Introduction to QCD and Small-x Physics

Lecture 2: Deep Inelastic Scattering

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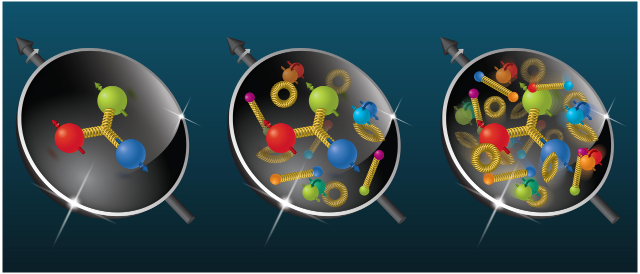
2026 CNUGS Summer School @ JLab

Understanding the Internal Structure of Proton



Credit:BNL

- **A proton is not simply a static uud bound state.**
- **The proton is a confined many-body system of quarks and gluons.**
- Besides the valence quarks, it contains gluons and sea quark pairs $q\bar{q}$.
- The apparent internal structure depends on the resolution scale of the probe.
- At higher energy / shorter distance, more partonic degrees of freedom become visible.

Understanding the Internal Structure of Proton

Credit: Rolf Ent (JLab) and Richard Milner (MIT).
Video source: `INT 2025 Program Precision QCD at EIC`

What Do We Want to Understand?

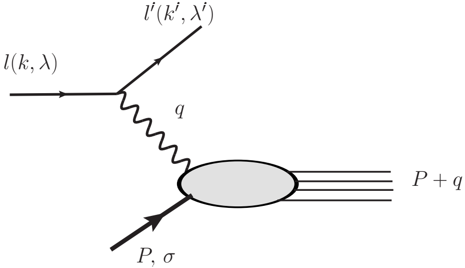
- How momentum is distributed among quarks and gluons?
- How do the mass and spin of the nucleon emerge from the quarks and gluons inside and their dynamics?
- How are the pressure and shear forces distributed inside the nucleon?
- How nonlinear many-body QCD dynamics emerges and whether gluon saturation occurs at very high density?
- How are hadrons formed from quarks and gluons produced in high-energy collisions?
- How does QCD generate the spectrum and structure of conventional and exotic hadrons?
- ...

Why Deep Inelastic Scattering (DIS)?

DIS provides a clean probe of the internal structure of hadrons.

- An energetic electron emits a highly virtual photon: $\gamma^*(q)$
- The virtual photon resolves short-distance structure inside the proton.
- The resolution scale is controlled by $Q^2 \equiv -q^2$.
- Large Q^2: short distances
- Small Q^2: long distances

$$l(k) + p(P) \to l(k') + X$$



DIS Experiments: A Timeline of Discovery

1960s – early 1970s
SLAC
(Fixed Target e–p)

KEY DISCOVERIES
- Point-like constituents
- Bjorken scaling
- Birth of parton model

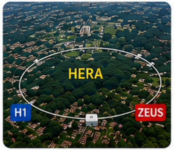
Q^2: 1 – 10 GeV^2
x: 0.1 – 0.6

1972 – early 1990s
CERN
(Muon–Nucleon DIS)

KEY DISCOVERIES
- Precision measurements of F_2, F_3
- Bjorken scaling confirmed
- Flavor separation of quarks
- Improved PDFs

Q^2: 1 – 50 GeV^2
x: 0.01 – 0.6

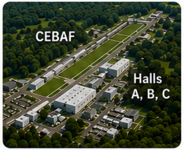
1992 – 2007
HERA (DESY)
($e^{\pm}$–p Collider)



KEY DISCOVERIES
- Small-x physics
- Gluon density rises rapidly at small x
- High-Q^2 regime
- Precise PDFs over wide x

Q^2: 1 – 10^5 GeV^2
x: 10^{-5} – 0.65

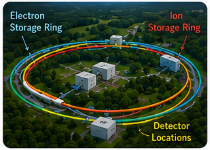
1995 – present
CEBAF (JLab)
(Fixed Target e–p/e–n/e–d/^{3}He)



KEY FOCUS
- Valence quark region
- Nucleon spin structure
- TMDs (transverse structure)
- GPDs (3D structure)
- 3D imaging of nucleons

