g}(\Delta^2) \right) e^{-i\Delta \cdot \mathbf{r}}$$

- ✧ $D_q(\Delta^2)$ and $\bar{c}_q(\Delta^2)$ form factors describe internal stresses!

Cauchy momentum equation gives average force felt by quarks or gluons:

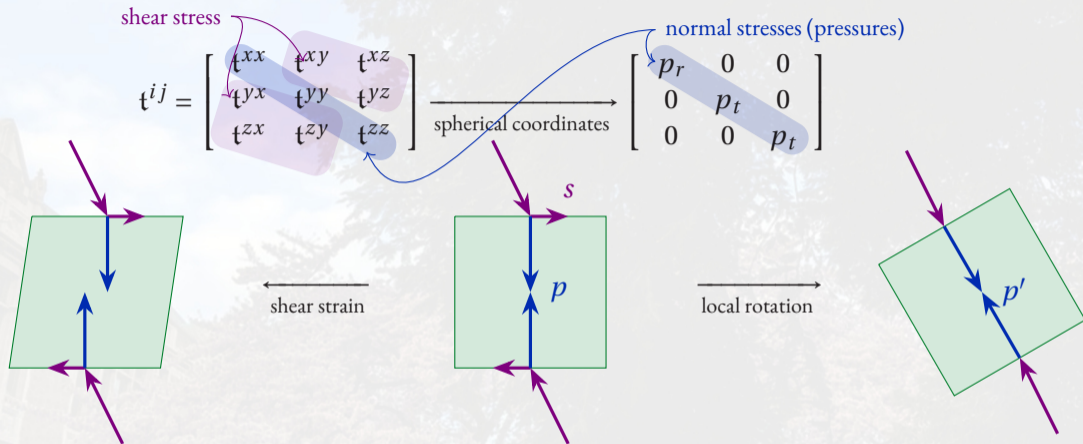
$$\langle f_{q,g}^j(\mathbf{r}) \rangle = -m \nabla_j \int \frac{d^3\Delta}{(2\pi)^3} \bar{c}_{q,g}(\Delta^2) e^{-i\Delta \cdot \mathbf{r}}$$

Polyakov, *Phys Lett B* 555 (2003) 57

Polyakov & Schweitzer, *Int J Mod Phys A* 33 (2018) 1830025

Polyakov & Son, *JHEP* 09 (2018) 156

Internal stresses: Cartesian vs. spherical coordinates



- Normal stresses induce compression or expansion (tension)
- Shear stresses cause medium to shear
- Shear stresses eliminated by diagonalization—get **principal stresses**

Simple pressure model

Start with a closed system (sum quarks & gluons—no \bar{c}):

$$t^{ij}(\mathbf{r}) = \int \frac{d^3\Delta}{(2\pi)^3} \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4m} D(\Delta^2) e^{-i\Delta \cdot \mathbf{r}} = \frac{-\nabla_i \nabla_j + \delta_{ij} \nabla^2}{4m} \underbrace{\int \frac{d^3\Delta}{(2\pi)^3} D(\Delta^2) e^{-i\Delta \cdot \mathbf{r}}}_{\equiv \mathcal{D}(r)}$$

Polyakov pressure potential

Simple tripole (denominator cubed) model:

$$D(\Delta^2) = \frac{D_0}{(1 + \Delta^2/\Lambda^2)^3} \quad \Rightarrow \quad \mathcal{D}(r) = \frac{D_0 \Lambda^3}{32\pi} (1 + r\Lambda) e^{-r\Lambda}$$

◆ Pretend this is a homework problem

(Hint: Fourier transform a dipole first, then differentiate with respect to Λ)

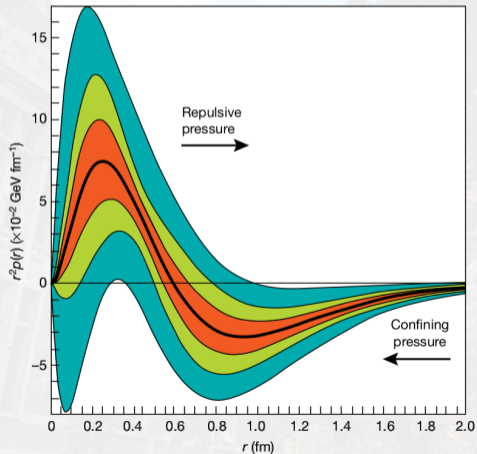
◆ Tripole from perturbative QCD; D_0 & Λ from experiment or lattice

Stress tensor becomes:

$$t^{ij}(\mathbf{r}) = -\frac{D_0 \Lambda^5}{128\pi m} \left\{ \underbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}_{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \underbrace{\delta^{ij}}_{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$

Isotropic pressure

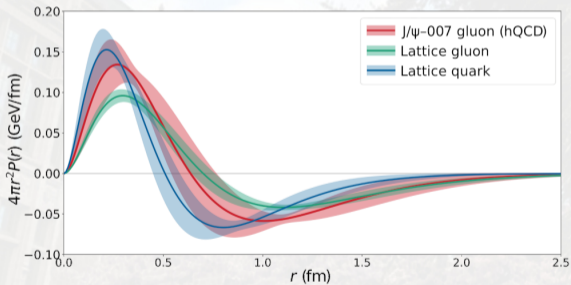
$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}^{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$



The JLab Nature paper

- 🌿 Burkert, Elouadrhiri & Girod, *Nature* (2018)
- 🌿 Assumed tripole form for $D(\Delta^2)$
- 🌿 Fit $D_0^{(q)} \approx -1.66$ from CLAS DVCS data
- 🌿 **Isotropic pressure:** average over directions
- 🌿 **Caution:** has only quarks, but missing $\bar{c}_q(\Delta^2)$

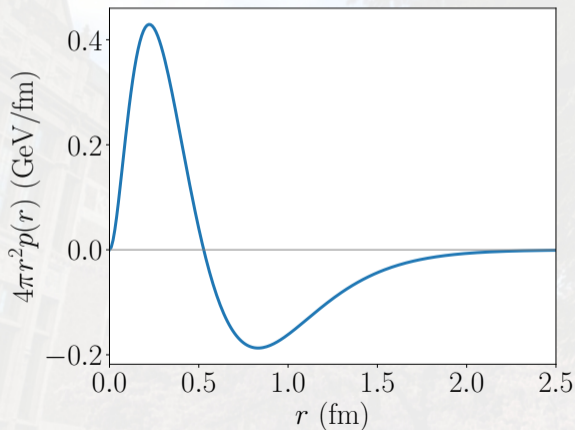
$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}^{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$



The other JLab Nature paper

- 🌿 Duran &al., [Nature \(2023\)](#)
- 🌿 Gets the gluon $D_g(\Delta^2)$!
- 🌿 $D_0^{(g)} \approx -1.8$, $\Lambda \approx 1.12$ GeV from Hall C data
(using Mamo-Zahed approach)
- 🌿 Figure from Joosten &al., [2602.14416](#)
(same experiment— J/ψ -007)

$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}^{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$



Combined tripole form

$$p(r) = -\frac{D_0\Lambda^5}{192\pi m} (3 - \Lambda r) e^{-\Lambda r}$$

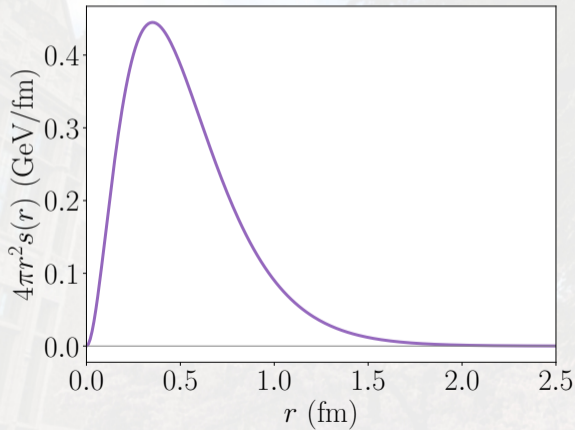
$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}^{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$



Combined tripole form

$$s(r) = -\frac{D_0\Lambda^5}{128\pi m} e^{-\Lambda r}$$

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

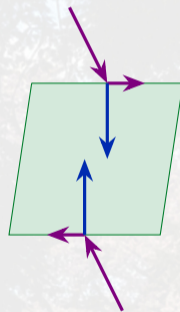
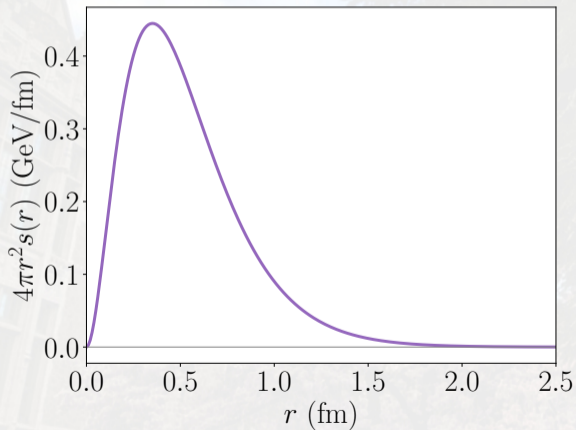
$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

Shear stress

$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{\left(\hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} \right)}^{\text{shear stresses}} \Lambda r + \frac{2}{3} (3 - \Lambda r) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} \right\} e^{-\Lambda r}$$



Recall: **shear stress** comes from misalignment with principal axes

Radial pressure

$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{2\hat{r}^i\hat{r}^j}^{\text{radial stress}} + (2-\Lambda r) \overbrace{(\delta^{ij} - \hat{r}^i\hat{r}^j)}^{\text{tangential stresses}} \right\} e^{-\Lambda r}$$

Combined tripole form

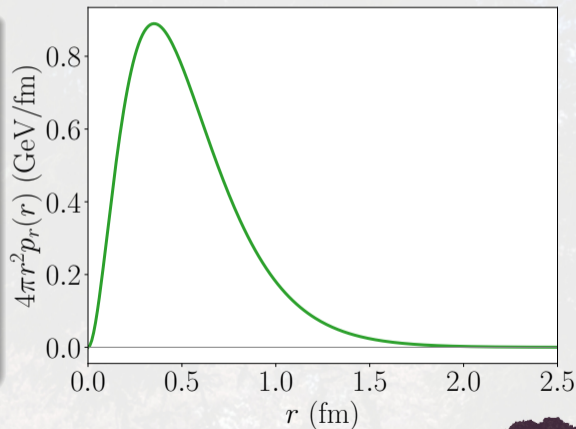
$$p_r(r) = -\frac{D_0\Lambda^5}{64\pi m} e^{-\Lambda r}$$

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)



🌿 **Radial pressure** is strictly positive in the proton (but not in all systems)

Tangential pressure and tension

$$t^{ij}(\mathbf{r}) = -\frac{D_0\Lambda^5}{128\pi m} \left\{ \overbrace{2\hat{r}^i\hat{r}^j}^{\text{radial stress}} + (2-\Lambda r) \overbrace{(\delta^{ij} - \hat{r}^i\hat{r}^j)}^{\text{tangential stresses}} \right\} e^{-\Lambda r}$$

Combined tripole form

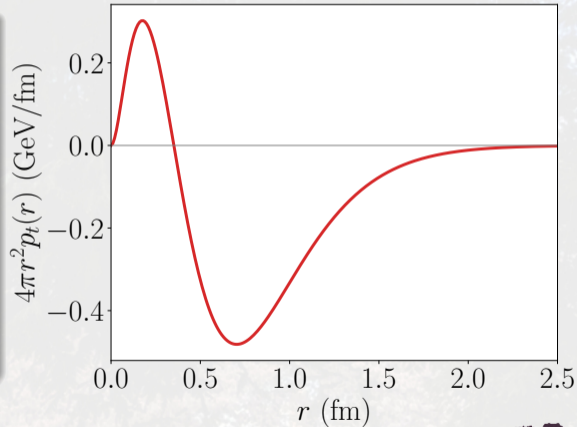
$$p_t(r) = -\frac{D_0\Lambda^5}{128\pi m} (2-\Lambda r) e^{-\Lambda r}$$

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

$$\Lambda = 1.12 \text{ GeV}$$

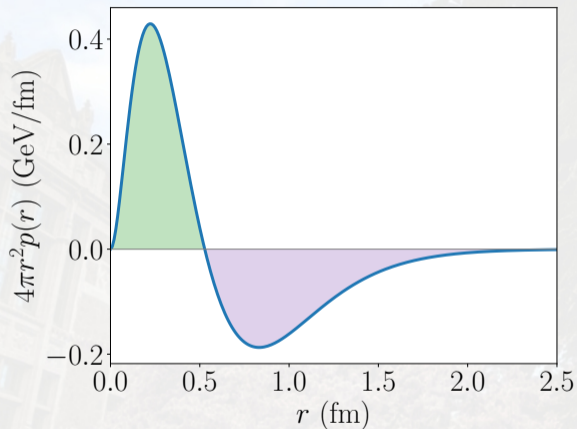
Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)



