Introduction to QCD and Small-x Physics

Lecture 1: Fundamentals of QCD

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Outline of the Lectures

- **Lecture 1: Fundamentals of QCD**
- **Lecture 2: Deep Inelastic Scattering in the Bjorken Limit I**
- **Lecture 3: Deep Inelastic Scattering in the Bjorken Limit II**
- **Lecture 4: Deep Inelastic Scattering in the Regge Limit (Small-x)**

References

1. John Collins, *“Foundations of Perturbative QCD”*, 2011, Cambridge.
2. George Sterman, *“An Introduction to Quantum Field Theory”*, 1993, Cambridge.
3. Yuri V. Kovchegov and Eugene Levin, *“Quantum Chromodynamics at High Energy”*, 2012, Cambridge.
4. Michael E. Peskin and Daniel V. Schroeder, *“An Introduction to Quantum Field Theory”*, 1995, Addison-Wesley.
5. Jianwei Qiu’s HUGS 2021 Lecture, *“Introduction to QCD”*.
6. Particle Data Group (PDG), *“The Review of Particle Physics (2025)”.*
7. ChatGPT.

The Fundamental Degrees of Freedom of QCD

Quark Fields

$$\psi_i^f(x)$$

- Spin-$\frac{1}{2}$ fermion field
- Flavor index: $f = u, d, s, c, b, t$
- Color index: $i = 1, 2, 3$
- Fundamental representation of $SU(3)$, 3 dimensional

Gluon Fields

$$A_\mu^a(x)$$

- Spin-1 vector boson field
- Lorentz index: $\mu = 0, 1, 2, 3$
- Adjoint color index: $a = 1, \cdots, 8$
- Adjoint representation of $SU(3)$, 8 dimensional

From Abelian to Non-Abelian Gauge Theory

QED: Abelian symmetry $U(1)$ associated with electric charge.
QCD: non-Abelian symmetry $SU(3)$ associated with color charge.

$SU(3)$ Global Symmetry

Quark Fields

Quarks $\psi_i(x)$ transform under global $SU(3)$:

$$\psi(x) \to U\,\psi(x), \qquad U = e^{i\theta^a t^a}.$$

Generators:

$$[t^a, t^b] = i f^{abc} t^c.$$

Quarks belong to the fundamental representation with U being a 3×3 matrix.

Gluon Fields

The gluon field is $A_\mu(x) = A_\mu^a(x) T^a$. Under a global $SU(3)$ transformation:

$$A_\mu \to U A_\mu U^\dagger, \qquad U = e^{i\theta^a T^a}.$$

The adjoint generators satisfy

$$(T^a)_{bc} = -i f^{abc}.$$

Gluons belong to the adjoint representation with U being a 8×8 matrix.

- *Different $SU(3)$ transformations generally do not commute with one another, unlike the Abelian $U(1)$ group.*
- QCD is invariant under global $SU(3)$ color symmetry.

Group Structure of $SU(3)$

$SU(3)$ is the group of special unitary transformations. (Recall the $SU(2)$ group for spin)

- $\mathbf{U}^\dagger \mathbf{U} = \mathbf{1}$: unitary transformations preserve probability.
- $\det \mathbf{U} = \mathbf{1}$: "special" condition removing the overall $U(1)$ phase.

There are eight generators $t^a \ (a = 1, 2, \ldots 8)$ satisfying the commutation relation

$$[t^a, t^b] = i f^{abc} t^c, \qquad [T^a, T^b] = i f^{abc} T^c.$$

The coefficients f^{abc} are the totally antisymmetric structure constants.
Color matrix identities (for a general $SU(N_c)$ group):

$$\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab}, \qquad t^a t^b = C_F \mathbf{1},$$

$$\mathrm{tr}[T^a T^b] = C_A \delta^{ab}, \qquad T^a T^b = C_A \mathbf{1}$$

with $C_F = (N_c^2 - 1)/2N_c$ and $C_A = N_c$.

Gell–Mann Matrices

The generators of the fundamental representation are $t^a = \frac{\lambda^a}{2}$.
The Gell–Mann matrices are

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

- Generalization of the $SU(2)$ group where Pauli matrices are the generators.

