Introduction to QCD and Small-x Physics

Lecture 3: Inclusive DIS in the Bjorken Limit: Next-to-Leading Order

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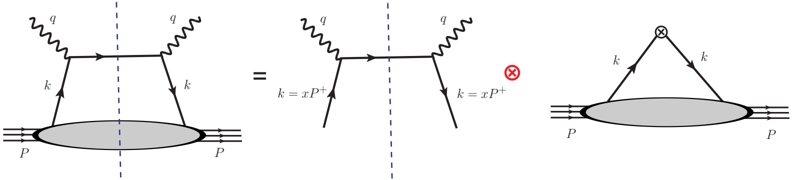
Collinear Factorization at Leading Order

The leading power approximation for $W_{\mu\nu}$ is,

$$W_{\mu\nu}(x_B, Q^2) = \sum_j \int \frac{d\xi}{\xi} H^{j,(0)}_{\mu\nu}(\xi) f_{j/P}(\xi) + \mathcal{O}(\frac{\Lambda^2}{Q^2}) \qquad \longleftarrow \text{Collinear Factorization}$$

$$\equiv \sum_j H^{j,(0)}_{\mu\nu} \otimes f_{j/P} + \mathcal{O}(\frac{\Lambda^2}{Q^2})$$



- Notation for convolution integral

$$(f \otimes g)(x) = \int_x^1 \frac{dy}{y} f(\frac{x}{y}) g(y)$$

Gauge Invariant Parton Distribution Function

Quark PDF (not gauge-invariant)

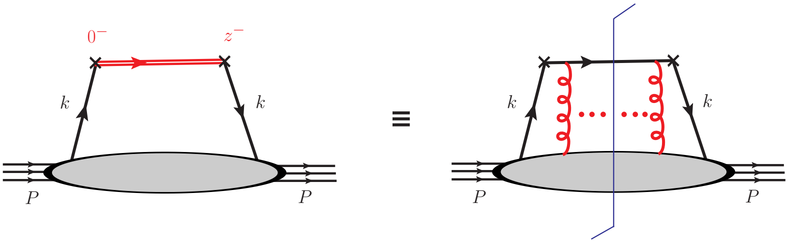
$$f_{j/P}(\xi) = \int \frac{dz^-}{2\pi} e^{-i\xi P^+ z^-} \left\langle P \left| \bar{\psi}^{(j)}(0^+, z^-, \mathbf{0}_\perp) \frac{\gamma^+}{2} \psi^{(j)}(0) \right| P \right\rangle .$$

Gauge Invariant Quark PDF

$$f_{j/P}(\xi) = \int \frac{dz^-}{2\pi} e^{-i\xi P^+ z^-} \left\langle P \left| \bar{\psi}^{(j)}(0^+, z^-, \mathbf{0}_\perp) W(z^-, 0^-) \frac{\gamma^+}{2} \psi^{(j)}(0) \right| P \right\rangle .$$

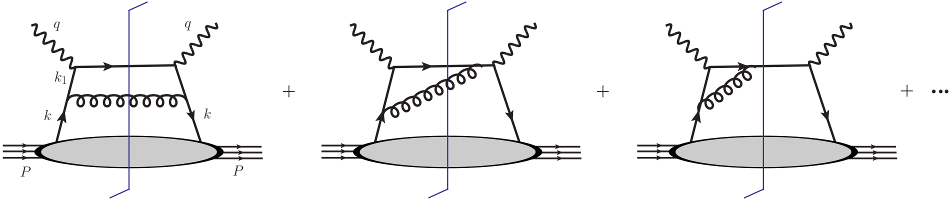
Collinear Wilson Line:

$$W(z^-, 0^-) = \mathcal{P}\exp\left\{ -ig \int_{0^-}^{z^-} dw^- A_a^+(0^+, w^-, \mathbf{0}_\perp) t^a \right\}$$



NLO Corrections and Collinear Divergences

Beyond the Born approximation, DIS receives QCD radiative corrections.