- Diffuse **tangential tension**
- Proton is like a liquid drop with a *very* fuzzy, diffuse boundary

Stability conditions: von Laue condition



von Laue condition

$$\int d^3r p(\mathbf{r}) = 0$$

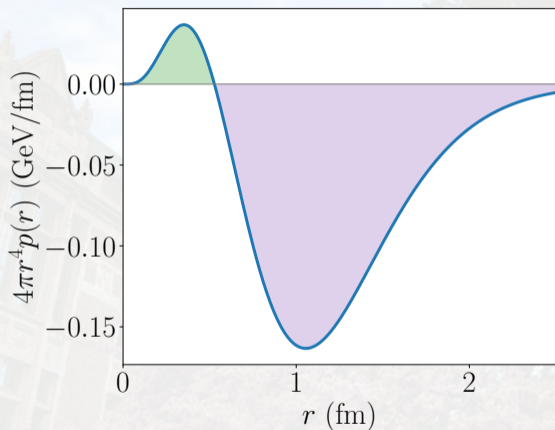
- ✓ Satisfied for every hadron
- ✓ Can be shown from:

$$\langle \mathbf{p} | T^{ij}(0) | \mathbf{p} \rangle = 0$$

von Laue, *Annalen Phys* 340 (1911) 524

Polyakov & Schweitzer, *Int J Mod Phys A* 33 (2018) 1830025

Stability conditions: Polyakov's conjecture



Polyakov's conjecture

$$D(0) = m \int d^3 r r^2 p(r) < 0$$

- Seems to be true for hadrons
- Seems intuitive:
 - Compression against collapse
 - Tension against disintegration
- Is false for some systems!**

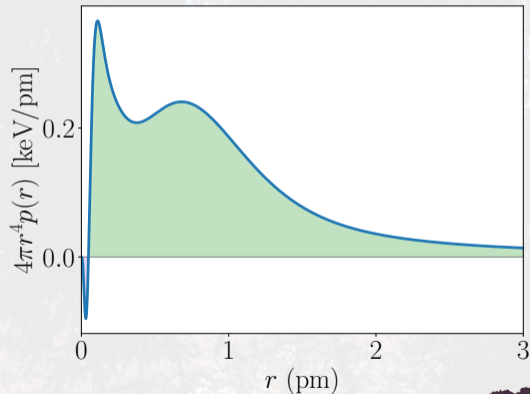
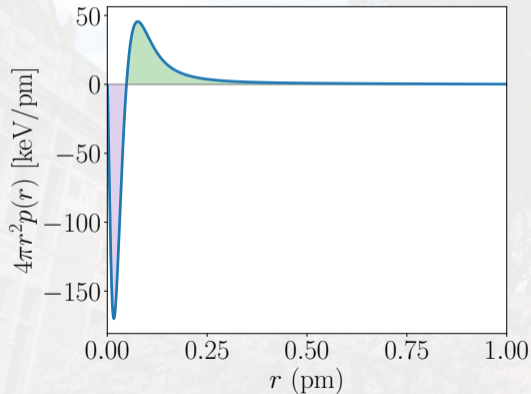
Perevalova, Polyakov & Schweitzer, *PRD* 94 (2016) 054024

Polyakov & Schweitzer, *Int J Mod Phys A* 33 (2018) 1830025

Sign of the D-term

Hydrogen atom

$$D(0) = \frac{M}{\mu} - 1 > 0$$



Ji, Yang & Liu, *PRD* 110 (2024) 114045

AF, *Phys Rev D* 111 (2025) 034047

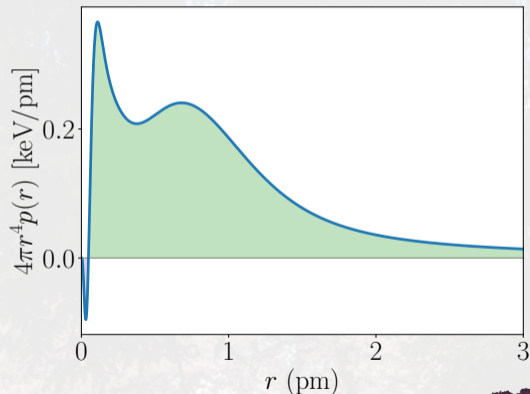
Sign of the D-term

Hydrogen atom

$$D(0) = \frac{M}{\mu} - 1 > 0$$

Other counterexamples

- ☛ Photon
AF & Cosyn, [PRD 106 \(2022\) 114014](#)
- ☛ Electron in QED
Metz, Pasquini & Rodini, [PLB 820 \(2021\) 136501](#)
- ☛ Proton with QED turned on!
Mejia & Schweitzer, [PRD 113 \(2026\) 054016](#)



Ji, Yang & Liu, [PRD 110 \(2024\) 114045](#)

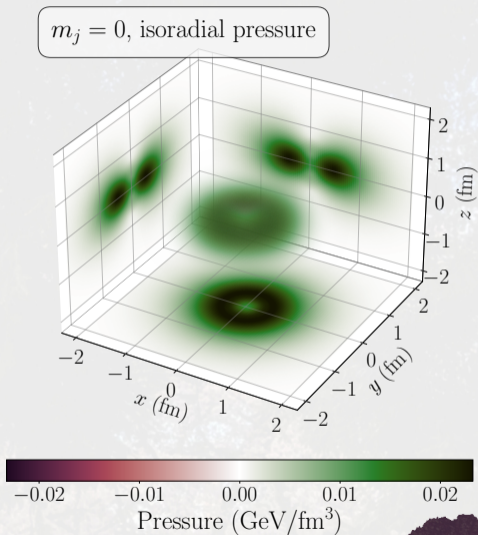
AF, [Phys Rev D 111 \(2025\) 034047](#)

A close-up photograph of a tree branch covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and others as buds. The background is a soft-focus view of a large tree in full cherry blossom, creating a dreamy, ethereal atmosphere. The text 'Stress in the Deuteron' is written in a white, elegant cursive font across the center of the mossy branch.

Stress in the Deuteron

Why stress in the deuteron?

- Deuteron is a non-relativistic system
 - Can put off debate about relativistic densities
 - Can use old-fashioned quantum mechanics
- Spin one – richer stress tensor
- Connection to nuclear force
- Visualizations are really cool



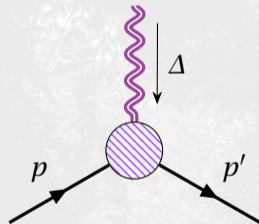
Mechanical form factors: spin-zero

- Energy-momentum tensor parametrized using **mechanical form factors** (MFFs)

$$\langle \mathbf{p}' | \hat{T}_{q,g}^{\mu\nu}(0) | \mathbf{p} \rangle = 2m \left\{ \frac{P^\mu P^\nu}{m} A_{q,g}(\Delta^2) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4m} D_{q,g}(\Delta^2) + m g^{\mu\nu} \bar{c}_{q,g}(\Delta^2) \right\}$$

- General form from Lorentz covariance
- MFFs encode hadron structure

$$P^\mu = \frac{1}{2}(p + p')^\mu$$
$$\Delta^\mu = (p' - p)^\mu$$



Kobzarev & Okun, *Sov Phys JETP* 16 (1963) 1343

H. Pagels, *Phys Rev* 144 (1966) 1250

X. Ji, *PRL* 78 (1997) 610

Mechanical form factors: spin-half

Spin-half has **new vector-polarized/dipole** Mechanical form factors:

$$\langle \mathbf{p}', s' | \hat{T}_{q,g}^{\mu\nu}(0) | \mathbf{p}, s \rangle = \bar{u}(\mathbf{p}', s') \left\{ \frac{P^\mu P^\nu}{m} A_{q,g}(\Delta^2) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu} \Delta^2}{4m} D_{q,g}(\Delta^2) + m \bar{c}_{q,g}(\Delta^2) \right. \\ \left. + \frac{i P^{[\mu} \sigma^{\nu]\Delta}}{2m} J_{q,g}(\Delta^2) - \underbrace{\frac{i P^{[\mu} \sigma^{\nu]\Delta}}{2m} S_{q,g}(\Delta^2)}_{\text{antisymmetric structure}} \right\} u(\mathbf{p}, s)$$

✦ **New structures** related to spin/angular momentum

✦ **Antisymmetric part** related to elementary fermion spin

Spin-half Non-relativistic formula (for stress tensor):

$$\langle \mathbf{p}', s' | \hat{T}_q^{ab}(0) | \mathbf{p}, s \rangle = \frac{P^a P^b}{m} A_q(\Delta^2) \delta_{s's} + \frac{\Delta^a \Delta^b - \delta^{ab} \Delta^2}{4m} D_q(\Delta^2) \delta_{s's} - m \delta^{ab} \bar{c}_q(\Delta^2) \delta_{s's} \\ - \frac{i(\Delta \times \boldsymbol{\sigma}_{s's})^{\{a} P^{b\}}}{2m} J_q(\Delta^2) - \frac{i(\Delta \times \boldsymbol{\sigma}_{s's})^{[a} P^{b]}}{2m} S_q(\Delta^2)$$

Form factor breakdown: spin-one

$$\langle p', s'_d | \hat{T}^{ij}(0) | p, s_d \rangle = M_d \varepsilon_a \varepsilon_b'^* \left\{ \begin{aligned} & \frac{P^i P^j}{M_d^2} \left[\delta^{ab} A_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} A_T(\Delta^2) \right] + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a] } P^{j]} J(\Delta^2) \right. \\ & + \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M_d^2} \left[\delta^{ab} D_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} D_{T1}(\Delta^2) \right] - \delta^{ij} \left[\delta^{ab} \bar{c}_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} \bar{c}_{T1}(\Delta^2) \right] \\ & + \frac{1}{2M_d^2} \left[Q^{jlab} Y_2^{il}(\hat{\Delta}) + Q^{liab} Y_2^{lj}(\hat{\Delta}) - Q^{klab} Y_2^{kl}(\hat{\Delta}) \delta^{ij} - \frac{1}{3} Q^{ijab} \right] D_{T2}(\Delta^2) - Q^{ijab} \bar{c}_{T2}(\Delta^2) \\ & \left. + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a] } P^{j]} S(\Delta^2) + \frac{\Delta^2}{4M_d^2} \left(\delta^{a[i} Y_2^{j]b}(\hat{\Delta}) + \delta^{b[i} Y_2^{j]a}(\hat{\Delta}) \right) \bar{s}(\Delta^2) \right) \right\} \\ & \underbrace{\hspace{15em}}_{\text{antisymmetric structure}}
 \end{aligned} \right.$$

- 🍃 Unpolarized/monopole ($j \geq 0$)
- 🍃 vector-polarized/dipole ($j \geq \frac{1}{2}$)
- 🍃 tensor-polarized/quadrupole ($j \geq 1$)

$$Y_2^{ij}(\hat{r}) = \hat{r}^i \hat{r}^j - \frac{1}{3} \delta^{ij} : \text{harmonic tensor}$$

$$Q^{ijab} = \frac{1}{3} \delta^{ij} \delta^{ab} - \frac{1}{2} \delta^{a[i} \delta^{j]b} : \text{quadrupole tensor}$$

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$$\langle p', s'_d | \hat{T}^{ij}(0) | p, s_d \rangle = M_d \varepsilon_a \varepsilon_b'^* \left\{ \begin{aligned} & \frac{P^i P^j}{M_d^2} \left[\delta^{ab} A_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} A_T(\Delta^2) \right] + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} J(\Delta^2) \\ & + \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M_d^2} \left[\delta^{ab} D_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} D_{T1}(\Delta^2) \right] - \delta^{ij} \left[\delta^{ab} \bar{c}_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} \bar{c}_{T1}(\Delta^2) \right] \\ & + \frac{1}{2M_d^2} \left[Q^{jlab} Y_2^{il}(\hat{\Delta}) + Q^{liab} Y_2^{lj}(\hat{\Delta}) - Q^{klab} Y_2^{kl}(\hat{\Delta}) \delta^{ij} - \frac{1}{3} Q^{ijab} \right] D_{T2}(\Delta^2) - Q^{ijab} \bar{c}_{T2}(\Delta^2) \\ & \underbrace{+ \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} S(\Delta^2) + \frac{\Delta^2}{4M_d^2} \left(\delta^{a[i} Y_2^{j]b}(\hat{\Delta}) + \delta^{b[i} Y_2^{j]a}(\hat{\Delta}) \right) \bar{s}(\Delta^2)}_{\text{antisymmetric structure}} \end{aligned} \right\}$$

