

High energy QCD: Evolution equations and particle production

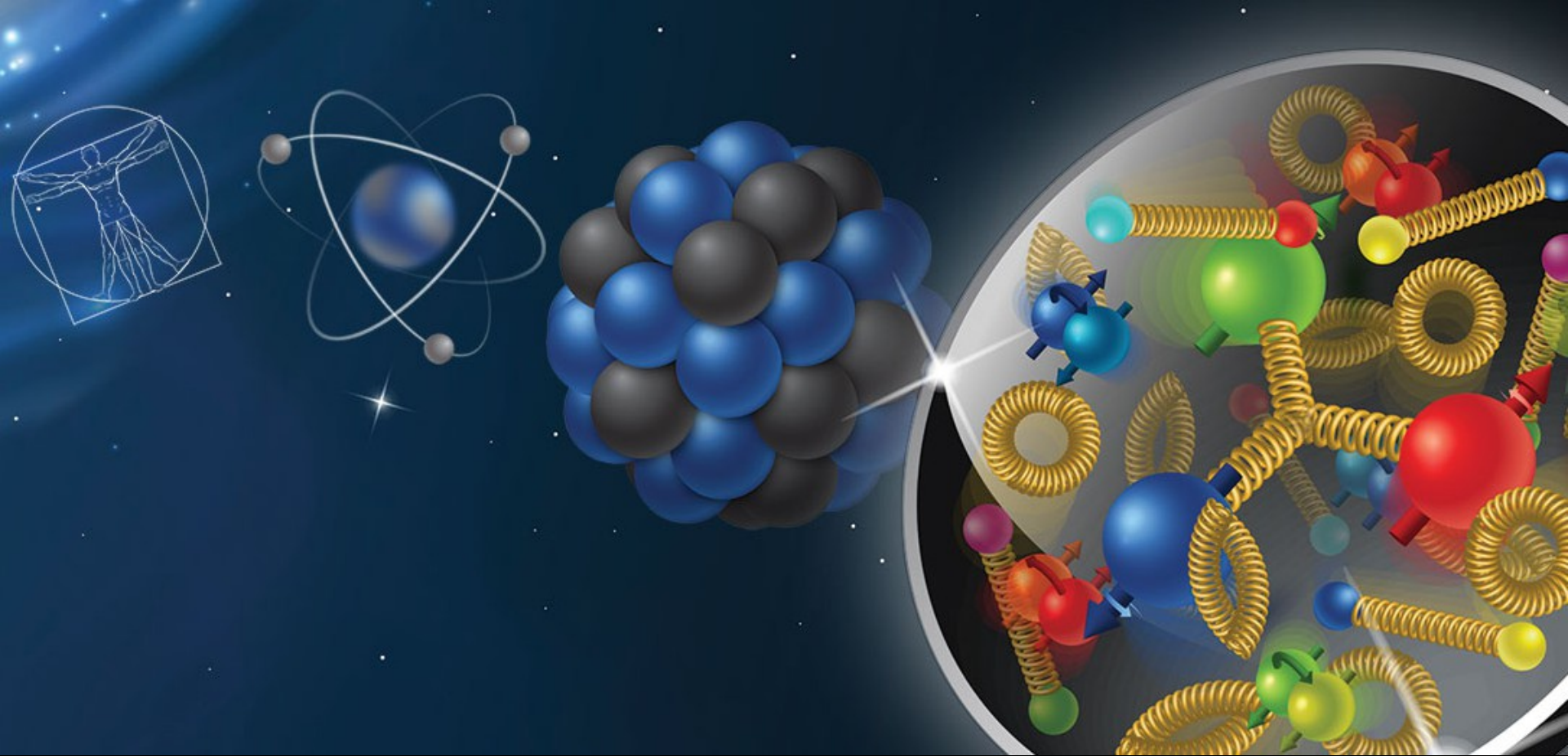
José Garrido* (USM & PUCV),
Carlos Contreras (USM),
Eugene Levin (TAU)

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Outline

- **General concepts of small-x physics and saturation:**
 - Quarks and gluons in a proton; DIS in the dipole picture; BFKL Pomeron; Unitarity; GGM/MV formula
- **Small-x evolution:**
 - The non-linear BK evolution equation; Solution of BK equation; geometric scaling
 - Single diffractive dissociation
- **Particle production at small-x**
 - AGK cross sections
 - Solution to the AGK evolution equations
 - Applications at EIC era: Multiplicity distribution and entanglement entropy for produced gluons

Based mainly on [arXiv:2603.21775 \[hep-ph\]](#),
[Phys. Rev. D 113, 114021 \(2026\)](#)



*General concepts of
small-x physics and
saturation*

The structure of matter

1

atom 10^{-8} cm

electron $< 10^{-16}$ cm

gluon $< 10^{-16}$ cm

nucleus 10^{-12} cm

nucleon (proton/
neutron) 10^{-13} cm

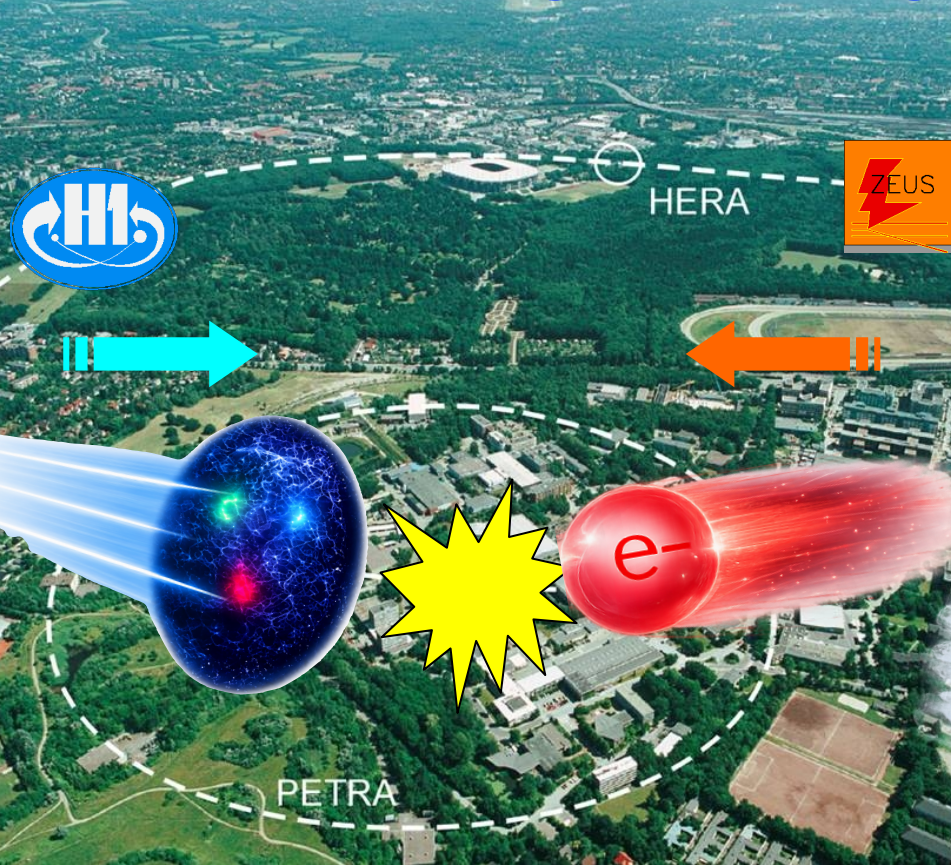
quark $< 10^{-16}$ cm

- We still don't have a clear picture of how quarks and gluons are distributed inside the proton.

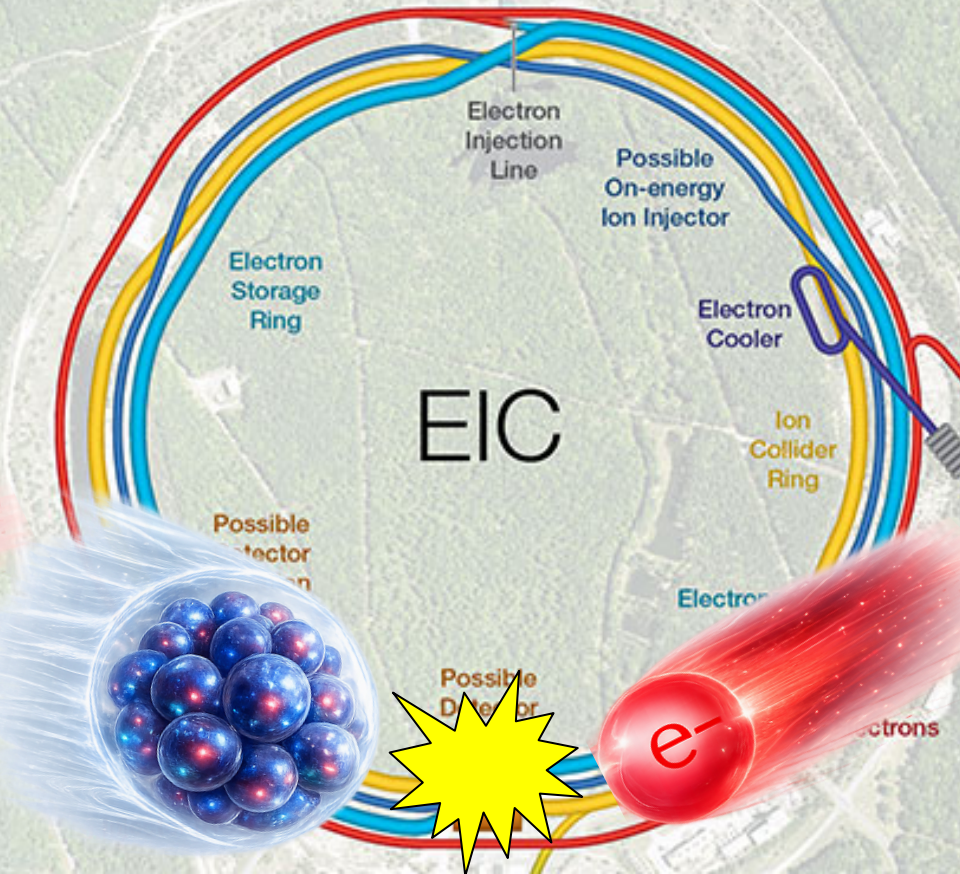
Quarks and Gluons in the Proton ²

- By colliding tiny electrons with relatively larger particles such as protons or nuclei in particle accelerators, it help us (nuclear physicists) to **unlock the mysteries of the strong force**, which binds atomic nuclei together.

DESY, Hamburg, Germany



BNL, Upton, NY



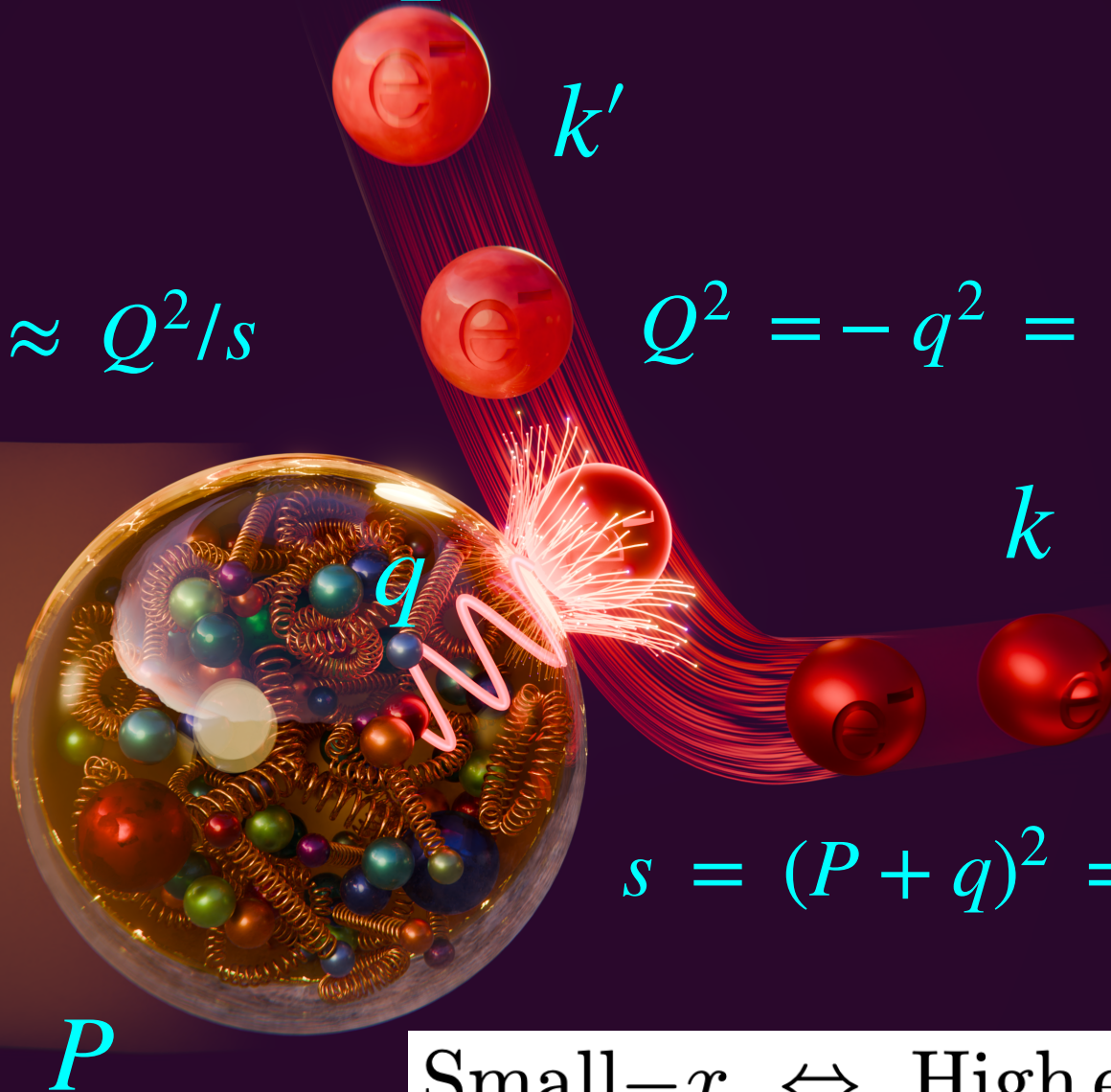
Deep inelastic scattering (DIS) 3 on a proton