The NLO graph contains both short-distance and long-distance physics. A typical contribution contains an integral of the form

$$\int \frac{dk_1^2}{k_1^2} \sim \ln \frac{Q^2}{\Lambda_{\rm QCD}^2}.$$

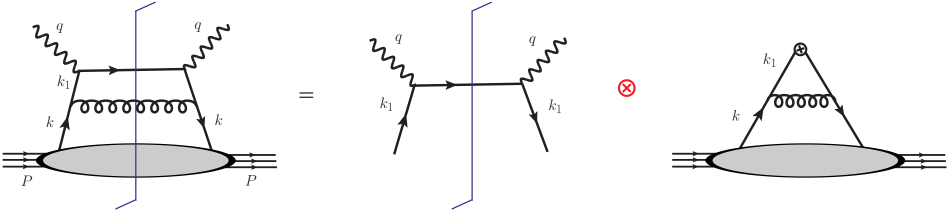
The integral is logarithmically enhanced when $Q^2 \gg \Lambda_{\rm QCD}^2$. (Note that UV divergences are taken care of by renormalization while infrared divergences cancel when adding real and virtual diagrams, collinear divergences remain.)

Introducing the Factorization Scale

Introduce an arbitrary factorization scale μ_F^2, separating nonperturbative collinear physics from perturbative hard scattering:

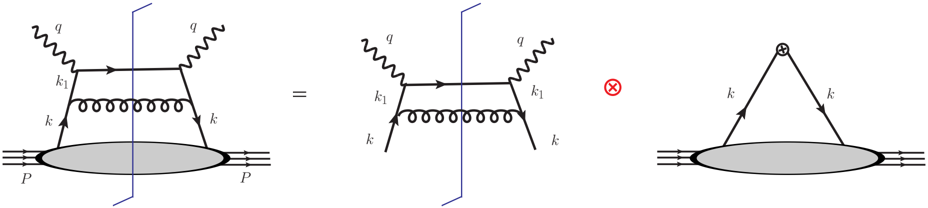
$$\int_0^{Q^2} \frac{dk_1^2}{k_1^2} = \int_0^{\mu_F^2} \frac{dk_1^2}{k_1^2} + \int_{\mu_F^2}^{Q^2} \frac{dk_1^2}{k_1^2}.$$

- Long-distance region $0 < k_1^2 < \mu_F^2$. Gluon emission is absorbed into the PDF: $H_{\mu\nu}^{(0)} \otimes f_{j/P}^{(1)}$



Introducing the Factorization Scale

- Short-distance region $\mu_F^2 < k_1^2 < Q^2$. Gluon emission belongs to the perturbative hard scattering process: $H_{\mu\nu}^{(1)} \otimes f_{j/P}^{(0)}$.



At order α_s,

$$W_{\mu\nu}^{(1)} = H_{\mu\nu}^{(1)} \otimes f_{f/P}^{(0)} + H_{\mu\nu}^{(0)} \otimes f_{f/P}^{(1)}.$$

Changing μ_F^2 reshuffles contributions between $H_{\mu\nu}$ and $f_{j/P}$, but the physical hadronic tensor $W_{\mu\nu}$ is independent of μ_F^2.

Perturbative Expansion of Collinear Factorization

The full hadronic tensor admits the factorized form

$$W_{\mu\nu} = H_{\mu\nu} \otimes f_{j/P}.$$

Expand perturbatively:

$$H_{\mu\nu} = H^{(0)}_{\mu\nu} + \alpha_s H^{(1)}_{\mu\nu} + \cdots ,$$

$$f_{j/P} = f^{(0)}_{j/P} + \alpha_s f^{(1)}_{j/P} + \cdots .$$

At order α_s,

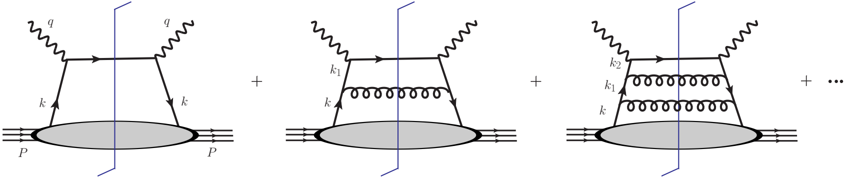
$$W^{(1)}_{\mu\nu} = H^{(1)}_{\mu\nu} \otimes f^{(0)}_{j/P} + H^{(0)}_{\mu\nu} \otimes f^{(1)}_{j/P}.$$

Interpretation:

- $H^{(0)}_{\mu\nu} \otimes f^{(1)}_{j/P}$: collinear gluon radiation absorbed into the PDF evolution.
- $H^{(1)}_{\mu\nu} \otimes f^{(0)}_{j/P}$: short-distance NLO correction to the hard scattering.

