Introduction to QCD and Small-x Physics

Lecture 4: Inclusive DIS in the Regge Limit

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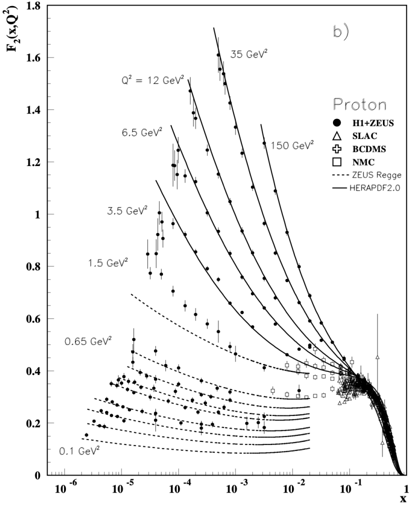
Proton Structure Function at Small Bjorken-x

Experimental observation:

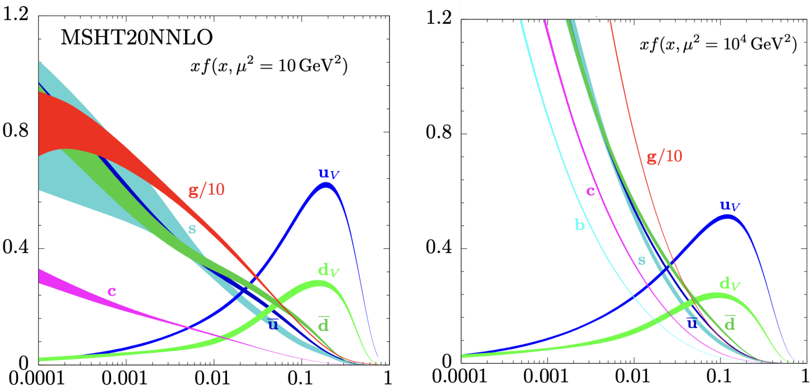
- $F_2(x, Q^2)$ increases rapidly as $x \to 0$.
- The growth becomes stronger at larger Q^2.

Implication:

- The proton contains a rapidly increasing density of small-x partons, especially gluons.



Parton Distribution Functions



(source: PDG)

Why Does the Gluon Density Grow at Small x?

The singlet DGLAP equations are

$$\frac{\partial \Sigma(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right)\Sigma(z,Q^2) + 2n_f P_{qg}\left(\frac{x}{z}\right) g(z,Q^2)\right],$$

$$\frac{\partial g(x,Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\left[P_{gq}\left(\frac{x}{z}\right)\Sigma(z,Q^2) + P_{gg}\left(\frac{x}{z}\right) g(z,Q^2)\right].$$

At small x, with parent momentum fraction $z \gg x$ and $\frac{x}{z} \ll 1$. The splitting kernels behave very differently:

$$P_{gg}\left(\frac{x}{z}\right) \simeq \frac{2C_A z}{x}, \qquad P_{gq}\left(\frac{x}{z}\right) \simeq \frac{2C_F z}{x}, \qquad P_{qq} \sim \mathcal{O}(1), \qquad P_{qg} \sim \mathcal{O}(1).$$

$P_{gg} \sim \frac{1}{x} \implies$ gluon evolution dominates at small x.

Small-x Approximation to Gluon DGLAP Evolution

At sufficiently small x, the gluon contribution dominates:

$$\frac{\partial g(x,Q^2)}{\partial \ln Q^2} \simeq \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} P_{gg}\left(\frac{x}{z}\right) g(z,Q^2) = \frac{\alpha_s C_A}{\pi}\frac{1}{x}\int_x^1 dz\, g(z,Q^2).$$

Define the gluon momentum density $G(x,Q^2) \equiv xg(x,Q^2)$.

$$\frac{\partial G(x,Q^2)}{\partial \ln Q^2} \simeq \bar{\alpha}_s \int_x^1 \frac{dz}{z} G(z,Q^2), \qquad \bar{\alpha}_s \equiv \frac{\alpha_s C_A}{\pi}.$$

The equation can be solved iteratively, yielding

$$G(x,Q^2) \sim \exp\left[2\sqrt{\bar{\alpha}_s \ln\frac{Q^2}{Q_0^2}\ln\frac{1}{x}}\,\right].$$

Therefore DGLAP predicts $xg(x,Q^2)$ grows rapidly as $x \to 0$. This explains qualitatively the strong rise of $F_2(x,Q^2)$ observed at HERA.