$$x = \frac{Q^2}{2P \cdot q} \approx Q^2/s$$

$$y = \frac{Q^2}{sx}$$

$$Q^2 = -q^2 = (k - k')^2$$

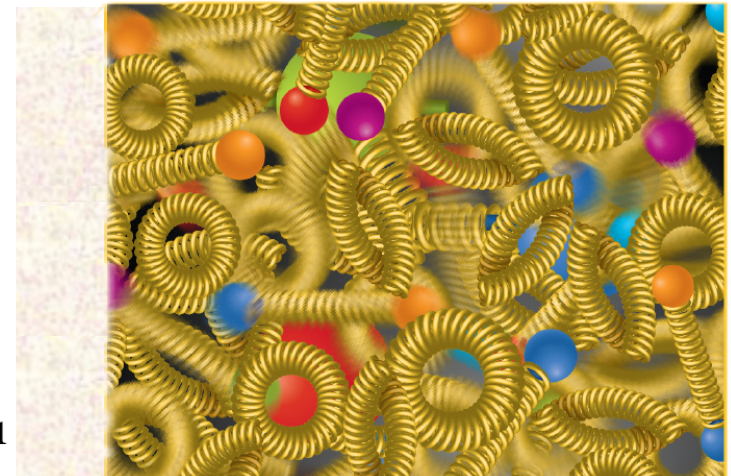
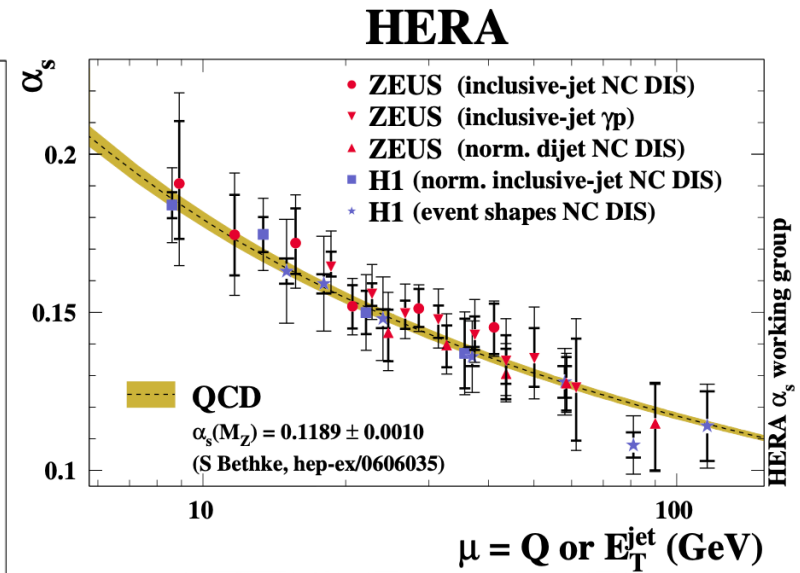
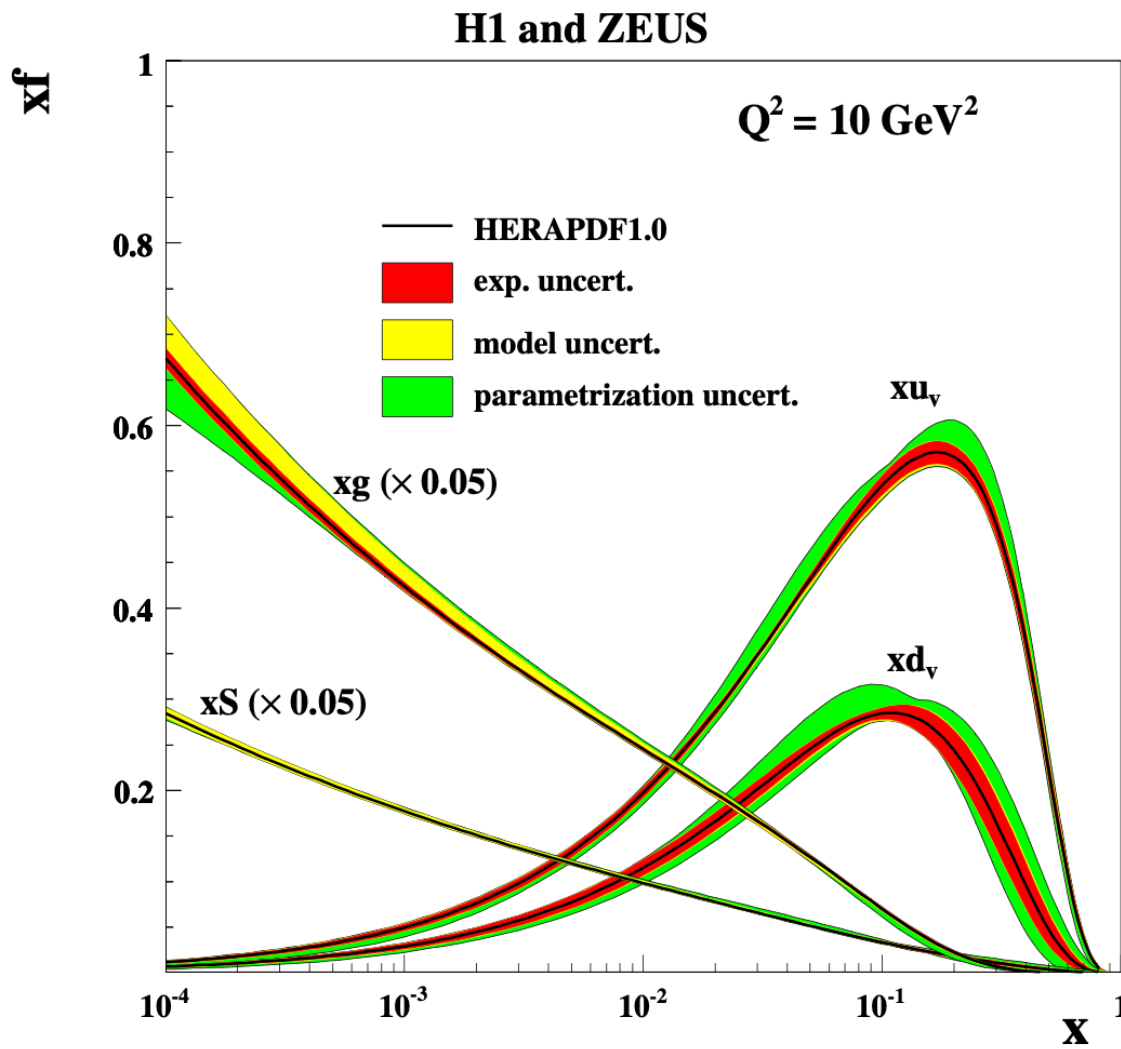
$$s = (P + q)^2 = W^2$$



Small- $x \Leftrightarrow$ High energy s

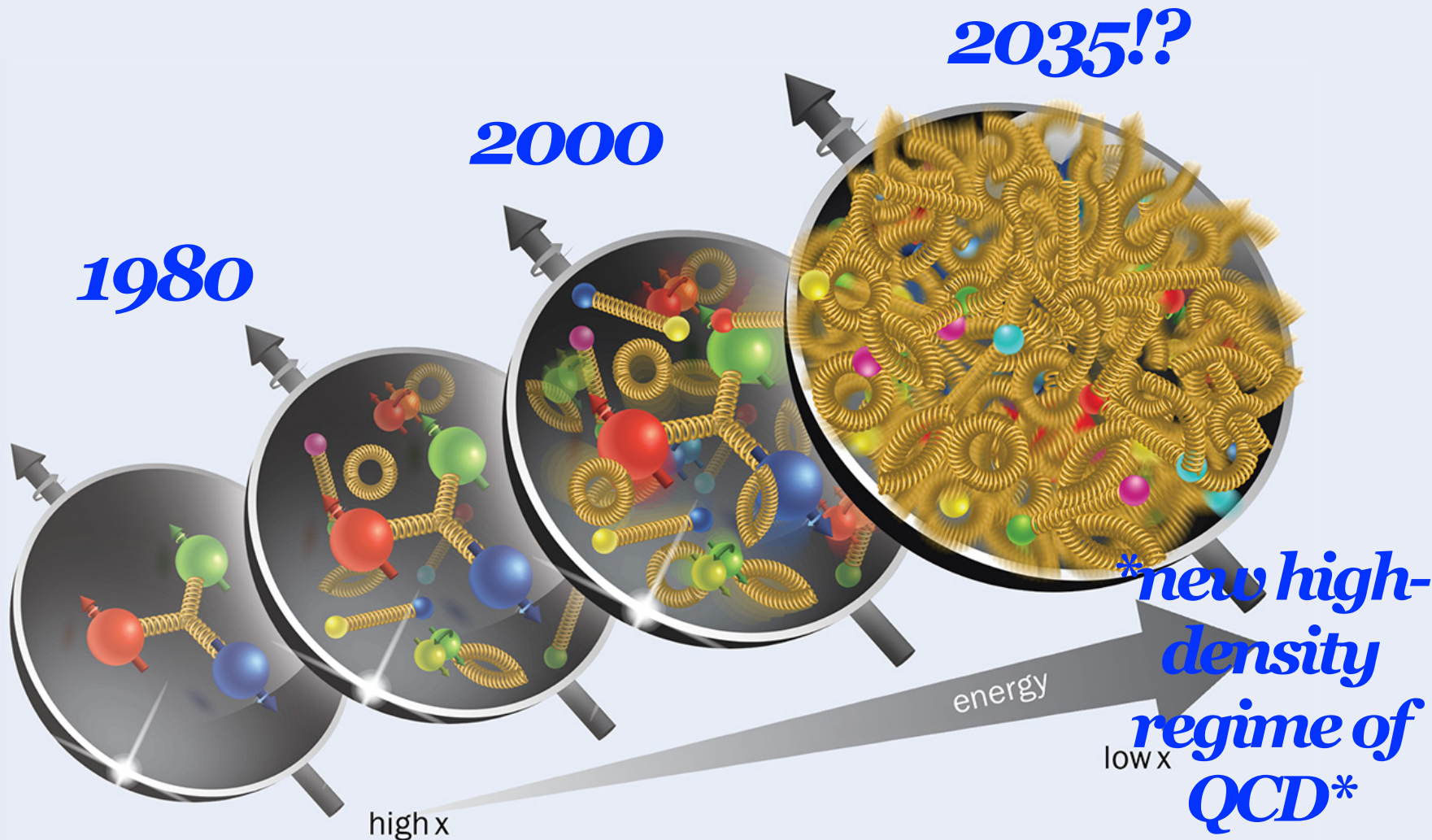
Quarks and Gluons in the Proton ⁴

- There is a large number of quarks, anti-quarks and gluons at small-x in a proton:



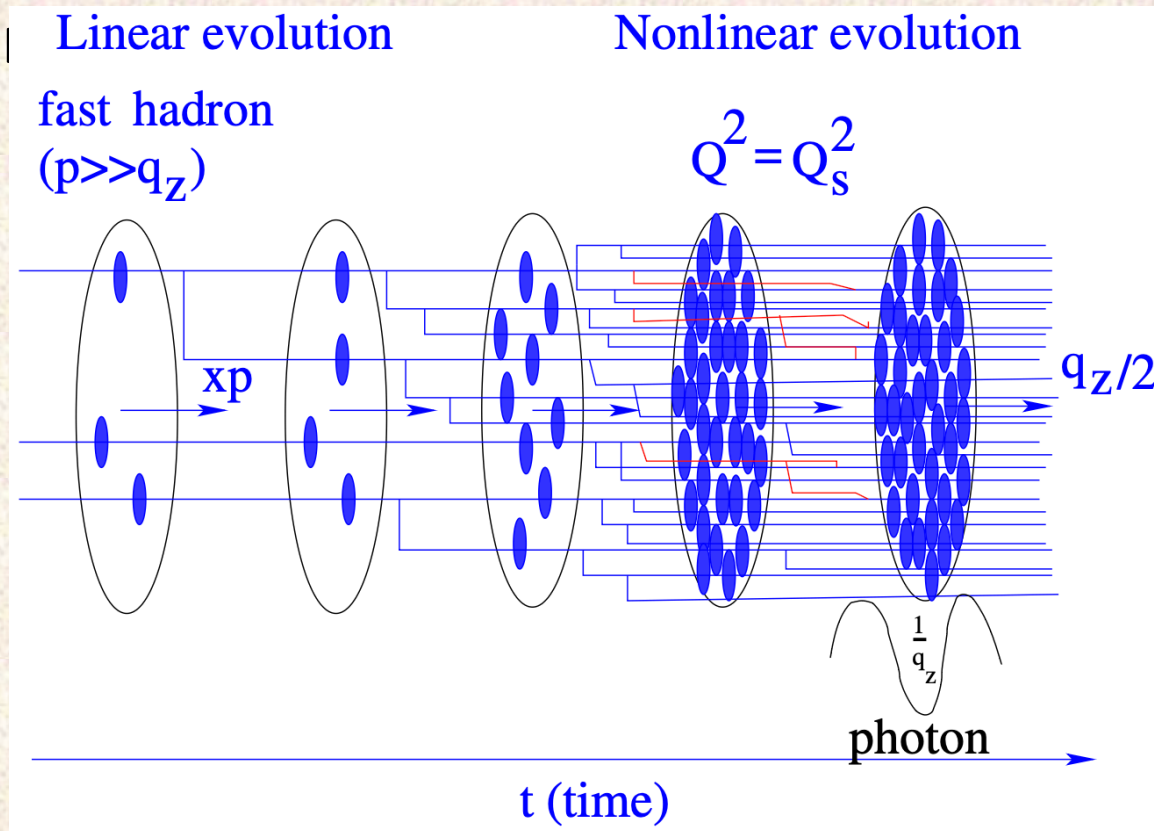
Quarks and Gluons in the Proton ⁵

- High number of gluons populates the transverse extent of the proton or nucleus, leading to a very dense saturated wave function known as the Color Glass Condensate (CGC)

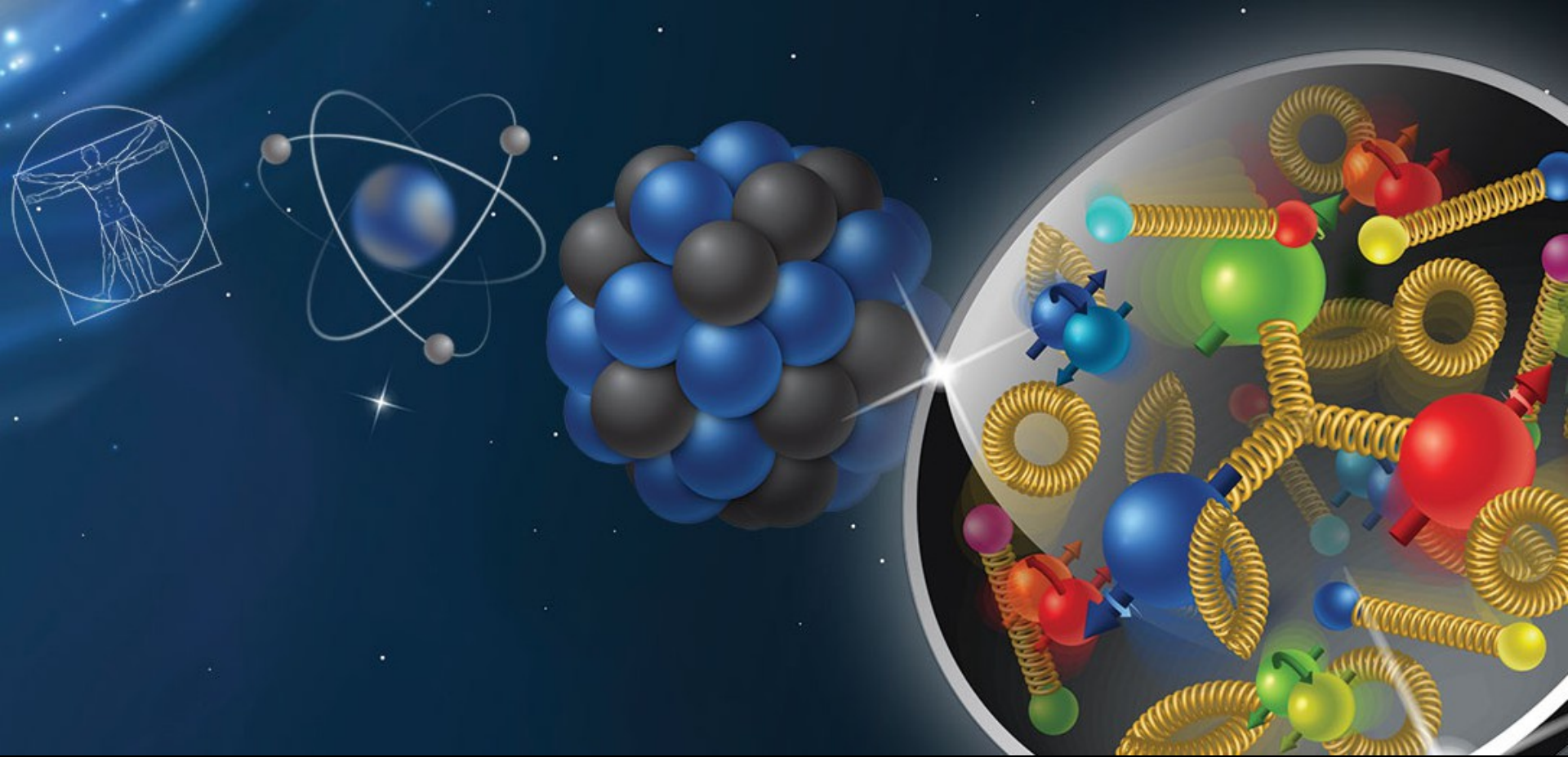


Quarks and Gluons in the Proton ⁶

- Qualitatively we understand that these extra quarks and gluons are emitted by the original three valence quarks in the proton