Q^2: 1 – 10 (tens) GeV^2
x: 0.1 – 0.7

2030s and beyond
Electron–Ion Collider
(EIC, BNL)
(Polarized $e^{\pm}$–p/A Collider)



KEY GOALS
- Gluon saturation
- Small-x frontier
- Precision QCD
- 3D imaging of hadrons and nuclei
- Polarized DIS & e–ion collisions to unravel nucleon spin structure

Q^2: 1 – 10^4 GeV^2
x: 10^{-4} – 1

★ DIS experiments across decades and facilities have revealed the quark and gluon structure of matter and continue to open new frontiers in QCD.

DIS Kinematic Variables

Consider the inclusive process $e(k) + p(P) \to e(k') + X$, with momentum transfer $q^\mu = k^\mu - k'^\mu$.
The basic Lorentz-invariant variables are:

- **Energy transfer**:

$$\nu = \frac{q \cdot P}{M}$$

  In the nucleon rest frame, $\nu = E - E'$, where E and E' are the initial and final lepton energies.

- **Photon virtuality**:

$$Q^2 \equiv -q^2.$$

  Neglecting lepton masses, $Q^2 \simeq 4EE' \sin^2 \frac{\theta}{2}$, where θ is the lepton scattering angle.

- **Bjorken variable**:

$$x_B = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2M\nu}.$$

  In the parton model, x_B is momentum fraction carried by the struck quark.

Additional DIS Invariants

- **Inelasticity**:
$$y = \frac{P \cdot q}{P \cdot k} = \frac{\nu}{E}.$$
In the nucleon rest frame, y is the fraction of lepton energy lost in the collision.
- **Invariant mass** of the hadronic final state:
$$W^2 = (P + q)^2 = M^2 + 2M\nu - Q^2.$$
Elastic scattering corresponds to $W^2 = M^2$, while DIS requires $W^2 \gg M^2$.
- **Center-of-mass energy** of the lepton–proton :
$$s = (k + P)^2.$$
Neglecting lepton masses, $s \simeq 2k \cdot P + M^2$.

Deep inelastic scattering corresponds to large Q^2 and large W^2.

Tree-Level DIS Scattering Amplitude (Unpolarized)

At leading order, deep inelastic scattering proceeds through single virtual photon exchange. Using the QED Feynman rules, the scattering amplitude is

$$i\mathcal{M} = \bar{u}(k')(-ie\gamma^{\mu})u(k)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\langle X|ieJ^{\nu}(0)|P,\sigma\rangle,$$

The electromagnetic current of quarks is

$$\boxed{J^{\mu}(x) = \sum_f e_f\,\bar{\psi}_f(x)\gamma^{\mu}\psi_f(x)}$$

with quark electric charges $e_u = \frac{2}{3}, e_d = e_s = -\frac{1}{3}, \cdots$

The leptonic part is perturbatively calculable, while the hadronic matrix element contains the nonperturbative structure of the nucleon.

Inclusive DIS Cross Section

For the inclusive process the tree-level differential cross section can be written as

$$\boxed{d\sigma = \frac{1}{F}\frac{d^3\mathbf{k}'}{(2\pi)^3\,2E_{\mathbf{k}'}}\frac{4\pi e^4}{Q^4}L^{\mu\nu}(k,k')\,W_{\mu\nu}(P,q)}$$

where the flux factor $F = 2\sqrt{(P\cdot k)^2 - M^2 m_e^2}$, and $Q^2 = -q^2$.
The leptonic tensor is

$$L^{\mu\nu} = \frac{1}{2}\sum_{\lambda,\lambda'}\bar{u}_{\lambda'}(k')\gamma^{\nu}u_{\lambda}(k)\,\bar{u}_{\lambda}(k)\gamma^{\mu}u_{\lambda'}(k').$$

The hadronic tensor is

$$W_{\mu\nu} = \frac{1}{4\pi}\frac{1}{2}\sum_{\sigma,X}(2\pi)^4\delta^{(4)}(P+q-P_X)\langle P,\sigma|J^{\dagger}_{\mu}(0)|X\rangle\langle X|J_{\nu}(0)|P,\sigma\rangle.$$