Deep Theoretical Message of HERA Small-x Data

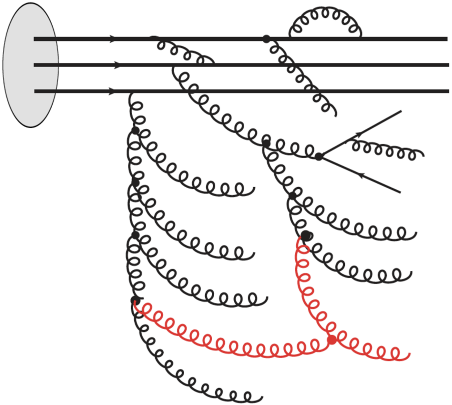
HERA measurements showed that $F_2(x, Q^2)$ rises rapidly as $x \to 0$. This implies a rapidly growing gluon distribution:

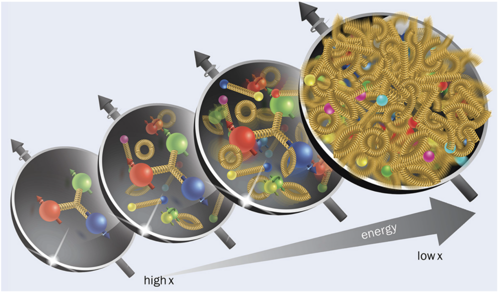
$$xg(x, Q^2) \uparrow \qquad \text{as} \qquad x \downarrow .$$

Linear evolution equations such as DGLAP and BFKL predict continued growth of parton densities at high energy. However, such growth cannot continue indefinitely. Otherwise scattering amplitudes eventually violate unitarity.

> The HERA small-x data suggest that QCD in the high energy limit evolves toward a regime of extremely high gluon density where nonlinear dynamics and gluon saturation must eventually become important.

Gluon Saturation





Credit: BNL

- In the high energy limit, a saturated gluon state is achieved when gluon splittings and recombinations reach a dynamic equilibrium.

Gluons in the DIS Hadronic Tensor

The DIS hadronic tensor is

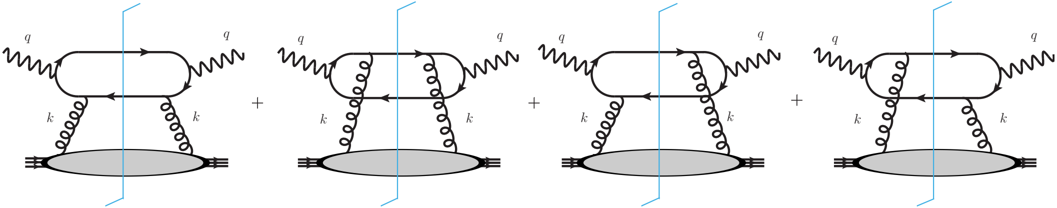
$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|J_\mu(z)J_\nu(0)|P\rangle.$$

with electromagnetic current

$$J_\mu(x) = e_f\,\bar{\psi}(x)\gamma_\mu\psi(x).$$

The current contains only quark fields, no gluon fields. To expose gluon contributions, contract the quark fields into a quark propagator in a background gluon field:

$$W_{\mu\nu} \simeq \frac{e_f^2}{4\pi}\int d^4z\, e^{iq\cdot z}\,\langle P|\mathrm{Tr}\left[S_A(-z)\gamma_\mu S_A(z)\gamma_\nu\right]|P\rangle.$$



Expansion in Background Gluon Fields

- The background-field expansion of quark propagators systematically generates gluon contributions to DIS.