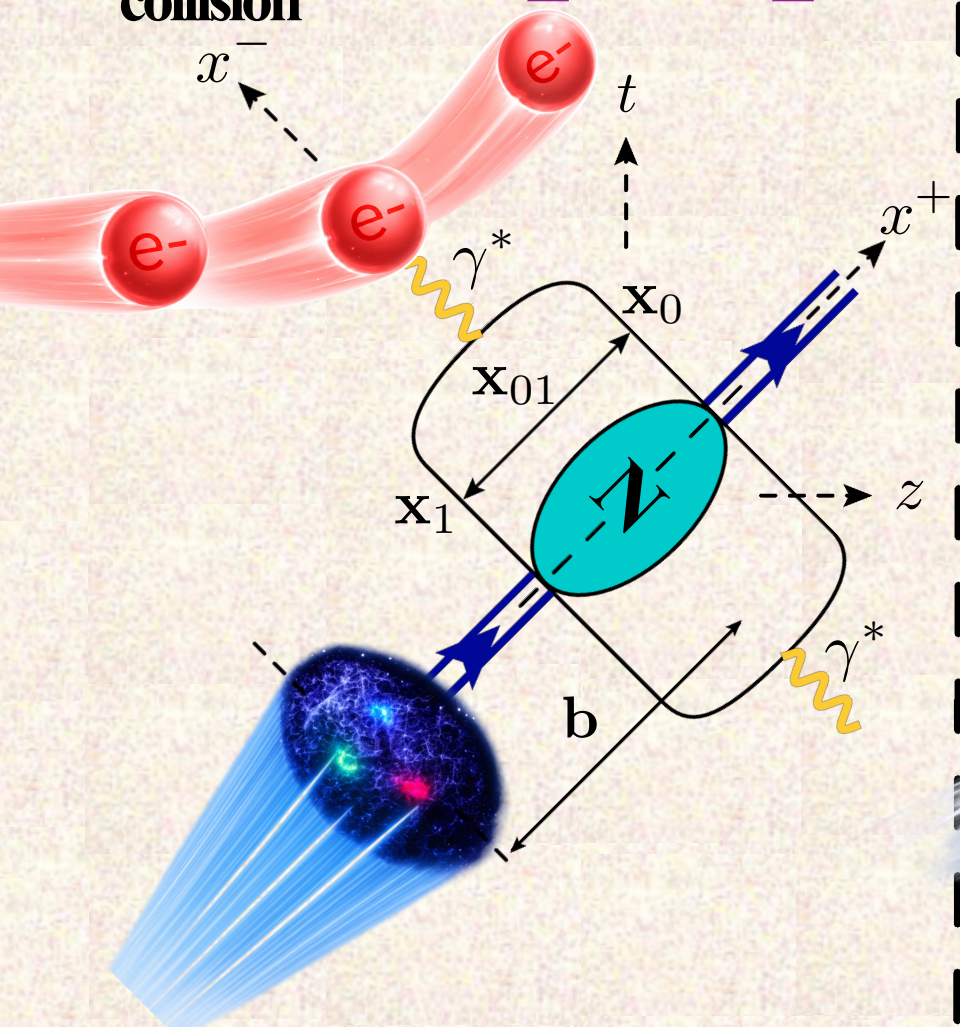


- The straight solid lines represent gluons.
- At some critical scale $Q=Q_s$, we start having not only emissions but also **recombination** of gluons

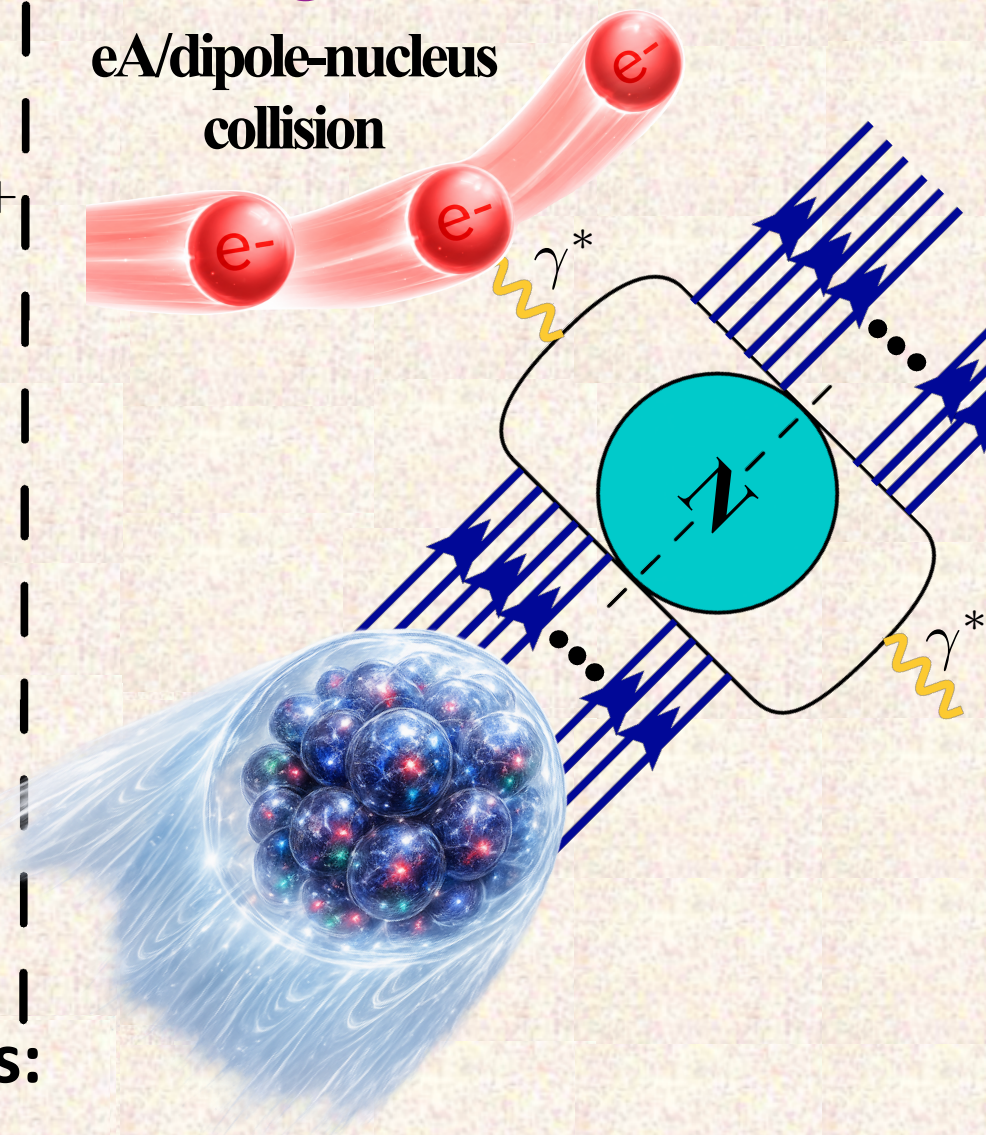


The dipole approach to QCD

ep/dipole-proton collision *Dipole picture of DIS*



eA/dipole-nucleus collision

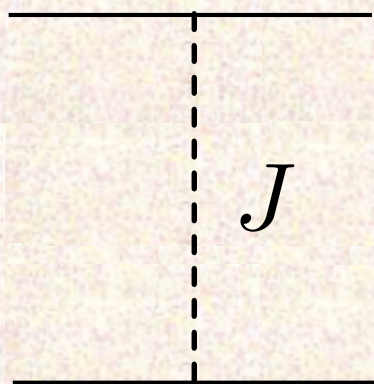


- The total DIS cross section is:

$$\sigma_{tot}^{\gamma^* p/A} = \int \frac{d^2 x_{01}}{2\pi} d^2 b \int_0^1 \frac{dz}{z(1-z)} |\Psi^{\gamma^* \rightarrow q\bar{q}}(\mathbf{x}_{01}, z, Q^2)| N(Y; \mathbf{x}_{01}, \mathbf{b})$$

Dipole picture of DIS

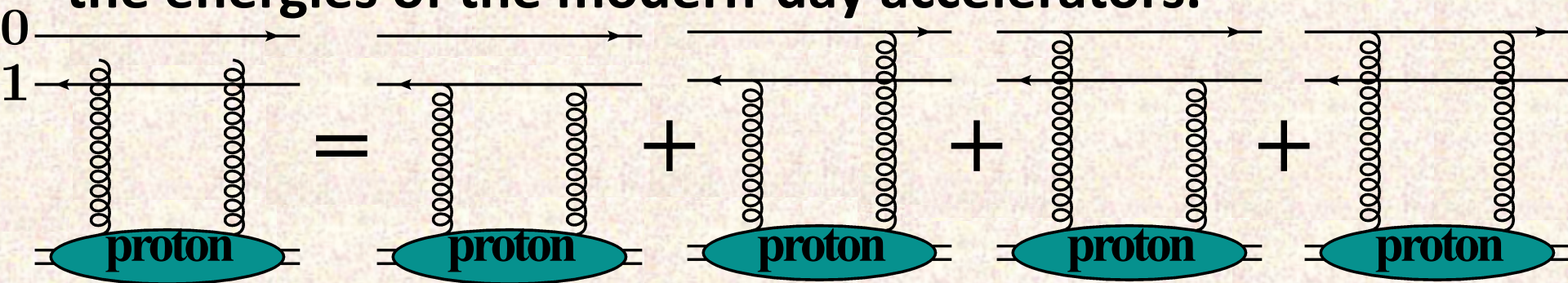
- How does the dipole interacts with the proton?



$$\sigma \sim \dots S^{J-1}$$

J = spin of exchanged particle

- Answer: the **(two) gluon exchange dominates (J=1)**. Spin-2 particles, such as gravitons, luckily are weakly-coupled at the energies of the modern-day accelerators.



$$N^{2-\text{gluons}}(Y=0, \mathbf{x}_{01}, \mathbf{b}) = \frac{1}{4} x_{01}^2 Q_s^2(Y=0, \mathbf{b})$$

The BFKL Pomeron

- The previous result does not include any energy dependence. Energy dependence comes in through **quantum corrections**, which bring in powers of $\alpha_S \ln(1/x)$

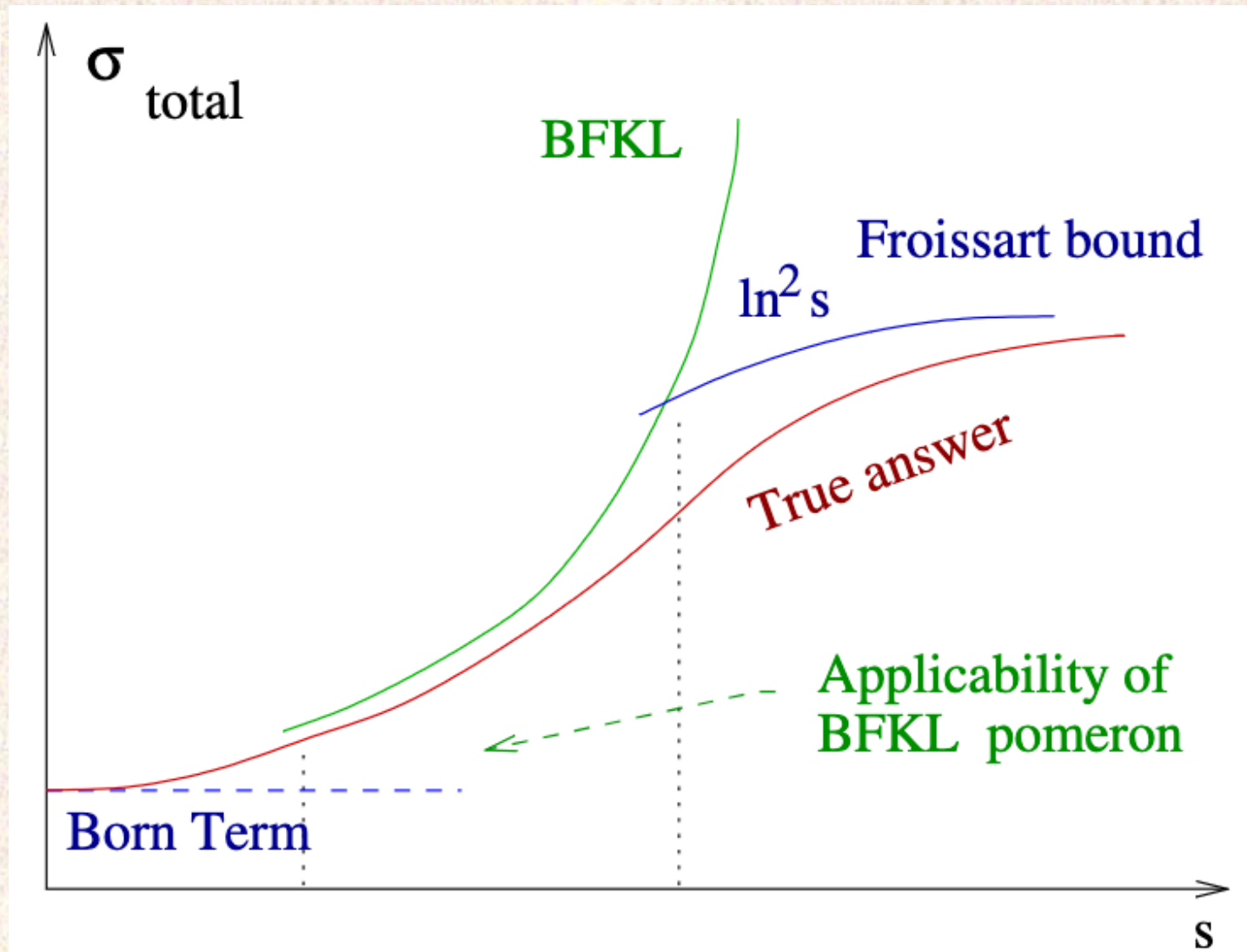
$$G_P(Y, \mathbf{k}_T, \mathbf{b}) =$$

The diagram illustrates the BFKL Pomeron evolution equation. On the left, a vertical chain of vertices (represented by black dots) is labeled 1, 2, 3, ..., n-1, n. A wavy line labeled $\mathbf{k}_T, Y$ enters from the top left. Horizontal wavy lines connect the vertices. At the bottom, a cylinder represents a target. On the right, a similar vertical chain is shown, but with a horizontal line at the top and a teal oval at the bottom, representing a different boundary condition. A large equals sign is between the two diagrams.