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Form factor breakdown: spin-one

$$\langle p', s'_d | \hat{T}^{ij}(0) | p, s_d \rangle = M_d \varepsilon_a \varepsilon_b'^* \left\{ \begin{aligned} & \frac{P^i P^j}{M_d^2} \left[\delta^{ab} A_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} A_T(\Delta^2) \right] + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} J(\Delta^2) \\ & + \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M_d^2} \left[\delta^{ab} D_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} D_{T1}(\Delta^2) \right] - \delta^{ij} \left[\delta^{ab} \bar{c}_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} \bar{c}_{T1}(\Delta^2) \right] \\ & + \frac{1}{2M_d^2} \left[Q^{jlab} Y_2^{il}(\hat{\Delta}) + Q^{liab} Y_2^{lj}(\hat{\Delta}) - Q^{klab} Y_2^{kl}(\hat{\Delta}) \delta^{ij} - \frac{1}{3} Q^{ijab} \right] D_{T2}(\Delta^2) - Q^{ijab} \bar{c}_{T2}(\Delta^2) \\ & \underbrace{+ \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} S(\Delta^2) + \frac{\Delta^2}{4M_d^2} \left(\delta^{a[i} Y_2^{j]b}(\hat{\Delta}) + \delta^{b[i} Y_2^{j]a}(\hat{\Delta}) \right) \bar{s}(\Delta^2)}_{\text{antisymmetric structure}} \end{aligned} \right\}$$

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$$\langle p', s'_d | \hat{T}^{ij}(0) | p, s_d \rangle = M_d \varepsilon_a \varepsilon'_b \left\{ \right.$$

$$\frac{P^i P^j}{M_d^2} \left[\delta^{ab} A_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} A_T(\Delta^2) \right] + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} J(\Delta^2)$$

$$+ \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M_d^2} \left[\delta^{ab} D_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} D_{T1}(\Delta^2) \right] - \delta^{ij} \left[\delta^{ab} \bar{c}_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} \bar{c}_{T1}(\Delta^2) \right]$$

$$+ \frac{1}{2M_d^2} \left[Q^{jlab} Y_2^{il}(\hat{\Delta}) + Q^{liab} Y_2^{lj}(\hat{\Delta}) - Q^{klab} Y_2^{kl}(\hat{\Delta}) \delta^{ij} - \frac{1}{3} Q^{ijab} \right] D_{T2}(\Delta^2) - Q^{ijab} \bar{c}_{T2}(\Delta^2)$$

$$\left. + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} S(\Delta^2) + \frac{\Delta^2}{4M_d^2} \left(\delta^{a[i} Y_2^{j]b}(\hat{\Delta}) + \delta^{b[i} Y_2^{j]a}(\hat{\Delta}) \right) \bar{s}(\Delta^2) \right\}$$

antisymmetric structure

- 🍃 A-like — mass density
- 🍃 J/S-like — spin density
- 🍃 D-like — pressures
- 🍃 c̄-like — forces

Form factor breakdown: spin-one

$$\langle p', s'_d | \hat{T}^{ij}(0) | p, s_d \rangle = M_d \varepsilon_a \varepsilon_b'^* \left\{ \begin{aligned} & \frac{P^i P^j}{M_d^2} \left[\delta^{ab} A_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} A_T(\Delta^2) \right] + \frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} J(\Delta^2) \\ & + \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4M_d^2} \left[\delta^{ab} D_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} D_{T1}(\Delta^2) \right] - \delta^{ij} \left[\delta^{ab} \bar{c}_U(\Delta^2) + Y_2^{ab}(\hat{\Delta}) \frac{\Delta^2}{2M_d^2} \bar{c}_{T1}(\Delta^2) \right] \\ & + \frac{1}{2M_d^2} \left[Q^{jlab} Y_2^{il}(\hat{\Delta}) + Q^{liab} Y_2^{lj}(\hat{\Delta}) - Q^{klab} Y_2^{kl}(\hat{\Delta}) \delta^{ij} - \frac{1}{3} Q^{ijab} \right] D_{T2}(\Delta^2) - Q^{ijab} \bar{c}_{T2}(\Delta^2) \\ & + \underbrace{\frac{1}{2M_d^2} \left(\delta^{a[i} \Delta^b - \delta^{b[i} \Delta^a \right) P^{j]} S(\Delta^2) + \frac{\Delta^2}{4M_d^2} \left(\delta^{a[i} Y_2^{j]b}(\hat{\Delta}) + \delta^{b[i} Y_2^{j]a}(\hat{\Delta}) \right) \bar{s}(\Delta^2)}_{\text{antisymmetric structure}} \end{aligned} \right\}$$

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🍃 D-like — pressures

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antisymmetric structure

- 🍃 A-like — mass density
- 🍃 J/S-like — spin density
- 🍃 D-like — pressures
- 🍃 \bar{c} -like — forces

🍃 \bar{c} -like give forces via Cauchy momentum equation:

$$f^j = \nabla_i T^{ij} = \bar{c}\text{-like terms}$$

Sum rules

☛ Momentum sum rule:

$$\sum_{\text{constituents}} A_U^{(n)}(0) = 1$$

☛ Angular momentum sum rule (generalized Ji sum rule):

$$\sum_{\text{constituents}} J^{(n)}(0) = 1$$

☛ Local momentum conservation:

$$\sum_n \bar{c}_U^{(n)}(\Delta^2) = \sum_n \bar{c}_{T1}^{(n)}(\Delta^2) = \sum_n \bar{c}_{T2}^{(n)}(\Delta^2) = \sum_n \bar{s}^{(n)}(\Delta^2) = 0$$

✦ **Violated for open sub-systems**

✦ Non-zero \bar{c} -like form factors **quantify forces**—Cauchy momentum equation!

$$\langle f_{s's}^j(\mathbf{b}) \rangle = \nabla_i \int \frac{d^3\Delta}{(2\pi)^3} \langle \mathbf{p}', s' | \hat{T}^{ij}(0) | \mathbf{p}, s \rangle e^{-i\Delta \cdot \mathbf{b}} \sim \bar{c}\text{-like terms}$$

✦ Called **non-conserved form factors** — quantify local conservation violation

Nucleon contributions to the deuteron

- Stress tensor breaks down into **one-body** and **two-body** contributions

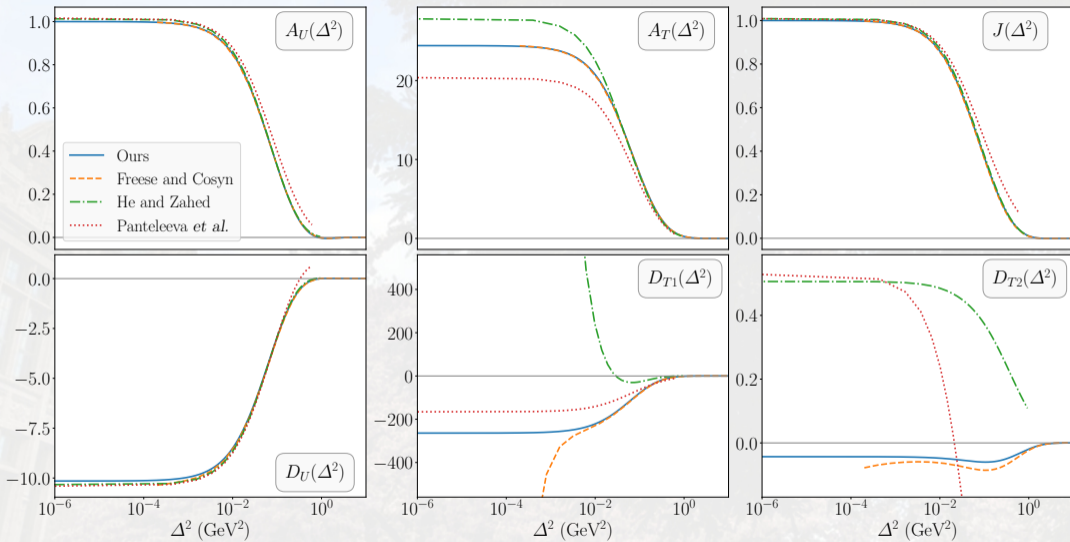
$$\hat{T}^{ij} = \underbrace{\hat{T}_p^{ij} + \hat{T}_n^{ij}}_{\text{one-body (impulse approximation)}} + \underbrace{\hat{T}_{\text{interaction}}^{ij}}_{\text{interaction contribution}}$$

- Only considering **one-body contributions** for now
 - Nucleons are an open sub-system of the deuteron** (neglects pions etc.)
 - Interaction contributions** (future work) include pion exchange, etc.
- Matrix element evaluated via two-particle completeness relations:

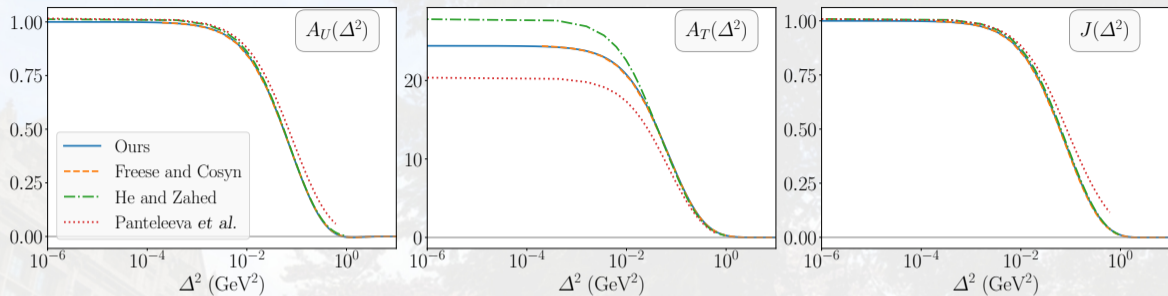
$$\langle \mathbf{p}'_d, s'_d | \hat{T}_p^{ij} | \mathbf{p}_d, s_d \rangle = \sum_{s_p, s_n, s'_p} \int d^3 r e^{i \frac{\mathbf{A} \cdot \mathbf{r}}{2}} \psi_d^{*(s'_d; s'_p, s_n)}(\mathbf{r}) \underbrace{\langle \mathbf{p}'_p, s'_p | \hat{T}_p^{ij} | \mathbf{p}_p, s_p \rangle}_{\text{nucleon stress tensor}} \psi_d^{(s_d; s_p, s_n)}(\mathbf{r}) \Big|_{\mathbf{P}_p = \frac{1}{2}(\mathbf{P}_d - i \vec{\nabla})}$$

- Use non-relativistic spin-half breakdown formula for nucleon matrix element
- Use realistic nucleon MFFs from lattice, holography, meson dominance, ...

Symmetric conserved form factors



Group comparisons: A-like form factors



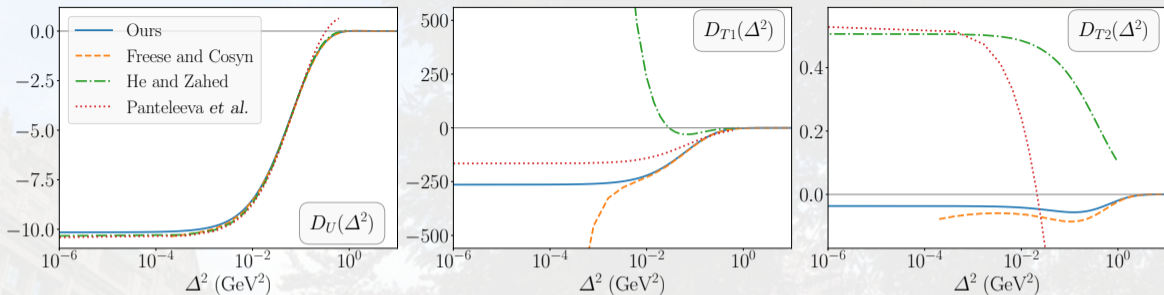
🍃 $A_U(0) = 1$ — momentum sum rule

🍃 $J(0) = 1$ — spin sum rule

🍃 $A_T(0) = (2m_N)^2 Q_d \approx 25$ — quadrupole moment

❖ Different wave functions \Rightarrow different quadrupole moments

Group comparisons: D -like form factors



🍃 $D_U(\Delta^2)$ all agree

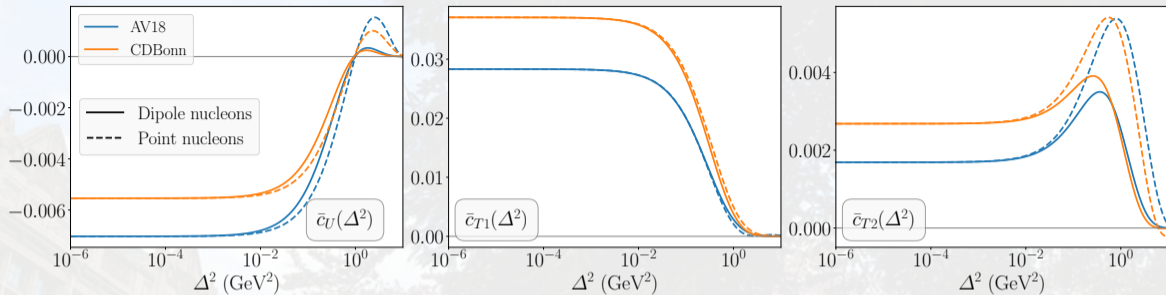
🍃 $D_{T1}(0) \approx 4D_N(0)(2m_N)^2Q_d$ — is finite