Expand the quark propagator in powers of the gluon field:

$$S_A = S_0 + S_0 V S_0 + S_0 V S_0 V S_0 + \cdots \,,$$

with the free quark propagator and the gluon field insertion:

$$S_0(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i\not{p}}{p^2+i0}, \qquad V(x) = ig\gamma^\rho t^a A^a_\rho(x),$$

The first-order term:

$$S_A^{(1)}(x,y) = ig \int d^4y_1\, S_0(x-y_1)\gamma^\rho t^a A^a_\rho(y_1) S_0(y_1-y).$$

The second-order term:

$$S_A^{(2)}(x,y) = (ig)^2 \int d^4y_1 d^4y_2\, S_0(x-y_1)\gamma^\rho t^a A^a_\rho(y_1) S_0(y_1-y_2)\gamma^\sigma t^b A^b_\sigma(y_2) S_0(y_2-y).$$

The gluon contribution to DIS first appears at order A^2.

Factorized Form Before Collinear Approximation

After Fourier transforming to momentum space:

$$W_{\mu\nu}^{(g)} = \int \frac{d^4k}{(2\pi)^4}\, \Phi_{ab}^{\rho\sigma}(k,P)\, H_{\mu\nu,\rho\sigma}^{ab}(k,q).$$

The nonperturbative gluon correlator is

$$\Phi_{ab}^{\rho\sigma}(k,P) = \int d^4y\, e^{-ik\cdot y}\langle P|A_a^\rho(y)A_b^\sigma(0)|P\rangle.$$

The hard part contains the short-distance photon-gluon scattering:

$$H_{\mu\nu,\rho\sigma}^{ab}(k,q).$$

At this stage the expression is exact (up to power corrections and truncation order).
DIS naturally separates into a hard scattering part and a gluon correlation function.

Collinear Approximation and Gluon PDF

In the Bjorken limit:

$$P^\mu \sim (P^+, 0, \mathbf{0}_T), \qquad k^\mu \approx xP^\mu.$$

The hard scattering becomes insensitive to k^-, k_T, k^2.
Projecting onto leading twist gives the gauge-invariant gluon PDF:

$$g(x,\mu_F) = \frac{1}{xP^+} \int \frac{dy^-}{2\pi} e^{ixP^+y^-} \langle P|F_a^{+i}(0) W_{\text{adj}}^{ab} F_{b,i}^{+}(y^-)|P\rangle.$$

The hadronic tensor factorizes:

$$W_{\mu\nu}^{(g)} = \int_{x_B}^{1} \frac{dx}{x} H_{\mu\nu,g}\left(\frac{x_B}{x}, Q^2, \mu_F\right) g(x,\mu_F).$$

This is the gluon version of collinear factorization.

Inclusive DIS in the Regge Limit

We now consider the high-energy **Regge limit:**

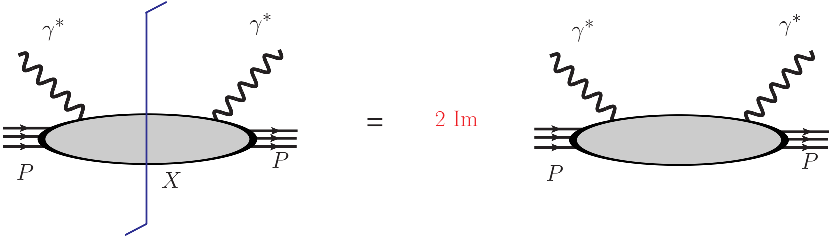
$$s \gg Q^2, \qquad x_B = \frac{Q^2}{s} \ll 1.$$

Instead of directly computing the hadronic tensor $W_{\mu\nu}$, it is more convenient to study the forward virtual Compton amplitude:

$$\boxed{T_{\mu\nu}(q) = \frac{i}{4\pi} \int d^4z\, e^{iq\cdot z} \langle P | T\{J_\mu(z) J_\nu(0)\} | P \rangle.}$$

Using the optical theorem,

$$W_{\mu\nu}(P,q) = 2\,\mathrm{Im}\,T_{\mu\nu}(P,q).$$



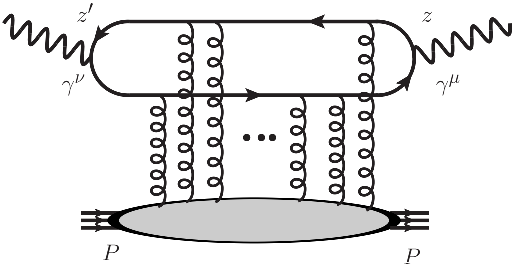
Forward Compton Amplitude in a Background Field