- This is the **Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron**.
- The Pomeron is a bound state of two reggeized gluons (due to evolution equation) and we denote it by G_p

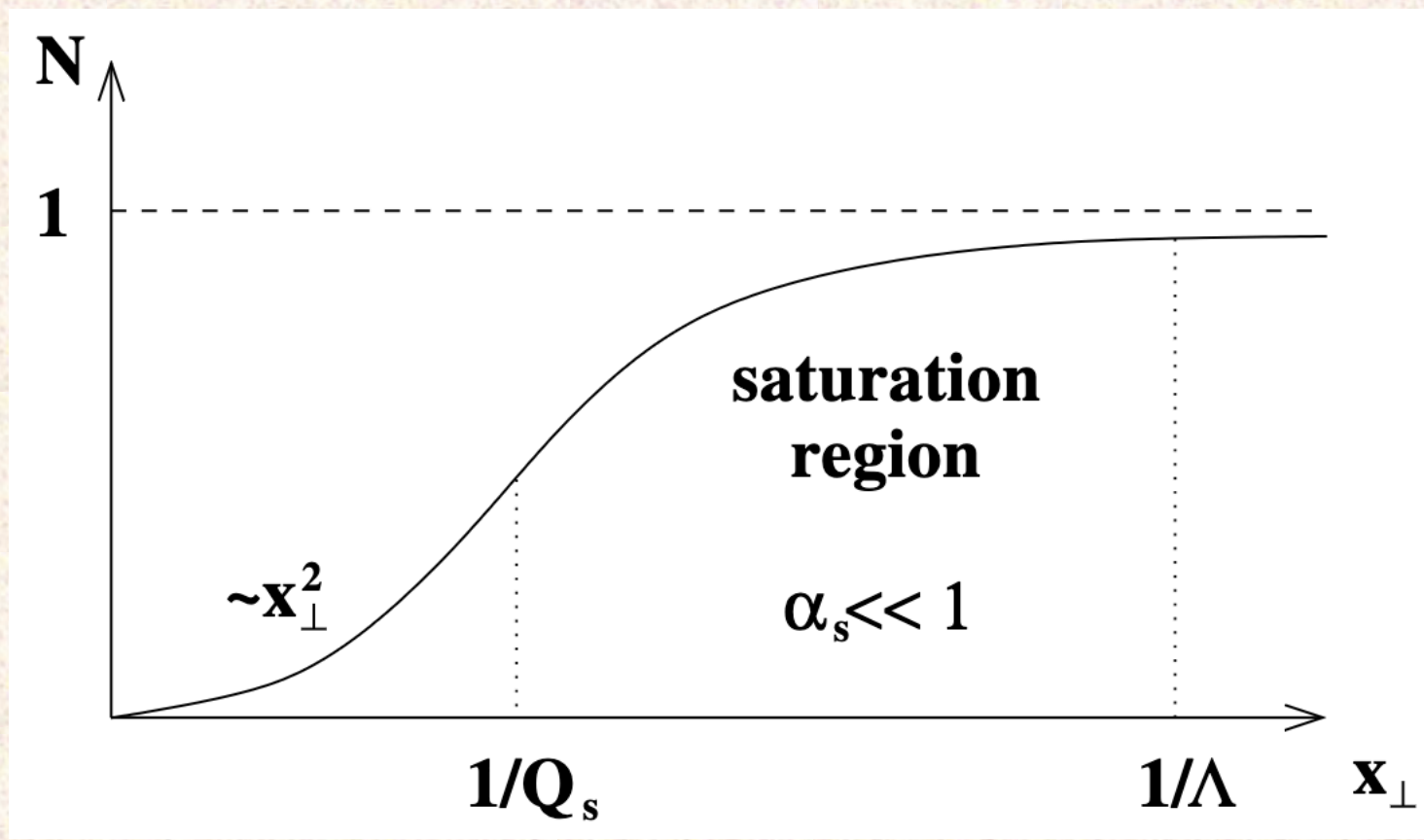
The BFKL Pomeron

- As we increase energy, the **single BFKL Pomeron** violates **unitarity** of the total cross section



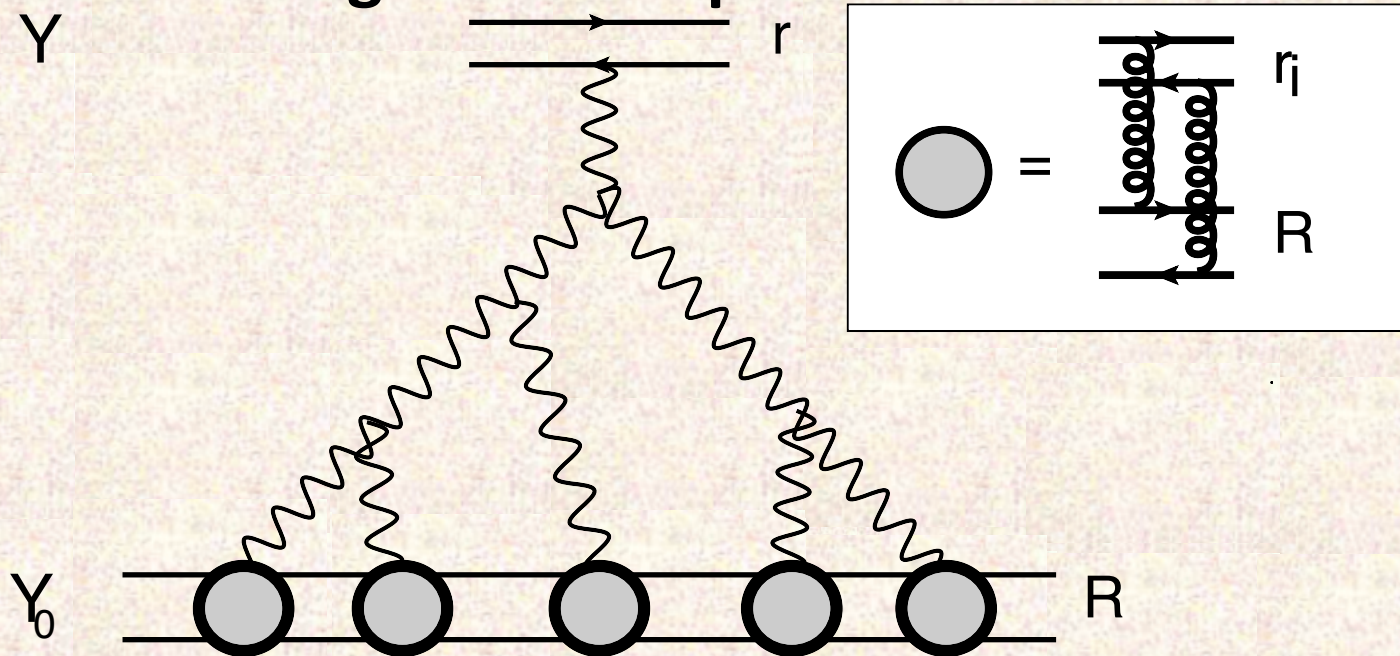
Fixing the problems

- Let us define the **saturation scale $Q_s(Y)$** like a transition point between the region where $N(x_\perp, Y)$ grows with $x_\perp$ and the region where this growth is tamed, that is, by requiring that $N(x_\perp = 1/Q_s(Y), Y) = \text{const}$, with some order-one constant.



Fixing the problems

- To reconcile with unitarity, we say that the dipole system in the saturation region develops the **BFKL Pomeron cascade**:

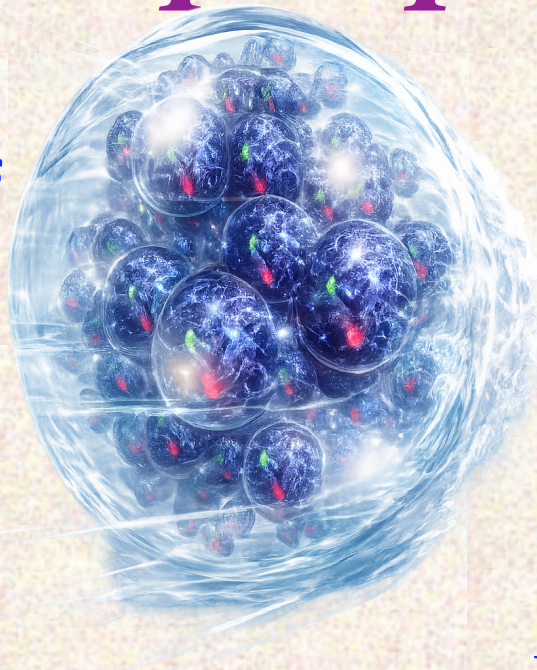


$$S_{p/A}^d(Y, \mathbf{x}_{01}, \mathbf{b}) = \sum_{n=0}^{\infty} (-1)^n C_n G_P^n(Y, \mathbf{x}_{01}, \mathbf{b}) = \oint_C \frac{dn}{2\pi i} \frac{\pi}{\sin(\pi n)} C_n G_P^n$$

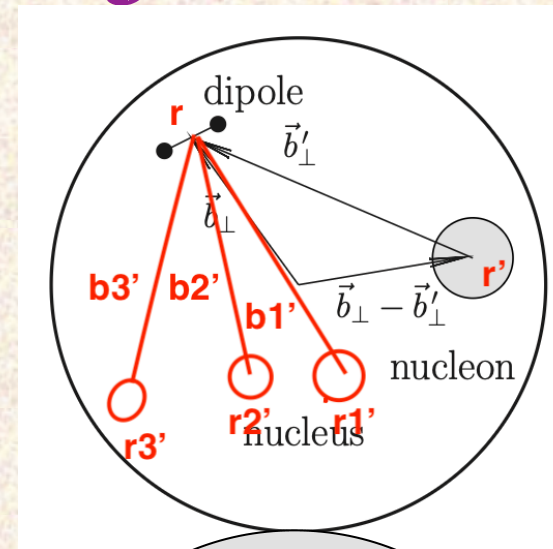
- S=1-N**. The wavy lines denote the BFKL Pomeron exchanges. The circles denote the Born amplitude.
- We don't know how to sum exactly!

Dipole picture of DLS

- Nucleus = Bag of quarks

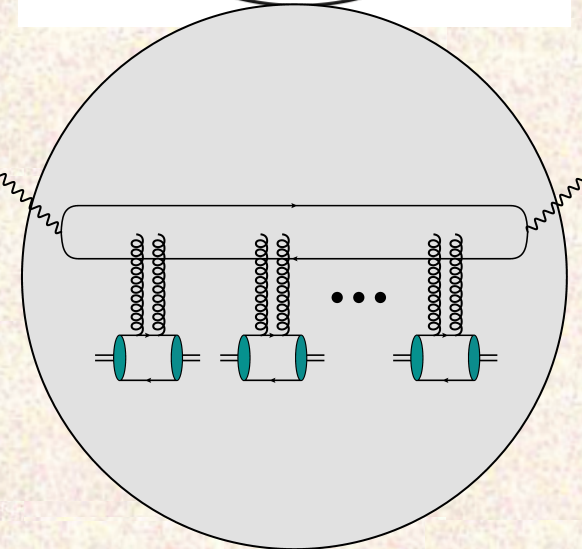


=



$\mathbf{b}_i = \mathbf{b}$

=

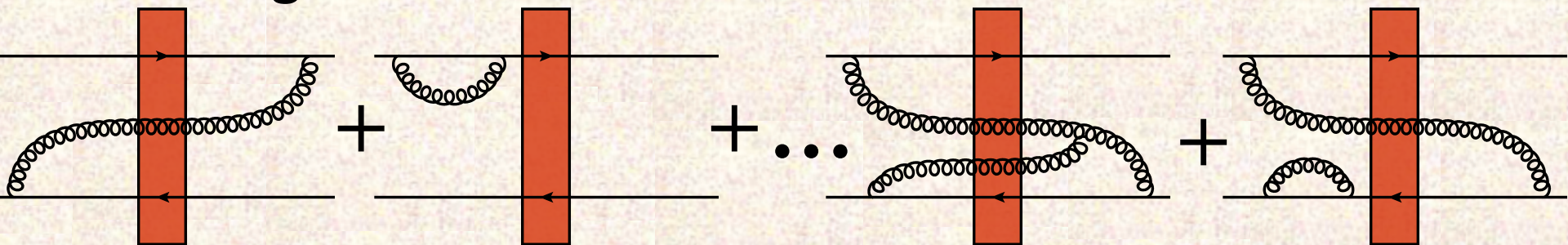


- For multiple exchanges simply exponentiate (Mueller '90)

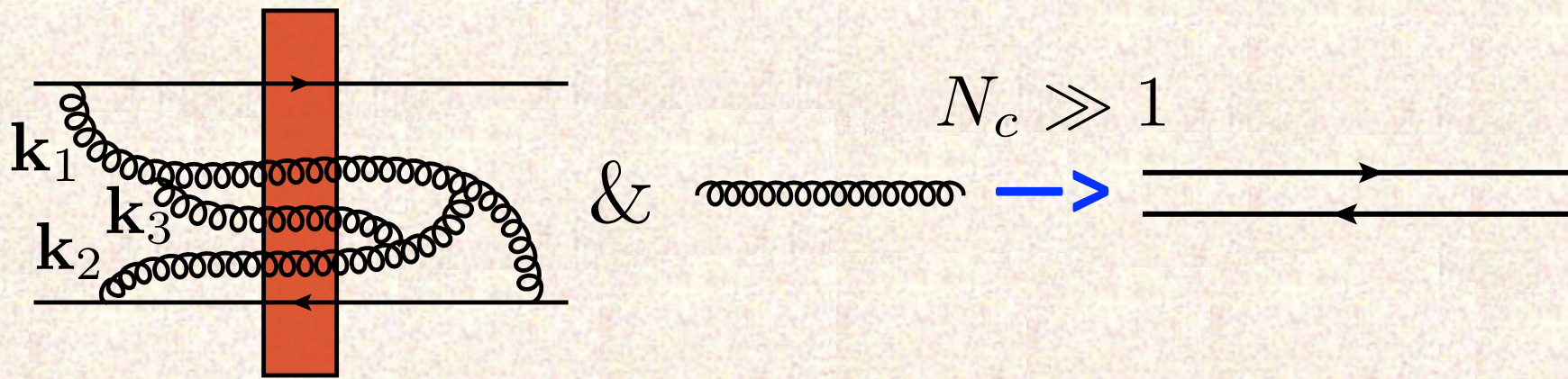
$$N^{(0)} = 1 - e^{-\frac{1}{4} x_{01}^2 Q_{s0}^2} \equiv \gamma(\mathbf{x}_{01}, \mathbf{b})$$

Small- x Evolution

- Let us denote by the red rectangle the GGM exchanges. The quantum corrections are the virtual and long-lived s -channel gluons:

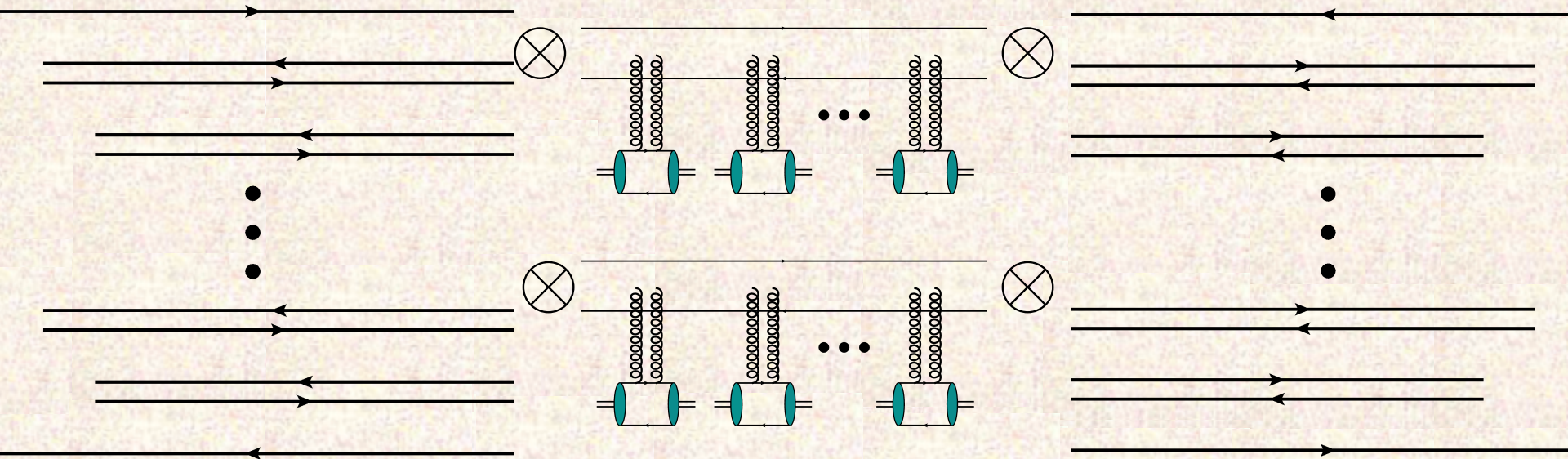


- In general we don't know how to sum this gluon cascade, only at leading $\log-1/x$ and next to leading log. Also due to color factors, still hard to resume. So we also employ the large- N_c limit ('t Hooft '77):