Gauge Principle in Quantum Field Theory

Fundamental interactions arise from demanding that global symmetries hold independently at every spacetime point.

Global Symmetry

$U \in SU(3), \qquad U =$ constant

Quark field: $\psi(x) \to U\psi(x)$

Gluon field: $A_\mu(x) \to U A_\mu(x) U^\dagger$

- The same color rotation acts everywhere in spacetime.

$$\Longrightarrow$$

Local Symmetry

$$U \to U(x)$$

$$\psi(x) \to U(x)\psi(x)$$

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

- The color transformation can rotate independently at each spacetime point.

Local Gauge Invariance and Quark–Gluon Interactions

- To make the free quark Lagrangian locally gauge invariant, replace ordinary derivatives with covariant derivatives.

$$\partial_\mu \quad \longrightarrow \quad D_\mu = \partial_\mu - igA_\mu^a t^a$$

The free quark Lagrangian becomes

$$\bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad \longrightarrow \quad \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.$$

Expanding the covariant derivative gives

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\,\bar{\psi}\gamma^\mu A_\mu^a t^a \psi.$$

Local gauge invariance automatically generates the interaction between quarks and gluons.

Local Gauge Invariance and Gluon Self-Interactions

- Local gauge invariance also determines the dynamics of the gluon field.

The Yang–Mills field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu .$$

The gluon Lagrangian is

$$\mathcal{L}_{\text{gluon}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} .$$

Gluon self-interactions: gluons carry color charge and therefore interact directly with other gluons.

The Full QCD Lagrangian

QCD Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \\
&= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\bar{\psi}\gamma^\mu A^a_\mu t^a \psi \\
&\quad - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) \\
&\quad - \frac{1}{2} g f^{abc}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{b\mu} A^{c\nu} - \frac{1}{4} g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}.
\end{aligned}
$$

- The last two terms describe three-gluon and four-gluon self-interactions.

Gauge Covariance of the Quark Derivative

Under a local $SU(3)$ transformation,

$$\psi'(x) = U(x)\psi(x), \qquad A'_\mu(x) = UA_\mu U^\dagger + \frac{i}{g}U(\partial_\mu U^\dagger).$$

Using $D_\mu = \partial_\mu - igA_\mu$, we compute

$$D'_\mu\psi' = (\partial_\mu - igA'_\mu)\psi'.$$

$$= (\partial_\mu U)\psi + U\partial_\mu\psi - ig\left[UA_\mu U^\dagger + \frac{i}{g}U(\partial_\mu U^\dagger)\right]U\psi.$$

$$= (\partial_\mu U)\psi + U\partial_\mu\psi - igUA_\mu\psi - (\partial_\mu U)\psi.$$

$$= U(\partial_\mu - igA_\mu)\psi = UD_\mu\psi$$

Since $UU^\dagger = 1$,

$$(\partial_\mu U)U^\dagger = -U(\partial_\mu U^\dagger),$$

$$\boxed{D'_\mu\psi' = UD_\mu\psi}$$

Gauge Transformation of the Field Strength Tensor

The non-Abelian field strength tensor can be written compactly as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

where $A_\mu = A_\mu^a t^a$. Under a local $SU(3)$ gauge transformation,

$$A_\mu \to A'_\mu = U A_\mu U^\dagger + \frac{i}{g} U(\partial_\mu U^\dagger).$$

Substituting into the definition of $F_{\mu\nu}$,

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu - ig[A'_\mu, A'_\nu] \\ &= \partial_\mu \left(U A_\nu U^\dagger + \frac{i}{g} U \partial_\nu U^\dagger \right) - \partial_\nu \left(U A_\mu U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger \right) - ig[A'_\mu, A'_\nu]. \end{aligned}$$

The derivative terms involving $U(x)$ cancel against contributions from the commutator term, yielding

$$\boxed{F'_{\mu\nu} = U F_{\mu\nu} U^\dagger}$$

Local Gauge Invariance of QCD

The quark part transforms as

$$\bar{\psi}'(i\gamma^\mu D'_\mu - m)\psi' = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.$$

The field strength transforms covariantly:

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^\dagger.$$

Therefore,

$$\mathrm{Tr}(F'_{\mu\nu} F'^{\mu\nu}) = \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}),$$

so the gluon part is also gauge invariant.