Computing the Leptonic Tensor

For **unpolarized DIS**, we average over the initial electron spin λ and sum over the final electron spin λ':

$$L^{\mu\nu}(k,k') = \frac{1}{2}\sum_{\lambda,\lambda'} \bar{u}_{\lambda'}(k')\gamma^\nu u_\lambda(k)\,\bar{u}_\lambda(k)\gamma^\mu u_{\lambda'}(k').$$

Using the spin sums for massless electrons, $\sum_\lambda u_\lambda(k)\bar{u}_\lambda(k) = \not{k}$, we obtain

$$L^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left[\not{k}'\gamma^\nu\not{k}\gamma^\mu\right].$$

Using the trace identity $\mathrm{Tr}\left[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu\right] = 4\left(g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\nu\beta}\right)$, we find

$$L^{\mu\nu} = 2\left[k'^\nu k^\mu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu}\right].$$

General Structure of the Hadronic Tensor

Starting from the inclusive sum over final states,

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\frac{1}{2}\sum_{\sigma,X}(2\pi)^4\delta^{(4)}(P+q-P_X)\langle P,\sigma|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|P,\sigma\rangle.$$

Using

$$(2\pi)^4\delta^{(4)}(P+q-P_X) = \int d^4z\, e^{i(P+q-P_X)\cdot z},$$

the completeness $\sum_X |X\rangle\langle X| = 1$, and the translation invariance $J_\mu^\dagger(z) = e^{i\hat{P}\cdot z}J_\mu^\dagger(0)e^{-i\hat{P}\cdot z}$, one obtains

$$\boxed{W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|J_\mu^\dagger(z)J_\nu(0)|P\rangle.}$$

- The average over the initial proton spin is implicit in the last final expression.

Lorentz Decomposition of the Hadronic Tensor

The hadronic tensor

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|J^\dagger_\mu(z)J_\nu(0)|P\rangle$$

depend only on $P^\mu, q^\mu, g^{\mu\nu}$. The most general decomposition consistent with parity is in the basis of $g_{\mu\nu}, P_\mu P_\nu, q_\mu q_\nu, P_\mu q_\nu, q_\mu P_\nu$. Further imposing electromagnetic current conservation $q^\mu W_{\mu\nu} = W_{\mu\nu}q^\nu = 0$ lead to the tensor structure

$$\boxed{W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1(x_B,Q^2) + \left(P_\mu - \frac{P\cdot q}{q^2}q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2}q_\nu\right)W_2(x_B,Q^2).}$$

The scalar functions

$$W_1(x_B,Q^2), \qquad W_2(x_B,Q^2)$$

are known as the *structure functions* and contain all nonperturbative information about the target.

Inclusive DIS Cross Section in Terms of W_1 and W_2

Using

$$L^{\mu\nu}W_{\mu\nu} = 2Q^2 W_1 + \left[\frac{1}{2}(s - M^2 - 2P\cdot q)(s - M^2) - M^2 Q^2\right] W_2,$$

In the proton rest frame, $P^\mu = (M, \mathbf{0})$, the inclusive DIS cross section becomes

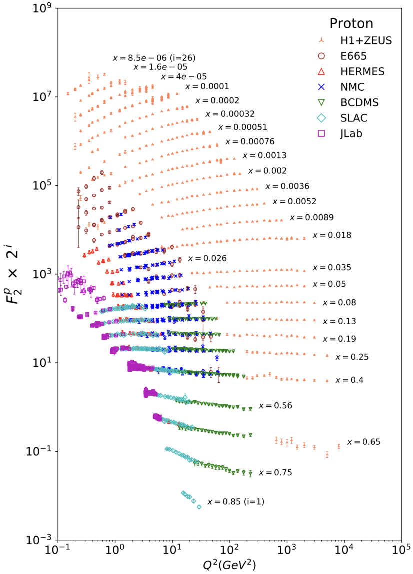
$$\boxed{\frac{d\sigma}{dE' d\Omega} = \frac{\alpha_{\rm em}^2}{4ME^2 \sin^4\frac{\theta}{2}}\left[2\sin^2\frac{\theta}{2}\, W_1(x, Q^2) + M^2\cos^2\frac{\theta}{2}\, W_2(x, Q^2)\right].}$$