Small- x Evolution

- The original dipole has developed a **dipole cascade**, each of which interacts with the nucleus via GGM scattering



$$S_{dA}(Y; \mathbf{x}_{01}, \mathbf{b}) = \sum_{n=0}^{\infty} (-1)^n \int \prod_{i=1}^n d^2 x_{01,i} d^2 b_i \times \gamma(\mathbf{x}_{01,i}, \mathbf{x}_{0'1'}, \mathbf{b}_i) \rho_n(Y; \{\mathbf{x}_{01,i}, \mathbf{b}_i\})$$

- Where $S=1-N$ and ρ denotes the dipole densities (probability to find n dipoles in the amplitude)

Small- x Evolution

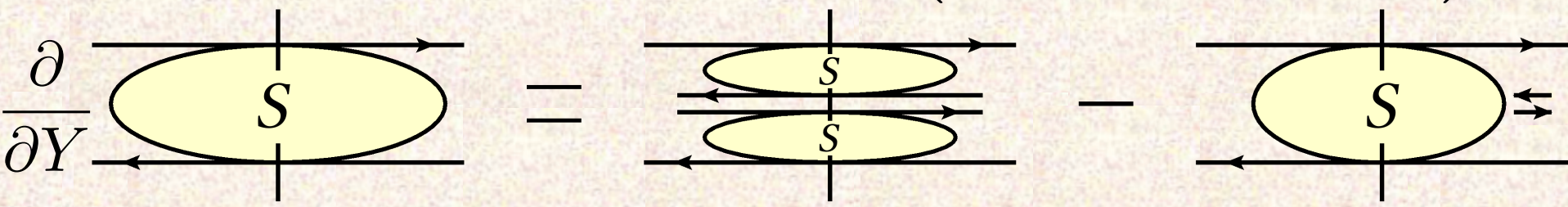
- In arriving to that expression, the dipole–nucleus S-matrix is found to obey the following **non-linear evolution equation** (Balitsky '96; Kovchegov '99)

$$\frac{\partial S_{dA}(Y, \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \times [S_{dA}(Y; \mathbf{x}_{12}, \mathbf{b}) S_{dA}(Y; \mathbf{x}_{02}, \mathbf{b}) - S_{dA}(Y; \mathbf{x}_{01}, \mathbf{b})]$$

where $Y = \ln 1/x$ and $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, $x_{ij} = |\mathbf{x}_{ij}|$

- The initial condition with energy is given by

$$S_{dA}(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) = \exp \left(-\frac{1}{4} x_{01}^2 Q_s^2(Y_A, \mathbf{b}) \right)$$



Solution of BK equation

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- For the leading twist BFKL kernel (E. Levin and K. Tuchin '00)

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \rightarrow \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y,b)}^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} + \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y,b)}^{x_{01}^2} \frac{dx_{12}^2}{x_{12}^2} \equiv \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{02} + \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{12}$$

where $\xi_{ij} = \ln(x_{ij}^2 Q_s^2(Y = Y_A, \mathbf{b}))$ and $\xi_s = \bar{\alpha}_S \kappa (Y - Y_A)$

- The BK evolution equation can be rewritten in terms of ξ_s and ξ as

$$\frac{\partial S_{dA}(\xi_s, \xi)}{\partial \xi_s} = \frac{1}{\kappa} S_{dA}(\xi_s, \xi) \int_{-\xi_s}^{\xi} d\xi' (1 - S_{dA}(\xi_s, \xi'))$$

- These equations can be solved by introducing the auxiliary function

$$S_{dA}(\xi_s, \xi) = \exp(-\Omega(\xi_s, \xi))$$

Solution of BK equation

- We can obtain the non-linear equation for $\Omega(\xi, \xi_s)$

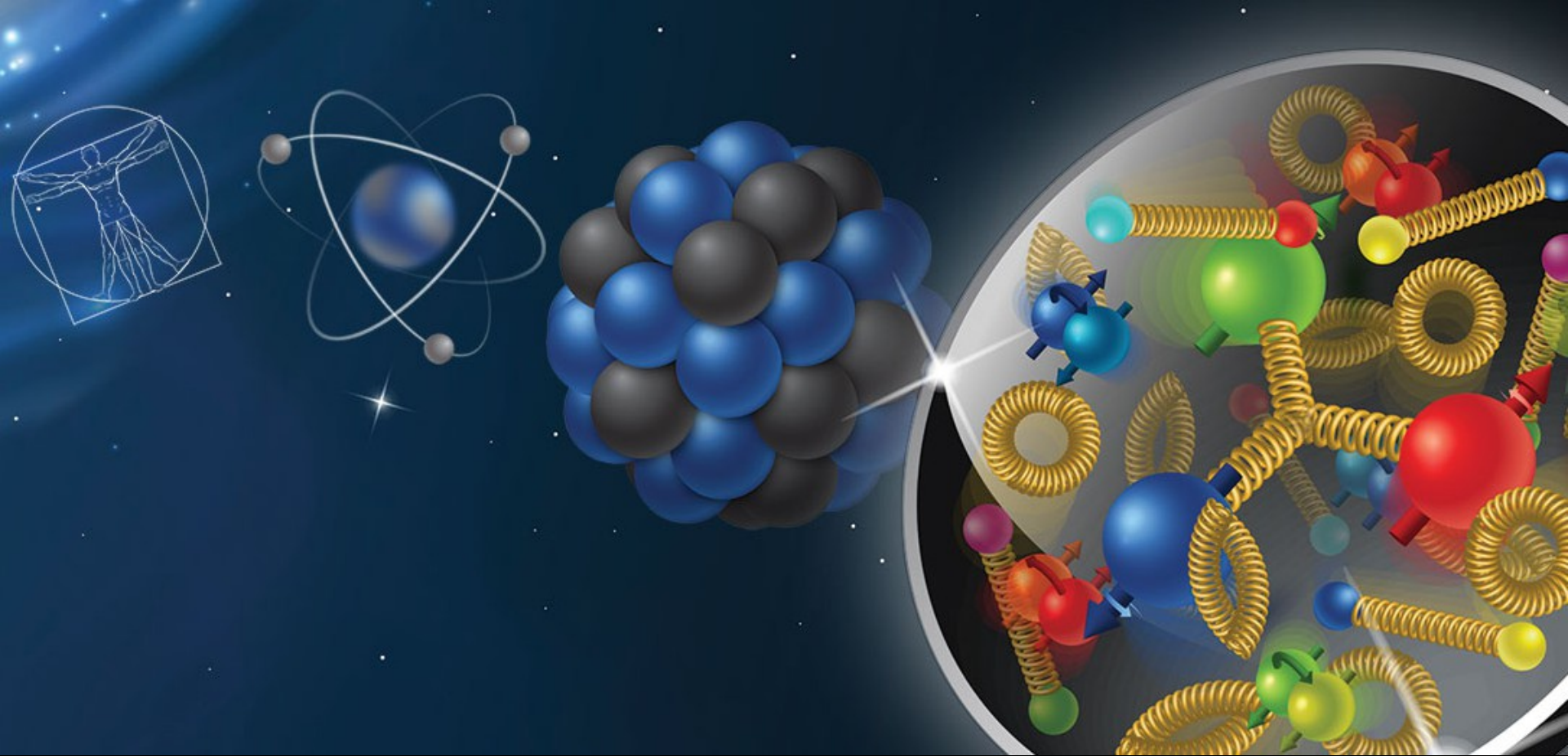
$$\frac{\partial^2 \Omega(\xi_s, \xi)}{\partial \xi_s \partial \xi} = \frac{1}{\kappa} \left(1 - e^{-\Omega(\xi_s, \xi)} \right)$$

- For the geometric scaling variable $z = \xi_s + \xi$ it reduces to

$$\frac{d^2 \Omega(z)}{dz^2} = \frac{1}{\kappa} \left(1 - e^{-\Omega(z)} \right)$$

$$\Rightarrow \int_{\Omega_0(\xi)}^{\Omega(z)} \frac{d\Omega'}{\sqrt{\Omega' + e^{-\Omega'} - 1 + \frac{\kappa}{2} C_1}} = \sqrt{\frac{2}{\kappa}} (z + C_2)$$

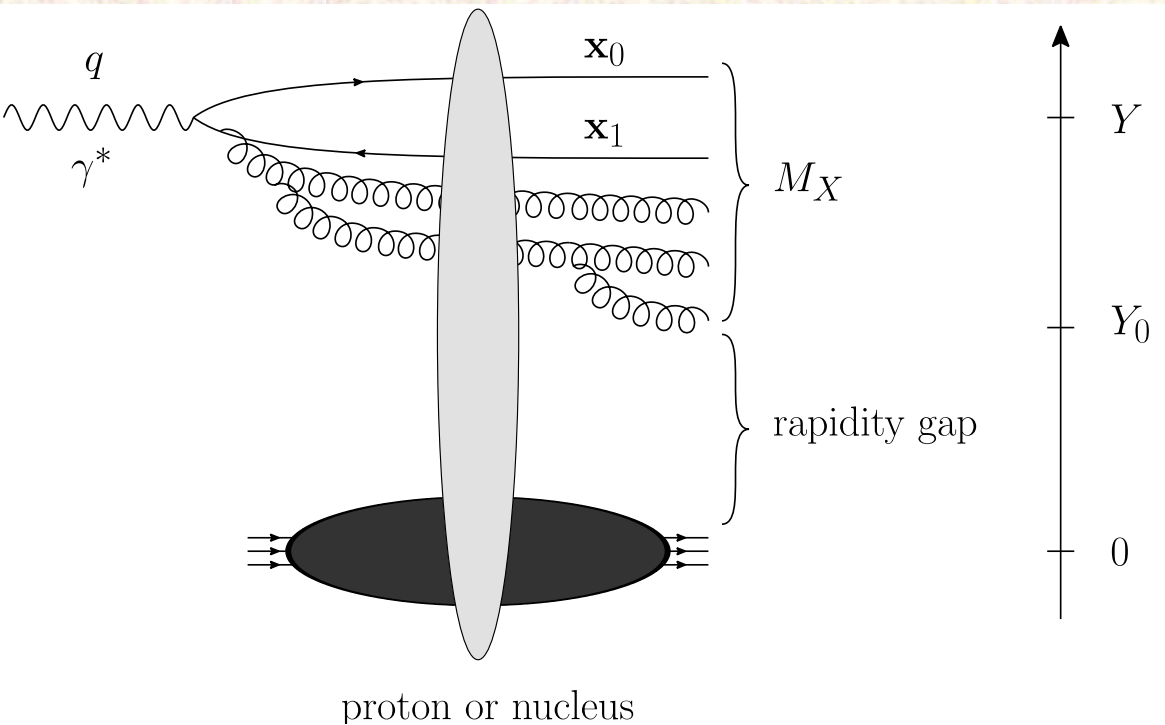
- Constants C_1 and C_2 can be found from matching with perturbative QCD region.
- Geometric scaling has been observed in DIS data by Stasto, Golec-Biernat, Kwiecinski in 00.



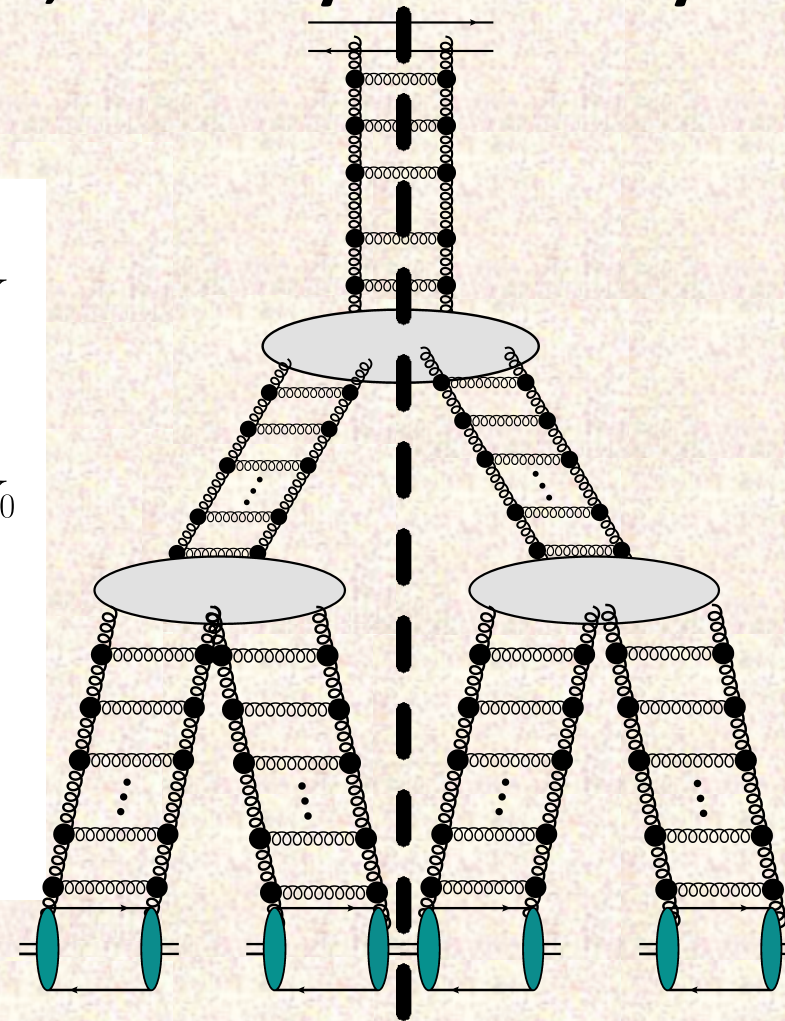
Single diffractive dissociation

Single diffractive dissociation 19

- At high M_X , in addition to the $q\bar{q}$, one may have many more gluons produced ($y > Y_0$):



The process



Diffraction evolution as
fan diagrams

Single diffractive dissociation 20

- The single diffractive cross section is

$$M_X^2 \frac{d\sigma_{diff}^{\gamma^* A}}{dM_X^2} = - \int d^2 x_0 d^2 x_1 \int_0^1 dz |\Psi^{\gamma^* \rightarrow q\bar{q}}(\mathbf{x}_{01}, z, Q^2)| \frac{\partial S^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})}{\partial Y_0}$$

- S^D is the diffractive S-matrix with the rapidity gap greater than or equal to Y_0 . It is descomposed as $S^D=1-2N+\sigma_{SD}$.
- S^D is a complicated object:

$$\begin{aligned} S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b}) &= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} (-1)^k \\ &\times \int \prod_{i=1}^n d^2 x_{01,i} d^2 b_i \prod_{j=1}^k d^2 x_{01,j} d^2 b_j \\ &\times \gamma^{BA}(\mathbf{x}_{01,i}, \mathbf{x}_{0'1'}, \mathbf{b}_i) \gamma^{BA}(\mathbf{x}_{01,j}, \mathbf{x}_{0'1'}, \mathbf{b}_j) \\ &\times \rho_{n,k}^D(Y_m, Y_0; \{\mathbf{x}_{01,i}, \mathbf{b}_i\}) \rho_n(Y_0; \{\mathbf{x}_{01,i}, \mathbf{b}_i\}) \end{aligned}$$

Diffractive S-matrix

- Fortunately, doing one step in evolution and using the following cancellations of final state interactions for gluons with $y > Y_0$ (Z. Chen, A. Mueller '95)

The diagram illustrates the cancellation of final state interactions for gluons. It consists of two rows of Feynman diagrams. The top row shows three diagrams with a gluon loop in the final state, each with a different vertical line configuration, summed to zero. The bottom row shows two diagrams with a gluon loop in the final state, summed to zero.