Leading-Power Factorization in DIS

Beyond LO, QCD corrections generate loop and real-emission momentum integrals $\int d^4 k_i$



Some internal partons can become nearly on shell $k_i^2 \to 0$ and collinear to the proton $k_i^\mu \parallel P^\mu$. These regions produce logarithmic enhancements $\alpha_s \ln \frac{Q^2}{\Lambda^2_{\mathrm{QCD}}}$.

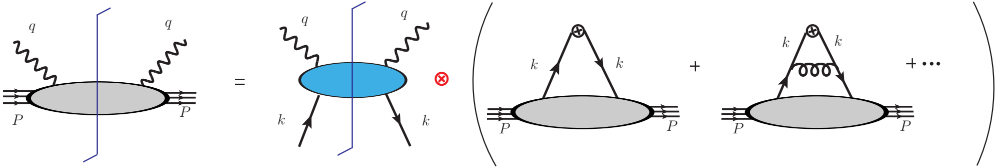
Leading-power factorization separates these collinear long-distance regions from the short-distance hard scattering.

Factorization Formula for F_2

Collinear Factorization for $F_2(x_B, Q^2)$

$$F_2(x_B,Q^2) = \sum_j \int_{x_B}^{1} \frac{dx}{x}\, C_j\left(\frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s\right) f_{j/P}(x,\mu_F^2) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{Q^2}\right).$$

- C_j: short-distance, perturbatively calculable coefficient function.
- $f_{j/P}$: long-distance, nonperturbative but universal PDF.
- μ_F: factorization scale separating collinear from non-collinear physics.
- Collinear gluon radiation is absorbed into the scale-dependent PDF $f_{j/P}(x,\mu_F^2)$.



Calculating C_j Using Partonic Targets

- The coefficient function is independent of the target state.
- To calculate the coefficient function, we **replace the proton by a perturbative parton state** $P \to q$.

For DIS on a single quark, the factorization formula becomes

$$F_{2q}(x, Q^2) = \sum_j C_j \otimes f_{j/q}.$$

Now both F_{2q} and $f_{j/q}$ are perturbatively calculable:

$$F_{2q} = F_{2q}^{(0)} + \alpha_s F_{2q}^{(1)} + \ldots,$$

$$f_{j/q} = f_{j/q}^{(0)} + \alpha_s f_{j/q}^{(1)} + \cdots.$$

Thus, the coefficient function can be obtained order-by-order perturbatively

$$C_j = C_j^{(0)} + \alpha_s C_j^{(1)} + \cdots,$$

Extracting the NLO Coefficient Function

At tree level, $f^{(0)}_{j/q}(x) = \delta(1-x)$. and therefore

$$C^{(0)}_j(x) = F^{(0)}_{2q}(x).$$

At one-loop order,

$$F^{(1)}_{2q} = C^{(1)}_j \otimes f^{(0)}_{j/q} + C^{(0)}_j \otimes f^{(1)}_{j/q}.$$

we obtain

$$\boxed{C^{(1)}_j = F^{(1)}_{2q} - C^{(0)}_j \otimes f^{(1)}_{j/q}.}$$

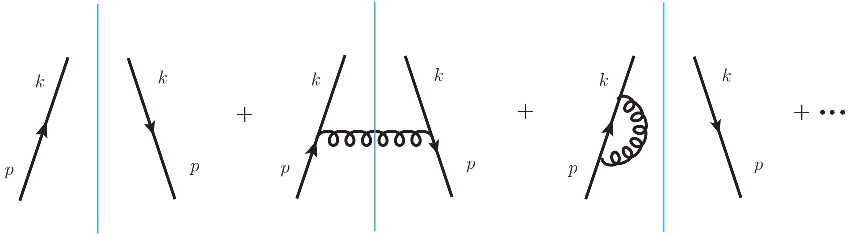
- $F^{(1)}_{2q}$: full one-loop DIS structure function.
- $C^{(0)}_j \otimes f^{(1)}_{j/q}$: one-loop quark PDF.
- Remaining finite part $C^{(1)}_j$. The coefficient function is obtained after subtracting collinear physics.

PDFs of a Quark

To calculate the quark PDF perturbatively, we change the external state: $|P\rangle \Longrightarrow |q(p)\rangle$. Then the PDF becomes

$$f_{j/q}(x) = \int \frac{dz^-}{2\pi} e^{-ixp^+z^-} \langle q(p)|\bar{\psi}^{(j)}(0^+, z^-, \mathbf{0}_\perp) W(z^-, 0^-) \frac{\gamma^+}{2} \psi^{(j)}(0)|q(p)\rangle.$$

which is calculable using Feynman diagrams.