- ✦ We agree with Panteleeva *et al.* (but have different Q_d)
- ✦ Older Freese & Cosyn result unstable for very small Δ^2
- ✦ Blowup in He & Zahed from their substitution:

$$P_N^\mu \rightarrow P_N^\mu - \frac{(\Delta \cdot P_N)}{\Delta^2} \Delta^\mu$$

🍃 $D_{T2}(\Delta^2)$ all disagree — sensitive to minor differences in dynamics

Non-conserved form factors



🌿 New calculation

🌿 Quantifies nuclear force on nucleons

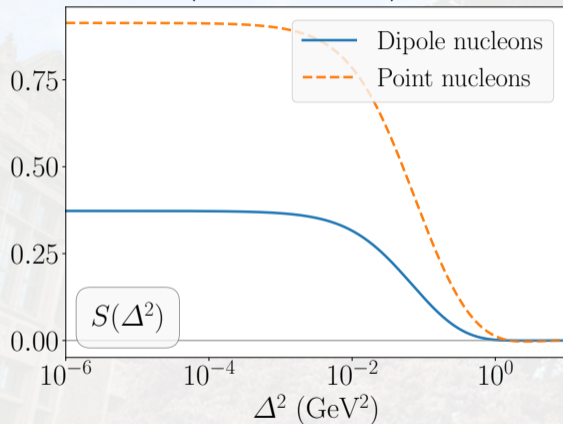
✦ Depends on deuteron wave function

🌿 Incorporates finite nuclear size (smearing)

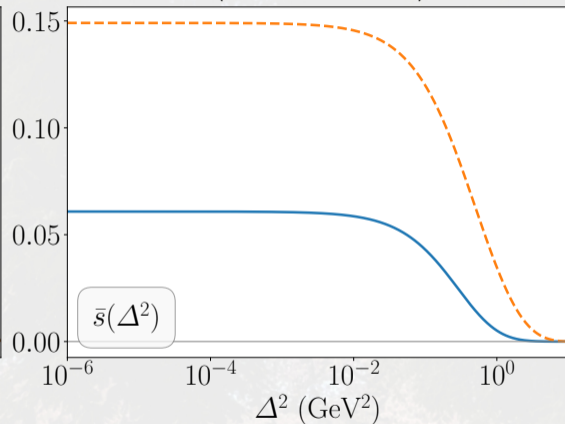
✦ Depends on nucleon form factors

Antisymmetric form factors

$S(\Delta^2)$ quantifies spin density
(via Fourier transform)



$\bar{s}(\Delta^2)$ quantifies torsion stress
(via Fourier transform)



Mass density: formulas

Mass density given by 3D Fourier transform:

$$\mathbf{a}_{s's}(\mathbf{b}) = \varepsilon_a \varepsilon_b'^* M_d \int \frac{d^3 \Delta}{(2\pi)^3} \left[\delta^{ab} A_U(\Delta^2) + \frac{\Delta^2}{2M_d^2} Y_2^{ab}(\hat{\Delta}) A_T(\Delta^2) \right] e^{-i\Delta \cdot \mathbf{b}}$$

Can be turned into 1D Bessel transforms:

$$\mathbf{a}_U(\mathbf{b}) = \frac{M_d}{2\pi^2} \int_0^\infty d\Delta \Delta^2 A_U(\Delta^2) j_0(b\Delta)$$

$$\mathbf{a}_T(\mathbf{b}) = \frac{M_d}{2\pi^2} \int_0^\infty d\Delta \frac{\Delta^4}{M_d^2} A_T(\Delta^2) j_2(b\Delta)$$

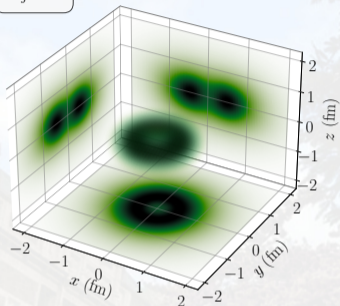
Mixtures of U and T give specific pure states:

$$\mathbf{a}_{m_j=\pm 1}(\mathbf{b}) = \mathbf{a}_U(\mathbf{b}) + \frac{1}{6} \left(\frac{3}{2} \cos^2(\theta_b) - \frac{1}{2} \right) \mathbf{a}_T(\mathbf{b})$$

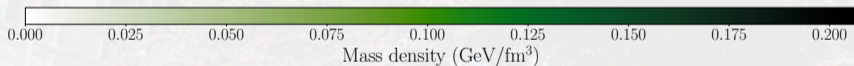
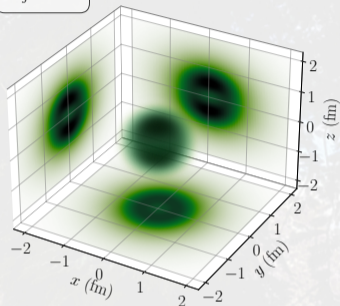
$$\mathbf{a}_{m_j=0}(\mathbf{b}) = \mathbf{a}_U(\mathbf{b}) - \frac{1}{3} \left(\frac{3}{2} \cos^2(\theta_b) - \frac{1}{2} \right) \mathbf{a}_T(\mathbf{b})$$

Mass density: 3D plots

$m_j = 0$



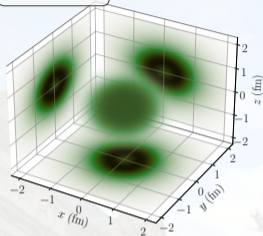
$m_j = \pm 1$



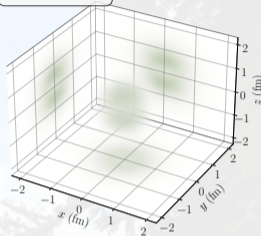
- $m_j = 0$ state has famous **donut shape**
- $m_j = \pm 1$ has slight dumbbell-like shape
- Shapes arise from quantum interference of *S* and *D* waves

Mass density: breakdown

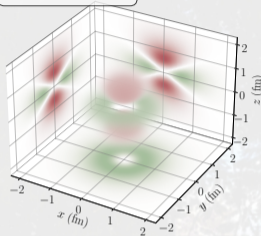
S-wave ($m_j = 0$)



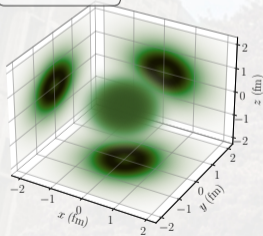
D-wave ($m_j = 0$)



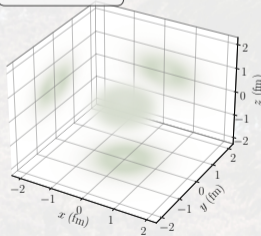
Interference ($m_j = 0$)



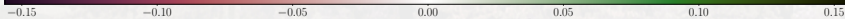
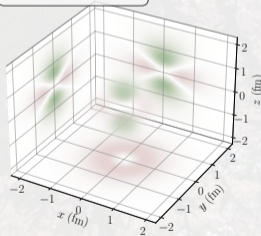
S-wave ($m_j = \pm 1$)



D-wave ($m_j = \pm 1$)



Interference ($m_j = \pm 1$)



Mass density contribution (GeV/fm³)

Momentum & mass flux densities: formulas

☛ Momentum & mass flux densities — as Bessel transforms

$$\mathbf{p}_{m_j=\pm 1}(\mathbf{b}) = \pm \frac{\hat{z} \times \hat{b}}{4\pi^2} \int_0^\infty d\Delta \Delta^3 (J(\Delta^2) - S(\Delta^2)) j_1(b\Delta)$$

$$\mathbf{f}_{m_j=\pm 1}(\mathbf{b}) = \pm \frac{\hat{z} \times \hat{b}}{4\pi^2} \int_0^\infty d\Delta \Delta^3 (J(\Delta^2) + S(\Delta^2)) j_1(b\Delta)$$

☛ Mass flux \neq momentum—for **asymmetric EMT**

☛ Momentum density has $J_N(\Delta^2) - S_N(\Delta^2)$

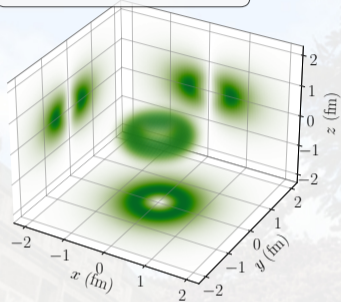
✧ Quark spin doesn't carry momentum

☛ Mass flux density has $J_N(\Delta^2) + S_N(\Delta^2)$

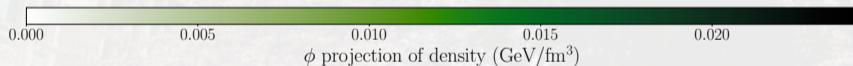
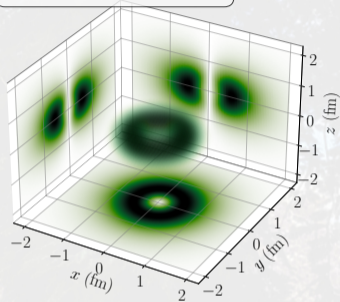
✧ Quark spin *does* transport mass (and energy)

Momentum & mass flux densities: 3D plots

Momentum density ($m_j = 1$)

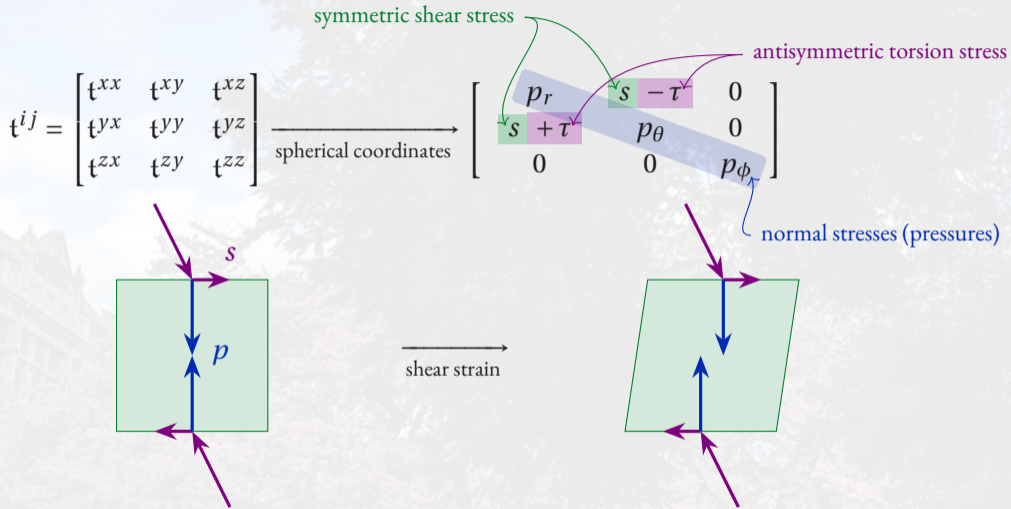


Mass flux density ($m_j = 1$)



- Momentum & mass flux concentrated in D -wave
- Mass flux \neq momentum—for **asymmetric EMT**
- Mass flux $>$ momentum—quark spin carries mass & energy flux without momentum

Internal stresses: Cartesian vs. spherical coordinates



- Normal stress induces compression or expansion (tension)
- Shear stress causes medium to shear

Principal stresses

☛ Symmetric part of stress tensor can be diagonalized

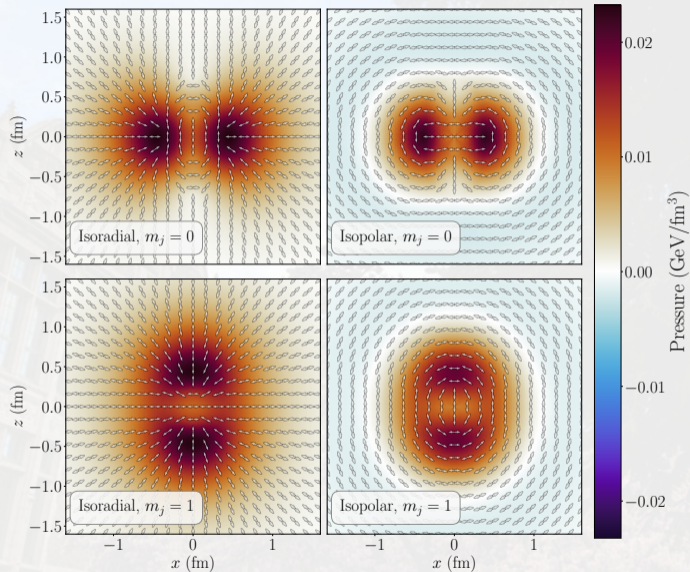
$$\begin{bmatrix} p_r & s & -\tau & 0 \\ s & +\tau & p_\theta & 0 \\ 0 & 0 & 0 & p_\phi \end{bmatrix} \xrightarrow{\text{local rotation}} \begin{bmatrix} p'_r & -\tau & 0 \\ +\tau & p'_\theta & 0 \\ 0 & 0 & p_\phi \end{bmatrix}$$