- The equation for the single diffraction production in QCD satisfies again the BK equation, but with different I.C.:

$$\frac{\partial S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \times [S_{dA}^D(Y, Y_0; \mathbf{x}_{12}, \mathbf{b}) S_{dA}^D(Y, Y_0; \mathbf{x}_{02}, \mathbf{b}) - S_{dA}^D(Y, Y_0; \mathbf{x}_{01}, \mathbf{b})]$$

I.C. $S_{dA}^D(Y = Y_0, Y_0; \mathbf{x}_{01}, \mathbf{b}) = S_{dA}^2(Y = Y_0; \mathbf{x}_{01}, \mathbf{b})$

Solution in the saturation region²²

- **Introducing** $S_{dA}^D(\xi, \xi_s) = \exp(-\Omega^D(\xi, \xi_s))$

$$\frac{\partial^2 \Omega^D(\xi_s, \xi)}{\partial \xi_s \partial \xi} = \frac{1}{\kappa} \left(1 - e^{-\Omega^D(\xi_s, \xi)} \right)$$

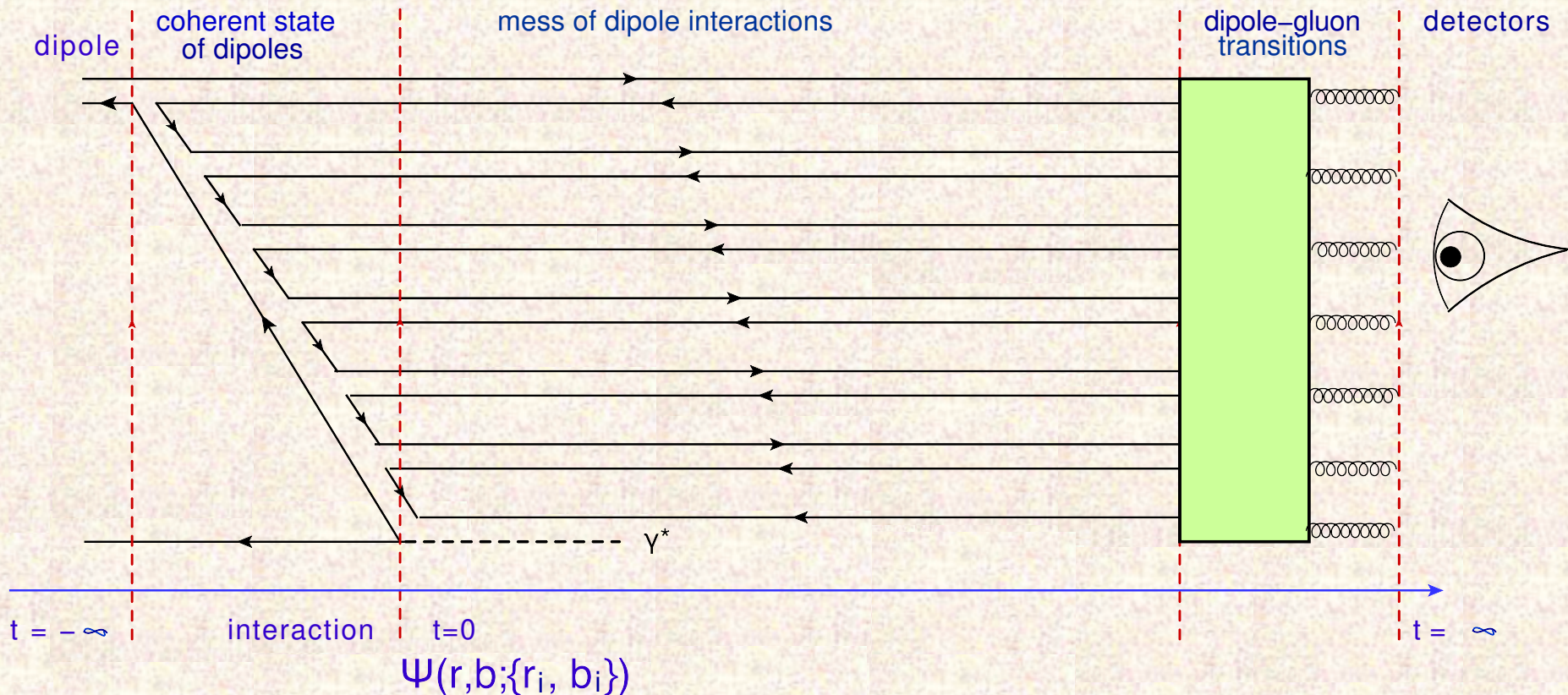
- **The solution to this equation has the form:**

$$S_{dA}^D(z) = S_{IP}^D(z_m) S_{dA}^2(z_0)$$

- **This notation reflects that the production of gluons with mass M_x occurs in the dipole-Pomeron scattering**



*Particle production
at small-x*

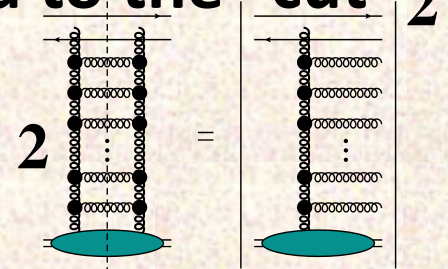


- The interaction of fast hadron (dipole) with the virtual photon (γ^*). The coherence of the partonic wave function of the fast hadron is destroyed at $t = 0$, while the gluons can be measured at $t = \infty$.

The AGK cutting rules

- The Pomeron or uncut Pomeron is related to the “cut Pomeron” through the optical theorem

$$2 N^{\text{BFKL}} = \sigma_{in}^{\text{BFKL}}$$



- The **Abramovsky-Gribov-Kancheli (AGK) cutting rules** allows us to calculate the cross sections for production of n-cut Pomerons if we know F_k : the contribution of the exchange of k-Pomerons to the cross section

$$\sigma_n^{\text{AGK}}(Y; \mathbf{r}, \mathbf{b}) = \sum_{k=n}^{\infty} \sigma_n^k(Y; \mathbf{r}, \mathbf{b})$$

where

$$\sigma_n^k(Y; \mathbf{r}, \mathbf{b}) = \begin{cases} (-1)^k \left(2^k - 2 \right) F_k(Y; \mathbf{r}, \mathbf{b}) & \text{for } n = 0 \\ (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k F_k(Y; \mathbf{r}, \mathbf{b}) & \text{for } n \geq 1 \end{cases}$$

and $F_k(Y; \mathbf{x}_{01}, \mathbf{b}) = C_k(\mathbf{x}_{01}, \mathbf{b}) (\text{Im } G_{\text{BFKL}}(Y; \mathbf{x}_{01}, \mathbf{b}))^k$

- The I.C. from the GGM formula

$$\begin{aligned}
 N^{(0)}(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) &= 1 - \exp \left(-\frac{1}{4} x_{01}^2 Q_s^2(Y_A, \mathbf{b}) \right) \\
 &= \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{1}{k!} \left(\frac{x_{01}^2 Q_s^2(Y_A, \mathbf{b})}{4} \right)^k}_{F_k}
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{n,01}^{\text{AGK}}(Y = Y_A) &= \sum_{k=n}^{\infty} (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k \frac{1}{k!} \left(\frac{x_{01}^2 Q_{s0}^2}{4} \right)^k \\
 &= \frac{\left(\frac{1}{2} x_{01}^2 Q_{s0}^2 \right)^n}{n!} \exp \left\{ -\frac{1}{2} x_{01}^2 Q_{s0}^2 \right\}
 \end{aligned}$$

- For $\xi = \ln(x_{01}^2 Q_{s0}^2)$

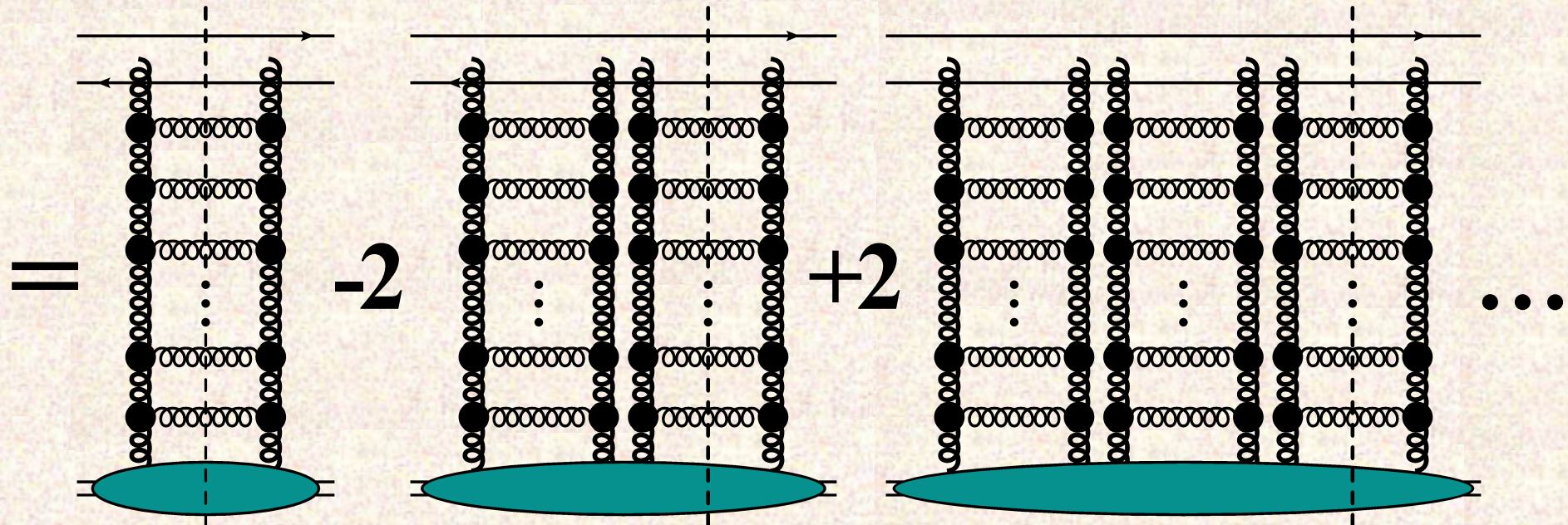
$$\sigma_n^{\text{AGK}}(Y = Y_A; \mathbf{x}_{01}, \mathbf{b}) = \frac{\left(\frac{1}{2} e^{\xi} \right)^n}{n!} \exp \left\{ -\frac{1}{2} e^{\xi} \right\}$$

The AGK cutting rules

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- General case $n=1$

$$\begin{aligned}\sigma_1^{\text{AGK}} &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k!}{(k-1)! 1!} 2^k C_k (\text{Im } G_{\text{BFKL}})^k \\ &= C_1 \sigma_{in}^{\text{BFKL}} + C_2 2 (2 \text{Im } G_{\text{BFKL}}) \sigma_{in}^{\text{BFKL}} \\ &\quad + C_3 3 (2 \text{Im } G_{\text{BFKL}})^2 \sigma_{in}^{\text{BFKL}} + \dots\end{aligned}$$



- To find the multiplicity distribution of particles in the final states one needs to convolute σ_k with the distribution of particles inside k-cut Pomerons

$$\sigma_n^{f.s.}(Y; \mathbf{x}_{01}, \mathbf{b}) = \sum_{k=1}^{\infty} \underbrace{\sigma_k^{\text{AGK}}(Y; \mathbf{x}_{01}, \mathbf{b})}_{\propto (\sigma_{in}^{\text{BFKL}})^k} \underbrace{\mathcal{P}_n^{\text{IP}}(k \Delta_{\text{BFKL}} Y)}_{\text{Poisson distribution}}$$

$$\xrightarrow{Y \gg 1} \sigma_{k=n/(\Delta_{\text{BFKL}} Y)}^{\text{AGK}}(Y, \mathbf{x}_{01}, \mathbf{b})$$

The AGK evolution equations

- The AGK cutting rules can be replaced by the evolution equations for σ_n . From the s-channel unitarity:

$$2 N(Y; \mathbf{x}_{01}, \mathbf{b}) = \sigma_{SD}(Y; \mathbf{x}_{01}, \mathbf{b}) + \sigma_{in}(Y; \mathbf{x}_{01}, \mathbf{b})$$

- We define a new quantity – the particle production S-matrix “ S_{in} ”: it includes σ_n along with the no-interaction term (=1) and all the interaction terms (N and σ_{sd}) on either side (or both sides) of the final state cut:

$$S_{in} = 1 - 2 N + \sigma_{SD} + \sum_{n=1}^{\infty} \sigma_n$$

- From the final state cancellations, the evolution equation for S_{in} is of BK type too

$$\underbrace{\frac{\partial}{\partial Y} S_{in}(Y; \mathbf{x}_{01}, \mathbf{b})}_{\text{select fixed multiplicity terms}} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \underbrace{[S_{in}(Y; \mathbf{x}_{12}, \mathbf{b}) S_{in}(Y; \mathbf{x}_{02}, \mathbf{b}) - S_{in}(Y; \mathbf{x}_{01}, \mathbf{b})]}_{\text{select fixed multiplicity terms}}$$

select fixed multiplicity terms

select fixed multiplicity terms

The AGK evolution equations

- Replacing $S_{in}=1-2N+\sigma_{SD}+\Sigma_n\sigma_n$ in the equation for the S_{in} -matrix, we obtain the evolution equation for σ_n :

$$\begin{aligned} \frac{\partial}{\partial Y} \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \\ &\times [\sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) - \sigma_n (Y; \mathbf{x}_{01}, \mathbf{b}) \\ &+ \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{02}, \mathbf{b}) + \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_{SD} (Y; \mathbf{x}_{12}, \mathbf{b}) \\ &+ \sum_{k=1}^{n-1} \sigma_{n-k} (Y; \mathbf{x}_{02}, \mathbf{b}) \sigma_k (Y; \mathbf{x}_{12}, \mathbf{b}) \\ &- 2 \sigma_n (Y; \mathbf{x}_{12}, \mathbf{b}) N (Y; \mathbf{x}_{02}, \mathbf{b}) - 2 \sigma_n (Y; \mathbf{x}_{02}, \mathbf{b}) N (Y; \mathbf{x}_{12}, \mathbf{b})] \end{aligned}$$

Evolution of σ_n at large N_c

$$\begin{aligned}
 \frac{\delta}{\delta Y} \sim & \left[\text{Diagram 1} \right] = \left[\text{Diagram 2} \right] + \left[\text{Diagram 3} \right] \\
 - & \left[\text{Diagram 4} \right] + \left[\text{Diagram 5} \right] + \left[\text{Diagram 6} \right] \\
 + \sum_{k=1}^{n-1} & \left[\text{Diagram 7} \right] - 2 \left[\text{Diagram 8} \right] - 2 \left[\text{Diagram 9} \right]
 \end{aligned}$$

The diagrams are represented as follows:

- Diagram 1:** A rectangle with a large blue circle labeled σ_n inside. The top and bottom edges have three vertical tick marks each. The top edge is labeled '0' and the bottom edge is labeled '1'.
- Diagram 2:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n with '0' above and '2' below it. Below the line is a blue circle labeled σ_n with '1' below it.
- Diagram 3:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n with '0' above and '2' below it. Below the line is a blue circle labeled σ_n with '1' below it.
- Diagram 4:** A rectangle with a large blue circle labeled σ_n inside. The top and bottom edges have three vertical tick marks each. The top edge is labeled '0' and the bottom edge is labeled '2'.
- Diagram 5:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_{SD} with '0' above and '2' below it. Below the line is a blue circle labeled σ_n with '1' below it.
- Diagram 6:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n with '0' above and '2' below it. Below the line is a blue circle labeled σ_{SD} with '1' below it.
- Diagram 7:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_{n-k} with '0' above and '2' below it. Below the line is a red circle labeled σ_k with '1' below it.
- Diagram 8:** A rectangle with a horizontal line. Above the line is a blue circle labeled N with '0' above and '2' below it. Below the line is a blue circle labeled σ_n with '1' below it.
- Diagram 9:** A rectangle with a horizontal line. Above the line is a blue circle labeled σ_n with '0' above and '2' below it. Below the line is a blue circle labeled N with '1' below it.