Gauge Fixing and Faddeev–Popov Ghosts

Local gauge symmetry is a redundancy of the field description, not a physical symmetry.

For a massless spin-1 particle:

$$A_\mu \quad \Rightarrow \quad 4 \text{ components}$$

but only two transverse polarizations are physical. Manifest Lorentz Invariance!

Gauge redundancy $\rightarrow$ gauge fixing
$\rightarrow$ Faddeev-Popov determinant $\rightarrow$ ghost fields

Ghost fields are not physical particles; they cancel unphysical gauge-field fluctuations in quantum loops.

Quantization: Gauge-Fixed QCD Lagrangian

Complete QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{c}^a\partial^\mu(D^{ab}_\mu c^b).$$

Here ξ is the gauge parameter:

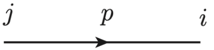
$\xi = 1$ Feynman gauge, $\xi = 0$ Landau gauge.

The Faddeev–Popov ghost term contains adjoint covariant derivative

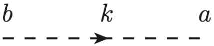
$$D^{ab}_\mu = \delta^{ab}\partial_\mu + gf^{aeb}A^e_\mu.$$

Ghost fields are auxiliary Grassmann fields that remove unphysical gauge fluctuations in quantum loops.

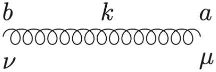
QCD Feynman Rules: Propagators

Quark propagator: 

$$= \frac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon}\,\delta^{ij},$$

Ghost propagator: 

$$= \frac{i}{k^2 + i\epsilon}\,\delta^{ab},$$

Gluon propagator: 

$$= \frac{-i\,D_{\mu\nu}(k)}{k^2 + i\epsilon}\,\delta^{ab},$$

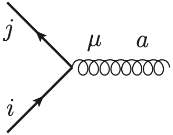
In a covariant Lorenz gauge,

$$D_{\mu\nu}(k) = g_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2}.$$

$\xi = 0:$ Landau gauge, $\xi = 1:$ Feynman gauge.

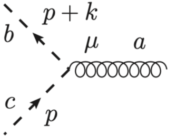
QCD Feynman Rules: Matter Vertices

Quark–gluon vertex:



$= ig\gamma^{\mu}(t^{a})_{ji}$,

Ghost–gluon vertex (Lorenz gauge only):



$= g(p+k)^{\mu} f^{abc}$

The quark–gluon interaction originates from

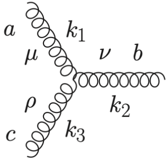
$$\mathcal{L}_{q\bar{q}g} = g\bar{\psi}\gamma^{\mu}A_{\mu}^{a}t^{a}\psi.$$

The ghost–gluon interaction arises from the Faddeev–Popov term

$$\mathcal{L}_{\text{ghost}} = -\bar{c}^{a}\partial^{\mu}(D_{\mu}^{ab}c^{b}).$$

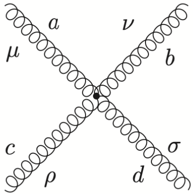
QCD Feynman Rules: Gluon Self-Interaction Vertices

Three-gluon vertex (all momenta flow into the vertex):



$$= -gf^{abc}\,[(k_1-k_3)^{\nu}g^{\mu\rho} + (k_2-k_1)^{\rho}g^{\mu\nu} + (k_3-k_2)^{\mu}g^{\nu\rho}]$$

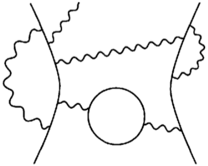
Four-gluon vertex:



$$= -ig^2\left[f^{abe}\,f^{cde}\,(g^{\mu\rho}\,g^{\nu\sigma} - g^{\mu\sigma}\,g^{\nu\rho}) + f^{ace}\,f^{bde}\,(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\sigma}\,g^{\nu\rho}) + f^{ade}\,f^{bce}\,(g^{\mu\nu}\,g^{\rho\sigma} - g^{\mu\rho}\,g^{\nu\sigma})\right]$$

Beyond Tree Level: Ultraviolet Divergences

Tree-level calculations are finite, but loop diagrams generally contain divergent momentum integrals.



$$\sim \int \frac{d^4k_1\, d^4k_2 \cdots d^4k_L}{(\not{k}_i - m)\cdots(k_j^2)\cdots(k_n^2)}.$$

As the loop momentum $k \to \infty$, these integrals can diverge.