It is relatively easier to perform the calculation in the light-cone gauge $A^+ = 0$ in which the gauge line $W(z^-, 0^-) = 1$. At the order $\mathcal{O}(\alpha_s)$ both real gluon emission diagram and virtual digram contribute.

PDFs of a Quark

- At tree level,
$$f_{j/q}^{(0)}(x) = \delta(1-x).$$
The incoming quark carries all of the longitudinal momentum $x = 1$.
- At order α_s, the one-loop correction contains logarithmic divergences:
$$\boxed{f_{j/q}^{(1)}(x) = C_F \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right] + \mathrm{UVCT}.}$$
The transverse momentum integral
$$\int \frac{dk_T^2}{k_T^2}$$
contains both collinear divergence ($k_T^2 \to 0$) and ultraviolet divergence ($k_T^2 \to \infty$). The ultraviolet counterterm (UVCT) renormalizes the PDF operator.

Partonic Structure Function

For massless parton $p^2 = 0$, the structure function $F_2(x, Q^2)$ can be projected out from the hadronic tensor $W_{\mu\nu}(x, Q^2)$.

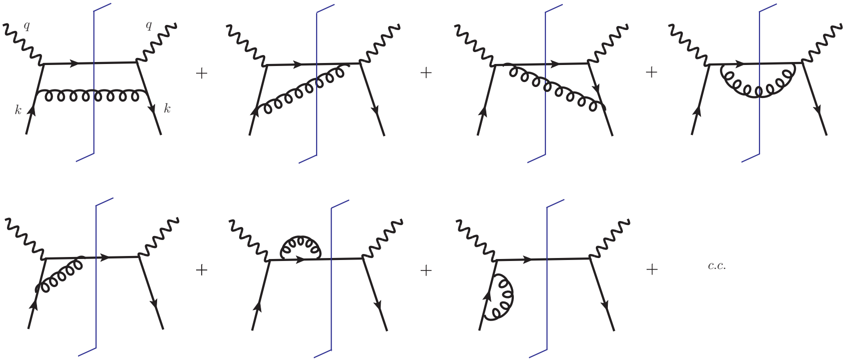
$$F_2(x, Q^2) = x\left(-g^{\mu\nu} + \frac{12x^2}{Q^2}p^\mu p^\nu\right)W_{\mu\nu}(x, Q^2).$$

To obtain $F_{2,q}(x, Q^2)$ up to the order $\mathcal{O}(\alpha_s)$, one needs to compute $W_{\mu\nu,q}(x, Q^2)$ to the order $\mathcal{O}(\alpha_s)$. At tree level,

$$F_{2q}^{(0)}(x) = xg^{\mu\nu}W_{\mu\nu,q}^{(0)} = e_q^2\, x\, \delta(1-x).$$

NLO Partonic Structure Function

At NLO, the following diagrams contribute to $W_{\mu\nu,q}$:



- First row contains real emission diagrams, representing gluon emission before and after the photon vertex and the interference terms.
- Second row contains virtual diagrams, representing vertex correction, quark self-energy correction.

NLO Partonic Structure Function

To regulate UV and collinear divergences, we work in $d = 4 - 2\epsilon$ dimensions. The projector for F_2 becomes

$$\boxed{(1-\epsilon)F_2 = x\left[-g^{\mu\nu} + (3-2\epsilon)\frac{4x^2}{Q^2}p^\mu p^\nu\right]W_{\mu\nu}.}$$

Strategy: computing separately

$$-g^{\mu\nu}W_{\mu\nu,q}, \qquad p^\mu p^\nu W_{\mu\nu,q}$$

.