- We can see the equation is linear but with complicated kernel ($K_{\text{BFKL}} + 2 - 4 N$) and non-homogeneous

Evolution of σ_n at large N_c

- **Noticing that**

$$\begin{aligned}
 &= \sigma_n^{02}(Y) (1 + \sigma_{SD}^{12}(Y) - 2 N_{12}(Y)) = \sigma_n^{02}(Y) S_{12}^D(Y, Y_0 = 2) \\
 &= \sigma_n^{12}(Y) (1 + \sigma_{SD}^{02}(Y) - 2 N_{02}(Y)) = \sigma_n^{12}(Y) S_{02}^D(Y, Y_0 = 2)
 \end{aligned}$$

- **we reduce to**

$$\begin{aligned}
 \frac{\partial}{\partial Y} \sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{12}^2 x_{02}^2} \\
 &\times \left[-\sigma_n(Y, \mathbf{x}_{01}, \mathbf{b}) + \sigma_n(Y, \mathbf{x}_{12}, \mathbf{b}) S^D(Y, Y_0 = 2, \mathbf{x}_{02}, \mathbf{b}) \right] \\
 &+ S^D(Y, Y_0 = 2, \mathbf{x}_{12}, \mathbf{b}) \sigma_n(Y, \mathbf{x}_{02}, \mathbf{b}) + \sum_{k=1}^{n-1} \sigma_{n-k}(Y, \mathbf{x}_{02}, \mathbf{b}) \sigma_k(Y, \mathbf{x}_{12}, \mathbf{b})
 \end{aligned}$$

- **For model LT kernel and GS scaling we have**

$$\begin{aligned}
 \kappa \frac{d\sigma_n(z)}{dz} &= - \underbrace{\left(z - \int dz' S^D(z', z_0 = 2) \right)}_{=T(z)} \sigma_n(z) \\
 &+ S^D(z, z_0 = 2) \int_{\xi_0^A}^z dz' \sigma_n(z') + \sum_{k=1}^{n-1} \int_{\xi_0^A}^z dz' \sigma_{n-k}(z') \sigma_k(z)
 \end{aligned}$$

Evolution of σ_n at large N_c

- Defining $\Sigma_n(z) = \int_z^{\xi_0^A} dz' \sigma_n(z')$, we can rewrite it as

$$\kappa \frac{d^2 \Sigma_n(z)}{dz^2} = - \frac{d}{dz} (T(z) \Sigma_n(z)) + \Sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z)$$

Rewriting the sum as $\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z) = \frac{1}{2} \frac{d}{dz} \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z)$

and introducing $\mathcal{S}_n(z) = \int_z^{\xi_0^A} dz' \Sigma_n(z')$, we have

$$\kappa \frac{d^2 \Sigma_n(z)}{dz^2} = - \frac{d}{dz} (T(z) \Sigma_n(z)) + \frac{d \mathcal{S}_n(z)}{dz} + \frac{1}{2} \frac{d}{dz} \left[\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z) \right]$$

The large $n \approx \langle n(z) \rangle$ Solution

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- **Rewriting the sum as**

$$\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z) = \frac{1}{2} \frac{d}{dz} \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z)$$

- **and introducing $\mathcal{S}_n(z) = \int_{\xi_0^A}^z dz' \Sigma_n(z')$, we have**

$$\begin{aligned} \kappa \frac{d^2 \Sigma_n(z)}{dz^2} = & - \frac{d}{dz} (T(z) \Sigma_n(z)) + \frac{d \mathcal{S}_n(z)}{dz} \\ & + \frac{1}{2} \frac{d}{dz} \left[\sum_{k=1}^{n-1} \Sigma_{n-k}(z) \Sigma_k(z) \right] \end{aligned}$$

The large $n \geq \langle n(z) \rangle$ Solution

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- We suggest that the solution has the following form

$$\Sigma_n(z) = \phi(z) \exp(-n \Phi(z))$$

- Functions $\Phi(z)$ and $\phi(z)$ we will find from the equation.
Replacing and taking the large z and n limit we find

$$\kappa \frac{d\Sigma_n}{dz} = -(z - \Sigma_\Delta(z)) \Sigma_n(z) + \mathcal{S}_n(z) + \frac{n-1}{2} \phi(z) \Sigma_n(z)$$

- We can now solve the equation and find for DIS:

$$\sigma_n(z) = \left(\phi'(z) - n \phi(z) \Phi'(z) \right) \exp(-n \Phi(z))$$

KNO structure

$$\xrightarrow{n \geq \langle n(z) \rangle} \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\xi = \frac{n}{\langle n(z) \rangle} \right)$$

but without KNO scaling

where $\Psi(\xi) = \xi e^{-\xi}$ and $\langle n(z) \rangle = 2 n_0 z e^{\frac{(z-z_\Delta)^2}{2\kappa}}$

Homotopy solution

- For σ_1 in region I (GS) (here $S^D = \Delta$)

$$\begin{aligned}
 \kappa \frac{d\sigma_1(z)}{dz} &= -z \sigma_1(z) + \sigma_1(z) \int_{\xi_0^A}^z dz' \Delta(z') + \Delta(z) \int_{\xi_0^A}^z dz' \sigma_1(z') \\
 &= - \left(z - \underbrace{\int_{\xi_0^A}^z dz' \Delta(z')}_{T(z)} \right) \sigma_1(z) + \Delta(z) \underbrace{\int_{\xi_0^A}^z dz' \sigma_1(z')}_{\int_{\xi_0^A}^{\infty} dz' \sigma_1(z') - \underbrace{\int_z^{\infty} dz' \sigma_1(z')}_{=\tilde{\Sigma}_1(z)}} \\
 &= -T(z) \sigma_1(z) + \sigma_{0,1} \Delta(z) - \Delta(z) \tilde{\Sigma}_1(z) \\
 \Rightarrow \underbrace{\kappa \frac{d\sigma_1(z)}{dz} + T(z) \sigma_1(z) - \sigma_{0,1} \Delta(z)}_{\mathcal{L}[\sigma_1]} + \underbrace{\Delta(z) \tilde{\Sigma}_1(z)}_{\mathcal{N}_{\mathcal{L}}[\sigma_1]} &= 0
 \end{aligned}$$

Homotopy solution

- We have divided the equation in two parts

$$\mathcal{L}[\sigma_1] = \kappa \frac{d\sigma_1(z)}{dz} + T(z) \sigma_1(z) - \sigma_{0,1} \Delta(z)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] = \Delta(z) \tilde{\Sigma}_1(z)$$

- As a solution, we introduce

$$\sigma_1 = \sigma_1^{(0)} + p \sigma_1^{(1)} + p^2 \sigma_1^{(2)} + \dots$$

$$\text{such that } \mathcal{L}[\sigma_1^{(p)}] + p \mathcal{N}_{\mathcal{L}}[\sigma_1^{(p)}] = 0$$

- The 0th iteration is

$$\mathcal{L}[\sigma_1^{(0)}]; \quad \kappa \frac{d\sigma_1^{(0)}(z)}{dz} = -T(z) \sigma_1^{(0)}(z) + \sigma_{0,1} \Delta(z)$$

- The solution is a sum of the solution to the homogeneous part of the equation and the particular solution of the non-homogeneous one. We obtain

$$\sigma_1^{(0,I)}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \left\{ \underbrace{\frac{\sigma_{0,1}}{\kappa} \int_{\xi_0^A}^z dz' \Delta(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)}_{\tilde{\sigma}_1^{(0,I)}} + C_\phi^{(1)} \right\}$$

where $C_\phi^{(1)} = \frac{1}{2} e^{\xi_0^A} \exp\left(-\frac{1}{2} e^{\xi_0^A}\right)$

- For the next homotopy iteration we need to account for the linear terms in p : $\mathcal{L}[\sigma_1^{(0)} + p \sigma_1^{(1)}] + p \mathcal{N}_{\mathcal{L}}[\sigma_1^{(0)}] = 0$
- The equation takes the form

$$\kappa \frac{d\sigma_1^{(1,I)}(z)}{dz} = -T(z) \sigma_1^{(1,I)}(z) - \Delta(z) \tilde{\Sigma}_1^{(0)}(z)$$

where $\tilde{\Sigma}_1^{(0)}(z) = \int_z^\infty dz' \sigma_1^{(0)}(z')$. The general solution is

$$\sigma_1^{(1,I)}(z) = -\frac{1}{\kappa} \exp \left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z') \right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(0)}(z') \exp \left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'') \right)$$

- In general, the equation for the p-iteration ($p \geq 1$) in region I takes the form

$$\kappa \frac{d\sigma_1^{(p,I)}(z)}{dz} = -T(z) \sigma_1^{(p,I)}(z) - \Delta(z) \tilde{\Sigma}_1^{(p-1)}(z)$$

with solution

$$\sigma_1^{(p,I)}(z) = -\frac{1}{\kappa} \exp \left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z') \right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(p-1)}(z') \exp \left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'') \right)$$

Homotopy solution for σ_n

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- Now we need to add to our definition of $\mathcal{L}[\sigma_1]$ the non-homogeneous term

$$\begin{aligned}\mathcal{L}[\sigma_n] &= \kappa \frac{d\sigma_n(z)}{dz} + T(z) \sigma_n(z) - \underbrace{\sigma_{0,n} \Delta(z) - \sum_{k=1}^{n-1} \Sigma_{n-k}(z) \sigma_k(z)}_{-U_n(z)} \\ \mathcal{N}_{\mathcal{L}}[\sigma_n] &= \Delta(z) \tilde{\Sigma}_n(z)\end{aligned}$$

- We solve the equation in a similar way to σ_1 , yielding

$$\sigma_n^{(0,I)}(z) = \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(n)} \right\}$$

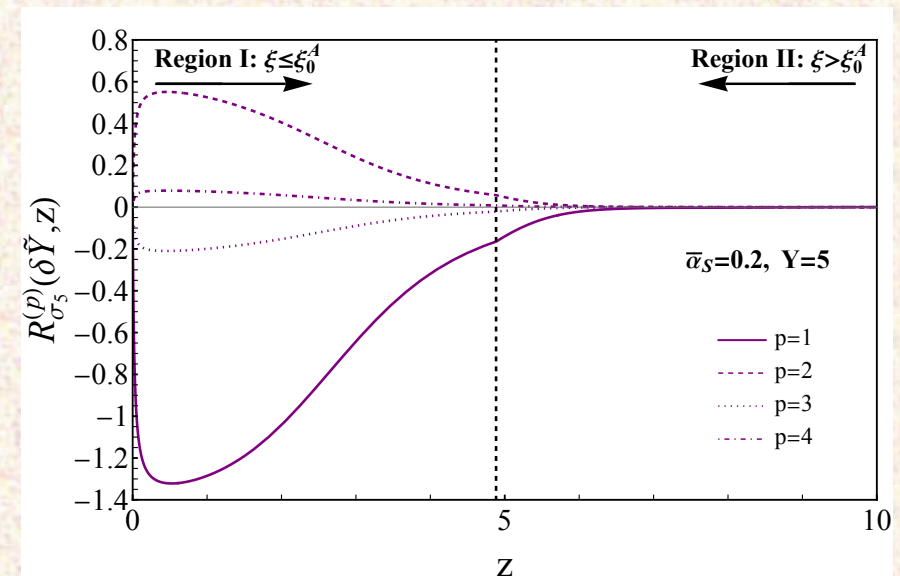
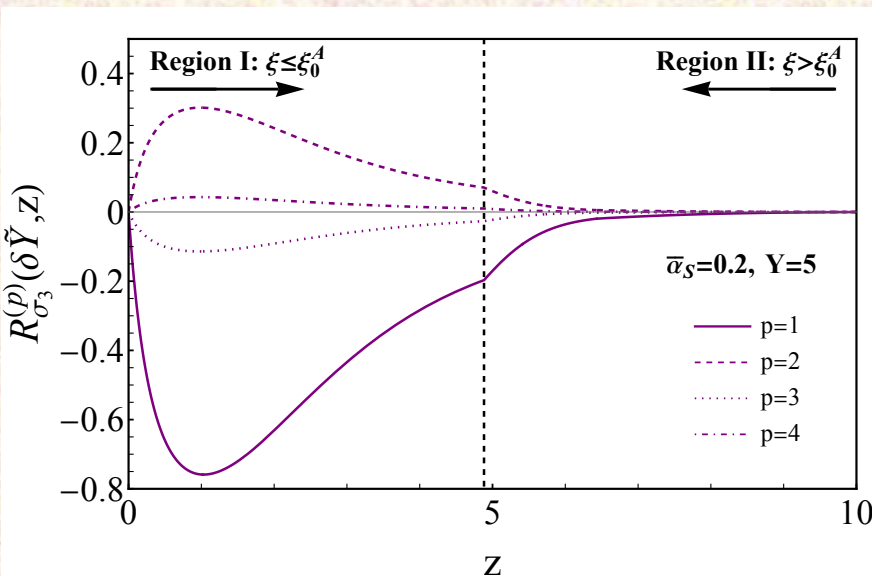
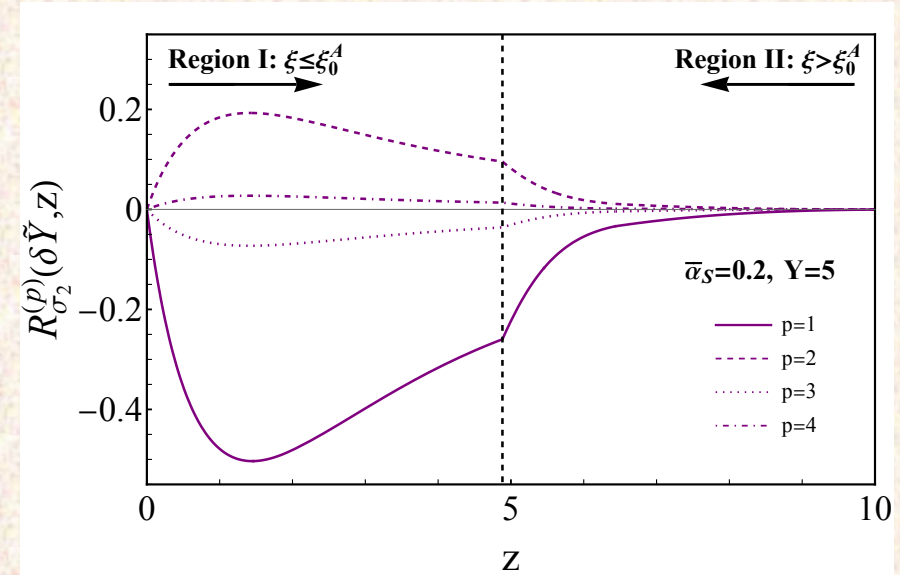
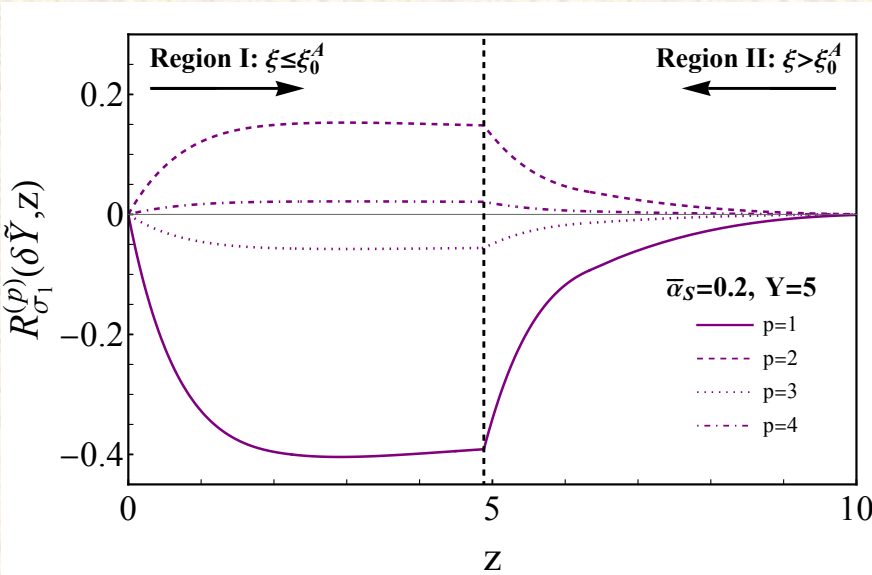
where $C_\phi^{(n)} = \frac{\left(\frac{1}{2}e^{\xi_0^A}\right)^n}{n!} e^{-\frac{1}{2}e^{\xi_0^A}}$. The formula for the p-iteration is

$$\sigma_n^{(p,I)}(z) = \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \tilde{\Sigma}_n^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(n)} \right\}$$