✦ Diagonalized stresses are **principal stresses**

☛ Amounts to local change of frame — eliminates shear stresses



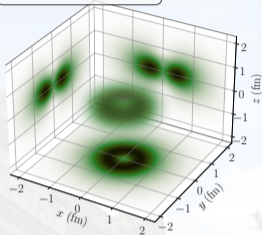
Principal axes of the stress tensor



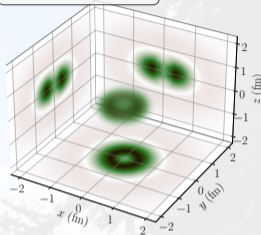
- Local frame deformed from spherical coordinates
- Deformation follows deuteron (donut/dumbbell) shape
- Isoradial principal stress (radial-like)
 - Positive (compressive)
 - Like radial pressure in proton
- Isopolar principal stress (polar-like)
 - Compressive near center
 - Tensile further out
 - Diffuse surface tension

Principal stresses: 3D plots

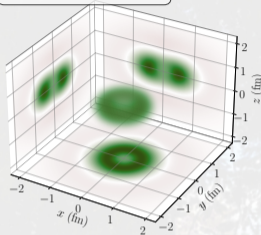
$m_j = 0$, isoradial pressure



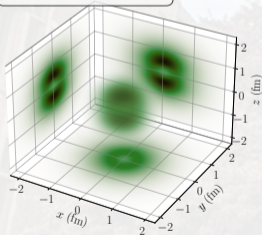
$m_j = 0$, isopolar pressure



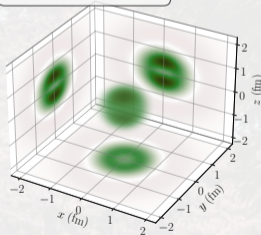
$m_j = 0$, azimuthal pressure



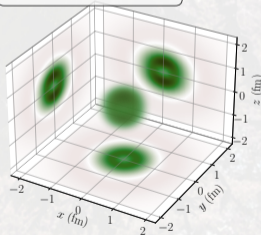
$m_j = \pm 1$, isoradial pressure



$m_j = \pm 1$, isopolar pressure



$m_j = \pm 1$, azimuthal pressure



-0.02

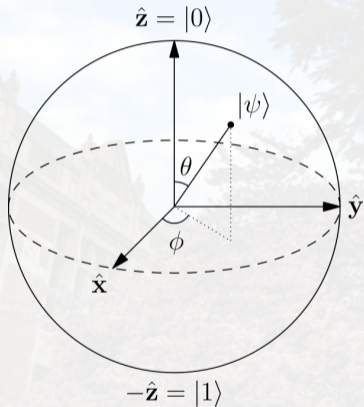
-0.01

0.00

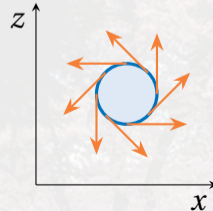
0.01

0.02

Pressure (GeV/fm^3)

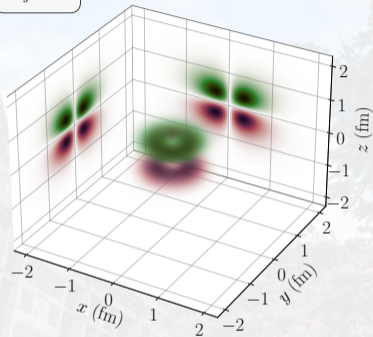


- Antisymmetric stresses **cannot** be diagonalized
- Antisymmetric stresses are non-normal
 - These stresses are *circulating*
 - Induce **torsion** strain
- Caused by reorientation of quark spins
 - Nucleon spin flips from *S*- to *D*-wave
 - Spin must continuously move through Bloch sphere

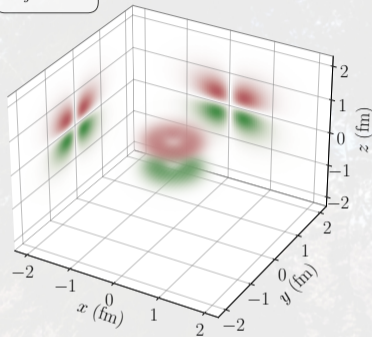


Torsion stress: 3D plot

$$m_j = 0$$



$$m_j = \pm 1$$



-0.0010

-0.0005

0.0000

0.0005

0.0010

Torsion stress (GeV/fm³)

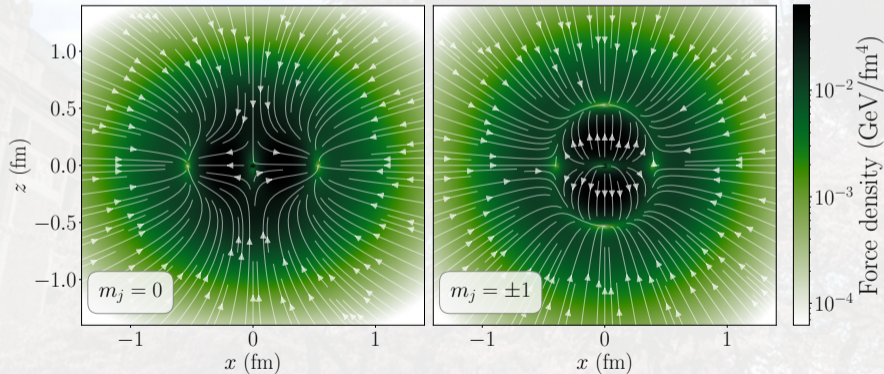
- Present at boundary of D -wave
- Reorients nucleons from $m_s = +1$ in S -wave to $m_s = -1$ in D -wave
(in $m_j = +1$ state, as an example)

Force distributions

- ☛ Nucleon is an **open system**
- ☛ **Cauchy momentum equation** gives average force distribution on nucleon:

$$\langle f^j(\mathbf{b}) \rangle = \nabla_i \langle T_{1\text{-body}}^{ij}(\mathbf{b}) \rangle$$

- ☛ Non-radial tensor and spin-orbit forces



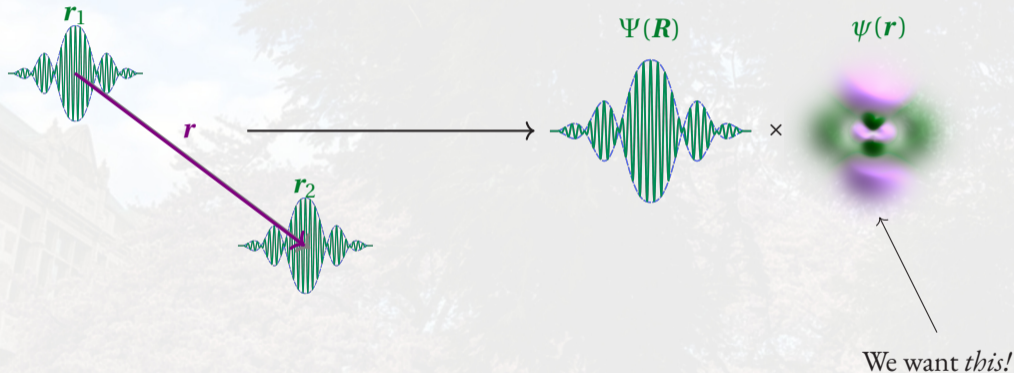
A close-up photograph of a tree trunk covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and some as buds. The background is a soft-focus view of a dense cherry blossom tree. The text 'Proton Densities' is written in a white, elegant cursive font, centered over the mossy trunk.

Proton Densities

Positions and wave packets

Quantum objects have two kinds of spatial extent:

- 1 Distance between constituents
- 2 Wave packet size



Reduce to overall wave packet and **internal structure**

$$\psi_{\text{total}}(\mathbf{r}_1, \mathbf{r}_2, t) = \Psi(\mathbf{R}, t)\psi(\mathbf{r}, t)$$

Internal structure is the interesting part

Imaging with Fourier transforms

- 🍃 We measure structure via scattering
- 🍃 This gives momentum info
- 🍃 Get position info with **Fourier transform**:

$$\psi(\mathbf{r}, t) = \int \frac{d^3 p}{(2\pi)^3} \tilde{\psi}(\mathbf{p}) e^{i(\mathbf{p} \cdot \mathbf{r} - Et)}$$

- 🍃 Only works when wave packets factorize:

$$\psi_{\text{total}}(\mathbf{r}_1, \mathbf{r}_2, t) = \Psi(\mathbf{R}, t) \psi(\mathbf{r}, t)$$

- 🍃 **Relativity** makes this break down

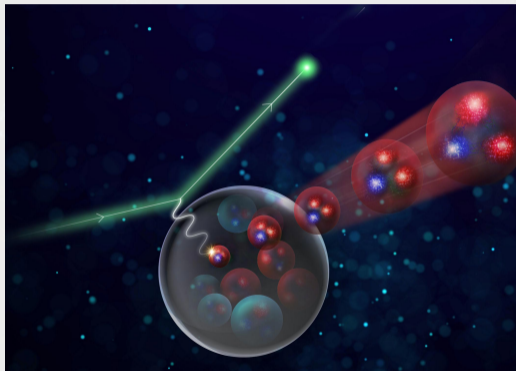


Image credit: Jefferson Lab

Dynamic and internal stress: non-relativistic case

$$\langle \Psi | \hat{T}^{ij}(\mathbf{x}, t) | \Psi \rangle = \sum_{s, s'} \int d^3 R \left\{ \overbrace{-\Psi_{s'}^*(\mathbf{R}, t) \frac{\vec{\nabla}^i \vec{\nabla}^j}{4m^2} \Psi_s(\mathbf{R}, t)}^{\text{dynamic pressures}} \underbrace{\alpha_{s's}(\mathbf{x} - \mathbf{R})}_{\text{internal mass density}} - \overbrace{\frac{i\Psi_{s'}^*(\mathbf{R}, t) \vec{\nabla}^i \Psi_s(\mathbf{R}, t)}{2m}}^{\text{barycentric flux}} \underbrace{p_{s's}^j(\mathbf{x} - \mathbf{R})}_{\text{internal momentum density}} \right. \\
 \left. - \overbrace{\frac{i\Psi_{s'}^*(\mathbf{R}, t) \vec{\nabla}^j \Psi_s(\mathbf{R}, t)}{2m}}^{\text{barycentric flux}} \underbrace{f_{s's}^i(\mathbf{x} - \mathbf{R})}_{\text{internal mass flux}} + \overbrace{\Psi_{s'}^*(\mathbf{R}, t) \Psi_s(\mathbf{R}, t) t_{s's}^{ij}(\mathbf{x} - \mathbf{R})}_{\text{barycentric probability}} \underbrace{t_{s's}^{ij}(\mathbf{x} - \mathbf{R})}_{\text{internal stresses}} \right\}$$



Image: dynamic pressure from wind

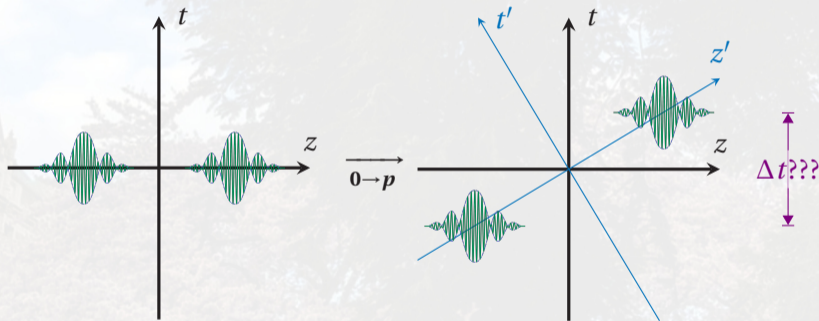
- 🍃 **Internal densities** smeared out by **barycentric distributions**
- 🍃 **Dynamic pressures** due to barycentric motion & wave packet dispersion
- 🍃 **Internal stress** due to internal motions *and* forces
 - ❖ Smeared by barycentric probability
- 🍃 **Strictly non-relativistic** breakdown
 - ❖ Afforded by Galilei symmetry (absoluteness of simultaneity)

AF, *Phys Rev D* 112 (2025) 034037

Cosyn, AF & Sosa, *Phys Rev C* 113 (2026) 055208

Relativity of simultaneity

- ✦ Momentum-space wave packet contains boosted versions of composite system
 - ✦ Boosts mix up planes of simultaneity
 - ✦ Internal constituents get boosted to **different times**



- ✦ Cannot decompose structure to overall wave packet \otimes internal structure at fixed t
- ✦ **Light front coordinates** fix this by defining a **new, boost-invariant time variable!**

Light front coordinates

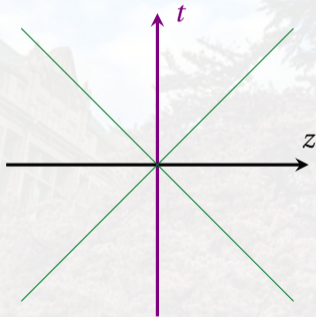
Light front coordinates reparametrize spacetime.

$$x^{\pm} = \frac{1}{\sqrt{2}} (t \pm z) \quad \mathbf{x}_{\perp} = (x, y)$$

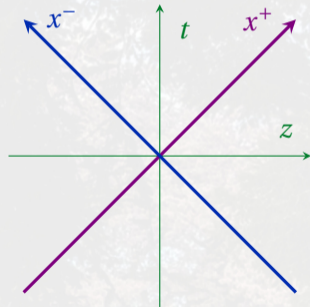
conventional factor

$$x^{+} = \frac{1}{\sqrt{2}} (t + z) \equiv \text{time}$$

Fixed-time surface given by light front moving in $-z$ direction.