We compute the forward amplitude

$$T_{\mu\nu}(q) = \frac{i}{4\pi}\int d^4z\, d^4z'\, e^{iq\cdot(z-z')}\theta(z^0 - z'^0)\langle P|J_\mu(z)J_\nu(z')|P\rangle.$$

In the background gluon fields, Wick contraction gives

$$T_{\mu\nu}[A] = -\frac{i}{4\pi}\sum_f e_f^2 \int d^4z\, d^4z'\, e^{iq\cdot(z-z')}\theta(z^0 - z'^0)\langle P|\,\mathrm{Tr}\,[S_A(z',z)\gamma_\mu S_A(z,z')\gamma_\nu]\,|P\rangle.$$



- The problem reduces to finding the quark propagator S_A in background gluon fields.
- In the Regge limit, the leading component of gluon fields is A^+ and multiple gluon exchange can be resummed into an eikonal Wilson-line.

Wilson-Line Resummation of the Quark Propagator

At small x, the target is approximated as a shockwave localized near $x^- = 0$. In covariant gauge, the leading background field is

$$A^\mu \simeq \delta^{\mu +} A^+(x^-, x_\perp).$$

The quark propagator in this background is approximated by

$$S_A(x,y) = S_0(x-y) + S_A^{\rm int}(x,y)$$

with

$$S_A^{\rm int}(x,y) = \theta(x^-)\theta(-y^-) \int d^4u\, d^4v\, S_0(x-u)\, \mathcal{V}(u,v)\, S_0(v-y).$$

The eikonal interaction vertex is

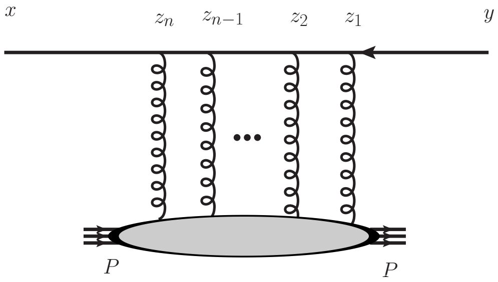
$$\mathcal{V}(u,v) = \gamma^-\delta(u^-)\delta(v^-)\delta(u^+ - v^+)\delta^{(2)}(u_\perp - v_\perp)\left[U(u_\perp) - 1\right].$$

The Wilson line is

$$\boxed{U(x_\perp) = P\exp\left[ig\int_{-\infty}^{+\infty} dx^-\, A_a^+(x^-, x_\perp) t^a\right].}$$

Wilson Line from Multiple Gluon Exchange

The quark interacts mainly through eikonal couplings to A^+. The relevant interaction vertex becomes $\gamma^\mu A_\mu \longrightarrow \gamma^- A^+$



Eikonalized free propagator

$$
\begin{aligned}
S_0(x-y) &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i\not{p}}{p^2+i\epsilon} \\
&\simeq \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i\gamma^+}{2(p^+ + i\epsilon)} \\
&= \frac{\gamma^+}{2}\delta(x^+ - y^+)\delta^{(2)}(x_\perp - y_\perp)\theta(x^- - y^-)
\end{aligned}
$$

We used $\not{p} \simeq p^-\gamma^+$ and $p^2 = 2p^+p^- - p_\perp^2 \simeq 2p^+p^-$. Intermediate quark propagators between gluon insertions are eikonalized but incoming and outgoing quark propagators are not eikonalized.

Dipole Factorization of Inclusive DIS at Small x

In the Regge limit, the inclusive DIS tensor factorizes as

Dipole Factorization

$$W_{\mu\nu} = \frac{2s}{4\pi e^2} \int \frac{d^2 r_\perp}{2\pi} \int_0^1 \frac{d\xi}{\xi(1-\xi)} \sum_f \sum_{s,s'} \Psi^f_{\nu,ss'}(\xi, r_\perp) \Psi^{f*}_{\mu,ss'}(\xi, r_\perp) N(r_\perp, Y) + \mathcal{O}\left(\frac{Q^2}{s}, \frac{m_f^2}{s}, \frac{M_P^2}{s}, \dots\right)$$

Here $s \equiv s_{\gamma^* P}$, the center-of-mass collision energy of $\gamma^* - P$.