Ultraviolet (UV) Divergences

UV divergences arise from short-distance quantum fluctuations at very large momentum.

Quantum field theory requires a systematic procedure to handle UV divergences.
The divergences are absorbed into redefinitions of fields, masses, and couplings.

Bare QCD Lagrangian

Bare Gauge-Fixed QCD Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F^a_{0\mu\nu} F_0^{a\mu\nu} + \bar{\psi}_0 (i\gamma^\mu D_{0\mu} - m_0)\psi_0 - \frac{1}{2\xi_0}(\partial_\mu A_0^{a\mu})^2 - \bar{c}_0^a\, \partial^\mu (D^{ab}_{0\mu} c_0^b).$$

with

$$D_{0\mu} = \partial_\mu - i g_0 A^a_{0\mu} t^a,$$

$$F^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}.$$

The Lagrangian obtained so far is written in terms of bare quantities:

$$\psi_0, \qquad A^a_{0\mu}, \qquad c^a_0, \qquad g_0, \qquad m_0.$$

These bare quantities are not directly measurable. Physical observables must instead be expressed in terms of renormalized fields and couplings:

$$\psi, \qquad A^a_\mu, \qquad c^a, \qquad g, \qquad m.$$

From Bare to Renormalized Quantities

Introduce renormalized fields and parameters:

$$\psi_0 = \sqrt{Z_2}\psi, \qquad A^a_{0\mu} = \sqrt{Z_3}A^a_\mu, \qquad c^a_0 = \sqrt{\widetilde{Z}_3}c^a,$$

$$m_0 = Z_m m, \qquad g_0 = Z_g g, \qquad \xi_0 = Z_3\xi.$$

Then the bare Lagrangian can be rewritten as

$$\mathcal{L}_0 = \mathcal{L}_{\rm ren} + \mathcal{L}_{\rm ct}.$$

Renormalized QCD Lagrangian

$$\mathcal{L}_{\rm ren} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{c}^a\partial^\mu(D^{ab}_\mu c^b).$$

where

$$D_\mu = \partial_\mu - igA^a_\mu t^a, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu.$$

Counterterm Lagrangian

Counterterm Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm ct} =& \delta Z_2\, \bar{\psi} i\gamma^\mu \partial_\mu \psi - \delta Z_m\, m\bar{\psi}\psi + \delta Z_1\, g\bar{\psi}\gamma^\mu A^a_\mu t^a \psi \\ & - \frac{1}{4}\delta Z_3\, G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2}\delta Z_{3g}\, g f^{abc} G^a_{\mu\nu} A^{b\mu} A^{c\nu} - \frac{1}{4}\delta Z_{4g}\, g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} \\ & - \delta\widetilde{Z}_3\, \bar{c}^a \partial^2 c^a - \delta\widetilde{Z}_1\, g f^{abc} (\partial^\mu \bar{c}^a) A^c_\mu c^b .\end{aligned}$$

Here $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$,

- The counterterms have the same operator structure as the original QCD Lagrangian.
- Their coefficients are chosen order by order to cancel UV divergences.
- Counterterms do not introduce new physics; they absorb UV divergences into field, mass, and coupling renormalizations.