Instant form coordinates



Light front coordinates

Coordinate conversions

☛ Contravariant (upper-index) four-vector:

$$(t; x, y, z) \rightarrow (x^0; x^1, x^2, x^3) \equiv x^\mu$$

☛ Conversion between instant form & light front coordinates:

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3)$$

$$x^- = \frac{1}{\sqrt{2}}(x^0 - x^3)$$

$$x^1 = x^1$$

$$x^2 = x^2$$

$$x^0 = \frac{1}{\sqrt{2}}(x^+ + x^-)$$

$$x^3 = \frac{1}{\sqrt{2}}(x^+ - x^-)$$

$$x^1 = x^1$$

$$x^2 = x^2$$

☛ Symmetric inversion formulas are why $\frac{1}{\sqrt{2}}$ is nice

P.A.M. Dirac, *Rev Mod Phys* 21 (1949) 392

AF, *A friendly introduction to the light front* (lecture series)

Four-products and the metric

🍃 Rule for four-products:

$$x \cdot y = x^0 y^0 - x^3 y^3 - x^1 y^1 - x^2 y^2$$

Four-products and the metric

🍃 Rule for four-products:

$$x \cdot y = \frac{1}{2}(x^+ + x^-)(y^+ + y^-) - \frac{1}{2}(x^+ - x^-)(y^+ - y^-) - x^1 y^1 - x^2 y^2$$

Four-products and the metric

☛ Rule for four-products:

$$x \cdot y = \frac{1}{2}(x^+ y^+ + x^- y^- + x^+ y^- + x^- y^+) - \frac{1}{2}(x^+ y^+ + x^- y^- - x^+ y^- - x^- y^+) - x^1 y^1 - x^2 y^2$$

Four-products and the metric

☛ Rule for four-products:

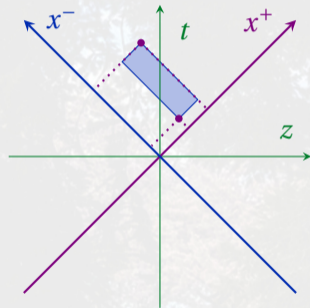
$$x \cdot y = x^+ y^- + x^- y^+ - \mathbf{x}_\perp \cdot \mathbf{y}_\perp$$

☛ Defines the metric in light front coordinates:

$$x \cdot y = g_{\mu\nu} x^\mu x^\nu$$

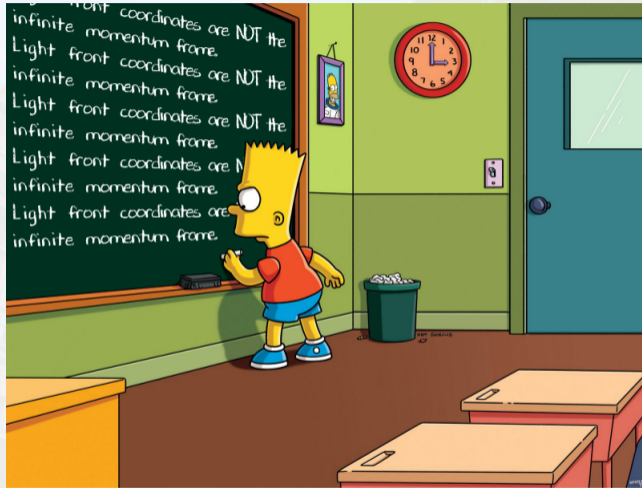
$$g_{\mu\nu}^{(\text{LF})} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- ✦ Metric is off-diagonal in x^+ and x^- !
- ✦ Normally list coordinates as $(x^+; x^1, x^2; x^-)$.



$$\Delta\tau^2 = 2\Delta x^+ \Delta x^-$$

Not the infinite momentum frame!



- Light front coordinates are valid in **any** frame.
 - They're not a reference frame.
- Light front coordinates are **not** the infinite-momentum frame.
 - A common misconception.
- You can't change the metric with boosts.
 - Lorentz transforms are isometries.
- Light front coordinates redefine **synchronization convention**.
 - What we mean by "simultaneous."

Equal-“time” surfaces are just a convention

- Relativity requires *round-trip* speed of light to be invariant.
- Convention that one-way speed of light be c is a *definition*, not an empirical fact.
 - ✦ Pointed out in Einstein’s original paper.
- Redefining “time” coordinate means changing this definition.
 - ✦ Light front coordinates do exactly this!

894

A. Einstein.

B durch einen in B befindlichen Beobachter möglich. Es ist aber ohne weitere Festsetzung nicht möglich, ein Ereignis in A mit einem Ereignis in B zeitlich zu vergleichen; wir haben bisher nur eine „ A -Zeit“ und eine „ B -Zeit“, aber keine für A und B gemeinsame „Zeit“ definiert. Die letztere Zeit kann nun definiert werden, indem man **durch Definition** festsetzt, daß die „Zeit“, welche das Licht braucht, um von A nach B zu gelangen, gleich ist der „Zeit“, welche es braucht, um von B nach A zu gelangen. Es gehe nämlich ein Lichtstrahl zur „ A -Zeit“ t_A von A nach B ab, werde zur „ B -Zeit“ t_B in B gegen A zu reflektiert und gelange zur „ A -Zeit“ t'_A nach A zurück. Die beiden Uhren laufen definitionsgemäß synchron, wenn

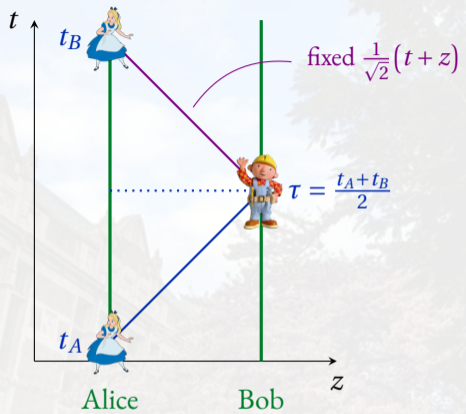
$$t_B - t_A = t'_A - t_B.$$

Einstein, *Annalen Phys* 17 (1905) 891

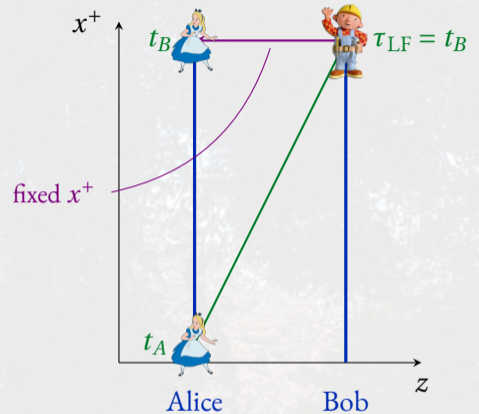
- **Didactic overview:** Veritasium, “Why No One Has Measured The Speed of Light” (YouTube)
- **Technical review:** Anderson, Stedman & Vetharaniam, *Phys. Rept.* 295 (1998) 93

Synchronization conventions

Einstein synchronization



Light front synchronization



- 🍃 **Einstein synchronization** defined to be isotropic.
- 🍃 **Light front synchronization** defines hyperplanes with fixed $t + z$ to be “simultaneous.”
 - ✦ Light travels instantaneously in $-z$ direction *by definition*.
 - ✦ We take what we see as literally happening now.

A. Einstein, *Annalen Phys* 17 (1905) 891
AF & Miller, *Phys Rev D* 107 (2023) 074036

Transverse boosts and Terrell rotations



☛ Lorentz-boosted objects *appear rotated*.

✧ **Terrell rotation**

✧ Optical effect: contraction + delay

☛ **Light front transverse boost**
undoes Terrell rotation:

$$B_x^{(\text{LF})} = \frac{1}{\sqrt{2}}(K_x + J_y)$$

✧ Standard boost + counter-rotation

✧ Leaves x^+ (time) invariant

✧ Part of the **Galilei subgroup**

Dice images by Ute Kraus,

<https://www.spacetime.travel.org/>

J. Terrell, *Phys Rev* 116 (1959) 1041

M. Burkardt, *Int J Mod Phys A* 18 (2003) 173

Galilei subgroup

☛ Poincaré group has a (2 + 1)D **Galilei subgroup**.

- ✦ x^+ is time and \mathbf{x}_\perp is space under this subgroup.
- ✦ $P^+ = E_p + p_z$ is the central charge.
- ✦ x^+ and P^+ are invariant under this subgroup!

☛ Light front time gives **fully relativistic** 2D picture that looks a lot like non-relativistic physics.



$$\frac{d\mathbf{P}_\perp}{dx^+} = P^+ \frac{d^2 \mathbf{x}_\perp}{dx^{+2}}$$

$$H = H_{\text{rest}} + \frac{P_\perp^2}{2P^+}$$

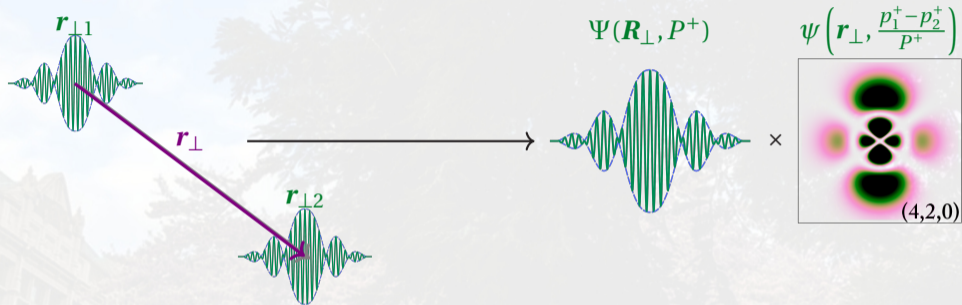
$$\mathbf{v}_\perp = \frac{P_\perp}{P^+}$$

G. Pinski, *J Math Phys* 9 (1986) 1927

M. Burkardt, *Int J Mod Phys A* 18 (2003) 173

Galilei subgroup and densities

Wave packet separation works for *transverse* spatial coordinates



Works thanks to the Galilei subgroup

Apparently stuck with 2D spatial densities

Can restore third dimension via **Abel transform**—in special cases

Need spherically symmetric & static density

Light front stress tensor

$$\int_{\mathbb{R}} dx^- \langle \Psi | \hat{T}^{ij}(x) | \Psi \rangle = \sum_{s,s'} \int d^2 R_{\perp} \left\{ \overbrace{-\Psi_{s'}^* \frac{\overleftrightarrow{\nabla}^i \overleftrightarrow{\nabla}^j}{4(P^+)^2} \Psi_s}_{\text{dynamic pressures}} \underbrace{\alpha_{s's}^i(\mathbf{x}_{\perp} - \mathbf{R}_{\perp})}_{\text{internal } P^+ \text{ density}} - \overbrace{\frac{i\Psi_{s'}^* \overleftrightarrow{\nabla}^i \Psi_s}{2P^+}}_{\text{barycentric flux}} \underbrace{p_{s's}^j(\mathbf{x}_{\perp} - \mathbf{R}_{\perp})}_{\text{internal momentum density}} \right. \\ \left. - \overbrace{\frac{i\Psi_{s'}^* \overleftrightarrow{\nabla}^j \Psi_s}{2P^+}}_{\text{barycentric flux}} \underbrace{f_{s's}^i(\mathbf{x}_{\perp} - \mathbf{R}_{\perp})}_{\text{internal } P^+ \text{ flux}} + \overbrace{\Psi_{s'}^* \Psi_s}_{\text{barycentric probability}} \underbrace{t_{s's}^{ij}(\mathbf{x}_{\perp} - \mathbf{R}_{\perp})}_{\text{internal stresses}} \right\}$$



Image: dynamic pressure from wind

- 🍃 Echoes non-relativistic breakdown—light front Galilei symmetry!
- 🍃 MFFs $D(\Delta_{\perp}^2)$ and $\bar{c}(\Delta_{\perp}^2)$ in stress tensor.
- 🍃 **Assuming spin along z -axis:**

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} \left(\frac{\Delta_{\perp}^i \Delta_{\perp}^j - \delta^{ij} \Delta_{\perp}^2}{4P^+} D(\Delta_{\perp}^2) - \frac{m^2}{P^+} \delta^{ij} \bar{c}(\Delta_{\perp}^2) \right) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

Simple pressure model revisited

Start with a closed system (sum quarks & gluons—no \bar{c}):

$$t_{\text{LF}}^{ij}(\mathbf{b}) = \int \frac{d^3 \Delta_{\perp}}{(2\pi)^2} \frac{\Delta^i \Delta^j - \delta^{ij} \Delta^2}{4P^+} D(\Delta_{\perp}^2) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}} = \frac{-\nabla_i \nabla_j + \delta_{ij} \nabla^2}{4P^+} \underbrace{\int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} D(\Delta_{\perp}^2) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}}_{\equiv \mathcal{D}_{\text{LF}}(b_{\perp})}$$

Simple tripole model (revisited):

$$D(\Delta_{\perp}^2) = \frac{D_0}{(1 + \Delta_{\perp}^2 / \Lambda^2)^3} \quad \Rightarrow \quad \mathcal{D}_{\text{LF}}(b_{\perp}) = \frac{D_0 \Lambda^4 b_{\perp}^2}{16\pi} \underbrace{K_2(\Lambda b_{\perp})}_{\text{modified Bessel function}}$$

✧ Bit harder than 3D case—extra credit problem?