Small-x DIS naturally separates into a perturbative photon wave function and a nonperturbative dipole scattering amplitude. The factorization happens in transverse coordinate space (transverse momentum). Not collinear factorization.

Ingredients of Dipole Factorization

- $r_\perp = x_\perp - y_\perp$, the transverse size of the $q\bar{q}$ dipole.
- $\xi, 1-\xi$, the longitudinal momentum fractions of the quark and antiquark.
- The photon wave function $\Psi^f_{\nu,ss'}(\xi, r_\perp)$ describes the perturbative splitting $\gamma^* \to q\bar{q}$

$$\Psi^f_{\nu,ss'}(\xi, l_\perp) = ee_f\, \xi(1-\xi)\, \frac{\bar{u}_s(l)\gamma_\nu v_{s'}(q-l)}{(l_\perp - \xi q_\perp)^2 + \xi(1-\xi)Q^2 + m_f^2}\delta_{ij}$$

- The dipole amplitude

$$\boxed{N(x_\perp, y_\perp; Y) = 1 - \frac{1}{N_c}\left\langle \mathrm{Tr}\, U(x_\perp) U^\dagger(y_\perp)\right\rangle_Y .}$$

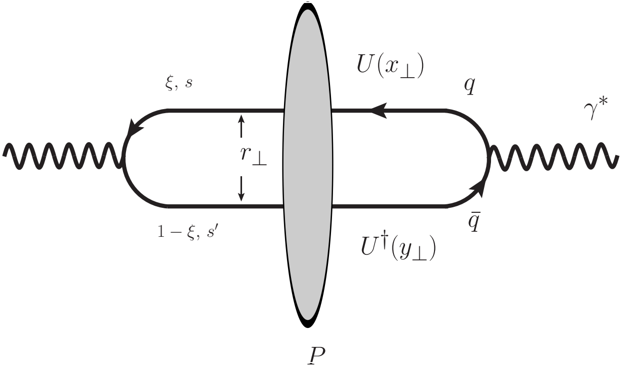
  The averaging over the target state is defined by

$$\langle \cdots \rangle = \frac{1}{2P^+V^-}\langle P| \cdots |P\rangle .$$

- $s_{\gamma^* P} \simeq 2P^+ q^-$, center-of-mass collision energy of the $\gamma^* - P$ system.

Physical Picture of Dipole Factorization

In the Regge limit, the virtual photon fluctuates into a quark-antiquark dipole $\gamma^* \to q\bar{q}$, which happens long before the interaction with the proton. The quark and antiquark each scatters eikonally off the proton acquiring Wilson line phases: $U(x_\perp), U^\dagger(y_\perp)$. The color structure naturally forms a dipole operator:



Physical Meaning of the Dipole Amplitude

In small-x DIS, the target dependence is encoded in

$$N(x_\perp, y_\perp; Y) = 1 - \frac{1}{N_c}\Big\langle \mathrm{Tr}\, U(x_\perp) U^\dagger(y_\perp) \Big\rangle_Y .$$

This looks like a Wilson-line correlator, but physically it is related to the gluon content of the proton. The target average is connected to a proton matrix element by

$$\langle \cdots \rangle = \frac{1}{2P^+V^-}\langle P| \cdots |P\rangle .$$

How is the dipole correlator related to a gluon distribution?

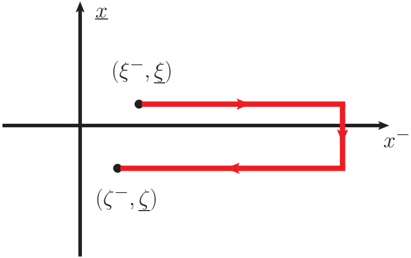
At small x, the answer is that the dipole correlator is the leading term in the $x \to 0$ expansion of a dipole-type gluon Transverse Momentum Distribution (TMD).