Computing $-g^{\mu\nu}W_{\mu\nu}$

- At tree-level:

$$-g^{\mu\nu}W^{(0)}_{\mu\nu,q} = e_q^2(1-\epsilon)\delta(1-x).$$

- The one-loop correction contains both virtual and real contributions:$W^{(1)}_{\mu\nu} = W^{(1)V}_{\mu\nu} + W^{(1)R}_{\mu\nu}$.
  Virtual contribution:

$$-g^{\mu\nu}W^{(1)V}_{\mu\nu,q} = e_q^2(1-\epsilon)\delta(1-x)\left(-\frac{\alpha_s}{\pi}\right)C_F\left[\frac{4\pi\mu^2}{Q^2}\right]^\epsilon \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times\left[\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+4\right].$$

  Real contribution:

$$-g^{\mu\nu}W^{(1)R}_{\mu\nu,q} = e_q^2(1-\epsilon)C_F\left(\frac{\alpha_s}{2\pi}\right)\left[\frac{4\pi\mu^2}{Q^2}\right]^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{x}{1-x}\right)^\epsilon \times\left\{-\frac{1-\epsilon}{\epsilon}\left[(1-x)+\left(\frac{2x}{1-x}\right)\left(\frac{1}{1-2\epsilon}\right)\right]+\frac{1-\epsilon}{2(1-2\epsilon)(1-x)}+\frac{2\epsilon}{1-2\epsilon}\right\}.$$

The Plus Distribution

In the real contribution, as $x \to 1$, the factor

$$\frac{1}{1-x}$$

becomes singular. These endpoint singularities are handled using the plus distribution.

$$\left(\frac{1}{1-x}\right)^{1+\epsilon} = -\frac{1}{\epsilon}\delta(1-x) + \frac{1}{(1-x)_+} - \epsilon\left(\frac{\ln(1-x)}{1-x}\right)_+ + \mathcal{O}(\epsilon^2).$$

Definition of the plus distribution:

$$\int_z^1 dx\, \frac{f(x)}{(1-x)_+} = \int_z^1 dx\, \frac{f(x)-f(1)}{1-x} + f(1)\ln(1-z).$$

The plus prescription isolates the integrable singularity while preserving finite convolutions.

One-Loop Result and Splitting Function

Combining real and virtual contributions gives

$$
\begin{aligned}
-g^{\mu\nu} W^{(1)}_{\mu\nu,q} =& e_q^2 (1-\epsilon) \left(\frac{\alpha_s}{2\pi}\right) \Bigg\{ -\frac{1}{\epsilon} P_{qq}(x) + P_{qq}(x) \ln\left(\frac{Q^2}{\mu^2 (4\pi e^{-\gamma_E})}\right) \\
& + C_F \Bigg[(1+x^2) \left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x} \ln x + 3 - x \\
& - \left(\frac{9}{2} + \frac{\pi^2}{3}\right) \delta(1-x) \Bigg] \Bigg\}.
\end{aligned}
$$

The coefficient of the collinear pole defines the splitting function:

$$
\boxed{P_{qq}(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right].}
$$

One-Loop Contribution to $p^\mu p^\nu W_{\mu\nu}$

To reconstruct F_2, we also need the longitudinal projection $p^\mu p^\nu W_{\mu\nu}$.

- At one loop, the virtual diagrams do not contribute:

$$p^\mu p^\nu W^{(1)V}_{\mu\nu,q} = 0.$$

- The real diagrams give the contribution

$$p^\mu p^\nu W^{(1)R}_{\mu\nu,q} = e_q^2\, C_F \frac{\alpha_s}{2\pi} \frac{Q^2}{4x}.$$

One-Loop DIS Structure Function of a Quark

Combining the two tensor projections gives

$$F_{2q}^{(1)}(x,Q^2) = e_q^2\, x\, \frac{\alpha_s}{2\pi} \Bigg\{ \left(-\frac{1}{\epsilon}\right)_{\mathrm{CO}} P_{qq}(x)\left[1+\epsilon \ln\left(4\pi e^{-\gamma_E}\right)\right] + P_{qq}(x) \ln\frac{Q^2}{\mu^2}$$

$$+ C_F \Bigg[(1+x^2)\left(\frac{\ln(1-x)}{1-x}\right)_+ - \frac{3}{2}\left(\frac{1}{1-x}\right)_+ - \frac{1+x^2}{1-x}\ln x$$

$$+ 3 + 2x - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-x)\Bigg]\Bigg\}.$$

As $\epsilon \to 0$, the structure function diverges because of the collinear pole:

$$\left(-\frac{1}{\epsilon}\right)_{\mathrm{CO}} P_{qq}(x).$$

- The partonic DIS tensor is not infrared safe by itself (massless theory, long-distance singularity).