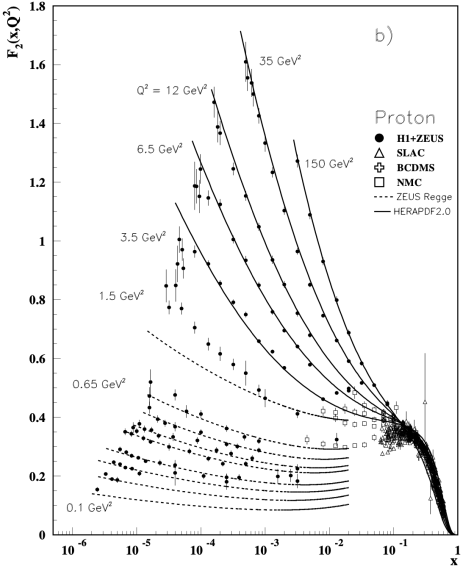
By measuring the energy and angular distribution of the scattered electron, one can extract the structure functions W_1 and W_2.

Dimensionless structure functions:

$$F_1(x_B, Q^2) \equiv W_1(x_B, Q^2), \qquad F_2(x_B, Q^2) = \nu W_2(x_B, Q^2). \tag{1}$$

Proton F_2 Data (Source: PDG)





Computing F_1 and F_2 is Difficult

The hadronic tensor is

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x}\langle P|J_\mu(x)J_\nu(0)|P\rangle.$$

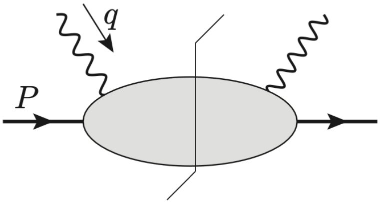
This is a **nonperturbative** correlation function in QCD.

- The proton is a strongly interacting bound state of quarks and gluons.
- At low energies, $\alpha_s \sim 1$, so perturbation theory is not directly applicable.
- The final hadronic state X contains infinitely many possible multiparticle states.
- The structure functions $F_1(x,Q^2), F_2(x,Q^2)$ therefore encode complicated nonperturbative information about the proton.

The Hadronic Tensor

Inclusive deep inelastic scattering is described by the hadronic tensor

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|J_\mu(z)J_\nu(0)|P\rangle.$$



- It is an integral over the whole spacetime region. Instead of asking how to calculate $W_{\mu\nu}(P,q)$, the right question to ask is what spacetime region z^μ dominates the integral.
- It depends on P^μ, q^μ that form Lorentz invariants $P^2, P\cdot q, q^2$. Equivalently it is characterized by values of $x_B = -q^2/2P\cdot q$ and $Q^2 = -q^2$.

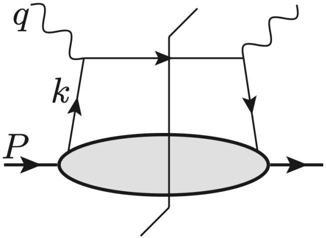
Different QCD descriptions emerge only after specifying the asymptotic kinematic limit (x_B, Q^2).

Bjorken Limit: Handbag Dominance

Bjorken limit

$$Q^2 \to \infty, \qquad x_B \text{ is fixed } (0 < x_B < 1). \qquad (s \sim Q^2)$$

- The dominant momentum region is **collinear to the proton**:
$$k^\mu \sim (Q, \Lambda^2_{\mathrm{QCD}}/Q, \Lambda_{\mathrm{QCD}})$$
- The struck quark propagator becomes highly off-shell $(k+q)^2 \sim Q^2$.
- The interaction is localized at short distances: $z^2 \sim 1/Q^2$
- This leads to the **collinear factorization**.



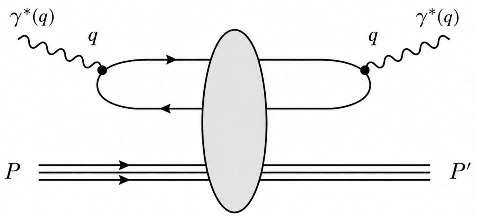
Handbag diagram

Regge Limit: Dipole Dominance

Regge / small-x limit

$$x_B \to 0, \qquad Q^2 \text{ is fixed}, \qquad (s \gg Q^2).$$

- The virtual photon fluctuates long before interacting with the target $\gamma^* \to q\bar{q}$.
- The fluctuation lifetime is parametrically long $\Delta x^+ \sim 1/x_B P^+$.
- The $q\bar{q}$ pair **eikonally** traverses the target gluon field.
- This leads to the **rapidity factorization**.