Combined tripole form

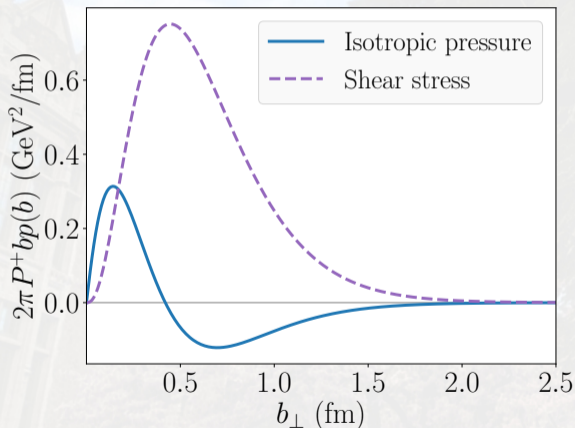
$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$
$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

Isotropic pressure and shear stress

$$t_{LF}^{ij}(\mathbf{b}_\perp) = \frac{D_0 \Lambda^5 b_\perp}{128\pi P^+} \left\{ \left(\Lambda b_\perp K_0(\Lambda b_\perp) - 2K_1(\Lambda b_\perp) \right) \overbrace{\delta^{ij}}^{\text{isotropic pressure}} - 2\Lambda b_\perp K_0(\Lambda b_\perp) \overbrace{\left(\hat{b}^i \hat{b}^j - \frac{1}{2} \delta^{ij} \right)}^{\text{shear stresses}} \right\}$$



Combined tripole form

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

$$\Lambda = 1.12 \text{ GeV}$$

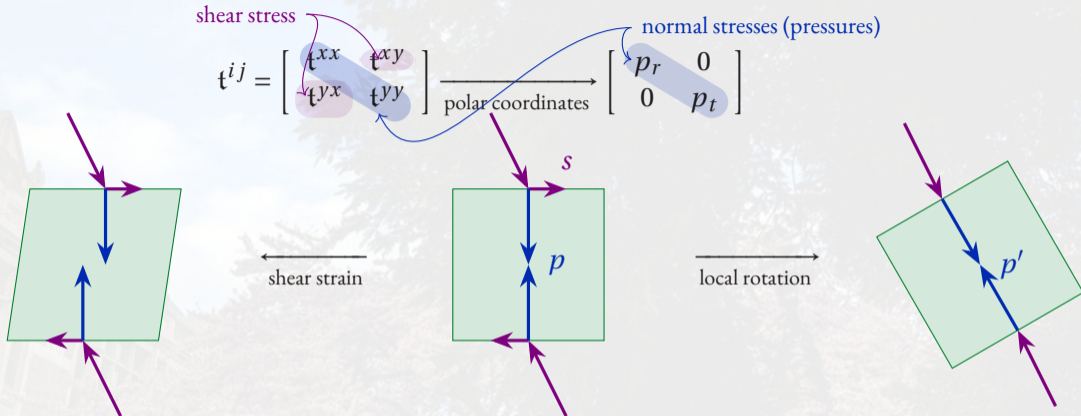
Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

Lorcé, Moutarde & Trawinski, *Eur Phys J C* 79 (2019) 89

AF & Miller, *Phys Rev D* 103 (2021) 094023

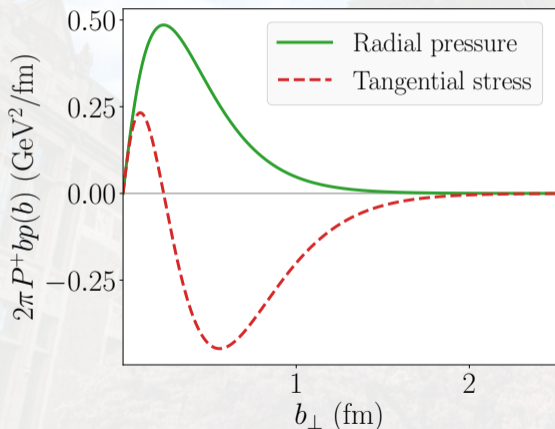
Principal stresses on the light front



- 🌿 **Normal stresses** induce compression or expansion (tension)
- 🌿 **Shear stresses** cause medium to shear
- 🌿 Shear stresses eliminated by diagonalization—get **principal stresses**

Principal stresses

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \frac{D_0 \Lambda^5 b_{\perp}}{64\pi P^+} \left\{ \overset{\text{radial pressure}}{-K_1(\Lambda b_{\perp}) \hat{b}^i \hat{b}^j} + \left(\Lambda b_{\perp} K_0(\Lambda b_{\perp}) - K_1(\Lambda b_{\perp}) \right) \overset{\text{tangential stresses}}{\left(\delta^{ij} - \hat{b}^i \hat{b}^j \right)} \right\}$$



Combined tripole form

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$

$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

- Radial pressure positive again
- Tangential tension again

Abel transform: relating 2D and 3D

☛ 3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} D(\Delta_{\perp}^2) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

AF & Miller, [Phys Rev D 105 \(2021\) 014003](#) (the notation)

Abel transform: relating 2D and 3D

☛ 3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int \frac{d^3 \Delta_{\perp}}{(2\pi)^3} D(\Delta^2) (2\pi) \delta(\Delta_z) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

AF & Miller, [Phys Rev D 105 \(2021\) 014003](#) (the notation)

Abel transform: relating 2D and 3D

☛ 3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \int \frac{d^3\Delta}{(2\pi)^3} D(\Delta^2) e^{-i\Delta \cdot \mathbf{b}}$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

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Abel transform: relating 2D and 3D

3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \mathcal{D}_{\text{3D}}(\mathbf{b})$$

Relationship given by **Abel transform**

Assuming the 3D case is spherically symmetric ...

$$\mathcal{D}_{\text{LF}}(b_{\perp}) = \int_{-\infty}^{\infty} dz \mathcal{D}_{\text{3D}}(r) \Big|_{r^2 = b_{\perp}^2 + z^2}$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

AF & Miller, [Phys Rev D 105 \(2021\) 014003](#) (the notation)

Abel transform: relating 2D and 3D

3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \mathcal{D}_{3\text{D}}(\mathbf{b})$$

Relationship given by **Abel transform**

Assuming the 3D case is spherically symmetric ...

$$\mathcal{D}_{\text{LF}}(b_{\perp}) = 2 \int_0^{\infty} dz \mathcal{D}_{3\text{D}}(r) \Big|_{r^2 = b_{\perp}^2 + z^2}$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

AF & Miller, [Phys Rev D 105 \(2021\) 014003](#) (the notation)

Abel transform: relating 2D and 3D

3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \mathcal{D}_{3\text{D}}(\mathbf{b})$$

Relationship given by **Abel transform**

Assuming the 3D case is spherically symmetric ...

$$\mathcal{D}_{\text{LF}}(b_{\perp}) = 2 \int_{b_{\perp}}^{\infty} dr \frac{dz}{dr} \mathcal{D}_{3\text{D}}(r) \Big|_{z^2=r^2-b_{\perp}^2}$$

Panteleeva & Polyakov, *Phys Rev D* 104 (2021) 014008 (the idea)

AF & Miller, *Phys Rev D* 105 (2021) 014003 (the notation)

Abel transform: relating 2D and 3D

3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \mathcal{D}_{\text{3D}}(\mathbf{b})$$

Relationship given by **Abel transform**

Assuming the 3D case is spherically symmetric ...

$$\mathcal{D}_{\text{LF}}(b_{\perp}) = 2 \int_{b_{\perp}}^{\infty} dr \frac{r}{\sqrt{r^2 - b_{\perp}^2}} \mathcal{D}_{\text{3D}}(r)$$

Panteleeva & Polyakov, [Phys Rev D 104 \(2021\) 014008](#) (the idea)

AF & Miller, [Phys Rev D 105 \(2021\) 014003](#) (the notation)

Abel transform: relating 2D and 3D

3D and light front are related:

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}) = \int_{-\infty}^{\infty} db_z \mathcal{D}_{3\text{D}}(\mathbf{b})$$

Relationship given by **Abel transform**

Assuming the 3D case is spherically symmetric ...

$$\mathcal{D}_{\text{LF}}(b_{\perp}) = 2 \int_{b_{\perp}}^{\infty} dr \frac{r}{\sqrt{r^2 - b_{\perp}^2}} \mathcal{D}_{3\text{D}}(r) \equiv \mathcal{A}[\mathcal{D}_{3\text{D}}(r)](b_{\perp})$$

This can be inverted:

$$\mathcal{D}_{3\text{D}}(r) = -\frac{1}{\pi} \int_r^{\infty} db_{\perp} \frac{d\mathcal{D}_{\text{LF}}(b_{\perp})}{db_{\perp}} \frac{1}{\sqrt{b_{\perp}^2 - r^2}}$$

✦ Proof is difficult; I won't reproduce here

Panteleeva & Polyakov, *Phys Rev D* 104 (2021) 014008 (the idea)

AF & Miller, *Phys Rev D* 105 (2021) 014003 (the notation)

Abel tomography

$$\mathcal{D}_{3D}(r) = -\frac{1}{\pi} \int_r^\infty db_\perp \frac{d\mathcal{D}_{LF}(b_\perp)}{db_\perp} \frac{1}{\sqrt{b_\perp^2 - r^2}}$$

- 🍃 If light front stress is valid, shouldn't its 3D reconstruction be too?
- 🍃 **Caution:** this doesn't always work—need spherical symmetry.



- ✧ Both of these shapes have 2D circular cross sections
- 🍃 Proton has a spin direction—might break spherical symmetry.
 - ✧ Shape of proton depends on its polarization direction!

Panteleeva & Polyakov, *Phys Rev D* 104 (2021) 014008 (the idea)

AF & Miller, *Phys Rev D* 105 (2021) 014003 (criticism of the idea)

Kogut-Soper spinors

$$t_{\text{LF}}^{ij}(\mathbf{b}_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \left(\frac{\Delta_\perp^i \Delta_\perp^j - \delta^{ij} \Delta_\perp^2}{4P^+} D(\Delta_\perp^2) - \frac{m^2}{P^+} \delta^{ij} \bar{c}(\Delta_\perp^2) \right) \frac{\bar{u}\left(P^+, \frac{\Delta_\perp}{2}, s'\right) u\left(P^+, -\frac{\Delta_\perp}{2}, s\right)}{2m} e^{-i\Delta_\perp \cdot \mathbf{b}_\perp}$$

🍃 I skipped the spin dependence before.