One-Loop Quark PDF of a Quark

The quark PDF operator between quark states gives $f^{(1)}_{j/q}(x, \mu_F^2)$. At one loop, using **dimensional regularization**:

$$f^{(1)}_{j/q}(x, \mu_F^2) = \left(\frac{\alpha_s}{2\pi}\right) P_{qq}(x) \left[\left(\frac{1}{\epsilon}\right)_{\mathrm{UV}} - \left(\frac{1}{\epsilon}\right)_{\mathrm{CO}}\right] + \mathrm{UVCT}.$$

Two distinct divergences appear:

- UV divergence: from renormalization of the PDF operator,
- Collinear divergence: from long-distance gluon radiation.

The UV counterterm removes the UV divergence:

$$\mathrm{UVCT} \sim -\left(\frac{1}{\epsilon}\right)_{\mathrm{UV}}.$$

After renormalization:

$$f^{\mathrm{ren}}_{j/q} \sim -\left(\frac{\alpha_s}{2\pi}\right) \frac{1}{\epsilon_{\mathrm{CO}}} P_{qq}(x).$$

Cancellation of Collinear Divergences

Recall the factorization formula:

$$F_{2q}^{(1)} = C_q^{(1)} + F_{2q}^{(0)} \otimes f_{j/q}^{(1)}.$$

The partonic DIS tensor contains:

$$F_{2q}^{(1)} \sim -\frac{1}{\epsilon_{\mathrm{CO}}} P_{qq}(x).$$

The renormalized PDF contains the same collinear pole:

$$f_{j/q}^{(1)} \sim -\frac{1}{\epsilon_{\mathrm{CO}}} P_{qq}(x).$$

The coefficient function:

$$C_j^{(1)} = F_{2q}^{(1)} - F_{2q}^{(0)} \otimes f_{j/q}^{(1)}$$

is finite after the cancellation of collinear divergences.

The same universal collinear divergence appears in both the partonic DIS tensor and the PDF.

Order $\mathcal{O}(\alpha_s)$ Coefficient Function

$$C_j^{(1)}(x,Q^2) = \frac{\alpha_s}{2\pi} e_q^2 x \Bigg[P_{qq}(x) \left[\ln \frac{Q^2}{\mu_F^2} + \gamma_E - \ln 4\pi \right] + C_F \Bigg((1+x^2) \left\{ \frac{\ln(1-x)}{(1-x)} \right\}_+ - \frac{3}{2} \left[\frac{1}{1-x} \right]_+ - (1+x^2) \frac{\ln x}{1-x} + 3 + 2x - \left(\frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \Bigg) \Bigg].$$

Remarks:

- The collinear subtraction of the PDF operator is not unique. Different subtraction prescriptions define different factorization schemes: $\overline{\mathrm{MS}}$, $\mathrm{DIS}, \cdots$.
- Once a factorization scheme is fixed (usually $\overline{\mathrm{MS}}$), PDFs become universal process-independent objects.

Evolution Equation from Factorization Scale Independence

The DIS structure function is physical and measurable $F_2(x_B, Q^2)$. Therefore it cannot depend on the arbitrary factorization scale:

$$\mu_F^2 \frac{d}{d\mu_F^2} F_2(x_B, Q^2) = 0.$$

Using collinear factorization,

$$F_2(x_B, Q^2) = \sum_j C_j \otimes f_j.$$

Differentiate the factorized expression:

$$\sum_j \left[\mu_F^2 \frac{d}{d\mu_F^2} C_j\right] \otimes f_j + \sum_j C_j \otimes \left[\mu_F^2 \frac{d}{d\mu_F^2} f_j\right] = 0.$$

The factorization scale appears in both objects: $C_j \sim \ln \frac{Q^2}{\mu_F^2}$, $f_j \sim \ln \frac{\mu_F^2}{\Lambda_{\mathrm{QCD}}^2}$,

Changing μ_F reshuffles collinear radiation between the coefficient function and the PDF.