Dipole picture

Lowest-Order Collinear Factorization

In the **Bjorken limit**, the dominant diagram is the handbag diagram. Starting from

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|J_\mu^\dagger(z)J_\nu(0)|P\rangle,$$

and using $J^\mu(z) = \sum_f e_f\bar{\psi}_f(z)\gamma^\mu\psi_f(z)$, we obtain

$$\langle P|J_\mu^\dagger(z)J_\nu(0)|P\rangle = \sum_f e_f^2\left\langle P\left|\bar{\psi}_f(z)\gamma_\mu\psi_f(z)\bar{\psi}_f(0)\gamma_\nu\psi_f(0)\right|P\right\rangle.$$

At lowest order, the outgoing quark is contracted across the cut, using the cut propagator:

$$S(z) = \langle 0|\psi_f(z)\bar{\psi}_f(0)|0\rangle = \int\frac{d^4k'}{(2\pi)^4}e^{-ik'\cdot z}\frac{i\not{k}'}{k'^2+i\epsilon} \Longrightarrow \int\frac{d^4k'}{(2\pi)^4}e^{-ik'\cdot z}\, i\not{k}'\left[-2\pi i\,\delta(k'^2)\right],$$

where

$$\mathrm{Disc}\left(\frac{1}{k'^2+i\epsilon}\right) = -2\pi i\,\delta(k'^2),$$

Cut Quark Propagator and Factorized Form

The hadronic tensor becomes

$$W_{\mu\nu} = \frac{1}{4\pi} \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta((k+q)^2) \sum_f e_f^2 \mathrm{Tr}\left[\gamma_\mu (\not{k} + \not{q}) \gamma_\nu \, \Phi^f(k,P)\right]$$

$$= \frac{1}{4\pi} \sum_f \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\mathcal{H}^f_{\mu\nu}(k,q) \Phi^f(k,P)\right]$$

The quark correlation function is

$$\boxed{\Phi^f_{ij}(k,P) = \int d^4z \, e^{-ik\cdot z} \langle P | \bar{\psi}_{f,j}(z) \psi_{f,i}(0) | P \rangle.}$$

and the hard scattering tensor:

$$\boxed{\mathcal{H}^f_{\mu\nu}(k,q) = e_f^2 \gamma_\mu (\not{k} + \not{q}) \gamma_\nu (2\pi)\delta((k+q)^2).}$$

Bjorken Kinematics and Collinear Scaling

Using the light-cone coordinates ($v^\mu = (v^+, v^-, \mathbf{v}_\perp)$) and in the **Breit frame** (A Lorentz frame in which the space-like photon has zero energy and its largest 3-momentum is along the $-z$ direction), the momenta can be parameterized by

$$q^\mu = \left(-x_B P^+, \frac{Q^2}{2x_B P^+}, \mathbf{0}_\perp\right), \qquad P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_\perp\right),$$

$$k^\mu = \left(\xi P^+, k^-, \mathbf{k}_\perp\right), \qquad k'^\mu = k^\mu + q^\mu = \left((\xi - x_B)P^+, \frac{Q^2}{2x_B P^+} + k^-, \mathbf{k}_\perp\right).$$

with $x_B P^+ = Q/\sqrt{2}$ in the Breit frame.