Renormalization Constants in QCD

The counterterm coefficients

$$\delta Z_2 \equiv Z_2 - 1, \qquad \delta Z_m \equiv Z_2 Z_m - 1, \qquad \delta Z_3 \equiv Z_3 - 1, \qquad \delta \widetilde{Z}_3 \equiv \widetilde{Z}_3 - 1,$$

$$\delta Z_1 \equiv Z_1 - 1, \qquad \delta Z_{3g} \equiv Z_{3g} - 1, \qquad \delta Z_{4g} \equiv Z_{4g} - 1, \qquad \delta \widetilde{Z}_1 \equiv \widetilde{Z}_1 - 1,$$

The vertex renormalization constants are defined as

$$Z_1 = Z_g Z_2 Z_3^{1/2}, \qquad Z_{3g} = Z_g Z_3^{3/2}, \qquad Z_{4g} = Z_g^2 Z_3^2, \qquad \widetilde{Z}_1 = Z_g \widetilde{Z}_3 Z_3^{1/2}.$$

- *Gauge invariance imposes nontrivial relations among these renormalization constants.*

Dimensional Regularization and the Scale μ?

- In $d = 4 - 2\epsilon$ dimensions, the gauge coupling is no longer dimensionless.
- Using the fact that the action $S = \int d^dx\mathcal{L}$ is dimensionless, the mass dimension of the quantities are

$$[\psi_0] = \frac{d-1}{2} = \frac{3}{2} - \epsilon, \qquad [A_0^{a,\mu}] = \frac{d-2}{2} = 1 - \epsilon, \qquad [m_0] = 1, \qquad [g_0] = \frac{4-d}{2} = \epsilon$$

- To define a dimensionless renormalized coupling, introduce an arbitrary mass scale μ:

$$g_0 = \mu^\epsilon Z_g g.$$

- The factor μ^ϵ keeps the renormalized coupling g dimensionless.
- Physical observables cannot depend on the arbitrary scale μ. This requirement leads to renormalization group equations.

Bare and Renormalized Parameters

The scale dependence of the renormalized coupling is governed by the beta function.

Beta Function Definition

$$\mu \frac{d}{d\mu} \alpha_s(\mu) = \beta(\alpha_s),$$

In QCD, in $d = 4 - 2\epsilon$, the bare coupling is related to the renormalized coupling by

$$\alpha_{s,0} = \mu^{2\epsilon} Z_\alpha(\mu)\, \alpha_s(\mu),$$

The bare coupling constant is independent of the scale:

$$\mu \frac{d}{d\mu} \alpha_{s,0} = 0.$$

Generalized Beta Function in $d = 4 - 2\epsilon$

Using $\alpha_{s,0} = \mu^{2\epsilon} Z_\alpha(\mu)\,\alpha_s(\mu)$, and $\mu \frac{d}{d\mu}\alpha_{s,0} = 0$, one obtains

$$0 = \mu^{2\epsilon} Z_\alpha \alpha_s \left[2\epsilon + Z_\alpha^{-1} \frac{dZ_\alpha}{d\ln\mu} + \frac{1}{\alpha_s} \frac{d\alpha_s}{d\ln\mu} \right].$$

Define the generalized beta function:

$$\boxed{\beta(\alpha_s, \epsilon) \equiv \frac{d\alpha_s}{d\ln\mu}}$$

Then

$$\boxed{\beta(\alpha_s, \epsilon) = \alpha_s \left[-2\epsilon - Z_\alpha^{-1} \frac{dZ_\alpha}{d\ln\mu} \right].}$$

The limit $\epsilon \to 0$ recovers the ordinary QCD beta function.

Calculating the Beta Function

In the MS scheme, the renormalization factor contains only pole terms and the generalized beta function is expanded in powers of ϵ:

$$Z_\alpha = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_\alpha^{[k]}(\alpha_s), \qquad \beta(\alpha_s, \epsilon) = \beta(\alpha_s) + \sum_{k=1}^{\infty} \epsilon^k \beta^{[k]}(\alpha_s).$$

Requiring cancellation of all $\frac{1}{\epsilon^n}$ pole terms leads to remarkable exact relations:

$$\beta^{[1]}(\alpha_s) = -2\alpha_s, \qquad \beta^{[k]}(\alpha_s) = 0, \qquad k \geq 2.$$

Most importantly,

$$\boxed{\beta(\alpha_s) = 2\alpha_s^2 \frac{d}{d\alpha_s} Z_\alpha^{[1]}(\alpha_s).}$$

The QCD beta function is determined entirely by the coefficient of the single $1/\epsilon$ pole in Z_α.