Numerical estimates

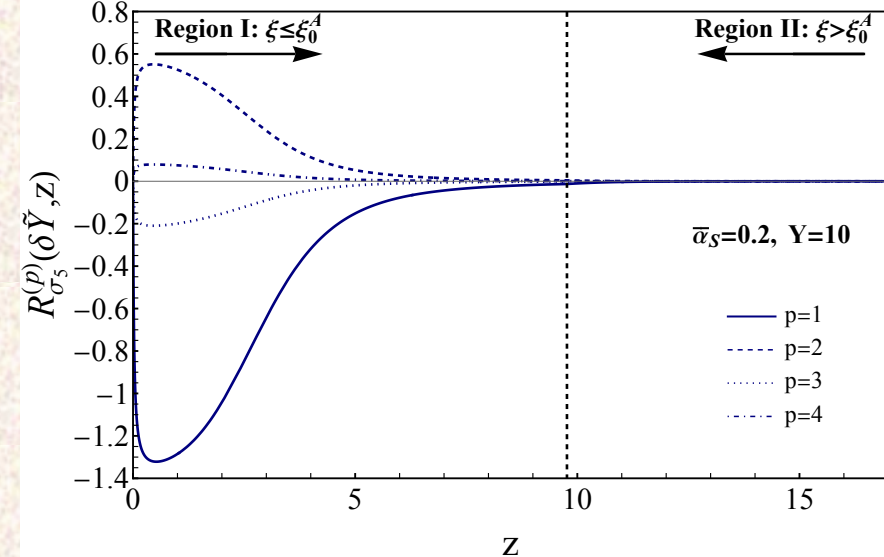
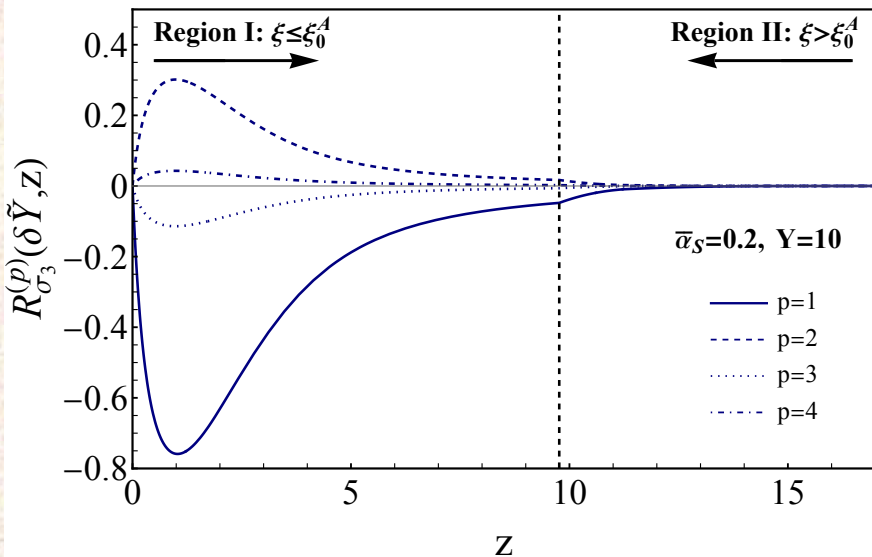
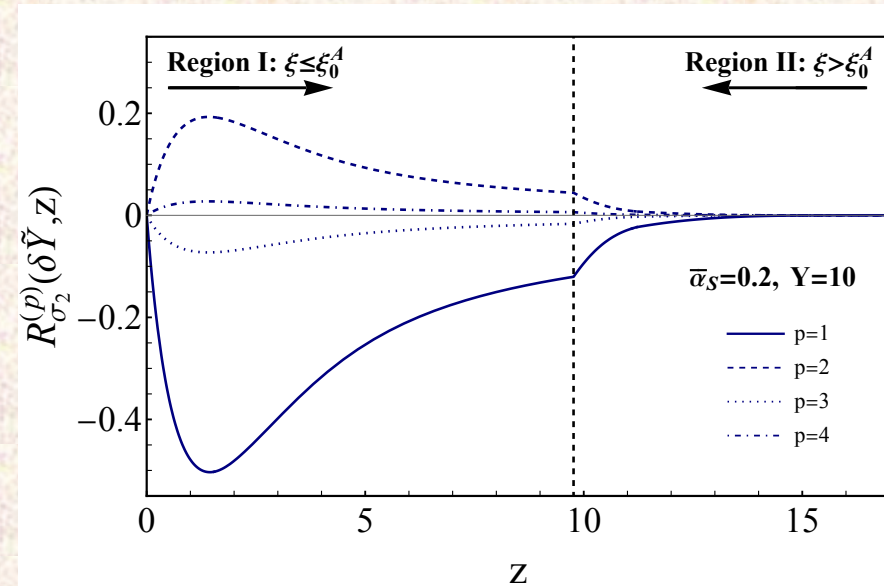
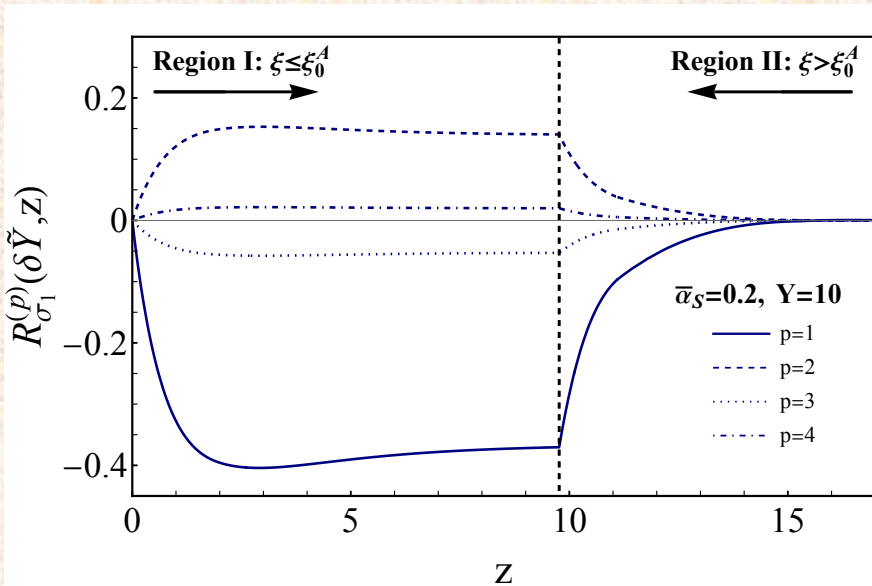
40

- Ratios of $\sigma_1, \sigma_2, \sigma_3$ and σ_5 for $\xi_0^A=0, \alpha_S=0.2, z_0=2$.



Numerical estimates

- Ratios of $\sigma_1, \sigma_2, \sigma_3$ and σ_5 for $\xi_0^A=0, \alpha_S=0.2, z_0=2$.



Numerical estimates

- One can see that the first iteration turns out to be rather large in the small- z region which corresponds to region I.
- Despite the fact that after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n . For this reason, we use this method only for small values of n .
- We solve the equation in the large- n limit, and study the possible matching with this solution.

The background of the slide features a stylized atomic model. A central nucleus, composed of yellow and orange spheres, is surrounded by a blue, translucent electron cloud. Several small green spheres represent electrons, positioned along thin, glowing yellow lines that indicate their orbits. The entire scene is set against a dark blue background with a subtle pattern of lighter blue and yellow bokeh-like spots, suggesting a cosmic or scientific theme.

Applications at EIC era

Applications at EIC era

- Having these cross sections, we can calculate the multiplicity moments using Mueller's formula

$$\langle n^k(z) \rangle = \sigma_k^{\text{AGK}}(z) \langle n_P(z) \rangle^k$$

where $\langle n_P(z) \rangle = \bar{\gamma} z$

- And therefore, all cumulants can be calculated:

$$C_1(z) = \langle n(z) \rangle \text{ Mean}$$

$$C_2(z) = \langle n^2(z) \rangle - \langle n(z) \rangle^2 \text{ Variance}$$

$$C_3(z) = \langle n^3(z) \rangle - 3 \langle n^2(z) \rangle \langle n(z) \rangle \text{ Third central moment of our distributions}$$

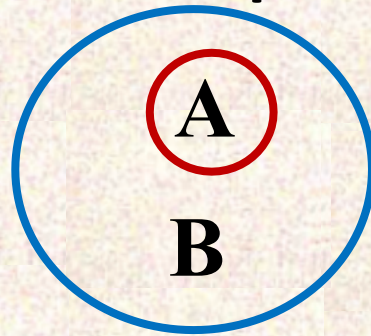
- Finally we can obtain the multiplicity distribution of the produced gluons in the saturation region as follows

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{C_n}{n!} (\lambda - 1)^n ; P_n(z) = \oint \frac{d\lambda}{2\pi i} \frac{e^{f(\lambda)}}{\lambda^{n+1}}$$

Entropy of produced gluons

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- DIS probes only a part of the proton's wave function (region A).



$$r^2 \sim 1/Q^2$$

- The probability to have n-cut BFKL Pomerons in the final state is equal to

$$P_n^{\text{AGK}}(z) \equiv \frac{\sigma_n^{\text{AGK}}(z)}{\sum_{n=1}^{\infty} \sigma_n^{\text{AGK}}(z)} = \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\frac{n}{\langle n(z) \rangle} \right)$$

- The probability to have n-gluons in the final state is

$$\begin{aligned} P_n^{\text{gluons}}(Y; \mathbf{x}_{01}, \mathbf{b}) &= \sum_{k=1}^{\infty} P_k^{\text{AGK}}(Y; \mathbf{x}_{01}, \mathbf{b}) P_n^{\text{Poisson}}(k \Delta_{\text{BFKL}} Y) \\ &= \frac{2}{\kappa} \frac{z^2}{\langle n(z) \rangle} \Psi \left(\frac{\frac{n}{\bar{\gamma} z}}{\langle n(z) \rangle} \right) \end{aligned}$$

Entropy of produced gluons

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- The new observable of DIS — the entanglement entropy — is given by the von Neumann's formula

$$\begin{aligned} S_E(z) &= - \sum_n P_n(z) \ln(P_n(z)) \\ &= \ln \langle n(z) \rangle \end{aligned}$$

$$\xrightarrow{z \gg 1} \frac{z^2}{2\kappa}$$

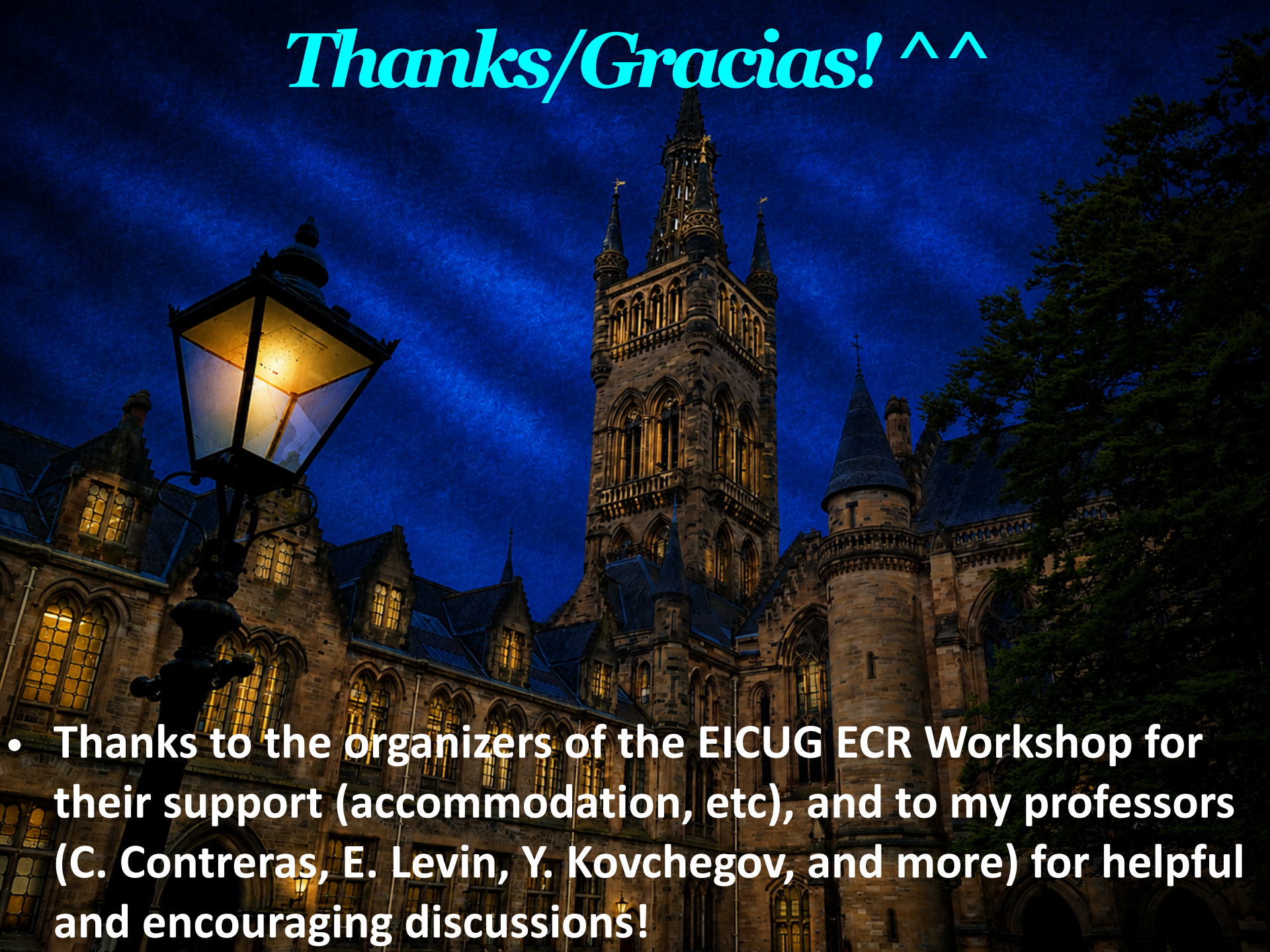
In agreement with
Kharzeev and Levin
(2017) predictions.

- A relation between the S_{hadron} and the initial-state parton entropy, S_{gluon} , due to “parton liberation” and “local parton-hadron duality (LPHD)”, is described by:

$$S_{\text{hadron}} \approx S_{\text{gluons}}$$

Conclusions and outlook

- We calculated the entropy and the multiplicity distributions directly from QCD evolution equations and the s-channel unitarity constraints. These results are in agreement with the predictions of Kharzeev and Levin.
- σ_1 , σ_2 and σ_3 are cross sections for 1, 2 and 3 cut Pomeron production which are closely related to the moments and cumulants of the multiplicity distributions for DIS experiment. Calculations of the moments $\langle n^k \rangle$ for small k will be subject of next papers.



Thanks/Gracias! ^^

- Thanks to the organizers of the EICUG ECR Workshop for their support (accommodation, etc), and to my professors (C. Contreras, E. Levin, Y. Kovchegov, and more) for helpful and encouraging discussions!