For a nearly on-shell collinear quark,

$$k^2 = 2k^+k^- - k_\perp^2 \sim \Lambda_{\rm QCD}^2.$$

Its momentum scales as

$$\boxed{k^\mu \sim \left(Q, \frac{\Lambda_{\rm QCD}^2}{Q}, \Lambda_{\rm QCD}\right).}$$

Leading-Power Approximation

For the outgoing quark momentum, $k' = k + q$, using

$$
\begin{aligned}
(k+q)^2 &= 2(k^+ + q^+)(k^- + q^-) - k_\perp^2 \\
&\simeq 2(\xi - x_B)P^+ q^- + \mathcal{O}(\Lambda_{\mathrm{QCD}}^2).
\end{aligned}
$$

One gets

$$
\boxed{\delta((k+q)^2) \simeq \frac{1}{2P^+q^-}\delta(\xi - x_B) + \mathcal{O}(\Lambda_{QCD}^2/Q^2).}
$$

For the numerator,

$$
\begin{aligned}
\not{k} + \not{q} =& (k^+ + q^+)\gamma^- + (k^- + q^-)\gamma^+ - k_\perp^i \gamma^i \\
=& (\xi - x_B)P^+\gamma^- + (k^- + q^-)\gamma^+ - k_\perp^i \gamma^i.
\end{aligned}
$$

Using $q^- \sim Q$ and $k^- \sim \Lambda_{\mathrm{QCD}}^2/Q$, the leading term is

$$
\boxed{\not{k} + \not{q} \simeq q^-\gamma^+ + \mathcal{O}(\Lambda_{QCD}).}
$$

Leading Dirac Structure of $\Phi^f(k, P)$

The quark correlator is a Dirac matrix:

$$\Phi^f(k, P) = S + V_\mu \gamma^\mu + \frac{1}{2} T_{\mu\nu} \sigma^{\mu\nu}.$$

For an unpolarized proton, $V_\mu = AP_\mu + Bk_\mu$. The leading contribution comes from the large component $P^+\gamma^-$. Therefore,

$$\boxed{\Phi^f(k, P) \simeq \frac{1}{4} \mathrm{Tr}[\gamma^+ \Phi^f(k, P)] \gamma^-.}$$

Collinear Factorization and Parton Distribution Functions

The leading approximations for $W_{\mu\nu}$,

$$W_{\mu\nu} = \sum_f \int d\xi \, H^{f,(0)}_{\mu\nu}(\xi) f_{f/P}(\xi) + \mathcal{O}(\frac{\Lambda^2_{QCD}}{Q^2})$$

$\longleftarrow$ Collinear Factorization

Quark PDF

$$f_{f/P}(\xi) = \int \frac{dk^- d^2\mathbf{k}_\perp}{(2\pi)^4} \mathrm{Tr}\left[\frac{\gamma^+}{2}\Phi^f(k,P)\right]\Big|_{k^+=\xi P^+}$$

$$= \int \frac{dz^-}{2\pi} e^{-i\xi P^+ z^-} \left\langle P \middle| \bar{\psi}_f(0^+, z^-, \mathbf{0}_\perp)\frac{\gamma^+}{2}\psi_f(0) \middle| P \right\rangle .$$

The hard scattering coefficient is

$$H^{f,(0)}_{\mu\nu}(\xi) = e^2\delta(\xi - x_B)\,\frac{1}{8}\mathrm{Tr}[\gamma_\mu\gamma^+\gamma_\nu\gamma^-],$$

At leading power, DIS factorizes into a perturbative hard scattering coefficient and universal parton distribution functions.

Structure Functions in the Parton Model

Using the projection to structure functions

$$F_1(x_B, Q^2) = \frac{1}{2}\left(-g^{\mu\nu} + \frac{4x^2}{Q^2}P^\mu P^\nu\right)W_{\mu\nu},$$
$$F_2(x_B, Q^2) = x\left(-g^{\mu\nu} + \frac{12x^2}{Q^2}P^\mu P^\nu\right)W_{\mu\nu},$$

one finds

$$F_1(x_B, Q^2) = \sum_f \frac{1}{2}e_f^2 f_{f/P}(x_B), \qquad F_2(x_B, Q^2) = \sum_f e_f^2 x_B f_{f/P}(x_B).$$

One then obtains the Callan-Gross relation

$$\boxed{F_2(x_B, Q^2) = 2x_B F_1(x_B, Q^2).}$$

Bjorken Scaling

The proton structure function F_2^p (source: PDG).

Parton model predicts that at fixed x the structures functions are independent of Q, This is called “Bjorken Scaling”. It is violated after allowing for QCD interactions.

- Bjorken scaling is approximately true at moderate x.
- The scaling violation becomes more prominent as x gets closer to 1 or 0.