Non-singlet DGLAP Equation

The coefficient function up to one-loop order

$$C_j = e_q^2 x \left(\delta(1-x) + \frac{\alpha_s}{2\pi} \left[P_{qq}(x) \ln \frac{Q^2}{\mu_F^2} + \text{finite} \right] + \cdots \right)$$

Differentiation with respect to the factorization scale:

$$\left(\mu_F^2 \frac{dC_j}{d\mu_F^2} \right) = \frac{\alpha_s}{2\pi} (-P_{qq})$$

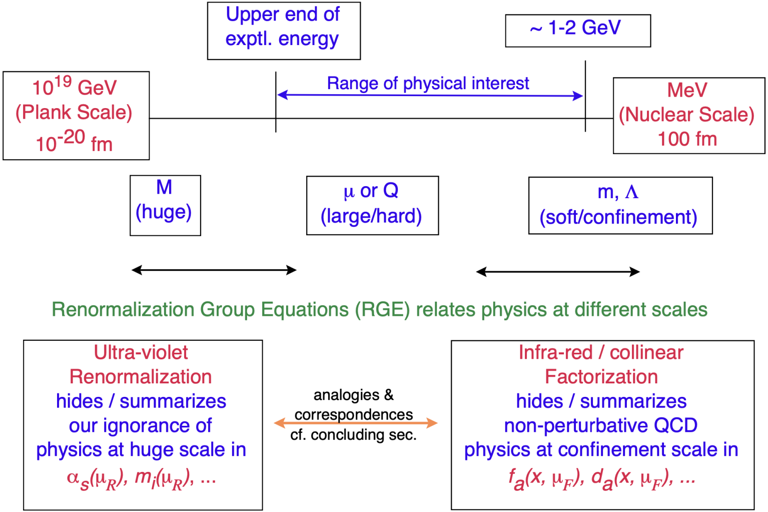
As a result

$$\boxed{\mu_F^2 \frac{\partial}{\partial \mu_F^2} f_j(x, \mu_F^2) = \frac{\alpha_s}{2\pi} P_{qq} \otimes f_j(x, \mu_F^2).}$$

This is the flavor non-singlet Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation. The leading-order quark splitting kernel is

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right].$$

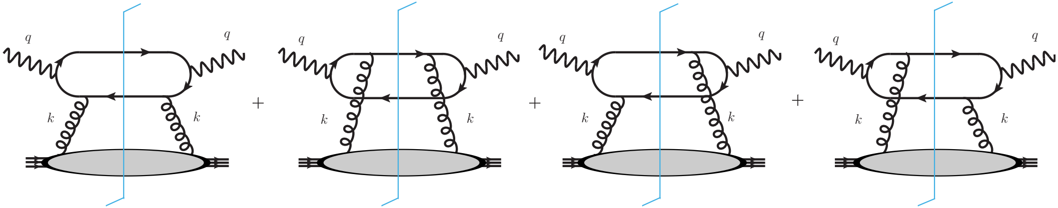
The importance of ***Scales*** -- Renormalization and Factorization



(Credit: Wu-Ki Tung)

Including Gluons in DIS Factorization

- At leading order, the photon couples directly to a quark $\gamma^* q \to q$.
- At next-to-leading order, gluons also contribute through $\gamma^* g \to q\bar{q}$.



Therefore the leading-power factorization formula for F_2 becomes

$$F_2(x_B, Q^2) = \sum_j C_j^q \otimes f_{j/P} + C^g \otimes f_{g/P} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{Q^2}\right).$$

Full Flavor Singlet DGLAP Evolution

For the quark flavor singlet distribution,

$$\Sigma(x,\mu_F^2) = \sum_j \left[q_j(x,\mu_F^2) + \bar{q}_j(x,\mu_F^2)\right],$$

the quark and gluon PDFs mix:

Singlet DGLAP Equation

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \begin{pmatrix} \Sigma \\ G \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ G \end{pmatrix}.$$

The kernels describe parton branchings:

$$P_{qq} : \; q \to qg, \qquad P_{qg} : \; g \to q\bar{q},$$

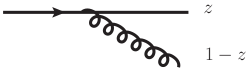
$$P_{gq} : \; q \to gq, \qquad P_{gg} : \; g \to gg.$$

At NLO and beyond, quarks and gluons evolve together as a coupled system.

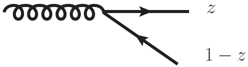
Leading-Order Splitting Functions

At leading order,

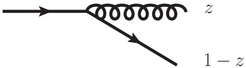
$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right],$$



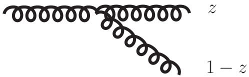
$$P_{qg}(z) = T_F\left[z^2 + (1-z)^2\right],$$



$$P_{gq}(z) = C_F\frac{1+(1-z)^2}{z},$$



$$P_{gg}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{\beta_0}{2}\delta(1-z).$$



where

$$C_F = \frac{N_c^2-1}{2N_c}, \qquad C_A = N_c, \qquad T_F = \frac{1}{2}, \qquad \beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f.$$