Dipole Gluon TMD

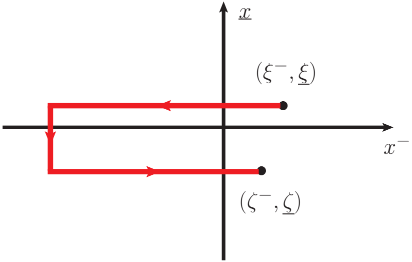
The dipole-type gluon TMD is defined by

$$g_1^{G,\mathrm{dip}}(x,k_T^2) = \frac{2\delta^{ij}}{xP^+}\int\frac{d\xi^- d^2\xi_\perp}{(2\pi)^3}e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp}\langle P|\mathrm{tr}\left[F^{+i}(0)\mathcal{U}^{[+]}[0,\xi]F^{+j}(\xi)\mathcal{U}^{[-]}[\xi,0]\right]|P\rangle.$$

- future-pointing gauge link $\mathcal{U}^{[+]}$



- past-point gauge link $\mathcal{U}^{[-]}$.



Dipole Correlator as the Small-x Gluon Distribution

Expanding in powers of x yields

$$xg_1^{G,\text{dip}}(x,k_T^2) = \frac{4}{2P^+V^-(2\pi)^3}\frac{k_T^2}{g^2}\int d^2\xi_\perp d^2\zeta_\perp\, e^{-ik_\perp\cdot(\xi_\perp-\zeta_\perp)}\langle P|\text{tr}[V^\dagger(\zeta_\perp)V(\xi_\perp)]|P\rangle.$$

$$\boxed{\int d^2\xi_\perp d^2\zeta_\perp\, e^{-ik_\perp\cdot(\xi_\perp-\zeta_\perp)}\Big\langle \text{Tr}[V^\dagger(\zeta_\perp)V(\xi_\perp)]\Big\rangle = \frac{g^2}{4k_T^2}\, xg_1^{G,\text{dip}}(x,k_T^2).}$$

The dipole amplitude

$$N(x_\perp,y_\perp) = 1-\frac{1}{N_c}\Big\langle \text{Tr}\, U(x_\perp)U^\dagger(y_\perp)\Big\rangle$$

is therefore directly connected to the small-x gluon distribution.

Small-x DIS probes gluons through Wilson lines. The dipole amplitude is a physical gluon distribution in the high-energy limit, expressed in transverse coordinate space.

Beyond Tree-Level Dipole Factorization

- At leading order, inclusive DIS at small x factorizes into

$$W_{\mu\nu} \sim |\Psi_{\gamma^* \to q\bar{q}}|^2 \boxtimes N(x_\perp, y_\perp).$$

  The nonperturbative target structure is encoded in the dipole amplitude $N(x_\perp, y_\perp; Y)$. This result is obtained from $\gamma^* \to q\bar{q}$, followed by eikonal scattering through Wilson lines.

- Beyond tree level, additional soft gluon emissions appear:

$$q\bar{q} \to q\bar{q}g \to q\bar{q}gg \to \cdots$$

  These emissions generate large high-energy logarithms $\alpha_s \ln \frac{1}{x}$.

- At sufficiently high energy,

$$\alpha_s \ln \frac{1}{x} \sim 1,$$

  and fixed-order perturbation theory breaks down. The large logarithms must be resummed to all order.

Rapidity Evolution of the Dipole Amplitude

- In collinear factorization large transverse logs $\alpha_s \ln Q^2$ is absorbed into the scale dependence of PDFs $f(x, Q^2)$, leading to DGLAP evolution.
- At small x, the large logarithms $\alpha_s \ln \frac{1}{x}$ are instead absorbed into Wilson-line correlators. The dipole amplitude therefore becomes rapidity dependent:

$$N(x_\perp, y_\perp; Y), \qquad Y \equiv \ln \frac{1}{x}.$$

The resummation of high-energy logarithms becomes an evolution equation in rapidity.