🍃 On light front, use **Kogut-Soper spinors**

$$u(p, \uparrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} \sqrt{2}p^+ \\ p_x + ip_y \\ m \\ 0 \end{bmatrix}$$

$$u(p, \downarrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} 0 \\ m \\ -p_x + ip_y \\ \sqrt{2}p^+ \end{bmatrix}$$

- ✦ For spin up & down **along z-axis**
- ✦ Found via light front boosts of rest frame spinors

Kogut & Soper, *Phys Rev D* 1 (1970) 2901 (mistakenly calls light front the “infinite momentum frame”, but great paper anyway)

AF, [A friendly introduction to the light front](#) (lecture series)

Matrix elements of Kogut-Soper spinors

$$u(p, \uparrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} \sqrt{2}p^+ \\ p_x + ip_y \\ m \\ 0 \end{bmatrix} \quad u(p, \downarrow) = \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{bmatrix} 0 \\ m \\ -p_x + ip_y \\ \sqrt{2}p^+ \end{bmatrix} \quad \gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Spinor Elements

$$\bar{u}\left(P^+, \frac{\Delta_\perp}{2}, \uparrow\right) u\left(P^+, -\frac{\Delta_\perp}{2}, \uparrow\right) = 2m$$

$$\bar{u}\left(P^+, \frac{\Delta_\perp}{2}, \uparrow\right) u\left(P^+, -\frac{\Delta_\perp}{2}, \downarrow\right) = \Delta_x - i\Delta_y$$

$$\bar{u}\left(P^+, \frac{\Delta_\perp}{2}, \downarrow\right) u\left(P^+, -\frac{\Delta_\perp}{2}, \uparrow\right) = -\Delta_x - i\Delta_y$$

$$\bar{u}\left(P^+, \frac{\Delta_\perp}{2}, \downarrow\right) u\left(P^+, -\frac{\Delta_\perp}{2}, \downarrow\right) = 2m$$

Transverse polarization

$$|s_x = \pm \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle)$$

$$|s_y = \pm \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle)$$

Matrix elements of Kogut-Soper spinors

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} \left(\frac{\Delta_{\perp}^i \Delta_{\perp}^j - \delta^{ij} \Delta_{\perp}^2}{4P^+} D(\Delta_{\perp}^2) - \frac{m^2}{P^+} \delta^{ij} \bar{c}(\Delta_{\perp}^2) \right) \frac{\bar{u}\left(P^+, \frac{\Delta_{\perp}}{2}, s'\right) u\left(P^+, -\frac{\Delta_{\perp}}{2}, s\right)}{2m} e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

Spinor Elements

$$\bar{u}\left(P^+, \frac{\Delta_{\perp}}{2}, \uparrow\right) u\left(P^+, -\frac{\Delta_{\perp}}{2}, \uparrow\right) = 2m$$

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Transverse polarization

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$$|s_y = \pm \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle)$$

Transverse polarization: the Polyakov pressure potential

For **any** polarization:

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} \left(\frac{\Delta_{\perp}^i \Delta_{\perp}^j - \delta^{ij} \Delta_{\perp}^2}{4P^+} D(\Delta_{\perp}^2) - \frac{m^2}{P^+} \delta^{ij} \bar{c}(\Delta_{\perp}^2) \right) \left(1 - \frac{i(\Delta_{\perp} \times \hat{z}) \cdot \hat{s}}{2m} \right) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

When $\hat{s} = \hat{z}$, the extra term drops out

Combined tripole form

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$
$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)

Duran & al., *Nature* (2023)

AF & Miller, *Phys Rev D* 104 (2021) 014024

AF & Miller, *Phys Rev D* 107 (2023) 074036

Polyakov pressure potential:

$$\mathcal{D}(b) = \frac{D_0 \Lambda^4 b_{\perp}^2}{16\pi} \underbrace{K_2(\Lambda b_{\perp})}_{\text{modified Bessel function}}$$

Transverse polarization: the Polyakov pressure potential

For **any** polarization:

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- When $\hat{s} = \hat{z}$, the extra term drops out
- Let's assume sum over quarks & gluons (drop \bar{c})

Combined tripole form

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Transverse polarization: the Polyakov pressure potential

For **any** polarization:

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \frac{-\nabla_i \nabla_j + \delta^{ij} \nabla^2}{4P^+} \left(1 + \frac{(\nabla_{\perp} \times \hat{z}) \cdot \hat{s}}{2m} \right) \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} D(\Delta_{\perp}^2) e^{-i\Delta_{\perp} \cdot \mathbf{b}_{\perp}}$$

- ✦ When $\hat{s} = \hat{z}$, the extra term drops out
- ✦ Let's assume sum over quarks & gluons (drop \bar{c})

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- When $\hat{s} = \hat{z}$, the extra term drops out
- Let's assume sum over quarks & gluons (drop \bar{c})

Combined tripole form

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$
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Transverse polarization: the Polyakov pressure potential

For **any** polarization:

$$t_{\text{LF}}^{ij}(\mathbf{b}_{\perp}) = \frac{-\nabla_i \nabla_j + \delta^{ij} \nabla^2}{4P^+} \underbrace{\left(1 + \frac{(\nabla_{\perp} \times \hat{z}) \cdot \hat{s}}{2m} \right)}_{\equiv \mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}, \hat{s}) \text{ (we're gonna look at this)}} \mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp})$$

- When $\hat{s} = \hat{z}$, the extra term drops out
- Let's assume sum over quarks & gluons (drop \bar{c})

Combined tripole form

$$D_0 = D_0^{(q)} + D_0^{(g)} = -3.46$$
$$\Lambda = 1.12 \text{ GeV}$$

Burkert, Elouadrhiri & Girod, *Nature* (2018)
Duran & al., *Nature* (2023)

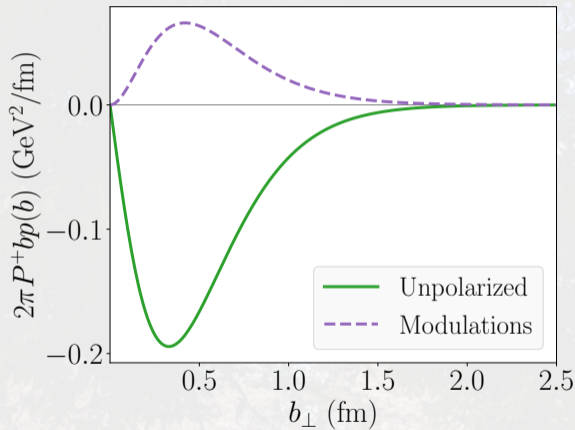
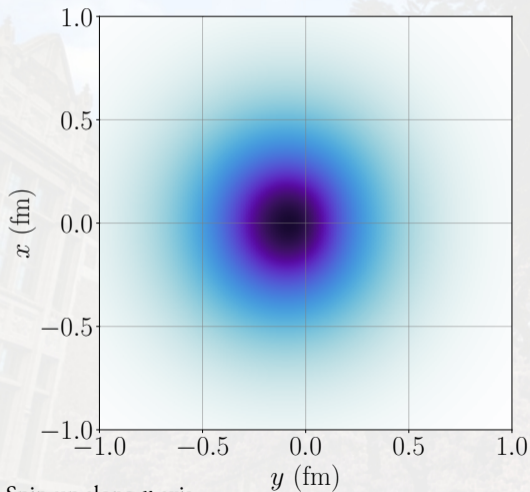
AF & Miller, *Phys Rev D* 104 (2021) 014024
AF & Miller, *Phys Rev D* 107 (2023) 074036

Polyakov pressure potential:

$$\mathcal{D}(b) = \frac{D_0 \Lambda^4 b_{\perp}^2}{16\pi} \underbrace{K_2(\Lambda b_{\perp})}_{\text{modified Bessel function}}$$

Transverse polarization: angular modulations

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}, \hat{s}) = \frac{D_0 \Lambda^4 b_{\perp}^2}{16\pi} \left\{ K_2(\Lambda b_{\perp}) - \frac{\Lambda}{2m} (\hat{\mathbf{b}}_{\perp} \times \hat{\mathbf{z}}) \cdot \hat{s} K_1(\Lambda b_{\perp}) \right\}$$

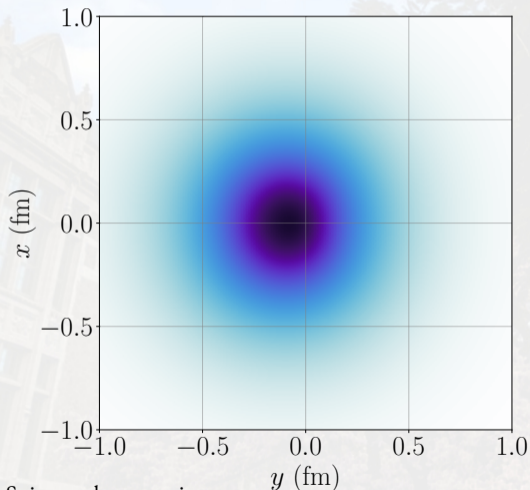


Spin-up along x -axis

AF & Miller, *Phys Rev D* 104 (2021) 014024

Transverse polarization: angular modulations

$$\mathcal{D}_{\text{LF}}(\mathbf{b}_{\perp}, \hat{s}) = \frac{D_0 \Lambda^4 b_{\perp}^2}{16\pi} \left\{ K_2(\Lambda b_{\perp}) - \frac{\Lambda}{2m} (\hat{b}_{\perp} \times \hat{z}) \cdot \hat{s} K_1(\Lambda b_{\perp}) \right\}$$



Spin-up along x -axis

AF & Miller, *Phys Rev D* 104 (2021) 014024

🍃 **Significant** (up to 54%) modification

$$\Lambda = 1.12 \text{ GeV}$$

$$m = 0.938 \text{ GeV}$$

$$\frac{\Lambda}{2m} \approx 0.6$$

🍃 Proton isn't spherically symmetric

🍃 Abel tomography doesn't work

- ✦ Is at best an approximation
- ✦ Works better when $m \gg \Lambda$
- ✦ Large m : massive target
- ✦ Small Λ : non-relativistic system

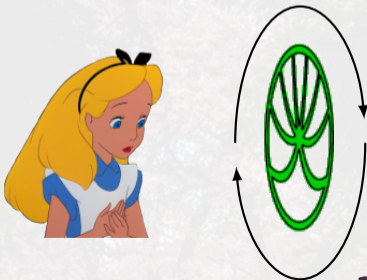
The relativistic wheel

- 🍃 **Static wheel** has regularly-placed spokes
- 🍃 **Spinning wheel** has distortions
- 🍃 Spokes moving away are **redshifted**.
 - ✦ *Appear to move slower, pile up*
- 🍃 Spokes moving towards are **blueshifted**.
 - ✦ *Appear to move faster, become sparse*
- 🍃 These same distortions are present in the proton!
 - ✦ **The proton is a relativistic wheel!**
- 🍃 Also see videos at:
<https://www.spacetime.travel/rad>
(green wheel is relevant case)

Static wheel



Spinning wheel



Various perspectives on relativistic densities

🍃 **Polyakov & Schweitzer:** *define* the static EMT densities in terms of Breit frame:

$$t_{\text{static}}^{\mu\nu}(\mathbf{r}) \equiv \int \frac{d^3\Delta}{(2\pi)^3} \frac{\langle \frac{\Delta}{2} | \hat{T}^{\mu\nu}(0) | -\frac{\Delta}{2} \rangle}{2m\sqrt{1 + \frac{\Delta^2}{4m^2}}} e^{-i\Delta\cdot\mathbf{r}}$$

- ✦ Most widely-used approach.
- ✦ [hep-ph/0207153](#), [PLB 555 \(2003\) 57](#), [IJMPA \(2018\) 1830025](#)

🍃 **Lorcé *et al.*:** use Wigner phase-space formalism to set $\mathbf{R} = 0$ and $\mathbf{P} = 0$.

- ✦ Gives Polyakov & Schweitzer's static EMT.
- ✦ [EPCJ 79 \(2019\) 89](#)
- ✦ Negative probabilities \implies energy condition violations; see [Dumitru & Noronha, 2505.09720](#)

🍃 **Yang Li *et al.*:** expand expectation values as tower of multipole moment densities:

$$\langle \Psi | \hat{T}^{\mu\nu}(\mathbf{x}, t) | \Psi \rangle \approx \int d^3R \left(\Psi^*(\mathbf{R}, t) i \overleftrightarrow{\partial}_t \Psi(\mathbf{R}, t) \right) t_{\text{static}}^{\mu\nu}(\mathbf{b}) + \text{corrections}$$

- ✦ Use spatially diffuse wave packet, zero average momentum.
- ✦ Gives Polyakov & Schweitzer's static EMT as leading contribution.
- ✦ [PLB 838 \(2023\) 137676](#), [2405.06892](#)

More perspectives on relativistic EMT densities

🍃 **Lorcé *et al.* / AF & Miller:** use light front densities.

$$t_{\text{LF}}^{\mu\nu}(\mathbf{b}_{\perp}) \equiv \int \frac{d^2\Delta_{\perp}}{(2\pi)^2} \frac{\langle P^+, \frac{\Delta_{\perp}}{2} | \hat{T}^{\mu\nu}(0) | P^+, -\frac{\Delta_{\perp}}{2} \rangle}{2P^+} e^{-i\Delta \cdot \mathbf{b}}$$

- ❖ Only $i, j \in \{1, 2\}$; only get 2D densities.
- ❖ Galilei symmetry allows barycenter/internal separation.
- ❖ EPCJ 79 (2019) 89, PRD 103 (2021) 094023

🍃 **Panteleeva *et al.*:** use localized, zero average momentum wave packets.

- ❖ Similar result to light front, but restore third dimension.
- ❖ EPJC 83 (2023) 617, JHEP 07 (2023) 237

🍃 **AF & Miller:** use light front time + Cartesian space.

- ❖ Still only 2D densities, but get 3×3 stress tensor.
- ❖ Separate wave packet & internal densities with factorization/smearing relations.
- ❖ PRD 107 (2023) 074036, PRD 108 (2023) 094026

A close-up photograph of a tree trunk covered in vibrant green moss. Several clusters of light pink cherry blossoms are in various stages of bloom, some fully open and others as buds. The background is a soft-focus view of a large tree in full cherry blossom, creating a dreamy, ethereal atmosphere. The text "Wrapping Up" is written in a white, elegant cursive font, centered over the mossy trunk.

Wrapping Up

Summary

🍃 The **energy-momentum tensor** is central to many open questions & controversies

- 1 **Proton mass**
- 2 **Proton spin**
- 3 **Proton pressure**
- 4 **Nuclear forces**
- 5 **Proton densities**

🍃 I've given an overview & my own opinions

🍃 Reasonable people have opposing opinions

🍃 Science is exciting when exploring unknowns!

Energy density

Momentum densities

$$\begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Stress tensor

Thank you for your time!