One-Loop Beta Function of QCD

At one loop,

$$Z_\alpha = 1 - \frac{\beta_0}{4\pi\epsilon}\alpha_s + \mathcal{O}(\alpha_s^2),$$

we obtain

$$\boxed{\beta(\alpha_s) = -\frac{\beta_0}{2\pi}\alpha_s^2 + \mathcal{O}(\alpha_s^3).}$$

For QCD, $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, and for $SU(3)$, $C_A = 3, T_F = \frac{1}{2}$.

$$\boxed{\beta_0 = 11 - \frac{2}{3}n_f.}$$

The negative sign of the beta function leads to asymptotic freedom.

Leading-Order Running of the QCD Coupling

At one loop, the QCD beta function is

$$\frac{d\alpha_s(\mu)}{d\ln\mu} = -\frac{\beta_0}{2\pi}\alpha_s^2(\mu)$$

One-Loop Running Coupling

$$\alpha_s(\mu) = \frac{\alpha_s(Q)}{1 + \frac{\beta_0}{4\pi}\alpha_s(Q)\ln\frac{\mu^2}{Q^2}}$$

A common reference scale is

$$Q = m_Z \simeq 91.19\ \mathrm{GeV},$$

where experimentally

$$\alpha_s(m_Z) \approx 0.118.$$

Asymptotic Freedom vs QED Running

- For QCD, $\beta_0 > 0$ because $n_f < 17$. Therefore,

  $\alpha_s(\mu)$ decreases at high energies

  This phenomenon is called **asymptotic freedom**.
- In contrast, in QED, $\beta_0 < 0$, so the electromagnetic coupling slowly increases with energy.

QCD becomes weak at short distances, while QED becomes slightly stronger.

The discovery of asymptotic freedom in QCD by David J. Gross, H. David Politzer, and Frank Wilczek in 1973 was awarded the 2004 Nobel Prize in Physics.

The QCD Scale Λ_{QCD}

- The one-loop running coupling can be rewritten as

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln\left(\mu^2/\Lambda_{\mathrm{QCD}}^2\right)}$$

- The scale

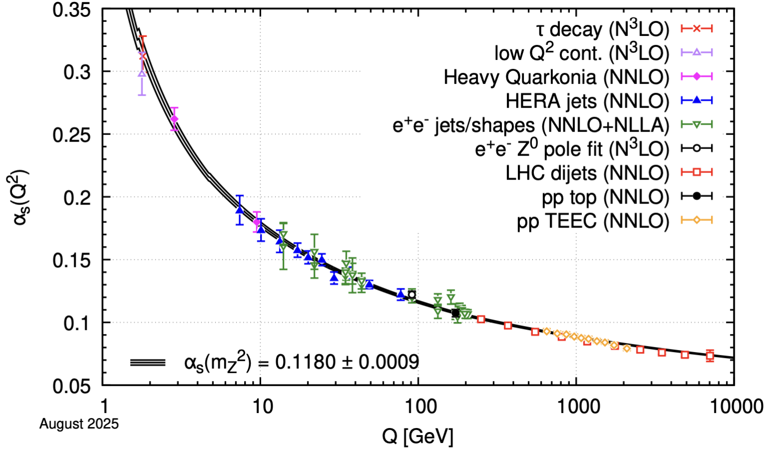
$$\Lambda_{\mathrm{QCD}}^2 \equiv Q^2 \exp\left(-\frac{4\pi}{\beta_0 \alpha_s(Q)}\right)$$

  is generated dynamically through renormalization, when quantum fluctuations become nonperturbatively important.
- At low energies, $\mu \to \Lambda_{\mathrm{QCD}}$, the coupling grows rapidly and perturbation theory breaks down. Typical value:

$$\Lambda_{\mathrm{QCD}} \sim 0.2\ \mathrm{GeV}.$$

Low-energy QCD becomes strongly coupled and quarks/gluons are confined inside hadrons.

Running Coupling Constant: Theory vs Experiments



- Theory: running coupling constant computed at five loops.