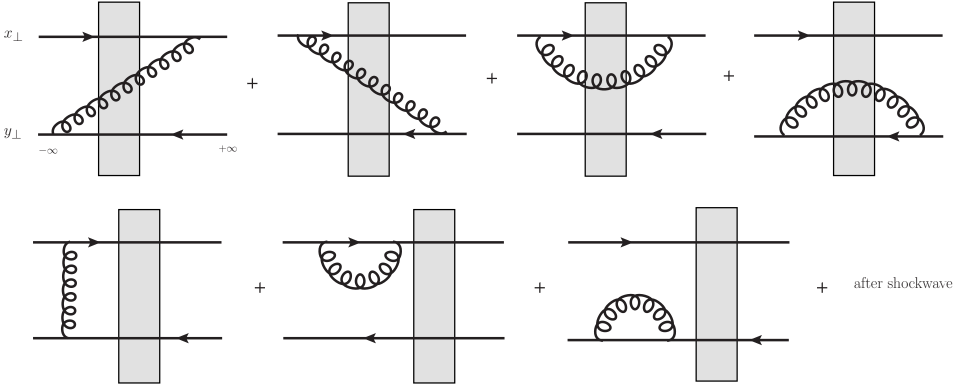
JIMWLK and BK Evolution Equations

- The most general high-energy evolution equation is the **JIMWLK equation** named after *Jalilian-Marian, Iancu, McLerran,Weigert, Leonidov, and Kovner*. The JIMWLK equation describes the rapidity evolution of the probability distribution of color gluon fields inside proton, or equivalently Wilson-line correlators.

- The JIMWLK equation generates an infinite hierarchy of coupled equations for Wilson-line correlators. In the large-N_c mean-field limit, this hierarchy closes into a single equation for the dipole amplitude **Balitsky–Kovchegov (BK) equation**.

Derivation of BK Equation

- Real gluon emission diagrams and virtual gluon emission diagram.



BK Equation

The Balitsky–Kovchegov equation

$$\frac{\partial N(x_\perp, y_\perp; Y)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \Big[N(x_\perp, z_\perp; Y) + N(z_\perp, y_\perp; Y) - N(x_\perp, y_\perp; Y) - N(x_\perp, z_\perp; Y) N(z_\perp, y_\perp; Y) \Big].$$

- The kernel describes dipole splitting $(x_\perp, y_\perp) \to (x_\perp, z_\perp) + (z_\perp, y_\perp)$.
- The linear terms $N(x,z) + N(z,y) - N(x,y)$ describe **gluon splitting** and growth of the dipole density.
- The nonlinear term $-N(x,z)\,N(z,y)$ describes **gluon recombination** and multiple scattering, responsible for **Gluon Saturation**, which prevents unlimited growth of the gluon density.

BK evolution balances gluon emission against gluon recombination.

Dilute Limit: Recovering the BFKL Equation

In the dilute regime:

$$N(x_\perp, y_\perp; Y) \ll 1,$$

multiple scattering is weak. The nonlinear term becomes negligible $N(x,z)N(z,y) \ll N$.
The BK equation reduces to

$$\boxed{\begin{aligned}\frac{\partial N(x_\perp, y_\perp; Y)}{\partial Y} = & \frac{\alpha_s N_c}{2\pi^2} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ & \Big[N(x_\perp, z_\perp; Y) + N(z_\perp, y_\perp; Y) - N(x_\perp, y_\perp; Y) \Big].\end{aligned}}$$

Fourier transforming into momentum space, precisely reproduces the **Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation**.

$$\boxed{\text{BFKL} = \text{linear small-}x\text{ evolution,}}$$

$$\boxed{\text{BK} = \text{nonlinear extension including saturation.}}$$

The Saturation Scale from BK Evolution

- The BK equation contains two competing effects: the linear terms generate growth of gluon density while the nonlinear term suppresses the growth when the gluon density becomes large.
- As rapidity increases, the BK solution develops a dynamical momentum scale $Q_s(Y)$. called the **saturation scale**. It is defined implicitly by

$$N\left(r_\perp = \frac{1}{Q_s(Y)}, Y\right) = \frac{1}{2}.$$

- The saturation scale is the boundary between linear and nonlinear QCD dynamics.
- For fixed coupling, the asymptotic BK solution gives

$$\boxed{Q_s^2(x) = Q_{s0}^2 \left(\frac{x_0}{x}\right)^{\lambda_s}.}$$

The evolution exponent is $\lambda_s = \bar{\alpha}_s \frac{\chi(\gamma_c)}{1-\gamma_c}$ with $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$. Numerically $\gamma_c \simeq 0.3725, \lambda_s \simeq 4.88\,\bar{\alpha}_s$

QCD Evolutions